Brief project report

Project: P 23664-N13

The d-bar Neumann problem

The $\overline{\partial}$-Neumann operator associated to a bounded pseudoconvex domain is a central object in complex analysis. The corresponding operator on weighted spaces $L^2(\mathbb{C}^n, e^{-\varphi})$ opens interesting connections to Schrödinger operators with magnetic field and to the Witten Laplacian.

a) The $\overline{\partial}$-Neumann operator of a bounded pseudoconvex domain

Let $\Omega \subset \mathbb{C}^n$ be a domain. The $\overline{\partial}$-complex is the complex

$$0 \longrightarrow L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(p,1)}(\Omega) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} L^2_{(p,n)}(\Omega) \longrightarrow 0,$$

where $L^2_{(p,q)}(\Omega)$ is the space of (p,q)-forms on Ω with L^2-coefficients. The $\overline{\partial}$ operator considered here is understood in the sense of distributions, defined on the L^2-space of forms with square integrable coefficients. As a densely defined closed operator, $\overline{\partial}$, at each form level, has a Hilbert space adjoint $\overline{\partial}^*$.
The complex Laplacian $\Box_q = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, with domain

$$\mathrm{dom}(\Box_q) = \{u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*) : \overline{\partial}u \in \mathrm{dom}(\overline{\partial}^*)\,,\, \overline{\partial}^*u \in \mathrm{dom}(\overline{\partial})\}$$

is an unbounded self-adjoint operator. If Ω is a bounded pseudoconvex domain, the complex Laplacian $\Box_q$ has a bounded inverse, the $\overline{\partial}$-Neumann operator $N_q : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q)}(\Omega)$. More precisely, on a bounded pseudoconvex domain, one has the basic estimate

$$\|u\|^2 \le C(\|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2), \tag{1}$$

for each $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$, where $C > 0$ is a constant, and (1) implies that $\Box_q$ has a bounded inverse. By the way, the converse is also true: if $\Box_q$ has a bounded inverse then the basic estimate (1) holds.
In his paper "Sobolev inequalities and the $\overline{\partial}$-Neumann operator", arXiv, 1409.2732, J. of Geom. Analysis 26 (2016), 287-293, the project leader points out, that if one has a slightly better estimate than (1), namely

$$\|u\|_{L^r} \le C(\|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2)^{1/2}, \tag{2}$$

for each $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$, where $C > 0$ is a constant and $r > 2$, then the $\overline{\partial}$-Neumann operator $N_q : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q)}(\Omega)$ is compact.
Now suppose that $0 < \epsilon \le 1/2$ and that

$$\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*) \subseteq W^{\epsilon}_{(0,q)}(\Omega),$$

and that there exists a constant $C > 0$ such that

$$\|u\|_{\epsilon,\Omega} \le C(\|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2)^{1/2}, \tag{3}$$

for all $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$, where $W^\epsilon_{(0,q)}(\Omega)$ is the standard ϵ-Sobolev space of $(0,q)$-forms with the norm $\|.\|_{\epsilon,\Omega}$. This is a so-called subelliptic estimate.
Using the continuous imbedding for the space

$$W^\epsilon(\Omega) \hookrightarrow L^r(\Omega),$$

for $2 \le r \le 2n/(n-\epsilon)$, one sees that (3) implies that (2) holds for some $r > 2$. And it follows that the $\overline{\partial}$-Neumann operator N_q can be continuously extended as an operator

$$\tilde{N}_q : L^{\frac{2n}{n+\epsilon}}_{(0,q)}(\Omega) \longrightarrow L^{\frac{2n}{n-\epsilon}}_{(0,q)}(\Omega),$$

which means that there is a constant $C > 0$ such that

$$\|\tilde{N}_q u\|_{\frac{2n}{n-\epsilon}} \le C\, \|u\|_{\frac{2n}{n+\epsilon}}, \tag{4}$$

for each $u \in L^{\frac{2n}{n+\epsilon}}_{(0,q)}(\Omega)$.
The crucial step in the proof of this result is to use the fact that the $\overline{\partial}$-Neumann operator N_q can be written as

$$N_q = j_q \circ j_q^*, \tag{5}$$

where $j_q : \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*) \hookrightarrow L^2_{(0,q)}(\Omega)$, the intersection $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ is endowed with the graph norm $u \mapsto (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$ and j_q^* is the adjoint of j_q.

