Communications in Applied Analysis **8** (2004), no. 1, 87–108

APPROXIMATION OF A LINEAR AGE-DEPENDENT POPULATION MODEL WITH SPATIAL DIFFUSION

Waltraud Huyer
Institut für Mathematik
Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria
waltraud.huyer@univie.ac.at, huyer@cma.univie.ac.at

Communicated by G.F. Webb

ABSTRACT: In this paper we first address the question under which conditions it is possible to obtain from Trotter-Kato approximations of two infinitesimal generators of C_0-semigroups a Trotter-Kato approximation of the semigroup generated by the sum operator. This cannot be true in general since the sum operator might not even be a generator, and we derive sufficient conditions. However, when wellposedness is already established, it may be more convenient to verify the conditions of the Trotter-Kato theorem directly, exploiting the special structure of the problem. We consider an approximation scheme for the semigroup generated by an age-dependent population model with spatial diffusion, and numerical examples are presented to demonstrate feasibility of the scheme.

AMS (MOS) Subject Classification. 47D06, 65J10, 92D25

1. INTRODUCTION

Approximation schemes for the semigroups of age-, size- and space-structured population models have been considered in several papers, where the proof of convergence uses one or the other version of the Trotter-Kato theorem in semigroup theory (see, for instance, Ackleh and Fitzpatrick [1], Banks and Kappel [2], Huyer [7], Ito et al [12] and Kappel and Zhang [14]). In Ito and Kappel [10, 11] and Ito et al [12] variational formulations of the Trotter-Kato theorem were derived. The characteristic feature of this approach is that instead of approximating the generator one rather approximates the associated sesquilinear form first and then gets the approximating generators from the approximating sesquilinear forms. In many cases the proof of consistency for a variational Trotter-Kato scheme is considerably simpler compared

Received August 30, 2001

to the one for an ordinary Trotter-Kato scheme. In Ito and Kappel [10], a special type of variational Trotter-Kato theorem for parabolic semigroups in a Hilbert space whose infinitesimal generator is associated with a V-coercive sesquilinear form was developed.

Motivated by the age-structured diffusion problem studied in Huyer [7, 8], we consider the case of two infinitesimal generators A and B on the same Banach space X, where $A - \omega I$ and $B - \rho I$ are m-dissipative for some $\omega, \rho \in \mathbb{R}$, and assume that Trotter-Kato approximations of the semigroups generated by A and B are given. We address the question whether it is possible to obtain an approximation for the semigroup generated by the sum $\mathcal{A}$ of A and B, defined on the intersection of their domains, from the approximations of the semigroups generated by A and B. This is not possible without imposing additional assumptions since $\mathcal{A}$ might not even be a generator. The question under which conditions the sum of two m-dissipative operators or at least its closure is again m-dissipative has been attracting considerable interest in the literature. Standard results that are also valid for nonlinear operators can be found in Barbu [4], for example. Da Prato and Grisvard [5] established a number of results for linear operators, and they distinguished the "parabolic" case that one of the operators is the generator of an analytic semigroup from the general "hyperbolic" case. A more recent result for the parabolic case can, for instance, be found in Monniaux and Prüss [15].

In Section 2 we state a general result on the convergence of the semigroups generated by the approximating sum operators to a semigroup on the original space. However, when wellposedness is already established, it is often most convenient to verify the Trotter-Kato theorem directly taking the special structure of the problem into account, which we do for our age-dependent diffusion problem in Section 3. We use the fact that the diffusion operator is the infinitesimal generator of an analytic semigroup associated with a bounded, coercive sesquilinear form. Finite elements are used to approximate the space structure, and the age structure is approximated by step functions. Finally, in Section 4 two numerical examples are presented to demonstrate applicability of the scheme.

2. APPROXIMATION OF THE SEMIGROUP GENERATED BY THE SUM OF TWO ω-M-DISSIPATIVE OPERATORS

We consider a version of the Trotter-Kato theorem (cf. Trotter [19] and Kato [13]) stated in Ito and Kappel [9, p. 94, Theorem 2.1]. It is not the most general version, but it is sufficient for our purposes since we have numerical approximation in mind, where the approximating spaces are usually subspaces of the original space. Let X be a Banach space and $\mathcal{A} : \operatorname{dom} \mathcal{A} \subset \mathcal{X} \to \mathcal{X}$ the infinitesimal generator of a strongly continuous semigroup $\mathcal{S}(t)$, $t \geq 0$, on X. Let X^N be subspaces of X and $\mathcal{A}^N : X^N \to X^N$ bounded linear operators with corresponding semigroups $\mathcal{S}^N(t) := \exp(\mathcal{A}^N t)$, $t \geq 0$, on X^N, let $P^N : X \to X^N$ be projections, and let $I^N : X^N \to X^N$ denote the identity mappings, $N \in \mathbb{N}$.

We consider the following assumptions:

(S) (*Stability hypothesis*). There exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that

$$\|\mathcal{S}^N(t)\| \leq Me^{\omega t}, \quad t \geq 0, \quad N \in \mathbb{N},$$

and

$$\|\mathcal{S}(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

(C) (*Consistency hypothesis*)

1. There exists a $\lambda \in \rho(\mathcal{A}) \cap \bigcap_{N=1}^{\infty} \rho(\mathcal{A}^N)$ such that

$$\lim_{N\to\infty} \|(\lambda I^N - \mathcal{A}^N)^{-1} P^N x - (\lambda I - \mathcal{A})^{-1} x\| = 0 \quad \text{for all } x \in X.$$

2. $\lim_{N\to\infty} P^N x = x \quad \text{for all } x \in X$.

(C′) (*Modified consistency hypothesis*)

1. There exists a set $D \subset \operatorname{dom} \mathcal{A}$ with $\overline{(\lambda I - \mathcal{A})D} = X$ for some $\lambda > \omega$, and for any $x \in D$ there exists $x^N \in X^N$, $N \in \mathbb{N}$, such that

$$\lim_{N\to\infty} \mathcal{A}^N x^N = \mathcal{A}x$$

and

$$\lim_{N\to\infty} x^N = x.$$

2. $\|P^N\|$ is bounded.

Hypothesis (S) is usually verified by establishing dissipativeness estimates.

Theorem 2.1. (Trotter-Kato) *The following statements are equivalent:*

1. *For any $x \in X$ and $T > 0$, $\lim_{N\to\infty} \mathcal{S}^N(t) P^N x = \mathcal{S}(t)x$ uniformly on $[0, T]$.*

2. *Hypotheses* (S) *and* (C) *are satisfied.*

3. *Hypotheses* (S) *and* (C′) *are satisfied.*

Let $A : \operatorname{dom} A \subset X \to X$ and $B : \operatorname{dom} B \subset X \to X$ be two linear, densely defined operators on X and $\omega, \rho \in \mathbb{R}$ such that $A - \omega I$ and $B - \rho I$ are m-dissipative and thus A and B generate C_0-semigroups $S(t)$, respectively $T(t)$ on X. Let X^N and P^N be as above, and let $A^N : X^N \to X^N$ and $B^N : X^N \to X^N$ be Trotter-Kato approximations of $S(t)$ and $T(t)$, respectively, i.e., assume that any (and therefore all) of the equivalent conditions of Theorem 2.1 is satisfied. Moreover, we assume that $A^N - \omega I^N$ and $B^N - \rho I^N$ are dissipative for all $N \in \mathbb{N}$.

We also consider the operator $\mathcal{A} : \operatorname{dom} \mathcal{A} \subset X \to X$, defined by

$$\operatorname{dom} \mathcal{A} = \operatorname{dom} A \cap \operatorname{dom} B, \tag{2.1}$$

$$\mathcal{A}x = Ax + Bx, \quad x \in \operatorname{dom} \mathcal{A}. \tag{2.2}$$

We ask whether $\mathcal{S}^N(t) := \exp((A^N + B^N)t)$, $t \geq 0$, $N \in \mathbb{N}$, is a Trotter-Kato approximation of the semigroup generated by $\mathcal{A}$. This cannot be true in general since $\mathcal{A}$ might not even be a generator. The first part of (S) is the only condition that is trivially fulfilled. If (C′), a) is satisfied for subsets $D_1 \subset \operatorname{dom} A$ and $D_2 \subset \operatorname{dom} B$, respectively, there is no straightforward recipe to construct a set $D \subset D_1 \cap D_2$ such that $\overline{(\lambda I - \mathcal{A})D} = X$ for a sufficiently large λ even if the same sequence (x^N) can be taken for $x \in D_1 \cap D_2$.

Next we turn to (C). The assumptions imply that we can write

$$(\lambda I^N - A^N - B^N)^{-1} = (\lambda I^N - A^N)^{-1}(I^N - B^N(\lambda I^N - A^N)^{-1})^{-1}$$

for $\lambda > \max(\omega, \omega + \rho)$ and $N \in \mathbb{N}$.

Proposition 2.2. *Let A, B, A^N, B^N, P^N and $\mathcal{S}^N(t)$, $t \geq 0$, be as above, $\lim_{N\to\infty} P^N x = x$ for all $x \in X$, and assume that for all $x \in X$ and a sufficiently large $\lambda_0 > \max(\omega, \omega + \rho)$,*

$$\lim_{N\to\infty} (I^N - B^N(\lambda_0 I^N - A^N)^{-1})^{-1} P^N x := R(\lambda_0)x \tag{2.3}$$

exists and that $\operatorname{range} R(\lambda_0)$ is dense in X. Then there exists a C_0-semigroup $\mathcal{S}(t)$, $t \geq 0$, on X such that

$$\lim_{N\to\infty} \mathcal{S}^N(t) P^N x = \mathcal{S}(t)x$$

for all $x \in X$, uniformly on bounded t-intervals.

