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WELL-POSEDNESS OF A LINEAR AGE-DEPENDENT POPULATION MODEL WITH SPATIAL DIFFUSION IN L^2

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Abstract. A linear model for an age-structured population with random diffusion in a bounded domain $\Omega \subset \mathbb{R}^n$ is studied in the semigroup framework in L^2. Three different approaches to establishing well-posedness of the problem are presented. Two of them involve applying known results on the m-dissipativeness of the sum of two m-dissipative operators to the population operator and the diffusion operator.

Key words. Linear age-dependent population dynamics with spatial diffusion, C_0-semigroups, sum of two m-dissipative operators, evolution systems.

1. Introduction

Mathematical models for age-structured populations with spatial diffusion were proposed by Gurtin (1973). Many authors have dealt with linear and nonlinear problems of this kind since then (cf. the bibliography Huyer (1992)). In this paper we consider a linear model for an age-structured population with random diffusion in a bounded domain $\Omega \subset \mathbb{R}^n$. Let $u(t,a,x)$ be the population density at time $t \geq 0$, where $a \geq 0$ denotes age and $x = (x_1, \ldots, x_n) \in \Omega$ is the spatial variable. The evolution of the population is described by the partial differential equation

$$\frac{\partial}{\partial t}u(t,a,x) + \frac{\partial}{\partial a}u(t,a,x) + \mu(a,x)u(t,a,x) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}u(t,a,x)\Big) = 0, \qquad t > 0, \quad a > 0, \quad x \in \Omega, \tag{1}$$

with the initial condition

$$u(0,a,x) = \varphi(a,x), \quad a \geq 0, \quad x \in \Omega, \tag{2}$$

and the birth law

$$u(t,0,x) = \int_0^\infty \beta(a,x)u(t,a,x)\,da, \quad t > 0, \quad x \in \Omega. \tag{3}$$

The homogeneous Neumann boundary condition

$$\sum_{i=1}^{n} k_i(a,x)\nu_i(x)\frac{\partial}{\partial x_i}u(t,a,x) = 0, \quad t > 0, \quad a > 0, \quad x \in \partial\Omega, \tag{4}$$

where $(\nu_1(x), \ldots, \nu_n(x))$ denotes the outward unit normal at $\partial\Omega$, means that the population does not leave the region Ω. Here $\mu(a,x)$ is the death rate, $\beta(a,x)$ is the birth rate and $k_i(a,x)$ are the diffusion coefficients. The maximum age is assumed to be ∞.

We want to treat this problem in the semigroup framework in $H := L^2((0,\infty)\times\Omega;\mathbb{R})$ and denote the norm and inner product on H by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$, respectively. A semigroup formulation of the problem is desirable since it allows one to use powerful techniques for qualitative investigations as well as for the construction of approximation schemes. Under the assumption that the solutions of (1)–(4) generate a C_0-semigroup on H and its infinitesimal generator takes a simple form, an approximation scheme was constructed in Huyer (1995) by applying a variational version of the Trotter-Kato theorem. The natural state space for our problem would be $L^1((0,\infty)\times\Omega;\mathbb{R})$ since the L^1-norm of a nonnegative age-space density is the corresponding total population, but the diffusion operator is easier to handle in L^2 and some of our considerations require a Hilbert space setting.

For the rest of this paper we make the following assumption about Ω:

$$\Omega \text{ is a bounded domain in } \mathbb{R}^n \ (n \in \mathbb{N}) \text{ of class } C^2. \tag{5}$$

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In Section 2 we characterize the population operator A and the diffusion operator B. The operators $A-\omega I$ and B are m-dissipative, where ω is an appropriately defined nonnegative constant, and the sum of these two operators defined on the intersection of their domains is trivially dissipative. Since each dissipative operator on a Hilbert space has an m-dissipative extension (cf. Tanabe (1979, Theorem 2.1.2)), we obtain existence of an infinitesimal generator $\mathcal{A}$ and well-posedness of our problem already under the assumptions of Section 2. However, this vague characterization of $\mathcal{A}$ is in general not sufficient for application of semigroup tools, and it is the aim of this paper to find conditions which allow a simple characterization of $\mathcal{A}$ on $\operatorname{dom}\mathcal{A}$ or at least a core. In the case of space-independent birth and mortality rates it is possible to apply a standard theorem on the m-dissipativeness of the sum of two m-dissipative operators to show that $A+B$ is the infinitesimal generator of a C_0-semigroup on H if appropriate regularity assumptions on the diffusion coefficients are made (cf. Huyer (1994) for $k_i(a,x)=k(a)$, $i=1,\dots,n$, but the approach can easily be extended to more general diffusion coefficients). The matter becomes more complicated for x-dependent birth and mortality rates, and we present three different approaches to proving well-posedness of our problem. We had to impose different technical conditions on the functions β, μ and k_i in order to succeed, but we avoided the trivial assumption of a bounded mortality rate. If $\mu\in L^\infty((0,\infty)\times\Omega;\mathbb{R})$, the death term in the operator A can be regarded as a bounded perturbation and no additional regularity conditions on μ are necessary.

In Section 3 we use the approach of Da Prato and Grisvard (1975) to deal with the sum of two closed operators satisfying the resolvent estimates, and in Section 4 we apply the "classical" approach to show that the sum of two m-dissipative operators is again m-dissipative if certain conditions are fulfilled. In Section 5 we use a completely different approach. We define an operator $\mathcal{A}$ appertaining to the full problem and show that $\mathcal{A}-\omega\mathcal{I}$ is m-dissipative for some $\omega>0$.

2. The population operator and the diffusion operator

Let the population operator $A:\operatorname{dom}A\subset H\to H$ be defined by

$$\begin{aligned}\operatorname{dom}A=\{\varphi\in H\mid{}&\varphi(\cdot,x)\text{ is locally absolutely continuous on }[0,\infty)\text{ and}\\&\varphi(0,x)=\textstyle\int_0^\infty\beta(a,x)\varphi(a,x)\,da\text{ for almost all }x\in\Omega,\ \tfrac{\partial}{\partial a}\varphi+\mu\varphi\in H\},\\(A\varphi)(a,x)={}&-\frac{\partial}{\partial a}\varphi(a,x)-\mu(a,x)\varphi(a,x),\quad\varphi\in\operatorname{dom}A,\end{aligned}$$

and the diffusion operator $B:\operatorname{dom}B\subset H\to H$ be defined by

$$\begin{aligned}\operatorname{dom}B=\{\varphi\in H\mid{}&\tfrac{\partial}{\partial x_i}\varphi,\tfrac{\partial^2}{\partial x_i^2}\varphi\in H\text{ for }i=1,\dots,n\text{ and }\sum_{i=1}^n k_i(a,x)\nu_i(x)\tfrac{\partial}{\partial x_i}\varphi(a,x)=0\text{ on }\partial\Omega\\&\text{for almost all }a\geq0\},\\(B\varphi)(a,x)={}&\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\Big),\quad\varphi\in\operatorname{dom}B.\end{aligned}$$

Then we define the operator $\mathcal{A}:\operatorname{dom}\mathcal{A}\subset\mathcal{H}\to\mathcal{H}$ appertaining to the full problem by

$$\operatorname{dom}\mathcal{A}=\operatorname{dom}\mathcal{A}\cap\operatorname{dom}\mathcal{B},\tag{6}$$

$$\mathcal{A}\varphi=\mathcal{A}\varphi+\mathcal{B}\varphi,\quad\varphi\in\operatorname{dom}\mathcal{A}.\tag{7}$$

First we consider the diffusion operator. If the functions k_i satisfy

$$\begin{aligned}&k_i(a,\cdot)\in C^1(\overline{\Omega};\mathbb{R})\text{ for almost all }a\geq0,\ k_i,\tfrac{\partial}{\partial x_i}k_i\in\mathbb{L}^\infty((0,\infty)\times\Omega;\mathbb{R}),\\&k_i(a,x)\geq k_0>0\text{ a.e. on }[0,\infty)\times\Omega,\ i=1,\dots,n,\end{aligned}\tag{8}$$

then it is possible to show similarly as in Huyer (1994) by means of a sesquilinear form (theory of "regularly dissipative operators"; cf. Tanabe (1979), for example) that B is self-adjoint and m-dissipative and generates an analytic semigroup on H.

