GLOBAL GRONWALL ESTIMATES FOR INTEGRAL CURVES ON RIEMANNIAN MANIFOLDS

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Abstract. We prove several Gronwall-type estimates for the distance of integral curves of smooth vector fields on a Riemannian manifold.

1. Introduction

Suppose that X is a complete smooth vector field on $\mathbb{R}^n$, let p_0, $q_0 \in \mathbb{R}^n$ and denote by $p(t)$, $q(t)$ the integral curves of X with initial values p_0 resp. q_0. In the theory of ordinary differential equations it is a well known consequence of Gronwall's inequality that in this situation we have

$$|p(t) - q(t)| \leq |p_0 - q_0| e^{C_T t} \qquad (t \in [0, T)) \tag{1}$$

with $C_T = \|DX\|_{L^\infty(K_T)}$ (K_T some compact convex set containing the integral curves $t \mapsto p(t)$ and $t \mapsto q(t)$) and DX the Jacobian of X (cf., e.g., [1], 10.5).

The aim of this paper is to derive estimates analogous to (1) for integral curves of vector fields on Riemannian manifolds. Apart from a purely analytical interest in this generalization, we note that Gronwall-type estimates play an essential role in the convergence analysis of numerical methods for solving ordinary differential equations. They are of central importance for all methods of solving ODEs in a verified way, i.e., with full control of roundoff errors (cf. [5]). The estimates presented in the sequel may be seen as a prerequisite for the generalization of such methods to the setting of Riemannian manifolds.

When replacing $\mathbb{R}^n$ by some Riemannian manifold M, new effects occur, in particular when M is not assumed to be geodesically complete. Typically, the topological structure of M has to be taken into account and a priori it is neither clear whether an inequality of the form (1) is attainable at all in general nor which geometric quantities (e.g., curvature) will enter the estimate (cf. Figure 1).

We shall see that in fact the covariant differential of X may serve as a substitute for DX in (1) and will derive topological conditions under which analogous estimates may be achieved in the general setting.

In the remainder of this section we fix some notation and terminology. Our basic references for Riemannian geometry are [2, 3, 4]. Let M be an n-dimensional $\mathcal{C}^\infty$ manifold with Riemannian metric g. The Riemannian distance between two points p and $q \in M$, i.e., the infimum of the length of all piecewise smooth curves which

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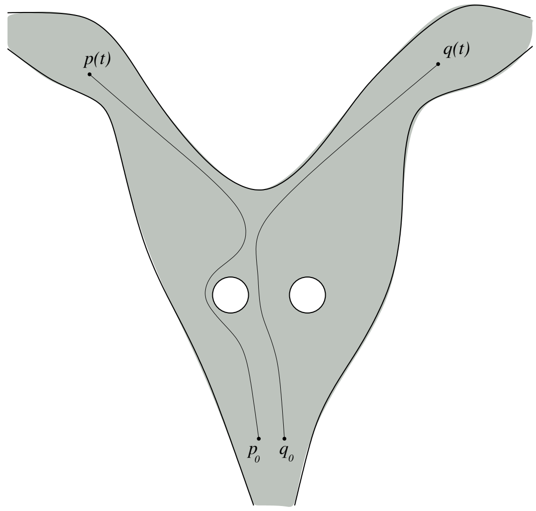

FIGURE 1.

connect p and q is denoted by $d(p,q)$. If N is a submanifold of M we write d_N for the distance function induced on N by the restriction of g to N.