Proof. Let $x \in X$ and $y := R(\lambda_0)x$. Then

$$\begin{aligned}
&\|(\lambda_0 I^N - A^N - B^N)^{-1} P^N x - (\lambda_0 I - A)^{-1} y\| \\
&= \|(\lambda_0 I^N - A^N)^{-1}(I^N - B^N(\lambda_0 I^N - A^N)^{-1})^{-1} P^N x - (\lambda_0 I - A)^{-1} y\| \\
&\leq \|(\lambda_0 I^N - A^N)^{-1}(I^N - B^N(\lambda_0 I^N - A^N)^{-1})^{-1} P^N x - (\lambda_0 I^N - A^N)^{-1} P^N y\| \\
&\quad + \|(\lambda_0 I^N - A^N)^{-1} P^N y - (\lambda_0 I - A)^{-1} y\| \\
&\leq \frac{1}{\lambda_0 - \omega} \|(I^N - B^N(\lambda_0 I^N - A^N)^{-1})^{-1} P^N x - P^N y\| \\
&\quad + \|(\lambda_0 I^N - A^N)^{-1} P^N y - (\lambda_0 I - A)^{-1} y\| \to 0 \quad \text{as } N \to \infty,
\end{aligned}$$

where we have used the definition of y and the assumptions, i.e., $\lim(\lambda_0 I^N - A^N - B^N)^{-1}P^N x = (\lambda_0 I - A)^{-1}R(\lambda_0)x$ for all $x \in X$. Next we prove that the range of $(\lambda_0 I - A)^{-1}R(\lambda_0)$ is dense in X. Let $x \in X$ and $\varepsilon > 0$. Since $\operatorname{dom} A$ is dense, there exists an $x_\varepsilon \in \operatorname{dom} A$ such that $\|x - x_\varepsilon\| < \varepsilon/2$. Let $y_\varepsilon := (\lambda_0 I - A)x_\varepsilon$, $x_\varepsilon = (\lambda_0 I - A)^{-1}y_\varepsilon$. Then by assumption there exists a z_ε such that $\|R(\lambda_0)z_\varepsilon - y_\varepsilon\| < \frac{1}{2}(\lambda_0 - \omega)\varepsilon$ and thus

$$\|x - (\lambda_0 I - A)^{-1}R(\lambda_0)z_\varepsilon\| \leq \|x - x_\varepsilon\| + \|(\lambda_0 I - A)^{-1}y_\varepsilon - (\lambda_0 I - A)^{-1}R(\lambda_0)z_\varepsilon\| < \varepsilon,$$

i.e., $\operatorname{range}(\lambda_0 I - A)^{-1}R(\lambda_0)$ is dense in X. By Pazy [17, p. 86, Theorem 4.3] there exists an operator $\mathcal{A}$ such that $(\lambda_0 I^N - A^N - B^N)^{-1}P^N x \to (\lambda_0 I - \mathcal{A})^{-1}x$ for all $x \in X$ and $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\mathcal{S}(t)$, $t \geq 0$, on X, and an application of the Trotter-Kato theorem finishes the proof. $\square$

Remark 2.3. Assume that we already know that $\mathcal{A}$, defined by (2.1) and (2.2), is the infinitesimal generator of a C_0-semigroup. If $\lim_{N\to\infty} P^N x = x$ and

$$\lim_{N\to\infty} (I^N - B^N(\lambda_0 I^N - A^N)^{-1})^{-1}P^N x = (I - B(\lambda_0 I - A)^{-1})^{-1}x \tag{2.4}$$

for all $x \in X$, we have convergence of the approximating semigroups. Moreover, (2.4) is satisfied if $\|(I^N - B^N(\lambda_0 I - A^N)^{-1})^{-1}\|$ is bounded and there exists a set $D \subset \operatorname{dom} B(\lambda_0 I - A)^{-1}$ (the natural domain) such that $\overline{(I - B(\lambda_0 I - A)^{-1})D} = X$ and that for each $x \in X$ there exists a sequence (x^N), $x^N \in X^N$, such that $\lim_{N\to\infty} x^N = x$ and

$$\lim_{N\to\infty} B^N(\lambda_0 I^N - A^N)^{-1}x^N = B(\lambda_0 I - A)^{-1}x. \tag{2.5}$$

However, (2.3), (2.4) and (2.5) are in general not so easily to be verified, so in some cases it might be better to go back to the original version of the Trotter-Kato theorem.

3. APPROXIMATION OF AN AGE-STRUCTURED POPULATION MODEL WITH SPATIAL DIFFUSION

In this section we consider the semigroup generated by a linear age-dependent population model with random diffusion in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2$ or 3, in the Hilbert space $H := L^2((0,\infty)\times\Omega;\mathbb{C})$. Let $u(t,a,x)$ be the population density at time $t \geq 0$, where $a \geq 0$ denotes age and $x = (x_1, \ldots, x_n) \in \Omega$ is the spatial variable. The evolution of the population is described by the partial differential equation

$$\begin{aligned}&\frac{\partial}{\partial t}u(t,a,x) + \frac{\partial}{\partial a}u(t,a,x) + \mu(a,x)u(t,a,x)\\ &\qquad - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}u(t,a,x)\Big) = 0, \quad t > 0, \quad a > 0, \quad x \in \Omega, \end{aligned}\tag{3.1}$$

with the initial condition

$$u(0,a,x)=\varphi(a,x), \quad a\ge 0, \quad x\in\Omega, \tag{3.2}$$

and the birth law

$$u(t,0,x)=\int_0^\infty \beta(a,x)u(t,a,x)\,da, \quad t>0, \quad x\in\Omega. \tag{3.3}$$

The homogeneous Neumann boundary condition

$$\sum_{i=1}^n k_i(a,x)\nu_i(x)\frac{\partial}{\partial x_i}u(t,a,x)=0, \quad t>0, \quad a>0, \quad x\in\partial\Omega, \tag{3.4}$$

where $(\nu_1(x),\dots,\nu_n(x))$ denotes the outward unit normal at $\partial\Omega$, means that the population does not leave the region Ω. Here $\mu(a,x)$ is the death rate, $\beta(a,x)$ is the birth rate and $k_i(a,x)$ are the diffusion coefficients. The maximum age is assumed to be ∞.

Let the population operator $A:\operatorname{dom}A\subset H\to H$ be defined by

$$\begin{aligned}\operatorname{dom}A=\{\varphi\in H \mid \varphi(\cdot,x) &\text{ is locally absolutely continuous on } [0,\infty) \text{ and}\\ \varphi(0,x)&=\textstyle\int_0^\infty\beta(a,x)\varphi(a,x)\,da \quad \text{for almost all} \quad x\in\Omega,\ \tfrac{\partial}{\partial a}\varphi+\mu\varphi\in H\},\\ (A\varphi)(a,x)&=-\frac{\partial}{\partial a}\varphi(a,x)-\mu(a,x)\varphi(a,x), \quad \varphi\in\operatorname{dom}A,\end{aligned}$$

and the diffusion operator $B:\operatorname{dom}B\subset H\to H$ be defined by

$$\begin{aligned}\operatorname{dom}B=\{\varphi\in H \mid \tfrac{\partial}{\partial x_i}\varphi,\ \tfrac{\partial^2}{\partial x_i^2}\varphi\in H &\text{ for } i=1,\dots,n \text{ and } \textstyle\sum_{i=1}^n k_i(a,x)\nu_i(x)\tfrac{\partial}{\partial x_i}\varphi(a,x)=0\\ &\text{on } \partial\Omega \text{ for almost all } a\ge 0\},\\ (B\varphi)(a,x)=\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(k_i(a,x)&\frac{\partial}{\partial x_i}\varphi(a,x)\Big), \quad \varphi\in\operatorname{dom}B.\end{aligned}$$