Next we turn to the operator A. In contrast to the case of space-independent β and μ, it does not immediately follow from the properties of the age-dependent population operator without space structure that A is the infinitesimal generator of a C_0-semigroup but we should say a few words about it. Let the functions μ and β satisfy

$$\mu\in L^\infty((0,a_1)\times\Omega;\mathbb{R})\text{ for each }a_1\geq0,\ \mu\geq0\text{ a.e.}\tag{9}$$

$$\beta(\cdot,x)\in L^2(0,\infty;\mathbb{R})\text{ for a.a. }x\in\Omega,\ \operatorname*{ess\,sup}_{x\in\Omega}\textstyle\int_0^\infty\beta(a,\cdot)^2\,a<\infty,\ \beta\geq0\text{ a.e.}\tag{10}$$

The mortality rate is allowed to increase unboundedly as $a\to\infty$. Then it is easy to see that $A-\omega I$ is dissipative for $\omega:=\frac{1}{2}\operatorname{ess\,sup}_{x\in\Omega}\int_0^\infty\beta(a,x)^2\,da$. Moreover, it can be shown that for $\lambda>\omega$ we have

$\operatorname{range}(\lambda I - A) \supset \{\psi \in H \mid \text{there exists an } a_1 \geq 0 \text{ such that } \psi(a,x) = 0 \text{ a.e. on } (a_1,\infty)\times\Omega\}$, which is dense in H, and the following lemma holds.

Lemma 1. *The operator A is closed.*

Proof. Let $\varphi_n \in \operatorname{dom} A$, $\psi_n := A\varphi_n$ be given with $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in H. By assumption (9) we have $\eta := -\psi - \mu\varphi \in L^2((0,a_1)\times\Omega;\mathbb{R})$ and $\int_0^{a_1}\int_\Omega |\frac{\partial}{\partial a}\varphi_n(a,x) - \eta(a,x)|^2\,dx\,da \to 0$ as $n \to \infty$ for each $a_1 > 0$. Then similarly as in Banks et al. (1991) we can argue that

$$\varphi(a,x) = \int_0^\infty \beta(b,x)\varphi(b,x)\,db + \int_0^a \eta(b,x)\,db$$

for almost all $x \in \Omega$ and $a \geq 0$, which proves $\varphi \in \operatorname{dom} A$ and $A\varphi = \psi$. □

Lemma 1 implies that $\operatorname{range}(\lambda I - A)$ is closed and thus $\operatorname{range}(\lambda I - A) = H$ for all $\lambda > \omega$, i.e., $A - \omega I$ is m-dissipative. Since H is a Hilbert space, A is the infinitesimal generator of a C_0-semigroup on H.

3. The approach of Da Prato and Grisvard

In this section (8), (9) and (10) are replaced with the following stronger assumptions:

$$\begin{aligned}
&\text{The functions } k_i \text{ satisfy (8), } k_i(a,x) = k(a,x)\tilde{k}_i(x),\ \tfrac{\partial}{\partial a}k,\ \tfrac{\partial^2}{\partial x_i^2}k \in L^\infty((0,\infty)\times\Omega;\mathbb{R}),\\
&\tilde{k}_i \in C^1(\overline{\Omega};\mathbb{R}),\ i = 1,\dots,n,\ \textstyle\sum_{i=1}^n \tilde{k}_i(x)\nu_i(x)\frac{\partial}{\partial x_i}k(a,x) = 0 \text{ on } \partial\Omega \text{ for almost all } a \geq 0.
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\mu,\ \tfrac{\partial^j}{\partial x_i^j}\mu\ (i = 1,\dots,n,\ j = 1,2) \in L^\infty((0,a_1)\times\Omega;\mathbb{R}) \text{ for each } a_1 \geq 0,\ \mu \geq 0 \text{ a.e.,}\\
&\text{there exist constants } c_1 \text{ and } c_2 \text{ such that } |\tfrac{\partial^j}{\partial x_i^j}\mu(a,x)| \leq c_1\mu(a,x) + c_2,\ i = 1,\dots,n,\\
&j = 1,2,\ \textstyle\sum_{i=1}^n k_i(a,x)\nu_i(x)\frac{\partial}{\partial x_i}\mu(a,x) = 0 \text{ on } \partial\Omega \text{ for a.a. } a \geq 0.
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\beta(\cdot,x),\ \tfrac{\partial^j}{\partial x_i^j}\beta(\cdot,x)\ (i = 1,\dots,n,\ j = 1,2) \in L^2(0,\infty;\mathbb{R}) \text{ for a.a. } x \in \Omega,\ \beta \geq 0 \text{ a.e.,}\\
&\operatorname*{ess\,sup}_{x\in\Omega}\textstyle\int_0^\infty \beta(a,x)^2\,da < \infty,\ \operatorname*{ess\,sup}_{x\in\Omega}\int_0^\infty \frac{\partial^j}{\partial x_i^j}\beta(a,x)^2\,da < \infty,\ i = 1,\dots,n,\ j = 1,2,\\
&\sum_{i=1}^n k_i(a,x)\nu_i(x)\tfrac{\partial}{\partial x_i}\beta(a,x) = 0 \text{ on } \partial\Omega \text{ for almost all } a \geq 0.
\end{aligned} \tag{13}$$

First we summarize the notations and results of Da Prato and Grisvard (1975) that we shall use in the following. Let X be a complex Banach space with norm $\|\cdot\|$.

Definition 1. A linear operator $A : \operatorname{dom} A \subset X \to X$ satisfies $\mathrm{H}(\varphi)$ for a $\varphi \in [0,\pi)$ if the resolvent set $\rho(A)$ of A contains a sector

$$\Sigma_A = \{z \in \mathbb{C} \mid -\pi + \varphi < \arg z < \pi - \varphi\}$$

and there exists an even convex function $C_A : (-\pi+\varphi, \pi-\varphi) \to \mathbb{R}$ such that

$$\|(zI - A)^{-1}\| \leq \frac{C_A(\theta)}{|z|}$$

for $\arg z = \theta$, $|\theta| < \pi - \varphi$.

Let A and B be two closed linear operators on X satisfying $\mathrm{H}(\theta_A)$ and $\mathrm{H}(\theta_B)$ with

$$\theta_A + \theta_B < \pi. \tag{14}$$

Definition 2. A and B satisfy $\mathrm{H}(A,B,\varphi)$ if

$$(\lambda I - A)^{-1}(\operatorname{dom} B) \subset \operatorname{dom} B \quad \text{for all } \lambda \in \rho(A)$$

and there exist two real-valued functions C and φ defined on $(-\pi+\theta_A, \pi-\theta_A)\times(-\pi+\theta_B,\pi-\theta_B)$ and $(0,\infty)\times(0,\infty)$, respectively, where C is even and convex in both variables and

$$\lim_{\lambda\to+\infty}\int_\gamma \varphi(|z+\lambda|,|z|)d|z| = 0,$$
$$\|[B,(z'I-A)^{-1}](z''I-B)^{-1}\| \leq C(\theta',\theta'')\varphi(|z'|,|z''|)$$

for $\theta' = \arg z'$, $\theta'' = \arg z''$, $|\theta'| < \pi - \theta_A$, $|\theta''| < \pi - \theta_B$. Here $[\cdot,\cdot]$ denotes the commutator of two operators and γ is a simply connected curve running from $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$, $\theta_B < \theta_0 < \pi - \theta_A$, and remaining in $(\Sigma_A - \lambda)\cap(-\Sigma_B)$.

Theorem 1 (Da Prato and Grisvard (1975, p. 350, Théorème 6.3)). *Let A and B be two closed, densely defined linear operators on X satisfying $\mathrm{H}(\theta_A)$ and $\mathrm{H}(\theta_B)$, respectively, with (14) and $\mathrm{H}(A, B, \varphi)$, and let L be the sum of A and B defined on the intersection of their domains. Then the closure $\overline{L}$ of L exists and there is an $\omega \geq 0$ such that $\rho(\overline{L}) \supset (\omega, \infty)$. If $C_A(0) = C_B(0) = 1$, we have $\omega = 0$.*

The authors introduce the following additional assumption:

$$\|[B,(z'I-A)^{-1}](z''I-B)^{-1}\|_{\mathcal{L}(\mathcal{D}_B(\theta,\in))} \leq C(\theta',\theta'')\varphi(|z'|,|z''|) \quad \text{for a } \theta \in (0,1) \tag{15}$$

for $\theta' = \arg z'$, $\theta'' = \arg z''$, $|\theta'| < \pi - \theta_A$, $|\theta''| < \pi - \theta_B$, where the interpolation space $D_B(\theta, p)$ is defined by

$$D_B(\theta,p) := (\operatorname{dom} B; X)_{1-\theta,p}, \quad p \in [1,\infty], \quad \theta \in (0,1),$$

for a closed linear operator B on X (cf. e.g. Triebel (1978) for the definition and properties of interpolation spaces). If X is a Hilbert space and B is positive and self-adjoint, then $D_B(\theta, 2) = \operatorname{dom} B^\theta$, the domain of the power θ of B (cf. Triebel (1978, Theorem 1.18.10)).