The Levi-Civita connection associated to g will be denoted by $D:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the space of smooth vector fields on M. The Christoffel-symbols of a coordinate chart $x=(x^1,\dots,x^n)$ on M will be written as Γ^k_{ij}, where $D_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial x^j})=\sum_k\Gamma^k_{ij}\frac{\partial}{\partial x^k}$. Given a smooth curve $\gamma:\mathbb{R}\supseteq I\to M$, let $\mathfrak{X}(\gamma)$ be the space of smooth vector fields on γ . For any $Z\in\mathfrak{X}(\gamma)$ we let $D_{\gamma'}Z$ denote the induced covariant derivative of Z along γ. Here $\gamma':\mathbb{R}\to TM$ is the derivative of γ. If the components of Z with respect to x are Z^k $(1\le k\le n)$, then

$$D_{\gamma'}Z(t)=\sum_k\left(\frac{dZ^k}{dt}(t)+\sum_{i,j}\Gamma^k_{ij}(\gamma(t))\frac{d\gamma^i}{dt}(t)Z^j(t)\right)\frac{\partial}{\partial x^k}\bigg|_{\gamma(t)} \tag{2}$$

We use the notation $\|\,.\,\|_g$ for the norm induced by g on the fibers of TM, the tangent bundle over M. For any $X\in\mathfrak{X}(M)$ we write DX for the covariant differential of X with respect to g. The mapping norm of $DX(p):(T_pM,\|\,.\,\|_g)\to(T_pM,\|\,.\,\|_g)$ is written as $\|DX(p)\|_g$. We denote the flow of X by $\mathrm{Fl}^X:U\subseteq\mathbb{R}\times M\to M$ and write Fl^X_t for $x\mapsto\mathrm{Fl}^X(t,x)$. Thus, for any $(t,x)\in U$ we have

$$\begin{aligned}\frac{d}{dt}\mathrm{Fl}^X_t(x)&=X(\mathrm{Fl}^X_t(x))\\ \mathrm{Fl}^X_0(x)&=x\end{aligned}$$

Finally, we note that although we have chosen to formulate all results in the $\mathcal{C}^\infty$-smooth setting, in fact lower degrees of regularity suffice to establish our estimates (in particular, only $\mathcal{C}^1$-differentiability has to be assumed for vector fields).

2. ESTIMATES

In the proof of the technical backbone (Proposition 1) of our Gronwall estimates we shall make use of the following auxiliary result:

Lemma 1. *Let $U\subseteq\mathbb{R}^2$ open, $c:U\to M$, $(t,\tau)\mapsto c(t,\tau)$ smooth. Set $c_t:\tau\mapsto c(t,\tau)$, ${}_\tau c:t\mapsto c(t,\tau)$, $\dot c_t(\tau):=\frac{d}{d\tau}c(t,\tau)$, and ${}_\tau c'(t):=\frac{d}{dt}c(t,\tau)$. Then for each $(t,\tau)\in U$ we have*

$$(D_{\dot c_t}\,{}_\tau c')(\tau)=(D_{{}_\tau c'}\dot c_t)(t)\,. \tag{3}$$

Note: In (3), the left hand side denotes (for fixed t) the induced covariant derivative along the curve $\tau\mapsto c_t(\tau)$ of the vector field $\tau\mapsto {}_\tau c'(t)\in\mathfrak{X}(c_t)$, evaluated at τ, and mutatis mutandis for the right hand side.

Proof. Using (2) we calculate in any given chart:

$$\begin{aligned}
(D_{\dot c_t}\,{}_\tau c')(\tau) &= \sum_k\left(\frac{d({}_\tau c'(t))^k}{d\tau}+\sum_{i,j}\Gamma^k_{ij}(c(t,\tau))\frac{dc_t^i}{d\tau}(\tau)({}_\tau c'(t))^j\right)\frac{\partial}{\partial x^k}\bigg|_{c(t,\tau)} =\\
&= \sum_k\left(\frac{d^2c^k}{dt d\tau}(t,\tau)+\sum_{i,j}\Gamma^k_{ij}(c(t,\tau))\frac{dc^i}{d\tau}(t,\tau)\frac{dc^j}{dt}(t,\tau)\right)\frac{\partial}{\partial x^k}\bigg|_{c(t,\tau)} =\\
&= \sum_k\left(\frac{d(\dot c_t(\tau))^k}{dt}+\sum_{i,j}\Gamma^k_{ij}(c(t,\tau))\frac{d({}_\tau c^i)}{dt}(t)(\dot c_t(\tau))^j\right)\frac{\partial}{\partial x^k}\bigg|_{c(t,\tau)} =\\
&= (D_{{}_\tau c'}\dot c_t)(t),
\end{aligned}$$