We make the following assumptions about the domain Ω, the diffusion coefficients k_i, the mortality rate μ and the birth rate β:

$$\Omega \text{ is a bounded domain in } \mathbb{R}^n \ (n = 1,\ 2 \text{ or } 3) \text{ of class } C^2. \tag{3.5}$$

$$k_i(a,\cdot) \in C^1(\overline{\Omega};\mathbb{R}) \text{ for almost all } a \geq 0,\ k_i, \tfrac{\partial}{\partial x_i} k_i \in L^\infty((0,\infty)\times\Omega;\mathbb{R}), \tag{3.6}$$

$$k_i(a,x) \geq k_0 > 0 \text{ a.e. on } [0,\infty)\times\Omega,\ i = 1,\ldots,n.$$

$$\mu \in L^\infty((0,a_1)\times\Omega;\mathbb{R}) \text{ for each } a_1 \geq 0,\ \mu \geq 0 \text{ a.e.} \tag{3.7}$$

$$\beta(\cdot,x), \tfrac{\partial^j}{\partial x_i^j}\beta(\cdot,x)\ (i = 1,\ldots,n,\ j = 1,2) \in L^2(0,\infty;\mathbb{R}) \text{ for a.a. } x \in \Omega,$$

$$\operatorname{ess\,sup}_{x\in\Omega} \textstyle\int_0^\infty \beta(a,x)^2\,da < \infty,\ \operatorname{ess\,sup}_{x\in\Omega} \int_0^\infty \frac{\partial^j}{\partial x_i^j}\beta(a,x)^2\,da < \infty, \tag{3.8}$$

$$i = 1,\ldots,n,\ j = 1,2,\ \beta \geq 0 \text{ a.e.}$$

Then $A - \omega I$ is m-dissipative for $\omega := \frac{1}{2}\operatorname{ess\,sup}\int_0^\infty \beta(a,x)^2\,da$ and B is m-dissipative and generates an analytic semigroup. Let V be the Hilbert space

$$V := \{\varphi \in H \mid \tfrac{\partial}{\partial x_i}\varphi \in H \text{ for } i = 1,\ldots,n\},$$

endowed with the inner product

$$\langle\varphi,\psi\rangle_V := \int_0^\infty\int_\Omega \overline{\varphi(a,x)}\psi(a,x)\,dx\,da + \sum_{i=1}^n \int_0^\infty\int_\Omega \overline{\varphi_{x_i}(a,x)}\psi_{x_i}(a,x)\,dx\,da,\ \ \varphi,\psi \in V,$$

where φ_{x_i} denotes the generalized derivative of $\varphi \in V$ with respect to x_i, and the corresponding norm. V is densely and continuously embedded in H. We define the sesquilinear form $\sigma_2 : V\times V \to \mathbb{C}$ by

$$\sigma_2(\varphi,\psi) := \sum_{i=1}^n \int_0^\infty\int_\Omega k_i(a,x)\overline{\varphi_{x_i}(a,x)}\psi_{x_i}(a,x)\,dx\,da, \quad \varphi,\psi \in V.$$

Then

$$|\sigma_2(\varphi,\psi)| \leq \|k\|_\infty\|\varphi\|_V\|\psi\|_V, \quad \varphi,\psi \in V, \tag{3.9}$$

$$\operatorname{Re}\sigma_2(\varphi,\varphi) = \sigma_2(\varphi,\varphi) \geq \rho(\|\varphi\|_V^2 - \|\varphi\|_H^2), \quad \varphi \in V, \quad 0 \leq \rho \leq k_0, \tag{3.10}$$

and

$$\operatorname{dom} B = \{\varphi \in V \mid \text{There exists an } \alpha \geq 0 \text{ such that}$$

$$|\sigma_2(\psi,\varphi)| \leq \alpha\|\psi\|_H \text{ for all } \psi \in V\},$$

$$\langle\psi, B\varphi\rangle_H = -\sigma_2(\psi,\varphi), \quad \psi \in V, \quad \varphi \in \operatorname{dom} B.$$

Since B is the infinitesimal generator of an analytic semigroup associated with the bounded, coercive sesquilinear form σ_2, it is more convenient to consider approximation of the sesquilinear forms instead of approximation of the generators. We also define the sesquilinear form $\sigma_1 : H \times \operatorname{dom} A \to \mathbb{C}$ by

$$\sigma_1(\psi, \varphi) = \langle \psi, A\varphi \rangle_H, \quad \psi \in H, \quad \varphi \in \operatorname{dom} A.$$

Note that σ_1 and σ_2 are sesquilinear forms of different kind since σ_2 is one appertaining to the generator of an analytic semigroup and σ_1 is an "ordinary" one.

If the birth and mortality rates are independent of x, a standard result about the m-dissipativeness of the sum of two m-dissipative operators can be applied to show that $\mathcal{A} - \omega I$ is m-dissipative, where the operator $\mathcal{A}$ is defined by (2.1) and (2.2); cf. Huyer [7] for the case $k_i(a, x) = k(a)$, which can easily be extended to more general diffusion coefficients. The general case is more complicated, and different approaches involving different additional assumptions on k_i, μ and β (that we shall not write down here) were investigated in Huyer [8] that guarantee that $\mathcal{A} - \omega I$ is m-dissipative, where we avoided the trivial assumption that μ is bounded and the mortality term can be regarded as a bounded perturbation. We assume in the following that $\mathcal{A} - \omega I$ is m-dissipative, and let $\mathcal{S}(t)$, $t \geq 0$, be the C_0-semigroup generated by $\mathcal{A}$.

Lemma 3.1. *Let D be defined by*

$$D := \{\varphi \in \operatorname{dom} \mathcal{A} \mid \exists\, a_1 \geq 0 \text{ such that } \varphi(a, x) = 0 \text{ a.e. on } (a_1, \infty) \times \Omega\}.$$

Then D is a dense subspace of H and $\mathcal{S}(t)D \subset D$ for all $t \geq 0$.

Proof. We prove density by generalizing and modifying the proof of Banks et al [3, Lemma 2.1]. The set $D_1 := \{\varphi \in C^2([0, \infty) \times \overline{\Omega}; \mathbb{R}) \mid \operatorname{supp} \varphi \subset [0, a_1] \times \Omega \text{ for an } a_1 \geq 0\}$ is dense in H. Therefore it suffices to show that any $\varphi \in D_1$ can be approximated by $\varphi_\varepsilon \in D$. Let $\varphi \in D_1$. Then by assumption (3.8) there exists an $\varepsilon_0 > 0$ such that

$$\int_0^\varepsilon \beta(a, x)\Big(1 - \frac{a}{\varepsilon}\Big)\, da \leq \frac{1}{2}$$

for almost all $x \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$. For $0 < \varepsilon \leq \varepsilon_0$ we define

$$\varphi_\varepsilon(a, x) := \begin{cases} \gamma_\varepsilon(x)(1 - \frac{a}{\varepsilon}) + \frac{a}{\varepsilon}\varphi(\varepsilon, x), & 0 \leq a \leq \varepsilon, \quad x \in \Omega, \\ \varphi(a, x), \quad a > \varepsilon, \quad x \in \Omega, \end{cases}$$

where γ_ε is given by

$$\gamma_\varepsilon(x) := \Big(1 - \int_0^\varepsilon \beta(a, x)\Big(1 - \frac{a}{\varepsilon}\Big)\, da\Big)^{-1} \times \Big(\varepsilon^{-1}\varphi(\varepsilon, x) \int_0^\varepsilon \beta(a, x) a\, da + \int_\varepsilon^\infty \beta(a, x)\varphi(a, x)\, da\Big).$$

Then $\gamma_\varepsilon \in L^\infty(\Omega;\mathbb{R})$ and

$$\|\gamma_\varepsilon\|_\infty \le 2\Big(\|\varphi\|_\infty\Big(\frac{\varepsilon_0}{3}\operatorname*{ess\,sup}_{x\in\Omega}\int_0^{\varepsilon_0}\beta(a,x)^2\,da\Big)^{1/2} + \operatorname*{ess\,sup}_{x\in\Omega}\Big(\int_0^\infty \beta(a,x)^2\,da\Big)^{1/2}\sup_{x\in\Omega}\Big(\int_0^\infty \varphi(a,x)^2\,da\Big)^{1/2}\Big)$$

for $0<\varepsilon\le\varepsilon_0$. Assumption (3.8) and the fact that $\varphi\in D_1$ imply $\varphi_\varepsilon\in D$, and we have $\|\varphi-\varphi_\varepsilon\|_H\to 0$ as $\varepsilon\to 0$.

The properties of the age and the diffusion semigroups imply that for $\varphi\in H$ with $\varphi(a,x)=0$ a.e. on $(a_1,\infty)\times\Omega$ we have $(\mathcal{S}(t)\varphi)(a,x)=0$ a.e. on $(a_1+t,\infty)\times\Omega$ for all $t\ge 0$, and $\operatorname{dom}\mathcal{A}$ is trivially invariant under $\mathcal{S}(t)$, $t\ge 0$. □

The Core Theorem (cf. Nagel [16, p. 47, Corollary 1.34], for example) implies that D is a core for $\mathcal{A}$, i.e., $\mathcal{A}$ is the closure of $\mathcal{A}|_D$, which entails that $(\lambda I-\mathcal{A})D$ is dense in H for each $\lambda\in\rho(\mathcal{A})$.

Let $(h_N)_{N\in\mathbb{N}}\subset(0,\infty)$ and $(l_N)_{N\in\mathbb{N}}\subset\mathbb{N}$ be sequences such that $\lim_{N\to\infty}h_N=0$ and $\lim_{N\to\infty}l_Nh_N=\infty$. For $N\in\mathbb{N}$ let E_i^N, $i=1,\dots,l_N$, be the characteristic function of the interval $[(i-1)h_N, ih_N)$ and B_i^N, $i=0,\dots,l_N$, be the first order spline with $B_i^N(jh_N)=\delta_{ij}$, $j=0,\dots,l_N+1$. Moreover, let $(m_N)_{N\in\mathbb{N}}\subset\mathbb{N}$ and ψ_j^N, $j=1,\dots,m_N$, $N\in\mathbb{N}$, be finite elements in $H^1(\Omega;\mathbb{R})$ such that for each $\varphi\in H^2(\Omega;\mathbb{C})$ there exists a sequence $(\varphi^N)_{N\in\mathbb{N}}$ such that $\varphi^N\in\operatorname{span}(\psi_1^N,\dots,\psi_{m_N}^N)$,

$$\|\varphi-\varphi^N\|_{L^2(\Omega;\mathbb{C})}\le Ch_N^2\|\varphi\|_{H^2(\Omega;\mathbb{C})} \tag{3.11}$$

and

$$\|\varphi-\varphi^N\|_{H^1(\Omega;\mathbb{C})}\le Ch_N\|\varphi\|_{H^2(\Omega;\mathbb{C})} \tag{3.12}$$

for $N\in\mathbb{N}$, where the constant C depends only on Ω and the sequence of finite elements but not on φ (cf. Hackbusch [6], for example). Note that the same discretization parameter h_N appears in the age and space approximations, i.e., the fineness of the space grid tends to zero at least as fast as the fineness of the age grid, a fact that will be used in the proof of Lemma 3.4.