Theorem 2 (Da Prato and Grisvard (1975, p. 356, Théorème 6.8)). *Let A and B be two closed linear operators with dense domains on a Hilbert space X. Assume that A and B satisfy $\mathrm{H}(\theta_A)$ and $\mathrm{H}(\theta_B)$, respectively, with (14), $\mathrm{H}(A, B, \varphi)$ and (15) with $\varphi(|z + \lambda|, |z|) = O(|z|^{-1})$ for fixed λ. Assume in addition that $D_B(\theta, 2) = D_{B^*}(\theta, 2)$. Then the operator L defined as in Theorem 1 is closed and there exists an $\omega \geq 0$ such that $\rho(L) \supset (\omega, \infty)$.*

Now we return to our population diffusion problem. Since complex spaces are needed for this approach, we replace H with its complexification. Moreover, for the rest of the paper we shall occasionally use the abbreviations φ_a and φ_{x_i} for the generalized derivatives of φ. Let A, B and ω be as in Section 2. We cannot apply the results of Da Prato and Grisvard immediately to the operators A and B in the case that the diffusion coefficients depend on a since then the operator "multiplication with k_i" does not commute with the resolvent of A. Therefore we consider the two auxiliary operators A_1 and B_1 on H, defined by

$$(A_1\varphi)(a,x) := -k(a,x)^{-1}\frac{\partial}{\partial a}\varphi(a,x) - k(a,x)^{-1}\mu(a,x)\varphi(a,x), \quad \varphi \in \operatorname{dom} A_1 := \operatorname{dom} A,$$

and

$$(B_1\varphi)(a,x) := k(a,x)^{-1}(B\varphi)(a,x) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\tilde{k}_i(x)\frac{\partial}{\partial x_i}\varphi(a,x)\Big) + \sum_{i=1}^n \tilde{k}_i(x)\frac{k_{x_i}(a,x)}{k(a,x)}\frac{\partial}{\partial x_i}\varphi(a,x),$$
$$\varphi \in \operatorname{dom} B_1 := \operatorname{dom} B.$$

The coefficients of the second-order terms of B_1 are independent of a and we made the assumption $k_i(a,x) = k(a,x)\tilde{k}_i(x)$ in order to be able to achieve that.

Lemma 2. *$A_1 - \omega_1 I$ is m-dissipative for $\omega_1 := \frac{\omega}{k_1} + \frac{1}{2}\|k^{-2}\frac{\partial}{\partial a}k\|_\infty$, where $k_1 > 0$ is such that $k(a,x) \geq k_1$ a.e. on $[0,\infty) \times \Omega$ (e.g. $k_1 = k_0/\|\tilde{k}_i\|_\infty$ for some i).*

Proof. Let $\varphi \in \operatorname{dom} A_1$. For $\delta > 0$ we obtain through integration by parts

$$\begin{aligned}
&\operatorname{Re}\int_0^\delta\int_\Omega \overline{\varphi(a,x)}(A_1\varphi)(a,x)\,dx\,da \leq -\operatorname{Re}\int_0^\delta\int_\Omega k(a,x)^{-1}\frac{\partial}{\partial a}\varphi(a,x)\overline{\varphi(a,x)}\,dx\,da \\
&\leq \int_\Omega k(0,x)^{-1}|\varphi(0,x)|^2\,dx + \operatorname{Re}\int_0^\delta\int_\Omega \overline{\varphi(a,x)}\Big(k(a,x)^{-1}\frac{\partial}{\partial a}\varphi(a,x) - k(a,x)^{-2}\frac{\partial}{\partial a}k(a,x)\varphi(a,x)\Big)\,dx\,da.
\end{aligned}$$

The limit $\delta \to \infty$ is possible, and this together with the assumptions yields

$$\operatorname{Re}\langle\varphi, A_1\varphi\rangle_H \leq \Big(\frac{\omega}{k_1} + \frac{1}{2}\Big\|k^{-2}\frac{\partial}{\partial a}k\Big\|_\infty\Big)\|\varphi\|_H^2.$$

The range condition can be reduced to that of an ordinary population operator. ☐

Lemma 3. *$B_1 - \omega_2 I$ is m-dissipative for $\omega_2 := \frac{k_2}{2} + \frac{nK^2}{2k_2}$, where $k_2 := \min_{1\leq i\leq n, x\in\overline{\Omega}} \tilde{k}_i(x)$ and $K := \max_{1\leq i\leq n}\|\tilde{k}_i k^{-1}\frac{\partial}{\partial x_i}k\|_\infty$, and B_1 is the infinitesimal generator of an analytic semigroup.*

Proof. Let V be the complex Hilbert space $V := \{\varphi \in H \mid \varphi_{x_i} \in H \text{ for } i = 1,\ldots,n\}$, endowed with the inner product

$$\langle\varphi,\psi\rangle_V := \int_0^\infty\int_\Omega \overline{\varphi(a,x)}\psi(a,x)\,dx\,da + \sum_{i=1}^n\int_0^\infty\int_\Omega \overline{\varphi_{x_i}(a,x)}\psi_{x_i}(a,x)\,dx\,da, \quad \varphi,\psi \in V.$$

This space is densely and continuously embedded in H. We define the sesquilinear form c on $V \times V$ by

$$\begin{aligned} c(\varphi,\psi) &:= \sum_{i=1}^{n} \int_0^\infty \int_\Omega \tilde{k}_i(x)\overline{\varphi_{x_i}(a,x)}\psi_{x_i}(a,x)\,dx\,da \\ &\quad - \sum_{i=1}^{n} \int_0^\infty \int_\Omega \tilde{k}_i(x)\frac{k_{x_i}(a,x)}{k(a,x)}\overline{\varphi(a,x)}\psi_{x_i}(a,x)\,dx\,da, \quad \varphi,\psi \in V. \end{aligned}$$

Then we have

$$|c(\varphi,\psi)| \le \big(\max_{1\le i\le n} \|\tilde{k}_i\|_\infty + K\big)\|\varphi\|_V\|\psi\|_V, \quad \varphi,\psi \in V,$$

and as in Tanabe (1979, p. 30)

$$\begin{aligned} \operatorname{Re} c(\varphi,\varphi) &\ge k_2 \sum_{i=1}^{n} \int_0^\infty \int_\Omega |\varphi_{x_i}(a,x)|^2\,dx\,da - K\sum_{i=1}^{n} \int_0^\infty \int_\Omega |\varphi(a,x)\varphi_{x_i}(a,x)|\,dx\,da \\ &\ge \frac{k_2}{2} \sum_{i=1}^{n} \int_0^\infty \int_\Omega |\varphi_{x_i}(a,x)|^2\,dx\,da - \frac{nK^2}{2k_2}\int_0^\infty \int_\Omega |\varphi(a,x)|^2\,dx\,da \\ &= \frac{k_2}{2}\|\varphi\|_V^2 - \Big(\frac{nK^2}{2k_2} + \frac{k_2}{2}\Big)\|\varphi\|_H^2, \quad \varphi \in V, \end{aligned} \tag{16}$$

where we have used the inequality

$$|\varphi(a,x)\varphi_{x_i}(a,x)| \le \frac{K}{2k_2}|\varphi(a,x)|^2 + \frac{k_2}{2K}|\varphi_{x_i}(a,x)|^2,$$

i.e., c is bounded and coercive. Then Tanabe (1979, Theorems 2.2.2 and 3.6.1 and p. 29, Example 1) imply that B_1 is the unique operator such that

$$\operatorname{dom} B_1 = \{\varphi \in V \mid \text{there exists an } \alpha > 0 \text{ such that } |c(\psi,\varphi)| \le \alpha\|\psi\|_H \text{ for all } \psi \in V\},$$
$$c(\psi,\varphi) = -\langle \psi, B_1\varphi\rangle_H, \quad \varphi \in \operatorname{dom} B_1, \quad \psi \in V,$$

$B_1 - \omega_2 I$ is m-dissipative and B_1 is the infinitesimal generator of an analytic semigroup. □

Let $A_2 := A_1 - (\omega_1 + 1)I$ and $B_2 := B_1 - \omega_2 I$. Then A_2 satisfies $\mathrm{H}(\frac{\pi}{2})$ and B_2 satisfies $\mathrm{H}(\theta)$ for some $\theta \in [0, \frac{\pi}{2})$. We verify the assumption $\mathrm{H}(A_2, B_2, \varphi)$ in several steps.