by the symmetry of Γ^k_{ij} in i,j. □

Proposition 1. *Let $c_0:[a,b]\to M$ be some smooth curve joining two points p_0, $q_0\in M$, i.e., $c_0(a)=p_0$, $c_0(b)=q_0$. Let $X\in\mathfrak{X}(M)$ and denote by $c_t:[a,b]\to M$ the image of c_0 under the map Fl^X_t, i.e., $c_t(\tau)=c(t,\tau)=\mathrm{Fl}^X(t,c_0(\tau))$. Choose some $T>0$ such that Fl^X is defined on $[0,T]\times c_0([a,b])$. Then denoting by $l(t)$ the length of c_t, we have*

$$l(t)\le l(0)e^{C_T t}\qquad (t\in[0,T]) \tag{4}$$

where $C_T=\sup_{p\in K_T}||DX(p)||_g$ and $K_T=c([0,T]\times[a,b])$.

Proof. Without loss of generality we may suppose that c_0 is parametrized by arclength with $a=0$ and $b=l(0)$. By continuity we may choose $0<T'\le T$ small enough such that $g(\dot c_t(\tau),\dot c_t(\tau))>0$ for all $(t,\tau)\in[0,T']\times[0,l(0)]$. Then for $t'\in[0,T']$ we have

$$\begin{aligned}
l(t')-l(0) &= \int_0^{t'}\frac{dl}{dt}(t)\,dt=\int_0^{t'}\frac{d}{dt}\int_0^{l(0)}\sqrt{g(\dot c_t(\tau),\dot c_t(\tau))}\,d\tau\,dt\\
&= \int_0^{t'}\int_0^{l(0)}\frac{1}{2}\frac{\frac{d}{dt}g(\dot c_t(\tau),\dot c_t(\tau))}{\sqrt{g(\dot c_t(\tau),\dot c_t(\tau))}}\,d\tau\,dt=\int_0^{t'}\int_0^{l(0)}\frac{g((D_{{}_\tau c'}\dot c_t)(t),\dot c_t(\tau))}{\sqrt{g(\dot c_t(\tau),\dot c_t(\tau))}}\,d\tau\,dt\\
&= \int_0^{t'}\int_0^{l(0)}\frac{g((D_{\dot c_t}\,{}_\tau c')(\tau),\dot c_t(\tau))}{\sqrt{g(\dot c_t(\tau),\dot c_t(\tau))}}\,d\tau\,dt=\int_0^{t'}\int_0^{l(0)}\frac{g(D_{\dot c_t(\tau)}X,\dot c_t(\tau))}{\sqrt{g(\dot c_t(\tau),\dot c_t(\tau))}}\,d\tau\,dt,
\end{aligned}$$

by Lemma 1. Fix some $t\in[0,T']$. Since $\dot c_t(\tau)\neq 0$ for τ in $[0,l(0)]$ we may choose a smooth vector field Y_t, defined and nonvanishing in some neighborhood of $c_t([0,l(0)])$ such that $Y_t(c_t(\tau))=\dot c_t(\tau)$ for $\tau\in[0,l(0)]$ (cf. [4], p. 94).

Set $\hat{Y}_t := Y_t/\|Y_t\|_g$. Then $D_{\dot{c}_t(\tau)}X = (D_{Y_t}X)(c(t,\tau))$ and

$$\begin{aligned}\|(D_{Y_t}X)(c(t,\tau))\|_g &= \|Y_t(c(t,\tau))\|_g \|(D_{\hat{Y}_t}X)(c(t,\tau))\|_g \\ &= \sqrt{g(\dot{c}_t(\tau),\dot{c}_t(\tau))}\,\|(DX(\hat{Y}_t))(c(t,\tau))\|_g \\ &\leq \sqrt{g(\dot{c}_t(\tau),\dot{c}_t(\tau))}\sup_{p\in K_T}\|DX(p)\|_g\end{aligned}$$