Let $Q^N : L^2(\Omega;\mathbb{C})\to\operatorname{span}(\psi_1^N,\dots,\psi_{m_N}^N)$ be the orthogonal projection. Then $\|Q^N\|=1$ for all $N\in\mathbb{N}$, and (3.11) and the fact that $H^2(\Omega;\mathbb{C})$ is dense in $L^2(\Omega;\mathbb{C})$ imply

$$\|\varphi-Q^N\varphi\|_{L^2(\Omega;\mathbb{C})}\to 0 \quad \text{as } N\to\infty \text{ for all } \varphi\in L^2(\Omega;\mathbb{C}).$$

Let

$$H^N=\Big\{\varphi=\sum_{i=1}^{l_N}\sum_{j=1}^{m_N}\alpha_{ij}E_i^N\psi_j^N \,\Big|\, \alpha_{ij}\in\mathbb{C}\Big\}\subset H,$$

endowed with the H-norm, and $V^N = H^N \subset V$, endowed with the V-norm. We define the isomorphisms $i_1^N : D^N \to H^N$ and $i_2^N : V^N \to H^N$ by

$$i_1^N \sum_{i=0}^{l_N} \sum_{j=1}^{m_N} \alpha_{ij} B_i^N \psi_j^N = \sum_{i=1}^{l_N} \sum_{j=1}^{m_N} \alpha_{ij} E_i^N \psi_j^N,$$

$$i_2^N \varphi = \varphi, \quad \varphi \in V^N,$$

where

$$D^N = \{\varphi = \sum_{i=0}^{l_N} \sum_{j=1}^{m_N} \alpha_{ij} B_i^N \psi_j^N \mid \alpha_{ij} \in \mathbb{C},\ \varphi(0,x) = Q^N \int_0^\infty \beta(a,x)(i_1^N \varphi)(a,x)\, da\}$$

is another subspace of H. The sesquilinear forms $\sigma_1^N : H^N \times D^N \to \mathbb{C}$ and $\sigma_2^N : V^N \times V^N \to \mathbb{C}$ are given by

$$\sigma_1^N(\varphi, \psi) = -\langle \varphi, \frac{\partial}{\partial a}\psi + \mu i_1^N \psi \rangle_H, \quad \varphi \in H^N, \quad \psi \in D^N,$$

$$\sigma_2^N(\varphi, \psi) = \sigma_2(\varphi, \psi), \quad \varphi, \psi \in V^N.$$

The approximation of σ_1 is similar to the one of a size-structured population model in Ito et al [12]. Let $P^N : H \to H^N$ be the orthogonal projection and thus $\|P^N\| = 1$. Similarly as in Ito et al [12], we get

Lemma 3.2. *For all $\varphi \in D^N$ we have*

$$\mathrm{Re}\, \sigma_1^N(i_1^N \varphi, \varphi) \le \omega \|i_1^N \varphi\|_H^2,$$

where the constant ω is defined as before.

Proof. Let $\varphi = \sum_{i=0}^{l_N} \sum_{j=1}^{m_N} \alpha_{ij} B_i^N \psi_j^N \in D^N$. Then $i_1^N \varphi = \sum_{i=1}^{l_N} \sum_{j=1}^{m_N} \alpha_{ij} E_i^N \psi_j^N$ and

$$\|i_1^N \varphi\|_H^2 = h_N \sum_{i=1}^{l_N} \int_\Omega \Big| \sum_{j=1}^{m_N} \alpha_{ij} \psi_j^N(x) \Big|^2 dx,$$

and we obtain

$$\mathrm{Re}\, \sigma_1^N(i_1^N \varphi, \varphi) \le -\mathrm{Re} \int_0^\infty \int_\Omega \overline{(i_1^N \varphi)(a,x)} \frac{\partial}{\partial a} \varphi(a,x)\, dx\, da$$

$$= -\mathrm{Re} \sum_{i=1}^{l_N} \int_{(i-1)h_N}^{ih_N} \int_\Omega \Big(\sum_{j=1}^{m_N} \bar{\alpha}_{ij} \psi_j^N(x) \Big) \frac{\partial}{\partial a} \varphi(a,x)\, dx\, da$$

$$= -\mathrm{Re} \sum_{i=1}^{l_N} \int_\Omega \Big(\sum_{j=1}^{m_N} \bar{\alpha}_{ij} \psi_j^N(x) \Big) \Big(\sum_{j=1}^{m_N} (\alpha_{ij} - \alpha_{i-1,j}) \psi_j^N(x) \Big) dx$$

$$= -\sum_{i=1}^{l_N} \int_\Omega \Big| \sum_{j=1}^{m_N} \alpha_{ij} \psi_j^N(x) \Big|^2 dx + \mathrm{Re} \sum_{i=1}^{l_N} \int_\Omega \Big(\sum_{j=1}^{m_N} \bar{\alpha}_{ij} \psi_j^N(x) \Big) \Big(\sum_{j=1}^{m_N} \alpha_{i-1,j} \psi_j^N(x) \Big) dx.$$

This together with $\operatorname{Re} \bar{z}_1 z_2 = \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2 - \frac{1}{2}|z_1 - z_2|^2 \le \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$ for $z_1, z_2 \in \mathbb{C}$ yields

$$\operatorname{Re} \sigma_1^N(i_1^N \varphi, \varphi) \le -\frac{1}{2} \sum_{i=1}^{l_N} \int_\Omega \Big| \sum_{j=1}^{m_N} \alpha_{ij} \psi_j^N(x) \Big|^2 dx + \frac{1}{2} \sum_{i=1}^{l_N} \int_\Omega \Big| \sum_{j=1}^{m_N} \alpha_{i-1,j} \psi_j^N(x) \Big|^2 dx$$

$$\le \frac{1}{2} \int_\Omega \Big| \sum_{j=1}^{m_N} \alpha_{0j} \psi_j^N(x) \Big|^2 dx = \frac{1}{2} \int_\Omega \Big| Q^N \int_0^\infty \beta(a,x)(i_1^N \varphi)(a,x)\, da \Big|^2 dx \le \omega \| i_1^N \varphi \|_H^2,$$

where we have used the boundary condition for $\varphi \in D^N$. □

Ito and Kappel [10] proved that under the conditions fulfilled by σ_1^N and σ_2^N there exist unique linear operators $A^N : H^N \to H^N$, $B^N : H^N \to H^N$, $N \in \mathbb{N}$, such that

$$\sigma_1^N(\varphi, \psi) = \langle \varphi, A^N i_1^N \psi \rangle, \quad \varphi \in H^N, \quad \psi \in D^N, \tag{3.13}$$

$$\sigma_2^N(i_2^{-1} \varphi, i_2^{-1} \psi) = -\langle \varphi, B^N \psi \rangle, \quad \varphi, \psi \in H^N, \tag{3.14}$$

and $A^N - \omega I$ and $B^N - \rho I$ $(0 < \rho \le k_0)$ are dissipative on H^N. Then a simple computation shows that

$$A^N E_i^N \psi_j^N = -\frac{1}{h_N} E_i^N \Big(\psi_j^N + Q^N \Big(\int_{(i-1)h_N}^{ih_N} \mu(a,\cdot)\, da\, \psi_j^N \Big) \Big) + \frac{1}{h_N} E_{i+1}^N \psi_j^N$$

$$+ \frac{1}{h_N} E_1^N Q^N \Big(\int_{(i-1)h_N}^{ih_N} \beta(a,\cdot)\, da\, \psi_j^N \Big), \quad i = 1, \dots, l_N, \quad j = 1, \dots, m_N,$$

where $E_{m_N+1}^N := 0$. Let $C^N := (\langle \psi_i^N, \psi_j^N \rangle_{L^2(\Omega;\mathbb{C})})_{i,j=1,\dots,m_N}$ be the Gramian matrix for the basis $(\psi_j^N)_{j=1,\dots,m_N}$. Then C^N is nonsingular, and we obtain for B^N the representation

$$B^N E_i^N \psi_j^N = -\frac{1}{h_N} E_i^N (\psi_1^N, \dots, \psi_{m_N}^N)$$

$$\times (C^N)^{-1} \operatorname{col} \Big(\sum_{l=1}^n \int_{(i-1)h_N}^{ih_N} \int_\Omega k_l(a,x) \frac{\partial}{\partial x_l} \psi_m^N(x) \frac{\partial}{\partial x_l} \psi_j^N(x)\, dx\, da \Big)_{m=1,\dots,m_N}$$

for $i = 1, \dots, l_N$, $j = 1, \dots, m_N$.

Lemma 3.3. *For $N \in \mathbb{N}$ we define $R^N : H \to H$ by*

$$(R^N \varphi)(a,x) := \sum_{i=1}^{l_N} \frac{1}{h_N} E_i^N(a) \int_{(i-1)h_N}^{ih_N} \varphi(b,x)\, db.$$

Then $\|R^N\| = 1$ and $\|R^N \varphi - \varphi\|_H \to 0$ as $N \to \infty$ for all $\varphi \in H$.