Lemma 4. *Let $\varphi \in \operatorname{dom} B$ and $\lambda \in \rho(A_1)$. Then we have $(\lambda I - A_1)^{-1}\varphi \in \operatorname{dom} B$.*

Proof. This follows from the representation

$$\begin{aligned} &((\lambda I - A_1)^{-1}\varphi)(a,x) = \exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big) \\ &\times \Big(1 - \int_0^\infty \beta(a,x)\exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\,da\Big)^{-1} \\ &\times \int_0^\infty \beta(a,x)\int_0^a \exp\Big(-\lambda\int_b^a k(\sigma,x)\,d\sigma - \int_b^a \mu(\sigma,x)\,d\sigma\Big)k(b,x)\varphi(b,x)\,db\,da \\ &+ \int_0^a \exp\Big(-\lambda\int_b^a k(\sigma,x)\,d\sigma - \int_b^a \mu(\sigma,x)\,d\sigma\Big)k(b,x)\varphi(b,x)\,db \end{aligned} \tag{17}$$

and the assumptions on k, μ and β. □

Remark 1. The regularity assumptions on k, μ and β are needed to ensure that the generalized derivatives $\frac{\partial^j}{\partial x_i^j}(\lambda I - A_1)^{-1}\varphi$, $i = 1,\ldots,n$, $j = 1,2$, are in H for $\varphi \in \operatorname{dom} B$ and can be estimated in terms of resolvents, and the boundary conditions on k, μ and β guarantee that $(\lambda I - A_1)^{-1}\varphi$ satisfies the Neumann boundary condition for $\varphi \in \operatorname{dom} B$ and $\lambda \in \rho(A_1)$. We would not need any additional boundary conditions in the case of homogeneous Dirichlet boundary conditions.

Lemma 5. *For each θ' with $|\theta'| < \frac{\pi}{2}$ there exists an $M'(\theta') > 0$ such that*

$$\Big\|\frac{\partial}{\partial x_i}(\lambda I - A_2)^{-1}\varphi\Big\|_H \le \frac{M'(\theta')}{|\lambda|}\Big(\|\varphi\|_H + \Big\|\frac{\partial}{\partial x_i}\varphi\Big\|_H\Big)$$

and

$$\Big\|\frac{\partial^2}{\partial x_i^2}(\lambda I - A_2)^{-1}\varphi - (\lambda I - A_2)^{-1}\frac{\partial^2}{\partial x_i^2}\varphi\Big\|_H \le \frac{M'(\theta')}{|\lambda|}\Big(\|\varphi\|_H + \Big\|\frac{\partial}{\partial x_i}\varphi\Big\|_H\Big)$$

for $i = 1,\ldots,n$, all $\varphi \in \operatorname{dom} B$ and all λ with $\arg\lambda = \theta'$.

Proof. The representation (17) yields for $\lambda \in \rho(A_1)$

$$\begin{aligned}
&\frac{\partial}{\partial x_i}((\lambda I - A_1)^{-1}\varphi)(a,x) = (\lambda I - A_1)^{-1}\frac{\partial}{\partial x_i}\varphi(a,x) - \Big(\lambda\int_0^a \frac{\partial}{\partial x_i}k(\sigma,x)\,d\sigma + \int_0^a \frac{\partial}{\partial x_i}\mu(\sigma,x)\,d\sigma\Big)\\
&\quad\times \exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\Big(1 - \int_0^\infty \beta(a,x)\exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\,da\Big)^{-1}\\
&\quad\times \int_0^\infty \beta(a,x)\int_0^a \exp\Big(-\lambda\int_b^a k(\sigma,x)\,d\sigma - \int_b^a \mu(\sigma,x)\,d\sigma\Big)k(b,x)\varphi(b,x)\,db\,da\\
&- \int_0^a \Big(\lambda\int_b^a \frac{\partial}{\partial x_i}k(\sigma,x)\,d\sigma + \int_b^a \frac{\partial}{\partial x_i}\mu(\sigma,x)\,d\sigma\Big)\exp\Big(-\lambda\int_b^a k(\sigma,x)\,d\sigma - \int_b^a \mu(\sigma,x)\,d\sigma\Big)k(b,x)\varphi(b,x)\,db\\
&+ \exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\Big(1 - \int_0^\infty \beta(a,x)\exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\,da\Big)^{-1}\\
&\quad\times \int_0^\infty \frac{\partial}{\partial x_i}\beta(a,x)((\lambda I - A_1)^{-1}\varphi)(a,x)\,da\\
&- \exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\Big(1 - \int_0^\infty \beta(a,x)\exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\,da\Big)^{-2}\\
&\quad\times \int_0^\infty \beta(a,x)\Big(\lambda\int_0^a \frac{\partial}{\partial x_i}k(\sigma,x)\,d\sigma + \int_0^a \frac{\partial}{\partial x_i}\mu(\sigma,x)\,d\sigma\Big)\exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\,da\\
&\quad\times \int_0^\infty \beta(a,x)\int_0^a \exp\Big(-\lambda\int_b^a k(\sigma,x)\,d\sigma - \int_b^a \mu(\sigma,x)\,d\sigma\Big)k(b,x)\varphi(b,x)\,db\,da\\
&- \exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\Big(1 - \int_0^\infty \beta(a,x)\exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\,da\Big)^{-1}\\
&\quad\times \int_0^\infty \beta(a,x)\int_0^a \Big(\lambda\int_b^a \frac{\partial}{\partial x_i}k(\sigma,x)\,d\sigma + \int_b^a \frac{\partial}{\partial x_i}\mu(\sigma,x)\,d\sigma\Big)\exp\Big(-\lambda\int_0^a k(\sigma,x)\,d\sigma - \int_b^a \mu(\sigma,x)\,d\sigma\Big)\\
&\quad\times k(b,x)\varphi(b,x)\,db\,da + ((\lambda I - A_1)^{-1}k^{-1}k_{x_i}\varphi)(a,x).
\end{aligned}$$

For $0 \le b \le a$ and λ with $\arg\lambda = \theta'$, $|\theta'| < \frac{\pi}{2}$ we have by (11)–(13)

$$\begin{aligned}
&\Big|\Big((\lambda+\omega_1+1)\int_b^a \frac{\partial}{\partial x_i}k(\sigma,x)\,d\sigma + \int_b^a \frac{\partial}{\partial x_i}\mu(\sigma,x)\,d\sigma\Big)\exp\Big(-(\lambda+\omega_1+1)\int_b^a k(\sigma,x)\,d\sigma - \int_b^a \mu(\sigma,x)\,d\sigma\Big)\Big|\\
&\le \frac{2|\lambda+\omega_1+1|}{k_1 e(\operatorname{Re}\lambda+2)}\|k_{x_i}\|_\infty \exp\Big(-\Big(\frac12\operatorname{Re}\lambda+\omega_1\Big)\int_b^a k(\sigma,x)\,d\sigma\Big) + \frac{c_1}{e}\exp\Big(-(\operatorname{Re}\lambda+\omega_1)\int_b^a k(\sigma,x)\,d\sigma\Big)\\
&\quad + \frac{c_2}{k_1 e}\exp\Big(-(\operatorname{Re}\lambda+\omega_1)\int_b^a k(\sigma,x)\,d\sigma\Big) \le M_1(\theta')\exp\Big(-\Big(\frac12\operatorname{Re}\lambda+\omega_1\Big)\int_b^a k(\sigma,x)\,d\sigma\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty \beta(a,x)\Big|\exp\Big(-(\lambda+\omega_1+1)\int_0^a k(\sigma,x)\,d\sigma - \int_0^a \mu(\sigma,x)\,d\sigma\Big)\Big|\,da\\
&\le (2\omega)^{1/2}\Big(\int_0^\infty e^{-2(\omega_1+1)k_1 a}\,da\Big)^{1/2} \le \Big(\frac{\omega_1}{\omega_1+1}\Big)^{1/2} < 1,
\end{aligned}$$