Summing up, we conclude

$$\begin{aligned}l(t') - l(0) &= \int_0^{t'}\int_0^{l(0)} \frac{g(D_{Y_t}X(c(t,\tau)),\dot{c}_t(\tau))}{\sqrt{g(\dot{c}_t(\tau),\dot{c}_t(\tau))}}\,d\tau\,dt \\ &\leq \int_0^{t'}\int_0^{l(0)} \|D_{Y_t}X(c(t,\tau))\|_g\,d\tau\,dt \leq \sup_{p\in K_T}\|DX(p)\|_g \int_0^{t'} l(t)\,dt\end{aligned}$$

Hence by the classical Gronwall lemma we obtain $l(t) \leq l(0)e^{C_T t}$ for all $t \in [0,T']$.

To show that (4) holds on all of $[0,T]$, let T_* be the supremum of all T' such that (4) holds on $[0,T']$ and suppose that $T_* < T$. By continuity, (4) in fact holds on all of $[0,T_*]$. But then we may parametrize the curve c_{T_*} by arclength and repeat the above argument to establish the existence of some $0 < \tilde{T} \leq T - T_*$ with

$$l(T_* + t) \leq l(T_*)e^{C_T t} \leq l(0)e^{C_T(T_*+t)}$$

for $t \in [0,\tilde{T}]$. This contradicts the definition of T_*, thereby establishing our claim. □

We may utilize this proposition to prove our first main result:

Theorem 1. *Let (M,g) be a connected smooth Riemannian manifold, $X \in \mathfrak{X}(M)$ a complete vector field on M and let $p_0, q_0 \in M$. Let $p(t) = \mathrm{Fl}^X_t(p_0)$, $q(t) = \mathrm{Fl}^X_t(q_0)$ and suppose that $C := \sup_{p\in M} ||DX(p)||_g < \infty$. Then*

$$d(p(t),q(t)) \leq d(p_0,q_0)e^{Ct} \qquad (t \in [0,\infty)). \tag{5}$$

Proof. For any given $\varepsilon > 0$, choose a piecewise smooth curve $c_0 : [0,1] \to M$ connecting p_0 and q_0 such that $d(p_0,q_0) > l(0) - \varepsilon$. Using the notation of Proposition 1 it follows that

$$d(p(t),q(t)) \leq l(t) \leq l(0)e^{Ct} < (d(p_0,q_0) + \varepsilon)e^{Ct}$$

for $t \in [0,\infty)$. Since $\varepsilon > 0$ was arbitrary, the result follows. □

Example 1. (i) In general, when neither M nor X is complete, the conclusion of Theorem 1 is no longer valid:

Consider $M = \mathbb{R}^2 \setminus \{(0,y) \mid y \geq 0\}$, endowed with the standard Euclidean metric. Let $X \equiv (0,1)$, $p_0 = (-x_0,-y_0)$, and $q_0 = (x_0,-y_0)$ $(x_0 > 0,\ y_0 \geq 0)$ (cf. Figure 2). Then $p(t) = (-x_0,-y_0+t)$, $q(t) = (x_0,-y_0+t)$ and

$$d(p(t),q(t)) = \begin{cases} 2x_0 & t \leq y_0 \\ 2\sqrt{x_0^2 + (t-y_0)^2} & t > y_0 \end{cases}$$

On the other hand, $DX = 0$, so (5) is violated for $t > y_0$, i.e., as soon as the two trajectories are separated by the "gap" $\{(0,y) \mid y \geq 0\}$.

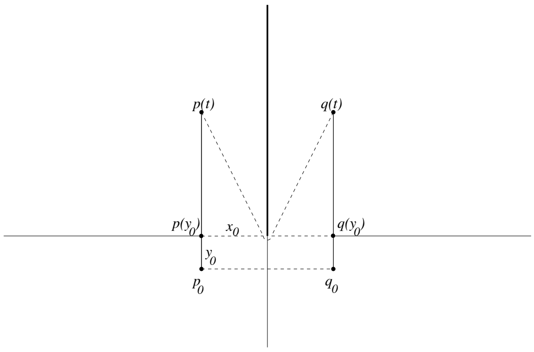

FIGURE 2.