Franz Berger, PhD student, supported by the funds of the project, considers a more general setting in his paper "Essential spectra of tensor product Hilbert complexes, and the $\overline{\partial}$-Neumann problem on product manifolds", arXiv:1508.01749, Journal of Functional Analysis, to appear. He investigates tensor products of Hilbert complexes, in particular the (essential) spectrum of their Laplacians. It is shown that the essential spectrum of the Laplacian associated to the tensor product complex is computable in terms of the spectra of the factors. Applications are given for the $\overline{\partial}$-Neumann problem on the product of two or more Hermitian manifolds, especially regarding (non-) compactness of the associated $\overline{\partial}$-Neumann operator.

b) Weighted spaces

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a plurisubharmonic $\mathcal{C}^2$-function and let

$$L^2(\mathbb{C}^n, e^{-\varphi}) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \text{ measurable} : \int_{\mathbb{C}^n} |f|^2 e^{-\varphi}\, d\lambda < \infty\}.$$

Now we consider the weighted $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi})$$

and we set

$$\Box_{\varphi,q} = \overline{\partial}\,\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial}.$$

Suppose that the sum $s_q(z)$ of the smallest q eigenvalues of the Levi matrix

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{j,k=1}^n$$

has the property

$$\lim_{|z|\to\infty} s_q(z) = +\infty. \tag{6}$$

Then the inverse operator

$$N_{\varphi,q} : L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$$

of $\Box_{\varphi,q}$ exists and is compact.
In some cases it is possible to determine the spectrum of $\Box_{\varphi,q}$. If $1 \le q \le n$ and $\Omega = \mathbb{C}^n$ and $\varphi(z) = \sum_{j=1}^n |z_j|^2$, then the spectrum of $\Box_{\varphi,q}$ consists of all integers $\{q, q+1, q+2, \dots\}$ each of which is of infinite multiplicity. The essential spectrum is non-empty, therefore $N_{\varphi,q}$ fails to be compact. This is the main result of the paper " Spectrum of the $\overline{\partial}$-Neumann Laplacian on the Fock space", arXiv:1301.7666, J. of Math. Anal. and Appl. 402 (2013), 739-744, by the project leader.

In the paper "On some spectral properties of the weighted $\overline{\partial}$-Neumann problem", by Franz Berger and the project leader (arXiv:1509.08741), necessary conditions for compactness of the weighted $\overline{\partial}$-Neumann operator on the space $L^2(\mathbb{C}^n, e^{-\varphi})$ for a plurisubharmonic function φ are studied.
Let $A^2(\mathbb{C}^n, e^{-\varphi})$ be the Bergman space of entire functions f belonging to $L^2(\mathbb{C}^n, e^{-\varphi})$. If one applies $\Box_{\varphi,q}$ to $(0,q)$-forms with coefficients belonging to $A^2(\mathbb{C}^n, e^{-\varphi})$, one finds out that $\Box_{\varphi,q}$ restricts to a multiplication operator on this space. In this way, one gets the following result: let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic C^2 function and suppose that the corresponding weighted space $A^2(\mathbb{C}^n, e^{-\varphi})$ of entire function is infinite dimensional. If there is $1 \le q \le n$ such that the $\overline{\partial}$-Neumann operator $N_{\varphi,q}$ is compact, then

$$\limsup_{|z|\to\infty} \operatorname{tr}(M_\varphi(z)) = +\infty. \tag{7}$$

But one can also use the fact that $\Box_{\varphi,n}$ has compact resolvent on $L^2_{0,n}(\mathbb{C}^n, e^{-\varphi})$ if and only if a certain magnetic Schrödinger operator $-\Delta_A + V$ has compact resolvent on $L^2(\mathbb{R}^{2n})$. From the theory of Schrödinger operators with magnetic field one obtains now a stronger necessary condition for compactness: if there is $1 \le q \le n$ such that $\Box_{\varphi,q}$ has compact resolvent, then

$$\lim_{|z|\to\infty} \int_{B(z,1)} \operatorname{tr}(M_\varphi)^2 \, d\lambda = +\infty. \tag{8}$$

For so-called decoupled weight functions

$$\varphi(z) = \varphi_1(z_1) + \cdots + \varphi_n(z_n),$$

where all φ_j are subharmonic and such that $\Delta\varphi_j$ defines a nontrivial doubling measure, one can use the tensor product structure and a formula for the essential spectrum of $\Box_{\varphi,q}$ to show that $N_{\varphi,q}$ fails to be compact for $1 \leq q \leq n-1$, and $N_{\varphi,n}$ is compact if and only if

$$\lim_{|z|\to\infty} \int_{B_1(z)} \operatorname{tr}(M_\varphi)\, d\lambda = \infty. \tag{9}$$

Up to now it was only known that decoupled weights are an obstruction for the compactness of the $\overline{\partial}$-Neumann operator $N_{\varphi,1}$, Franz Berger's formula for the essential spectrum provided the method to settle the problem also for $N_{\varphi,q}$, where $q > 1$.