where k_1 is defined as in Lemma 2, which together with the assumptions implies

$$\begin{aligned}
&\Big\|\frac{\partial}{\partial x_i}(\lambda I - A_2)^{-1}\varphi\Big\|_H \le \Big\|(\lambda I - A_2)^{-1}\frac{\partial}{\partial x_i}\varphi\Big\|_H + M_1(\theta')\Big\|\Big(\Big(\frac12\operatorname{Re}\lambda+\omega_1\Big)I - A_0\Big)^{-1}|\varphi|\Big\|_H\\
&+ (2k_1)^{-1/2}\big((\omega_1+1)^{1/2} - \omega_1^{1/2}\big)^{-1}\operatorname*{ess\,sup}_{x\in\Omega}\Big(\int_0^\infty \Big(\frac{\partial}{\partial x_i}\beta(a,x)\Big)^2\,da\Big)^{1/2}\|(\lambda I - A_2)^{-1}\varphi\|_H\\
&+ M_1(\theta')\Big(\frac{\omega}{k_1}\Big)^{1/2}\big((\omega_1+1)^{1/2} - \omega_1^{1/2}\big)^{-1}\Big\|\Big(\Big(\frac12\operatorname{Re}\lambda+\omega_1\Big)I - A_0\Big)^{-1}|\varphi|\Big\|_H + \|(\lambda I - A_2)^{-1}k^{-1}k_{x_i}\varphi\|_H,
\end{aligned}$$

where A_0 is the operator A_1 with the mortality rate replaced with $\mu \equiv 0$, and the assertion follows from the properties of A_2 and A_0. For $\frac{\partial^2}{\partial x_i^2}(\lambda I - A_2)^{-1}\varphi$ we obtain an even nastier formula and can make a similar estimate to obtain the desired result. ☐

Lemma 6. *There exists a constant $c_3 > 0$ such that for all $\varphi \in H$ and λ with $|\arg\lambda| < \pi - \theta$ we have*

$$\sum_{i=1}^n \Big\|\frac{\partial}{\partial x_i}(\lambda I - B_2)^{-1}\varphi\Big\|_H \le c_3\|\varphi\|_H^{1/2}\|(\lambda I - B_2)^{-1}\varphi\|_H^{1/2}.$$

Proof. Let $\varphi \in H$. Then (16) and the fact that $c(\psi,\psi) = -\langle \psi, B_1\psi\rangle_H$ for $\psi \in \operatorname{dom} B_1$ imply

$$\begin{aligned}\Big(\sum_{i=1}^n \Big\|\frac{\partial}{\partial x_i}(\lambda I - B_2)^{-1}\varphi\Big\|_H\Big)^2 &\le n\sum_{i=1}^n \Big\|\frac{\partial}{\partial x_i}(\lambda I - B_2)^{-1}\varphi\Big\|_H^2 \\ &\le -n\|(\lambda I - B_2)^{-1}\varphi\|_H^2 - \frac{2n}{k_2}\operatorname{Re}\langle(\lambda I - B_2)^{-1}\varphi, B_2(\lambda I - B_2)^{-1}\varphi\rangle_H \\ &\le \frac{2n}{k_2}\|(\lambda I - B_2)^{-1}\varphi\|_H\|B_2(\lambda I - B_2)^{-1}\varphi\|_H,\end{aligned}$$

and the fact that B_2 is m-dissipative and generates an analytic semigroup yields the assertion. □

Now we can prove

Lemma 7. *For $\varphi \in H$, $\lambda', \lambda'' \in \mathbb{C}$, $\arg\lambda' = \theta'$, $\arg\lambda'' = \theta''$, $|\theta'| < \frac{\pi}{2}$, $|\theta''| < \pi - \theta$ we have*

$$\|[B_2, (\lambda' I - A_2)^{-1}](\lambda'' I - B_2)^{-1}\| \le M(\theta',\theta'')\Big(\frac{1}{|\lambda'||\lambda''|} + \frac{1}{|\lambda'||\lambda''|^{1/2}}\Big).$$

Proof. Let $\psi := (\lambda'' I - B_2)^{-1}\varphi$ for some $\varphi \in H$. Then

$$\begin{aligned}&([B_2, (\lambda' I - A_2)^{-1}]\psi)(a,x) \\ &= \sum_{i=1}^n \tilde{k}_i(x)\frac{\partial^2}{\partial x_i^2}((\lambda' I - A_2)^{-1}\psi)(a,x) + \sum_{i=1}^n\Big(\frac{\partial}{\partial x_i}\tilde{k}_i(x) + \tilde{k}_i(x)\frac{k_{x_i}(a,x)}{k(a,x)}\Big)\frac{\partial}{\partial x_i}((\lambda' I - A_2)^{-1}\psi)(a,x) \\ &\quad - \sum_{i=1}^n \tilde{k}_i(x)(\lambda' I - A_2)^{-1}\frac{\partial^2}{\partial x_i^2}\psi(a,x) - \sum_{i=1}^n(\lambda' I - A_2)^{-1}\Big(\frac{\partial}{\partial x_i}\tilde{k}_i(x) + \tilde{k}_i(x)\frac{k_{x_i}(a,x)}{k(a,x)}\Big)\frac{\partial}{\partial x_i}\psi(a,x).\end{aligned}$$

Lemmas 5 and 6 and the assumptions imply

$$\|[B_2, (\lambda' I - A_2)^{-1}]\psi\|_H \le \frac{M_2(\theta')}{|\lambda'|}\Big(\|\psi\|_H + \sum_{i=1}^n\|\psi_{x_i}\|_H\Big) \le \frac{M_2(\theta')}{|\lambda'|}\Big(\frac{M_3(\theta'')}{|\lambda''|} + \frac{c_3 M_3(\theta'')^{1/2}}{|\lambda''|^{1/2}}\Big)\|\varphi\|_H,$$

which finishes the proof. □

Lemmas 4 and 7 prove $\mathrm{H}(A_2, B_2, \varphi)$, and an application of Theorem 1 gives

Lemma 8. *Let* (11)–(13) *be satisfied. Then the closure of the operator C, defined by $C\varphi := A_2\varphi + B_2\varphi$, $\varphi \in \operatorname{dom} C := \operatorname{dom} A_2 \cap \operatorname{dom} B_2$, is m-dissipative.*

Lemma 8 and the dissipativeness of C imply that $\operatorname{range}(\lambda I - C)$ is dense in H for all $\lambda > 0$. Since C is closable, the same holds for $\mathcal{A}$. The range condition for $\mathcal{A}$ can be reduced to the range condition for C, and we obtain the following result.

Theorem 3. *Let* (11)–(13) *be satisfied. Then the closure $\overline{\mathcal{A}}$ of the operator $\mathcal{A}$ defined by* (6) *and* (7) *is the infinitesimal generator of a C_0-semigroup on H.*

Since $\operatorname{dom} B_2 = \operatorname{dom} B_2^*$ (cf. Tanabe (1979)), the condition $D_{B_2}(\theta,2) = D_{B_2^*}(\theta,2)$ is satisfied. If suitable stronger assumptions on k_i, β and μ concerning regularity with respect to x are imposed, the conditions of Theorem 2 are satisfied and C and thus also $\mathcal{A}$ itself generates a strongly continuous semigroup on H.

4. The "classical" approach

In this section we assume that (8), (12) and (13) are satisfied, and in addition

$$\tfrac{\partial}{\partial a}k_i, \tfrac{\partial}{\partial x_i}\mu \in L^\infty((0,\infty)\times\Omega;\mathbb{R}),\ i = 1,\dots,n. \tag{18}$$

The disadvantage of this approach is that we have to assume that the gradient of μ is bounded, which is, however, an assumption that is often used in the literature on age-structured population models with spatial diffusion. On the other hand, the assumptions on the diffusion coefficients are weaker than in Section 3.

Let A, B and ω be defined as in Section 2, and let $C := A - \omega I$. Then $B + C$, defined on $\operatorname{dom} B \cap \operatorname{dom} C$, is dissipative. For verifying the range condition for $B + C$, the classical approach (cf. Barbu (1976), for example) is to consider for an arbitrary $\psi \in H$ the approximate equations

$$\varphi_\lambda - B\varphi_\lambda - C_\lambda\varphi_\lambda = \psi \tag{19}$$

for $\lambda > 0$, where $C_\lambda := C(I-\lambda C)^{-1} = \frac{1}{\lambda}(I-\lambda C)^{-1} - \frac{1}{\lambda}I$ is the Yosida approximation of C. Then it is well-known that (19) possesses a solution $\varphi_\lambda \in \operatorname{dom} B$ for each $\lambda > 0$ and $\|\varphi_\lambda\|_H$ is bounded as $\lambda \to 0$. Moreover, the equation

$$\varphi - B\varphi - C\varphi = \psi$$

has a unique solution $\varphi \in \operatorname{dom} B \cap \operatorname{dom} C$ and $\lim_{\lambda\to 0} \varphi_\lambda = \varphi$ if and only if $\|C_\lambda \varphi_\lambda\|_H$ is bounded as $\lambda \to 0$ (or, equivalently, $\|B\varphi_\lambda\|_H$ is bounded). We prove that in several steps.