Proof. For $\varphi \in H$ we have

$$\|R^N\varphi\|_H^2 = \frac{1}{h_N}\sum_{i=1}^{l_N}\int_\Omega\Big|\int_{(i-1)h_N}^{ih_N}\varphi(b,x)\,db\Big|^2\,dx \le \sum_{i=1}^{l_N}\int_\Omega\int_{(i-1)h_N}^{ih_N}|\varphi(a,x)|^2\,da\,dx \le \|\varphi\|_H^2.$$

Consider $\varphi \in H$ such that $\frac{\partial}{\partial a}\varphi \in H$. Then

$$\|\varphi - R^N\varphi\|_H^2 = \frac{1}{h_N^2}\sum_{i=1}^{l_N}\int_\Omega\int_{(i-1)h_N}^{ih_N}\Big|\int_{(i-1)h_N}^{ih_N}(\varphi(a,x)-\varphi(b,x))\,db\Big|^2\,da\,dx$$

$$+\int_{l_Nh_N}^{\infty}\int_\Omega|\varphi(a,x)|^2\,dx\,da =: I + II.$$

Since $\lim_{N\to\infty} l_Nh_N = \infty$ by assumption, II tends to 0 as $N \to \infty$. For I we obtain

$$I = \frac{1}{h_N^2}\sum_{i=1}^{l_N}\int_\Omega\int_{(i-1)h_N}^{ih_N}\Big|\int_{(i-1)h_N}^{ih_N}\int_c^b\frac{\partial}{\partial a}\varphi(a,x)\,da\,dc\Big|^2\,db\,dx$$

$$\le h_N^2\sum_{i=1}^{l_N}\int_\Omega\int_{(i-1)h_N}^{ih_N}\Big|\frac{\partial}{\partial a}\varphi(a,x)\Big|^2\,da\,dx \le h_N^2\Big\|\frac{\partial}{\partial a}\varphi\Big\|_H^2,$$

i.e., $\|R^N\varphi - \varphi\|_H \to 0$ as $N \to \infty$. Since the set $\{\varphi \in H \mid \frac{\partial}{\partial a}\varphi \in H\}$ is dense in H and $\|R^N\| = 1$ for all $N \in \mathbb{N}$, we have $\|R^N\varphi - \varphi\|_H \to 0$ as $N \to \infty$ for all $\varphi \in H$. $\square$

Lemma 3.4. *Let $\varphi \in V$ be such that $\frac{\partial}{\partial a}\varphi, \frac{\partial^2}{\partial x_i^2}\varphi \in H$, $i = 1,\dots,N$, and $\varphi(a,x) = 0$ a.e. on $(a_1,\infty)\times\Omega$ for some $a_1 \ge 0$. Then there exist a sequence $(\tilde\varphi^N)_{N\in\mathbb{N}}$, $\tilde\varphi^N \in H^N = V^N$ for all $N \in \mathbb{N}$, and a constant $C_1 = C_1(\varphi) > 0$ such that $\|\varphi - \tilde\varphi^N\|_V \to 0$ as $N \to \infty$ and*

$$\|\varphi - \tilde\varphi^N\|_H \le C_1h_N \tag{3.15}$$

for N sufficiently large.

Proof. Let φ satisfy the assumptions. Then from the proof of Lemma 3.3 we see that

$$\|\varphi - R^N\varphi\|_H \le h_N\Big\|\frac{\partial}{\partial a}\varphi\Big\|_H \tag{3.16}$$

for N sufficiently large. Moreover, $\varphi \in V$ implies $R^N\varphi \in V$, $\frac{\partial}{\partial x_i}R^N\varphi = R^N\frac{\partial}{\partial x_i}\varphi$, $i = 1,\dots,n$, and thus

$$\Big\|\frac{\partial}{\partial x_i}R^N\varphi - \frac{\partial}{\partial x_i}\varphi\Big\|_H \to 0 \quad \text{as } N \to \infty$$

for $i = 1,\dots,n$, i.e.,

$$\|\varphi - R^N\varphi\|_V \to 0 \quad \text{as } N \to \infty. \tag{3.17}$$

Let $R^N\varphi = \sum_{i=1}^{l_N} E_i^N \varphi_i^N$, where $\varphi_i^N(x) := \frac{1}{h_N}\int_{(i-1)h_N}^{ih_N} \varphi(a,x)\,da$. Then the assumptions imply $\varphi_i^N \in H^2(\Omega;\mathbb{C})$. By (3.11) and (3.12) there exist $\tilde{\varphi}_i^N \in \operatorname{span}(\psi_1^N,\dots,\psi_{m_N}^N)$, $i=1,\dots,l_N$, $N\in\mathbb{N}$, such that

$$\|\varphi_i^N - \tilde{\varphi}_i^N\|_{L^2(\Omega;\mathbb{C})} \le Ch_N^2 \|\varphi_i^N\|_{H^2(\Omega;\mathbb{C})}, \tag{3.18}$$

$$\|\varphi_i^N - \tilde{\varphi}_i^N\|_{H^1(\Omega;\mathbb{C})} \le Ch_N \|\varphi_i^N\|_{H^2(\Omega;\mathbb{C})}. \tag{3.19}$$

Let $\tilde{\varphi}^N := \sum_{i=1}^{l_N} E_i^N \tilde{\varphi}_i^N \in V^N$. Then we obtain by using (3.19)

$$\|R^N\varphi - \tilde{\varphi}^N\|_V^2 = h_N \sum_{i=1}^{l_N} \|\varphi_i^N - \tilde{\varphi}_i^N\|_{H^1(\Omega;\mathbb{C})}^2 \le C^2 h_N^3 \sum_{i=1}^{l_N} \|\varphi_i^N\|_{H^2(\Omega;\mathbb{C})}^2$$

$$= C^2 h_N \sum_{i=1}^{l_N} \Big\| \int_{(i-1)h_N}^{ih_N} \varphi(a,\cdot)\,da \Big\|_{H^2(\Omega;\mathbb{C})}^2 \le C^2 h_N^2 \int_0^\infty \|\varphi(a,\cdot)\|_{H^2(\Omega;\mathbb{C})}^2\,da.$$

This estimate together with (3.16) and (3.17) proves the assertions. $\square$

Remark 3.5. The proof of Lemma 3.4 shows that for φ satisfying the assumptions of Lemma 3.4 with $\varphi(a,x)=0$ a.e. on $(a_1,\infty)\times\Omega$ for some $a_1\ge 0$ we can choose $(\tilde{\varphi}^N)_{N\in\mathbb{N}}$ such that $\tilde{\varphi}^N(a,x)=0$ on $(a_2,\infty)\times\Omega$ for all $N\in\mathbb{N}$, where $a_2 := a_1 + \max_{N\in\mathbb{N}} h_N$.

Moreover, we state a trivial property of functions $\varphi\in H$.

Lemma 3.6. *Let $\varphi\in H$. Then we have*

$$\lim_{h\downarrow 0} \int_h^\infty \int_\Omega |\varphi(a,x) - \varphi(a-h,x)|^2\,dx\,da = 0.$$

We are now ready to prove the main theorem.

Theorem 3.7. *Let $\mathcal{A}$, $\mathcal{S}(t)$, $t\ge 0$, and $H^N, V^N, D^N, i_1^N, i_2^N, \sigma_1^N, \sigma_2^N, P^N$, $N\in\mathbb{N}$, be defined as before, let A^N and B^N be given by* (3.13) *and* (3.14)*, and let $\mathcal{S}^N(t)$, $t\ge 0$, be the semigroup generated by $A^N + B^N$ on H^N. Then for each $\varphi\in H$ we have*

$$\|\mathcal{S}^N(t)P^N\varphi - \mathcal{S}(t)\varphi\|_H \to 0 \quad \text{as } N\to\infty$$

uniformly on bounded t-intervals.

Proof. It suffices to show that

$$\|(\lambda_0 I - \mathcal{A}^N)^{-1}P^N\psi - (\lambda_0 I - \mathcal{A})^{-1}\psi\|_H \to 0 \quad \text{as } N\to\infty$$

for all ψ in a dense set and a $\lambda_0 > \omega$. Let $\lambda_0 > \omega + 1$ and D be as in Lemma 3.1, $\varphi\in D$, $\psi := (\lambda_0 I - \mathcal{A})\varphi$ and $\varphi^N := (\lambda_0 I - A^N - B^N)^{-1}P^N\psi \in H^N = V^N$. Then we have

$$\lambda_0\langle\chi,\varphi\rangle_H - \sigma_1(\chi,\varphi) + \sigma_2(\chi,\varphi) = \langle\chi,\psi\rangle_H$$

for all $\chi \in V$ and

$$\lambda_0 \langle \chi^N, i_2^N \varphi^N \rangle_H - \sigma_1^N(\chi^N, (i_1^N)^{-1}\varphi^N) + \sigma_2^N(\chi^N, \varphi^N) = \langle \chi^N, P^N \psi \rangle_H$$

for all $\chi^N \in V^N$. Let $(\tilde{\varphi}^N)_{N \in \mathbb{N}}$ be chosen for φ according to Lemma 3.4 and $\chi^N := \varphi^N - \tilde{\varphi}^N$. Then similarly as in Ito and Kappel [10] we obtain

$$\lambda_0 \|\varphi^N - \tilde{\varphi}^N\|_H^2 - \sigma_1^N(\varphi^N - \tilde{\varphi}^N, (i_1^N)^{-1}(\varphi^N - \tilde{\varphi}^N)) + \sigma_2(\varphi^N - \tilde{\varphi}^N, \varphi^N - \tilde{\varphi}^N)$$
$$= \lambda_0 \langle \chi^N, \varphi - \tilde{\varphi}^N \rangle_H + \sigma_1^N(\chi^N, (i_1^N)^{-1}\tilde{\varphi}^N) - \sigma_1(\chi^N, \varphi) + \sigma_2(\chi^N, \varphi - \tilde{\varphi}^N),$$

where we have used that $\langle \chi^N, P^N \psi \rangle_H = \langle \chi^N, \psi \rangle_H$ by definition of the orthogonal projection and $\sigma_2^N = \sigma_2\big|_{V^N \times V^N}$. From Lemma 3.2, we obtain

$$(\lambda_0 - \omega)\|\varphi^N - \tilde{\varphi}^N\|_H^2 + \sigma_2^N(\varphi^N - \tilde{\varphi}^N, \varphi^N - \tilde{\varphi}^N)$$
$$\leq \lambda_0 |\langle \chi^N, \varphi - \tilde{\varphi}^N \rangle_H| + |\sigma_1^N(\chi^N, (i_1^N)^{-1}\tilde{\varphi}^N) - \sigma_1(\chi^N, \varphi)| + |\sigma_2(\chi^N, \varphi - \tilde{\varphi}^N)|.$$