The project leader finished his monography

"The $\overline{\partial}$-Neumann problem and Schrödinger operators"

De Gruyter Expositions in Mathematics 59, 2014. This book is devoted to the spectral analysis of the complex Laplacian and to compactness of the $\overline{\partial}$-Neumann operator. It contains a detailed account of the application of the $\overline{\partial}$-methods to Schrödinger operators, Pauli and Dirac operators and to the Witten-Laplacians.

Duration of the project: 01.01.2012 – 31.08.2016

Personnel:
Bartokos Stephanie, PhD student, 01.01.2012–31.08.2012
Berger Franz, Master student, 01.10.2012–30.09.2014
Berger Franz, PhD student, 01.11.2014–31.08.2016
Preinerstorfer Tobias, Master student, 01.08.2013–31.12.2013
Ferizovic Damir, Master student, 01.05.2014–31.08.2016
Dall Ara Gian Maria, Postdoc, 01.01.2016–31.08.2016
Reiter Michael, Postdoc, 01.01.2016–30.04.2016

Personnel development

Stephanie Bartokos wrote a promising master thesis about different aspects of the Bergman kernel and started for a PhD project on the $\overline{\partial}$-Neumann problem. After some months she got an offer for a permanent position at the national library, decided to be on the safe side and accepted the offer.

Franz Berger is a highly gifted PhD student. He finished his impressing master thesis on C^*-algebra methods for certain problems in several complex variables in October 2014 and began his PhD studies some weeks later. He already obtained interesting new results on spectral analysis of the $\overline{\partial}$-Neumann operator, the first part of his thesis has already been accepted for publication in the Journal of Functional Analysis. He will soon be able to defend his thesis.

Tobias Preinerstorfer finished his master studies with an interesting master thesis on potential theoretic aspects of the $\overline{\partial}$-Neumann problem and decided then to concentrate on financial mathematics.

Damir Ferisovic wrote a promising master thesis about rather involved estimates for the Bergman kernel on weighted L^2-spaces, recently he got a PhD position at the Technical University of Graz.

Gian Maria Dall'Ara finished his PhD studies at Scuola Normale Superiore di Pisa under the supervision of Fulvio Ricci. His impressing thesis is closely related to the topic of the project, parts of it were published in Advances of Mathematics and in the Journal of Functional Analysis. His approach comes from partial differential equations in mathematical physics and will surely enhance the complex analytic aspects of the $\overline{\partial}$-Neumann problem.

Michael Reiter recently finished his PhD studies under supervision of Bernhard Lamel. He studies geometric aspects of CR manifolds and CR mappings. In May 2016 he got his own stand alone FWF grant.

Participation in conferences

Project leader: Lectures on the $\overline{\partial}$-Neumann problem at CIRM (Marseille), University of Nice (France), AMS-Meeting (San Diego), University of Toledo (USA), Sabanci University (Istanbul), AIM (Palo Alto, USA), BIRS (Oaxaca, Mexico), Jagiellonian University (Krakow), University of Ljubljana (Slovenia), DMV-ÖMG Meeting (Innsbruck), Universität Salzburg, DMV-PTM Meeting (Poznan, Poland), Kent University (Canterbury, United Kingdom), Erwin Schrödinger Institute Vienna (together with Franz Berger and Gian Maria Dall'Ara), University of Rome Tor Vergata, Tsinghua Sanya International Mathematics Forum (China) (together with Franz Berger and Gian Maria Dall'Ara), CSASC Meeting Barcelona (together with Franz Berger and Gian Maria Dall'Ara).

Organisation of Conferences

Common Seminars with the Jagiellonian Unversity , 2012 in Vienna, 2013 in Krakow, supported by WTZ-ÖAD travel grants.

Several Complex Variables and CR Geometry, Workshop Erwin Schrödinger Institute Vienna, Nov. 2015, with about 30 leading experts.

New approved FWF grant : "Spectral analysis of the $\overline{\partial}$-Neumann operator",
P28154-N35, € 327.127,50 , started April 2016.