Lemma 9. *There exist positive constants d_1 and d_2 such that for each $\varphi \in \operatorname{dom} A \cap \operatorname{dom} B$ with $A\varphi \in \operatorname{dom} B$ we have*

$$\langle A\varphi, B\varphi\rangle_H \geq -d_1\|\varphi\|_H^2 - d_2 \sum_{i=1}^n \Big\|\frac{\partial}{\partial x_i}\varphi\Big\|_H^2. \tag{20}$$

Proof. Let φ satisfy the assumptions of Lemma 9 and consider

$$\langle A\varphi, B\varphi\rangle_H = -\int_0^\infty \int_\Omega \Big(\frac{\partial}{\partial a}\varphi(a,x) + \mu(a,x)\varphi(a,x)\Big) \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\Big)\,dx\,da.$$

Let $\delta > 0$. Then by using the assumptions and integrating by parts we obtain

$$\begin{aligned}
&-\int_0^\delta \int_\Omega \Big(\frac{\partial}{\partial a}\varphi(a,x) + \mu(a,x)\varphi(a,x)\Big) \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\Big)\,dx\,da \\
&= \sum_{i=1}^n \int_0^\delta\int_\Omega k_i(a,x)\frac{\partial^2}{\partial a\partial x_i}\varphi(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\,dx\,da + \sum_{i=1}^n \int_0^\delta\int_\Omega k_i(a,x)\frac{\partial}{\partial x_i}(\mu(a,x)\varphi(a,x))\frac{\partial}{\partial x_i}\varphi(a,x)\,dx\,da \\
&\geq \sum_{i=1}^n \int_0^\delta\int_\Omega k_i(a,x)\frac{\partial^2}{\partial a\partial x_i}\varphi(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\,dx\,da + \sum_{i=1}^n \int_0^\delta\int_\Omega k_i(a,x)\frac{\partial}{\partial x_i}\mu(a,x)\varphi(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\,dx\,da \\
&=: I + II.
\end{aligned}$$

Since $\varphi \in \operatorname{dom} A$, we have

$$\varphi(0,x) = \int_0^\infty \beta(a,x)\varphi(a,x)\,da \quad \text{for almost all } x \in \Omega,$$

and (13) and the assumptions on φ imply for $i = 1,\dots,n$

$$\frac{\partial}{\partial x_i}\varphi(0,x) = \int_0^\infty \frac{\partial}{\partial x_i}\beta(a,x)\varphi(a,x)\,da + \int_0^\infty \beta(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\,da \quad \text{for almost all } x\in\Omega$$

and

$$\begin{aligned}
\Big(\frac{\partial}{\partial x_i}\varphi(0,x)\Big)^2 &\leq \bigg(\Big(\int_0^\infty \Big(\frac{\partial}{\partial x_i}\beta(a,x)\Big)^2 da\Big)^{1/2}\Big(\int_0^\infty \varphi(a,x)^2\,da\Big)^{1/2} \\
&\qquad + \Big(\int_0^\infty \beta(a,x)^2\,da\Big)^{1/2}\Big(\int_0^\infty \Big(\frac{\partial}{\partial x_i}\varphi(a,x)\Big)^2 da\Big)^{1/2}\bigg)^2 \\
&\leq 2\Big(\int_0^\infty \Big(\frac{\partial}{\partial x_i}\beta(a,x)\Big)^2 da\Big)\Big(\int_0^\infty \varphi(a,x)^2\,da\Big) + 2\Big(\int_0^\infty \beta(a,x)^2\,da\Big)\Big(\int_0^\infty\Big(\frac{\partial}{\partial x_i}\varphi(a,x)\Big)^2 da\Big).
\end{aligned} \tag{21}$$

For I we obtain through integration by parts

$$I \geq -\sum_{i=1}^n \int_\Omega k_i(0,x)\Big(\frac{\partial}{\partial x_i}\varphi(0,x)\Big)^2 dx - \sum_{i=1}^n \int_0^\delta\int_\Omega \frac{\partial}{\partial a}k_i(a,x)\Big(\frac{\partial}{\partial x_i}\varphi(a,x)\Big)^2 dx\,da - I,$$

and (21) yields

$$\begin{aligned}
I \geq{}& -\|\varphi\|_H^2 \sum_{i=1}^n \operatorname*{ess\,sup}_{x\in\Omega}\Big(k_i(0,x)\int_0^\infty \Big(\frac{\partial}{\partial x_i}\beta(a,x)\Big)^2 da\Big) - \sum_{i=1}^n \operatorname*{ess\,sup}_{x\in\Omega}\Big(k_i(0,x)\int_0^\infty \beta(a,x)^2\,da\Big)\Big\|\frac{\partial}{\partial x_i}\varphi\Big\|_H^2 \\
&- \frac{1}{2}\sum_{i=1}^n \Big\|\frac{\partial}{\partial a}k_i\Big\|_\infty \Big\|\frac{\partial}{\partial x_i}\varphi\Big\|_H^2.
\end{aligned}$$

The quantity II can be estimated as

$$II \geq -\|\varphi\|_H \sum_{i=1}^n \Big\|k_i\frac{\partial}{\partial x_i}\mu\Big\|_\infty \Big\|\frac{\partial}{\partial x_i}\varphi\Big\|_H,$$

and the estimates for I and II prove the assertion if we let $\delta \to \infty$. $\square$

Lemma 10. *For all* $\varphi \in \operatorname{dom} B$ *and* $\lambda > 0$ *we have*

$$\langle C_\lambda \varphi, B\varphi\rangle_H \geq -d_1 \|(I-\lambda C)^{-1}\varphi\|_H^2 - d_2 \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i}(I-\lambda C)^{-1}\varphi \right\|_H^2.$$

Proof. Let $\varphi \in \operatorname{dom} B$, $\lambda > 0$. Then (12) and (13) and the representation of $(\lambda I - C)^{-1}\varphi$ imply $(\lambda I - C)^{-1}\varphi \in \operatorname{dom} B$, and we obtain the estimate

$$\begin{aligned}
&\langle C_\lambda \varphi, B\varphi\rangle_H = \langle C(I-\lambda C)^{-1}\varphi, B(I-\lambda C)(I-\lambda C)^{-1}\varphi\rangle_H \\
&= \langle C(I-\lambda C)^{-1}\varphi, B(I-\lambda C)^{-1}\varphi\rangle_H - \lambda\langle C(I-\lambda C)^{-1}\varphi, BC(I-\lambda C)^{-1}\varphi\rangle_H \\
&\geq \langle (A-\omega I)(I-\lambda C)^{-1}\varphi, B(I-\lambda C)^{-1}\varphi\rangle_H \geq -d_1\|(I-\lambda C)^{-1}\varphi\|_H^2 - d_2\sum_{i=1}^n \left\|\frac{\partial}{\partial x_i}(I-\lambda C)^{-1}\varphi\right\|_H^2,
\end{aligned}$$

where we have used the dissipativeness of B and Lemma 9. □

Similarly as in Section 3 we have $\|\frac{\partial}{\partial x_i}(I-\lambda C)^{-1}\varphi\|_H \leq d_3(\|\varphi\|_H + \|\frac{\partial}{\partial x_i}\varphi\|_H)$ for all $\varphi \in \operatorname{dom} B$, $\lambda > 0$, $i = 1, \dots, n$, and $\sum_{i=1}^n \|\frac{\partial}{\partial x_i}\varphi\|_H^2 \leq d_4\|\varphi\|_H\|B\varphi\|_H$ for all $\varphi \in \operatorname{dom} B$, and this together with Lemma 10 implies

Lemma 11. *There exist positive constants* d_5 *and* d_6 *such that*

$$\langle C_\lambda\varphi, B\varphi\rangle_H \geq -d_5\|\varphi\|_H^2 - d_6\|\varphi\|_H\|B\varphi\|_H$$

for all $\lambda > 0$ *and all* $\varphi \in \operatorname{dom} B$.