We have

$$\lambda_0 |\langle \chi^N, \varphi - \tilde{\varphi}^N \rangle_H| \leq \lambda_0 \|\chi^N\|_H \|\varphi - \tilde{\varphi}^N\|_H,$$
$$|\sigma_2(\chi^N, \varphi - \tilde{\varphi}^N)| \leq \sigma_2(\chi^N, \chi^N)^{1/2} \sigma_2(\varphi - \tilde{\varphi}^N, \varphi - \tilde{\varphi}^N)^{1/2},$$

where $\|\varphi - \tilde{\varphi}^N\|_H \to 0$ and $\sigma_2(\varphi - \tilde{\varphi}^N, \varphi - \tilde{\varphi}^N)^{1/2} \to 0$ as $N \to \infty$ by Lemma 3.4.

The middle term satisfies the estimate

$$|\sigma_1^N(\chi^N, (i_1^N)^{-1}\tilde{\varphi}^N) - \sigma_1(\chi^N, \varphi)|$$
$$\leq |\langle \chi^N, \mu(\tilde{\varphi}^N - \varphi) \rangle_H| + \left| \left\langle \chi^N, \frac{\partial}{\partial a}(i_1^N)^{-1}\tilde{\varphi}^N - \frac{\partial}{\partial a}\varphi \right\rangle \right| =: I + II.$$

For I we have

$$I \leq \|\chi^N\|_H \operatorname*{ess\,sup}_{[0,a_2]\times\Omega} \mu(a,x)\, \|\tilde{\varphi}^N - \varphi\|_H,$$

where a_2 is defined as in Remark 3.5. Let $\chi^N = \sum_{i=1}^{l_N} \sum_{j=1}^{m_N} \alpha_{ij}^N E_i^N \psi_j^N$. Then we obtain

$$II = \Big|\sum_{i=1}^{l_N}\sum_{j=1}^{m_N}\bar{\alpha}_{ij}^N\int_{(i-1)h_N}^{ih_N}\int_\Omega \psi_j^N(x)\Big(\frac{\partial}{\partial a}\varphi(a,x) - \frac{\partial}{\partial a}((i_1^N)^{-1}\tilde{\varphi}^N)(a,x)\Big)\,dx\,da\Big|$$

$$= \Big|\sum_{i=1}^{l_N}\sum_{j=1}^{m_N}\bar{\alpha}_{ij}^N\int_\Omega \psi_j^N(x)(\varphi(ih_N,x) - ((i_1^N)^{-1}\tilde{\varphi}^N)(ih_N,x) - \varphi((i-1)h_N,x)$$

$$+ ((i_1^N)^{-1}\tilde{\varphi}^N)((i-1)h_N,x))\,dx\Big|$$

$$\le h_N^{-1/2}\|\chi^N\|_H\Big(\sum_{i=1}^{l_N}\int_\Omega |\varphi(ih_N,x) - ((i_1^N)^{-1}\tilde{\varphi}^N)(ih_N,x) - \varphi((i-1)h_N,x)$$

$$+ ((i_1^N)^{-1}\tilde{\varphi}^N)((i-1)h_N,x)|^2\,dx\Big)^{1/2}.$$

Let φ_i^N and $\tilde{\varphi}_i^N$ be as in the proof of Lemma 3.4. For $i=1$ we have

$$h_N^{-1/2}\Big(\int_\Omega |\varphi(h_N,x) - ((i_1^N)^{-1}\tilde{\varphi}^N)(h_N,x) - \varphi(0,x) + ((i_1^N)^{-1}\tilde{\varphi}^N)(0,x)|^2\,dx\Big)^{1/2}$$

$$\le h_N^{-1/2}\Big(\int_\Omega |\varphi(h_N,x) - \tilde{\varphi}_1^N(x)|^2\,dx\Big)^{1/2} + h_N^{-1/2}\Big(\int_\Omega\Big|\int_0^\infty \beta(a,x)\varphi(a,x)\,da$$

$$- Q^N\int_0^\infty \beta(a,x)\tilde{\varphi}^N(a,x)\,da\Big|^2\,dx\Big)^{1/2} =: III + IV.$$

Using (3.18) and $\frac{\partial}{\partial a}\varphi \in H$ we obtain

$$III \le h_N^{-1/2}\Big(\int_\Omega |\varphi(h_N,x) - \varphi_1^N(x)|^2\,dx\Big)^{1/2} + h_N^{-1/2}\Big(\int_\Omega |\varphi_1^N(x) - \tilde{\varphi}_1^N(x)|^2\,dx\Big)^{1/2}$$

$$\le h_N^{-3/2}\Big(\int_\Omega\Big|\int_0^{h_N}\int_b^{h_N}\frac{\partial}{\partial a}\varphi(a,x)\,da\,db\Big|^2\,dx\Big)^{1/2} + Ch_N\Big(\int_0^{h_N}\|\varphi(a,\cdot)\|_{H^2(\Omega;\mathbb{C})}\,da\Big)^{1/2}$$

$$= h_N^{-3/2}\Big(\int_\Omega\Big|\int_0^{h_N} a\frac{\partial}{\partial a}\varphi(a,x)\,da\Big|^2\,dx\Big)^{1/2} + Ch_N\Big(\int_0^{h_N}\|\varphi(a,\cdot)\|_{H^2(\Omega;\mathbb{C})}\,da\Big)^{1/2}$$

$$\le \frac{1}{\sqrt{3}}\Big(\int_0^{h_N}\int_\Omega\Big|\frac{\partial}{\partial a}\varphi(a,x)\Big|^2\,dx\,da\Big)^{1/2} + Ch_N\Big(\int_0^{h_N}\|\varphi(a,\cdot)\|_{H^2(\Omega;\mathbb{C})}\,da\Big)^{1/2},$$

which tends to 0 as $N \to \infty$. Moreover, (3.8), (3.11) and (3.15) yield the estimate

$$IV \le h_N^{-1/2}\Big(\int_\Omega \Big|\int_0^\infty \beta(a,x)\varphi(a,x)\,da - Q^N \int_0^\infty \beta(a,x)\varphi(a,x)\Big|^2\,dx\Big)^{1/2}$$

$$+ h_N^{-1/2}\Big(\operatorname*{ess\,sup}_{x\in\Omega} \int_0^\infty \beta(a,x)^2\,da\Big)^{1/2} \|\varphi - \tilde\varphi^N\|_H$$

$$\le C h_N^{3/2}\Big\|\int_0^\infty \beta(a,\cdot)\varphi(a,\cdot)\,da\Big\|_{H^2(\Omega;\mathbb{C})} + (2\omega)^{1/2} C_1 h_N^{1/2}$$

for N sufficiently large, i.e., $IV \to 0$ as $N \to \infty$.