(ii) Replace X in (i) by the complete vector field $(0, e^{-1/x^2+1})$ and set $x_0 = 1$, $y_0 = 0$. Then $C := \|DX\|_{L^\infty(\mathbb{R}^2)} = 3\sqrt{3/(2e)}$ and

$$d(p(t), q(t)) = 2\sqrt{1+t^2} \leq d(p_0, q_0)e^{Ct} = 2e^{Ct}$$

for all $t \in [0, \infty)$, in accordance with Theorem 1.

The following result provides a sufficient condition for the validity of a Gronwall estimate even if neither M nor X satisfies a completeness assumption.

Theorem 2. *Let (M, g) be a connected smooth Riemannian manifold, $X \in \mathfrak{X}(M)$ and let $p_0, q_0 \in M$. Let $p(t) = \mathrm{Fl}_t^X(p_0)$, $q(t) = \mathrm{Fl}_t^X(q_0)$ and suppose that there exists some relatively compact submanifold N of M containing p_0, q_0 such that $d(p_0, q_0) = d_N(p_0, q_0)$. Fix $T > 0$ such that Fl^X is defined on $[0, T] \times N$ and set $C_T := \sup\{||DX(p)||_g \mid p \in \mathrm{Fl}^X([0, T] \times N)\}$. Then*

$$d(p(t), q(t)) \leq d(p_0, q_0)e^{C_T t} \qquad (t \in [0, T]). \tag{6}$$

Proof. As in the proof of Theorem 1, for any given $\varepsilon > 0$ we may choose a piecewise smooth curve $c_0 : [0, 1] \to N$ from p_0 to q_0 such that $d(p_0, q_0) = d_N(p_0, q_0) > l(0) - \varepsilon$. The corresponding time evolutions c_t of c_0 then lie in $\mathrm{Fl}^X([0, T] \times N)$, so an application of Proposition 1 gives the result. □

Example 2. Clearly such a submanifold N need not exist in general. As a simple example take $M = \mathbb{R}^2 \setminus \{(0, 0)\}$, $p_0 = (-1, 0)$, $q_0 = (1, 0)$. In Example 1.(i) with $y_0 > 0$ the condition is obviously satisfied with N an open neighborhood of the straight line joining p_0, q_0 and the supremum of the maximal evolution times of such N under Fl^X is $T = y_0$, coinciding with the maximal time-interval of validity of (6). On the other hand, if there is no N as in Theorem 2 then the conclusion in general breaks down even for arbitrarily close initial points p_0, q_0: if we set $y_0 = 0$ in Example 1.(i) then no matter how small x_0 (i.e., irrespective of the initial distance of the trajectories) the estimate is not valid for any $T > 0$.

Let us single out some important special cases of Theorem 2:

Corollary 1. *Let M be a connected geodesically complete Riemannian manifold, $X \in \mathfrak{X}(M)$, and p_0, q_0, $p(t)$, $q(t)$ as above. Let S be a minimizing geodesic segment connecting p_0, q_0 and choose some $T > 0$ such that Fl^X is defined on $[0, T] \times S$.*

Then (6) holds with $C_T = \sup\{||DX(p)||_g \,|\, p \in \mathrm{Fl}^X([0,T]\times S)\}$. *In particular, if* X *is complete then for any* $T > 0$ *we have*

$$d(p(t), q(t)) \leq d(p_0, q_0) e^{C_T t} \qquad (t \in [0,T]). \tag{7}$$

Proof. Choose for N in Theorem 2 any relatively compact open neighborhood of S. The value of C_T then follows by continuity. □

Example 3. For $M = \mathbb{R}^n$ with the standard Euclidean metric, Corollary 1 reduces to our point of departure, the classical Gronwall estimate (1).

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