Now we can prove

Lemma 12. *Let* φ_λ *be the unique solution of* (19) *for* $\lambda > 0$. *Then* $\|B\varphi_\lambda\|_H$ *is bounded as* $\lambda \to 0$.

Proof. We take the inner product with $B\varphi_\lambda$ on both sides of (19) and obtain

$$\|B\varphi_\lambda\|_H^2 = \langle \varphi_\lambda, B\varphi_\lambda\rangle_H - \langle C_\lambda\varphi_\lambda, B\varphi_\lambda\rangle_H - \langle \psi, B\varphi_\lambda\rangle_H.$$

Lemma 11 and the dissipativeness of B imply

$$\|B\varphi_\lambda\|_H^2 \leq d_5\|\varphi_\lambda\|_H^2 + d_6\|\varphi_\lambda\|_H\|B\varphi_\lambda\|_H + \|\psi\|_H\|B\varphi_\lambda\|_H,$$

and the boundedness of $\|\varphi_\lambda\|_H$ as $\lambda \to 0$ proves the assertion. □

An application of Barbu (1976, p. 80, Lemma 3.2) yields the main result of this section.

Theorem 4. *Let the assumptions* (8), (12), (13) *and* (18) *be satisfied, and let the operator* $\mathcal{A}$ *be defined by* (6) *and* (7). *Then* $\mathcal{A} - \omega\mathcal{I}$ *is m-dissipative and* $\mathcal{A}$ *is the infinitesimal generator of a* C_0*-semigroup on* H.

5. The "range condition" approach

In this section we assume that β satisfies (10) and that the following assumptions hold:

$$\begin{aligned}
&k_i \in C([0,\infty)\times\overline{\Omega};\mathbb{R}),\ \text{ɘ} \mapsto \text{ℸב}(\text{ɘ},\cdot) \text{ is continuously differentiable } [\nvdash,\infty) \to \mathbb{L}^\infty(\Omega;\mathbb{R}), \\
&k_i(a,\cdot) \in C^1(\overline{\Omega};\mathbb{R}) \text{ for all } \text{ɘ} \geq \nvdash,\ \text{ℸב}, \tfrac{\partial}{\partial \frown\text{ב}}\text{ℸב} \in \mathbb{L}^\infty((\nvdash,\infty)\times\Omega;\mathbb{R}), \qquad (22)\\
&k_i(a,x) > 0 \text{ on } [0,\infty)\times\overline{\Omega},\ i = 1,\dots,n.
\end{aligned}$$

$$\mu \text{ satisfies (9)},\ a \mapsto \mu(a,\cdot) \text{ is continuously differentiable } [0,\infty) \to L^\infty(\Omega;\mathbb{R}). \qquad (23)$$

Note that k_i is not necessarily bounded away from zero, i.e., it might approach 0 as $a \to \infty$. Moreover, instead of regularity conditions with respect to x we now impose regularity conditions with respect to a on k_i and μ. In this section we do not apply any theorem concerning the sum of two m-dissipative operators but define an operator $\mathcal{A}$ appertaining to the full problem and show that $\mathcal{A} - \omega\mathcal{I}$ is m-dissipative for the constant ω defined in Section 2, where the difficult part is verification of the range condition. Such an approach was used by Guo and Chan (1994) in the case of constant diffusion coefficients $k_i(a,x) \equiv K > 0$, a finite age interval and homogeneous Dirichlet boundary conditions. They managed with weaker assumptions on the mortality rate, but the meaning of their operator $\mathcal{A}$ and its domain and some other statements are not entirely clear, and the paper contains a couple of misprints.

An elegant and general framework for age-structured population models with an additional structure in $L^1(0,\infty;E)$ was developed by Thieme (1991), where the Banach space E contains an additional structure of the population. Age-dependent population models with spatial diffusion are covered by that framework, and it is one of the few papers that allow the diffusion operators to depend on both age and space. With the aid of the theory of integrated semigroups on a larger space, first a rather general meaning was given to the infinitesimal generator of the problem with zero birth and death rates in terms of an evolution system on E, and its resolvent was represented by a formula similar to the one in Lemma 16 below. The birth and death terms were regarded as Lipschitz perturbations, which means that they were allowed to be nonlinear and very general, but the mortality rate had to be bounded.

Let the operator $\mathcal{A} : \operatorname{dom}\mathcal{A} \subset \mathcal{H} \to \mathcal{H}$ be defined by

$$\begin{aligned}
\operatorname{dom}\mathcal{A} = \{\varphi \in \mathcal{H} \mid \varphi(a,\cdot) \in \mathcal{H}^1(\Omega;\mathbb{R}) \text{ and } \textstyle\sum_{i=1}^{n} k_i(a,x)\nu_i(x)\frac{\partial}{\partial x_i}\varphi(a,x) = 0 \text{ on } \partial\Omega \text{ for a.a. } a \geq 0,\\
a \mapsto \varphi(a,\cdot) \text{ is continuous } [0,\infty) \to L^2(\Omega;\mathbb{R}) \text{ and } \varphi(0,x) = \textstyle\int_0^\infty \beta(a,x)\varphi(a,x)\,da \text{ for a.a. } x \in \Omega,\\
\tfrac{\partial}{\partial a}\varphi \in L^2((\varepsilon,\delta)\times\Omega;\mathbb{R}) \text{ for each } 0 < \varepsilon < \delta < \infty,\ -\tfrac{\partial}{\partial a}\varphi - \mu\varphi + \textstyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(k_i \frac{\partial}{\partial x_i}\varphi\big) \in \mathbb{H}\},
\end{aligned} \tag{24}$$

$$(\mathcal{A}\varphi)(a,x) = -\frac{\partial}{\partial a}\varphi(a,x) - \mu(a,x)\varphi(a,x) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\Big), \quad \varphi \in \operatorname{dom}\mathcal{A}. \tag{25}$$

Then integration by parts yields

Lemma 13. *The operator $\mathcal{A} - \omega\mathcal{I}$ is dissipative, where ω is defined as in Section 2.*

Now we turn to the range condition. The equation $(\lambda I - \mathcal{A})\varphi = \psi$, $\varphi \in \operatorname{dom}\mathcal{A}$, $\psi \in H$, $\lambda > \omega$, can be written as

$$\frac{\partial}{\partial a}\varphi(a,x) + (\mu(a,x)+\lambda)\varphi(a,x) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\Big) = \psi(a,x), \quad a > 0, \quad x \in \Omega, \tag{26}$$

$$\varphi(0,x) = \int_0^\infty \beta(a,x)\varphi(a,x)\,da, \quad x \in \Omega, \tag{27}$$

$$\sum_{i=1}^{n} k_i(a,x)\nu_i(x)\frac{\partial}{\partial x_i}\varphi(a,x) = 0, \quad x \in \partial\Omega, \quad a > 0. \tag{28}$$

We regard (26)–(28) as an abstract evolution equation in the new time variable a. In the following we need a complex space and replace H and the operator $\mathcal{A}$ with their complexifications. We consider the Hilbert spaces $X := L^2(\Omega;\mathbb{C})$ and $V := H^1(\Omega;\mathbb{C})$, endowed with the usual inner products. Then $H \cong L^2(0,\infty;X)$, and V is densely and continuously embedded in X. For $a \geq 0$ we define the sesquilinear forms $c(a;\cdot,\cdot)$ on $V \times V$ by

$$c(a;\varphi,\psi) := \sum_{i=1}^{n}\int_\Omega k_i(a,x)\overline{\varphi_{x_i}(x)}\psi_{x_i}(x)\,dx + \int_\Omega \mu(a,x)\overline{\varphi(x)}\psi(x)\,dx, \quad \varphi,\psi \in V.$$

Then

$$|c(a;\varphi,\psi)| \leq \big(\max_{1\leq i\leq n}\|k_i\|_\infty + \operatorname*{ess\,sup}_{[0,a_1]\times\Omega}\mu(a,x)\big)\|\varphi\|_V\|\psi\|_V, \quad \varphi,\psi \in V, \quad 0 \leq a \leq a_1,$$

and

$$\begin{aligned}
c(a;\varphi,\varphi) &= \sum_{i=1}^{n}\int_\Omega k_i(a,x)|\varphi_{x_i}(x)|^2\,dx + \int_\Omega \mu(a,x)|\varphi(x)|^2\,dx\\
&\geq \min_{1\leq i\leq n}\min_{[0,a_1]\times\Omega} k_i(a,x)(\|\varphi\|_V^2 - \|\varphi\|_H^2), \quad \varphi \in V, \quad 0 \leq a \leq a_1,
\end{aligned}$$

for each $a_1 > 0$. Moreover, the assumptions (22) and (23) imply that the mapping $a \mapsto c(a;\varphi,\psi)$ is continously differentiable on $[0,\infty)$ for each $\varphi,\psi \in V$, and we have

$$\overline{c(a;\varphi,\psi)} = c(a;\psi,\varphi), \quad \varphi,\psi \in V, \quad a \geq 0.$$