For $2 \le i \le l_N$ we obtain from (3.18),

$$\frac{1}{h_N}\int_\Omega |\varphi(ih_N,x) - \tilde\varphi_i^N(x) - \varphi((i-1)h_N,x) + \tilde\varphi_{i-1}^N(x)|^2\,dx$$

$$\le \frac{3}{h_N}\int_\Omega |\tilde\varphi_i^N(x) - \varphi_i^N(x)|^2\,dx + \frac{3}{h_N}\int_\Omega |\tilde\varphi_{i-1}^N(x) - \varphi_{i-1}^N(x)|^2\,dx$$

$$+ \frac{3}{h_N}\int_\Omega |\varphi(ih_N,x) - \varphi_i^N(x) - \varphi((i-1)h_N,x) + \varphi_{i-1}^N(x)|^2\,dx$$

$$\le 3C^2 h_N^2 \int_{(i-1)h_N}^{ih_N} \|\varphi(a,\cdot)\|^2_{H^2(\Omega;\mathbb{C})}\,da + 3C^2 h_N^2 \int_{(i-2)h_N}^{(i-1)h_N} \|\varphi(a,\cdot)\|^2_{H^2(\Omega;\mathbb{C})}\,da + V.$$

The term V satisfies the estimate

$$V = \frac{3}{h_N^3}\int_\Omega \Big|\int_{(i-1)h_N}^{ih_N}\int_b^{ih_N} \Big(\frac{\partial}{\partial a}\varphi(a,x) - \frac{\partial}{\partial a}\varphi(a-h_N,x)\Big)\,da\,db\Big|^2\,dx$$

$$= \frac{3}{h_N^3}\int_\Omega \Big|\int_{(i-1)h_N}^{ih_N} (a-(i-1)h_N)\Big(\frac{\partial}{\partial a}\varphi(a,x) - \frac{\partial}{\partial a}\varphi(a-h_N,x)\Big)\,da\Big|^2\,dx$$

$$\le \int_{(i-1)h_N}^{ih_N}\int_\Omega \Big|\frac{\partial}{\partial a}\varphi(a,x) - \frac{\partial}{\partial a}\varphi(a-h_N,x)\Big|^2\,dx\,da.$$

Thus we obtain

$$\frac{1}{h_N}\sum_{i=2}^{l_N}\int_\Omega |\varphi(ih_N,x) - \tilde\varphi_i^N(x) - \varphi((i-1)h_N,x) + \tilde\varphi_{i-1}^N(x)|^2\,dx$$

$$\le 6C^2 h_N^2 \int_0^\infty \|\varphi(a,\cdot)\|^2_{H^2(\Omega;\mathbb{C})}\,da + \int_{h_N}^\infty \int_\Omega \Big|\frac{\partial}{\partial a}\varphi(a,x) - \frac{\partial}{\partial a}\varphi(a-h_N,x)\Big|^2\,dx\,da,$$

which tends to 0 as $N \to \infty$ by Lemma 3.6. Putting all estimates together we get

$$(\lambda_0 - \omega)\|\varphi^N - \tilde\varphi^N\|_H^2 + \sigma_2(\varphi^N - \tilde\varphi^N, \varphi^N - \tilde\varphi^N)$$

$$\le c_N(\|\varphi^N - \tilde\varphi^N\|_H^2 + \sigma_2(\varphi^N - \tilde\varphi^N, \varphi^N - \tilde\varphi^N))^{1/2},$$

where $c_N \to 0$ as $N \to \infty$. Since $\lambda_0 > \omega + 1$, we obtain

$$\|\varphi^N - \tilde{\varphi}^N\|_H^2 + \sigma_2(\varphi^N - \tilde{\varphi}^N, \varphi^N - \tilde{\varphi}^N) \leq c_N^2,$$

and (3.10) implies

$$\|\varphi^N - \tilde{\varphi}^N\|_H \leq c_N.$$

Thus we obtain the final estimate

$$\|(\lambda_0 I - \mathcal{A})^{-1} P^N \psi - (\lambda_0 I - \mathcal{A})^{-1}\psi\|_H = \|\varphi^N - \varphi\|_H$$

$$\leq \|\varphi^N - \tilde{\varphi}^N\|_H + \|\tilde{\varphi}^N - \varphi\|_H \leq c_N + \|\tilde{\varphi}^N - \varphi\|_H,$$

which tends to zero as $N \to \infty$. □

Remark 3.8. Since $\mathcal{A}$ does not generate an analytic semigroup, it is not possible to give error estimates for $\|\mathcal{S}(t)\varphi - \mathcal{S}^N(t)P^N\varphi\|_H$ without imposing smoothness assumptions on φ. See Ito and Kappel [11] for general results on error estimates for Trotter-Kato approximations.

4. NUMERICAL EXAMPLES

In this section we present two numerical examples. We take $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$ and

$$H^{N,M} = \{\varphi \in H \,|\, \varphi(a,x,y) = \sum_{i=1}^{N} \sum_{j=1}^{(M+1)^2} a_{ij} E_i^N(a) \psi_j^M(x,y)\}, \quad N, M \in \mathbb{N},$$

where ψ_j^M, $j = 1, \ldots, (M+1)^2$, are piecewise linear finite elements on the triangulation $(\Delta_j^M)_{j=1,\ldots,2M^2}$ of Ω, $\Delta_{(i-1)M+j}^M := \{(x,y) \mid \frac{i-1}{M} \leq x \leq \frac{i}{M}, \frac{j-1}{M} \leq y \leq \frac{j-i}{M} + x\}$, $\Delta_{M^2+(i-1)M+j}^M := \{(x,y) \mid \frac{i-1}{M} \leq x \leq \frac{i}{M}, \frac{j-i}{M} + x \leq y \leq \frac{j}{M}\}$, $i, j = 1, \ldots, M$, and E_i^N is the characteristic function of the interval $[(i-1)h_N, ih_N)$, $i = 1, \ldots, N$. The corresponding approximate solutions are denoted by $u^{N,M}$. Of course, we cannot expect very fast convergence of the age densities since they are only approximated by step functions, but we obtain reasonable approximations of the space densities even for small values of M. The algorithm was implemented in Fortran 77 and the Adams routine `d02caf` from the NAG library turned out to be most successful in solving the approximating ODEs, whereas oscillations occurred with a Runge-Kutta algorithm for some examples.

In the first example we compare $u^{N,M}$ with the approximations obtained by the scheme developed in Huyer [7], which can also be applied to space-dependent birth and mortality rates as long as $k_i(a,x,y) = k(a)$, $i = 1, 2$, although the proof of convergence has only been established for space-independent birth and mortality rates. However, the scheme of Huyer [7] cannot be applied to more general diffusion coefficients and to complicated spatial domains where it is difficult to compute the eigenfunctions of the Laplacian. We denote the approximate solutions generated by the scheme of Huyer [7] by $\hat{u}^{N,M}$, $\hat{u}^{N,M} \in \hat{H}^{N,M}$, where

$$\hat{H}^{N,M} = \big\{\varphi \in H \,\big|\, \varphi(a,x,y) = \sum_{l=0}^{N}\sum_{m=0}^{M}\sum_{n=0}^{M} c_{lmn} e^{-a/2} L_l(a) \cos m\pi x \cos n\pi y\big\}$$

for $N, M \in \mathbb{N}_0$ and $L_l(a)$, $l \in \mathbb{N}_0$, is the lth Laguerre polynomial.

Example 4.1. This is problem (3.1)–(3.4) with

$$\mu(a,x,y) = 2(\sqrt{3}-1)(1+0.125\cos 2\pi x)(1+0.125\cos 2\pi y), \quad a \ge 0, \quad 0 < x,y < 1,$$

$$\beta(a,x,y) = 12ae^{-2a}(1-0.125\cos 2\pi x)(1-0.125\cos 2\pi y), \quad a \ge 0, \quad 0 < x,y < 1,$$

$$k_i(a,x,y) \equiv 0.015, \quad i = 1,2,$$

$$\varphi(a,x,y) = \begin{cases} (a/a_0-1)^2, & 0 \le a \le a_0, \quad 0 < x < 1, \quad x < y < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

where $a_0 = 1.8519$. The birth and mortality rates are chosen such that in the case without diffusion the population would become extinct for $(x,y) \in \Omega$ with $3\beta_0(x,y) < (1+(\sqrt{3}-1)\mu_0(x,y))^2$, where $\beta_0(x,y) := (1-0.125\cos 2\pi x)(1-0.125\cos 2\pi y)$, $\mu_0(x,y) := (1+0.125\cos 2\pi x)(1+0.125\cos 2\pi y)$, and increase unboundedly for $(x,y) \in \Omega$ with $3\beta_0(x,y) > (1+(\sqrt{3}-1)\mu_0(x,y))^2$. We took $a_{\max} = 5$ and $h_N = a_{\max}/N$. The age densities $\int_\Omega u^{35,20}(t,a,x,y)\,dx\,dy$ (solid line) and $\int_\Omega \hat{u}^{20,20}(t,a,x,y)\,dx\,dy$ (dashed line) for $t = 4$ are depicted in Figure 1. In Figures 2 and 3 $\int_0^\infty u^{35,20}(t,a,x,y)\,da$ and $\int_0^\infty (u^{35,20}(t,a,x,y) - u^{20,15}(t,a,x,y))\,da$ are shown at the same scale for $t = 4$.

Example 4.2. This is problem (3.1)–(3.4) with β as in Example 4.1,

$$\mu(a,x,y) = e^{a^2/10}(1+0.125\cos 2\pi x)(1+0.125\cos 2\pi y), \quad a \ge 0, \quad 0 < x,y < 1,$$

$$k_i(a,x,y) = 0.015(1+0.125\cos 2\pi x)(1+0.125\cos 2\pi y), \quad a \ge 0, \quad 0 < x,y < 1,$$

$$\varphi(a,x,y) = \begin{cases} (a/a_0-1)^2, & 0 \le a \le a_0, \quad 0 < x,y < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

where a_0 is defined as in Example 4.1, i.e., we consider an example with space-dependent diffusion coefficients, which cannot be treated with the approximation

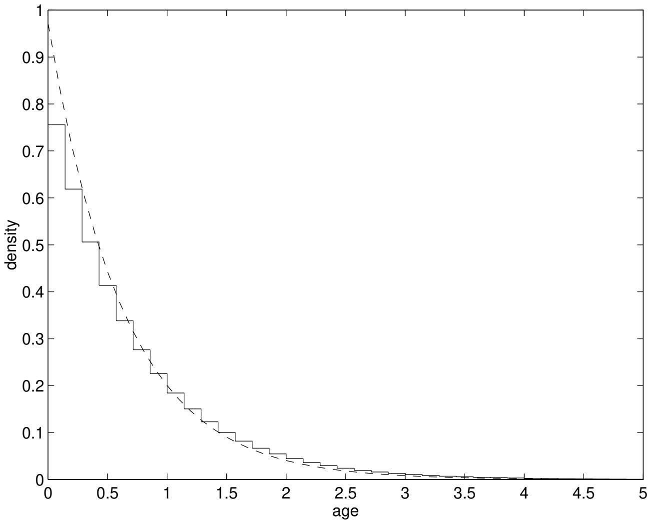

Figure 1: Age density of Example 4.1 at $t = 4$

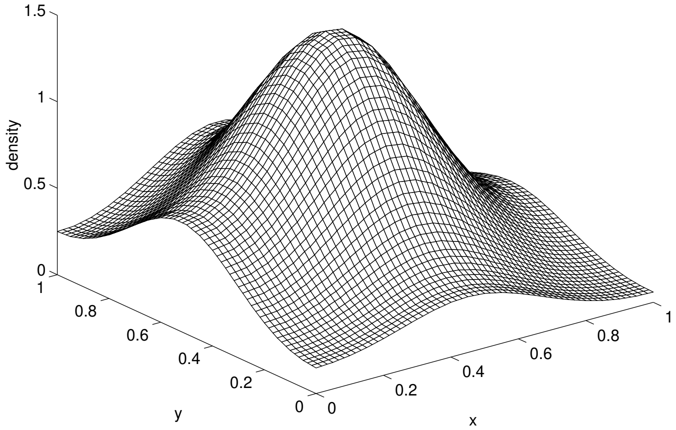

Figure 2: Space density of Example 4.1 at $t = 4$

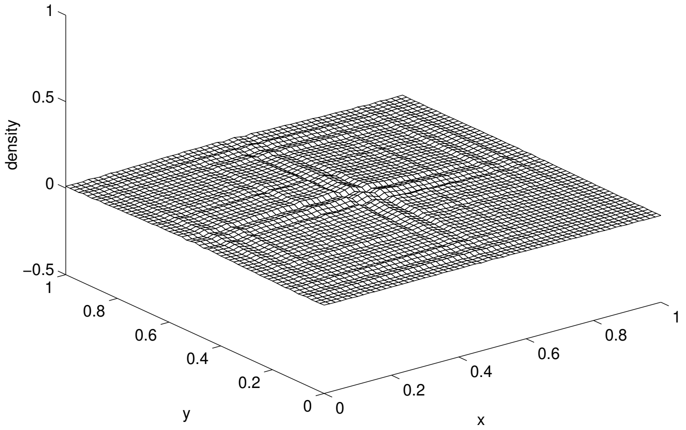

Figure 3: Difference between the space densities of Example 4.1 for $N = 35$, $M = 20$ and $N = 20$, $M = 15$ at $t = 4$

scheme of Huyer [7]. The age increments h_N were chosen as in Example 4.1. Figure 4 shows the age densities $\int_\Omega u^{N,M}(t, a, x, y)\, dx\, dy$ for $t = 4$, where $N = 35$, $M = 20$ (solid line) and $N = 20$, $M = 15$ (dashed line). In Figures 5 and 6 $\int_0^\infty u^{35,20}(t, a, x, y)\, da$ and $\int_0^\infty (u^{35,20}(t, a, x, y) - u^{20,15}(t, a, x, y))\, da$ are plotted at the same scale for $t = 4$.

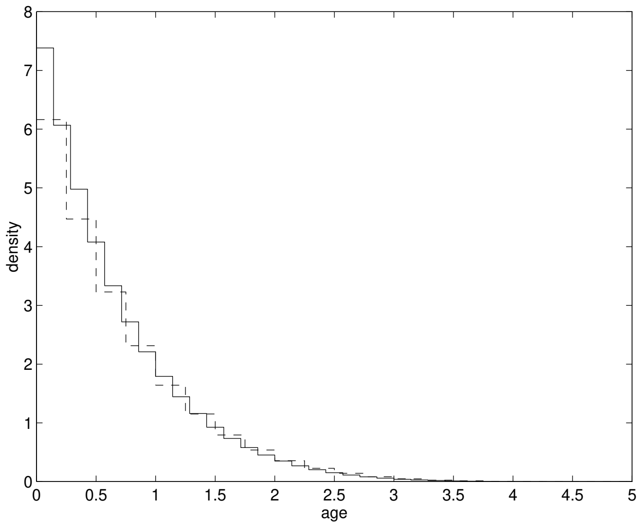

Figure 4: Age density of Example 4.2 at $t = 4$

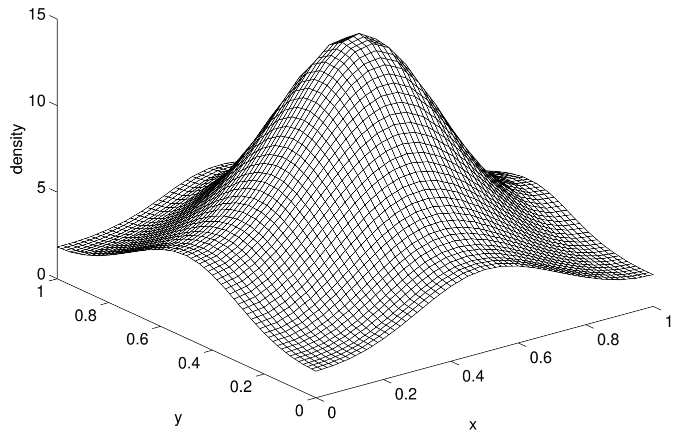

Figure 5: Space density of Example 4.2 at $t = 4$

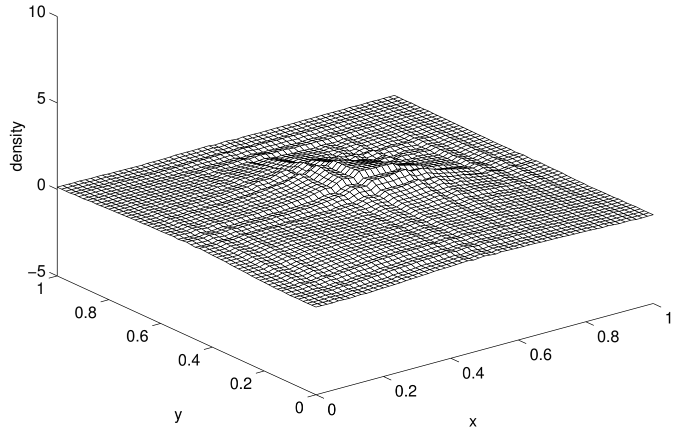

Figure 6: Difference between the space densities of Example 4.2 for $N = 35$, $M = 20$ and $N = 20$, $M = 15$ at $t = 4$

REFERENCES

[1] A.S. Ackleh and B.G. Fitzpatrick, Modeling aggregation and growth processes in an algal population model: analysis and computations, *J. Math. Biol.*, **35** (1997), 480–502.

[2] H.T. Banks and F. Kappel, Transformation semigroups and L^1-approximation for size structured population models, *Semigroup Forum*, **38** (1989), 141–155.

[3] H.T. Banks, F. Kappel and C. Wang, Weak solutions and differentiability for size structured population models, in *Estimation and Control of Distributed Parameter Systems* (Ed-s: W. Desch, F. Kappel and K. Kunisch), Birkhäuser, Basel, 1991, 35–50.

[4] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Space*, Noordhoff, Leyden, 1976.

[5] G. Da Prato and P. Grisvard, Sommes d'opérateurs linéaires et équations différentielles opérationnelles, *J. Math. Pures Appl.*, **54** (1975), 305–387.

[6] W. Hackbusch, *Elliptic Differential Equations*, Springer, Berlin, 1992.

[7] W. Huyer, Semigroup formulation and approximation of a linear age-dependent population problem with spatial diffusion, *Semigroup Forum*, **49** (1994), 99–114.

[8] W. Huyer, Well-posedness of a linear age-dependent population model with spatial diffusion in L^2, in *Advances in Mathematical Population Dynamics: Molecules, Cells and Man* (Ed-s: O. Arino, D. Axelrod and M. Kimmel), World Scientific, Singapore, 1997, 713–732.

[9] K. Ito and F. Kappel, Two families of approximation schemes for delay systems, *Result. Math.*, **1/2** (1992), 93–137.

[10] K. Ito and F. Kappel, On variational formulations of the Trotter-Kato theorem, Spezialforschungsbereich F 003 Report **39**, 1995.

[11] K. Ito and F. Kappel, The Trotter-Kato theorem and approximation of PDEs, *Math. Comput.*, **67** (1998), 21–44.

[12] K. Ito, F. Kappel and D. Salamon, A variational approach to approximation of delay systems, *Differential Integral Equations*, **4** (1991), 51–72.

[13] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, Berlin, 1995.

[14] F. Kappel and K. Zhang, Approximation of linear age-structured population models using Legendre polynomials, *J. Math. Anal. Appl.*, **180** (1993), 518–549.

[15] S. Monniaux and J. Prüss, A theorem of the Dore-Venni type for non-commuting operators, *Trans. Am. Math. Soc.*, **349** (1997), 4787–4814.

[16] R. Nagel (ed.), *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Mathematics, Vol. **1184**, Springer, Berlin, 1986.

[17] A. Pazy, *Semigroups of Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[18] H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.

[19] H.F. Trotter, Approximation of semi-groups of operators, *Pacific J. Math.*, **8** (1958), 887–919.