According to Tanabe (1979), for $a \geq 0$ the unique operator $A(a) : \operatorname{dom}A(a) \subset X \to X$ such that $c(a;\varphi,\psi) = -\langle\varphi, A(a)\psi\rangle_X$, $\varphi \in V$, $\psi \in \operatorname{dom}A(a)$, is given by

$$\operatorname{dom}A(a) = \{\varphi \in H^2(\Omega;\mathbb{C}) \mid \textstyle\sum_{i=1}^{n} k_i(a,x)\nu_i(x)\frac{\partial}{\partial x_i}\varphi(x) = 0 \text{ on } \partial\Omega\},$$

$$(A(a)\varphi)(x) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}\varphi(x)\Big) - \mu(a,x)\varphi(x), \quad \varphi \in \operatorname{dom}A(a).$$

The assumptions of Tanabe (1979, Theorem 5.4.2) and Lions (1961, p. 65, Théorème 6.1) are satisfied, and we obtain

Lemma 14. *There exists a unique evolution system $U(a,b)$, $0 \le b \le a < \infty$, such that*

a) *$U(a,b)$ is a bounded linear operator on X with $\|U(a,b)\| \le 1$ for $0 \le b \le a$, $U(a,a) = I$ for $a \ge 0$ and $U(a,b) : X \to \operatorname{dom} A(a)$ for $0 \le b < a$.*
b) *The mapping $a \mapsto U(a,b)$ is differentiable in $\mathcal{L}(\mathcal{X})$ on (b,∞) and $\frac{\partial}{\partial a} U(a,b) = A(a)U(a,b)$ for $0 \le b < a$.*
c) *Let $f \in L^2(0,\infty;X)$. Then there exists a unique u such that $u \in L^2(0,a_1;V)$, $u' \in L^2(0,a_1;H)$ for each $a_1 > 0$, $u(0) = 0$, $u(a) \in \operatorname{dom} A(a)$ for almost all $a \ge 0$ and $u'(a) = A(a)u(a) + f(a)$ a.e. on $[0,\infty)$, and u is given by*

$$u(a) = \int_0^a U(a,\sigma) f(\sigma)\, d\sigma, \quad a \ge 0.$$

Lemma 15. *Let the operator $B_\lambda : X \to X$ be defined by*

$$(B_\lambda \varphi)(x) := \int_0^\infty \beta(a,x) e^{-\lambda a} (U(a,0)\varphi)(x)\, da.$$

Then $\|B_\lambda\| < 1$ and therefore $1 \in \rho(B_\lambda)$ for $\lambda > \omega$.

Proof. The assumption (10) and Lemma 14 imply that B_λ is a well-defined bounded linear operator on X. For $\varphi \in X$, $\lambda > 0$ we obtain

$$\Big\| \int_0^\infty \beta(a,\cdot) e^{-\lambda a} U(a,0)\varphi\, da \Big\|_X^2 \le 2\omega \int_0^\infty e^{-2\lambda a} \int_\Omega |(U(a,0)\varphi)(x)|^2\, dx\, da \le \frac{\omega}{\lambda} \|\varphi\|_X^2,$$

where we have used Lemma 14, a) and Fubini's Theorem. □

Lemma 16. $\operatorname{range}(\lambda I - \mathcal{A}) = \mathcal{H}$ *for $\lambda > \omega$.*

Proof. Let $\lambda > \omega$, $\psi \in H$ and define

$$\chi(a,x) := \int_0^a e^{-\lambda(a-\sigma)} U(a,\sigma)\psi(\sigma,x)\, d\sigma, \quad (a,x) \in [0,\infty) \times \Omega.$$

Then χ is measurable, the mapping $a \mapsto \chi(a,\cdot)$ is continuous $[0,\infty) \to X$, and Lemma 14, c) implies $\frac{\partial}{\partial a}\chi \in L^2((0,\delta) \times \Omega; \mathbb{C})$ for each $\delta > 0$, $\chi(a,\cdot) \in \operatorname{dom} A(a)$ for almost all $a \ge 0$ and

$$\frac{\partial}{\partial a}\chi(a,x) = -\lambda\chi(a,x) + A(a)\chi(a,x) + \psi(a,x) \quad \text{a.e.}$$

Moreover, from Lemma 14 we obtain

$$\|\chi(a,\cdot)\|_X \le \int_0^a e^{-\lambda(a-\sigma)} \|\psi(a,\cdot)\|_X\, d\sigma, \quad a \ge 0,$$

where the right-hand side is the resolvent of the m-dissipative operator A_0 on $L^2(0,\infty;\mathbb{C})$, defined by $A_0\varphi = -\varphi'$, $\varphi \in \operatorname{dom} A_0 = \{\varphi \in H^1(0,\infty;\mathbb{C}) \mid \varphi(\nvdash) = \nvdash\}$, applied to an element of $L^2(0,\infty;\mathbb{C})$, i.e., it is in $L^2(0,\infty;\mathbb{C})$ and thus $\chi \in H$. The condition (10) entails that the function $\varphi_0 : \Omega \to \mathbb{C}$, defined by

$$\varphi_0(x) := \int_0^\infty \beta(a,x)\chi(a,x)\, da, \quad x \in \Omega,$$

is in X. We define $\varphi : [0,\infty) \times \Omega \to \mathbb{C}$ by

$$\varphi(a,x) := e^{-\lambda a}(U(a,0)(I - B_\lambda)^{-1}\varphi_0)(x) + \chi(a,x), \quad (a,x) \in [0,\infty) \times \Omega.$$

Then $\varphi \in H$, $a \mapsto \varphi(a,\cdot)$ is continuous $[0,\infty) \to X$ and

$$\int_0^\infty \beta(a,x)\varphi(a,x)\, da = \int_0^\infty \beta(a,x)e^{-\lambda a}(U(a,0)(I - B_\lambda)^{-1}\varphi_0)(x)\, da + \varphi_0(x) = ((I - B_\lambda)^{-1}\varphi_0)(x) = \varphi(0,x).$$

Moreover, Lemma 14 and the properties of χ imply $\frac{\partial}{\partial a}\varphi \in L^2((\varepsilon,\delta) \times \Omega; \mathbb{C})$ for each $0 < \varepsilon < \delta$, $\varphi(a,\cdot) \in \operatorname{dom} A(a)$ for a.a. $a > 0$ and

$$\begin{aligned} \frac{\partial}{\partial a}\varphi(a,x) &= -\lambda\varphi(a,x) + A(a)\varphi(a,x) + \psi(a,x) \\ &= -\lambda\varphi(a,x) - \mu(a,x)\varphi(a,x) + \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(k_i(a,x)\frac{\partial}{\partial x_i}\varphi(a,x)\Big) + \psi(a,x) \quad \text{a.e.,} \end{aligned}$$

from which we infer $\varphi \in \operatorname{dom} \mathcal{A}$ and $(\lambda I - \mathcal{A})\varphi = \psi$. □

Lemmas 13 and 16 yield the main result of this section.

Theorem 5. *Let* (10), (22) *and* (23) *be satisfied, let the operator* $\mathcal{A}$ *be given by* (24) *and* (25), *and let* ω *be as in Section 2. Then* $\mathcal{A} - \omega\mathcal{I}$ *is m-dissipative and* $\mathcal{A}$ *is the infinitesimal generator of a* C_0*-semigroup on* H.

Remark 2. Weaker assumptions (Hölder continuity of c in a with exponent $\alpha \in (\frac{1}{2}, 1])$ would already suffice to guarantee the existence of an evolution system satisfying Lemma 14, a) and b) (cf. Tanabe (1979, Theorem 5.4.2)). However, the property c) need not necessarily be satisfied, and in that case we can only show that range$(\lambda I - \mathcal{A})$ is dense in H (cf. Tanabe (1979, Theorem 5.4.3)).

6. References

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