THE NUMBER OF INTEGER POINTS IN RATIONAL POLYTOPES IS GIVEN BY QUASIPOLYNOMIALS IN THE DILATION PARAMETERS – AN ELEMENTARY APPROACH

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Abstract. We study the behavior of the counting function of integer points in rational polytopes under independent parallel translations of the bounding hyperplanes. This is a natural multivariate generalization of the situation dealt with by Ehrhart theory. It is known that this counting function is piecewise represented by quasipolynomials, and we start our paper by providing an elementary and self-contained proof of this result. We stress the fact that the domains of validity of the quasipolynomials overlap, which is useful for explaining linear factors appearing in many polynomial enumeration formulas. We then explore tools provided by the proof to effectively compute the quasipolynomials in the unimodular case. In this case, we also encounter a combinatorial reciprocity phenomenon, namely we are able to provide a combinatorial interpretation for certain signed sums of quasipolynomials outside of their original domains of validity. Finally, we apply the theory to a class of directed graph polytopes, which includes the order polytopes of posets as special instances.

1. Introduction

1.1. **The problem.** Given an $n \times d$ integer matrix A with columns $a_1, a_2, \ldots, a_d$, a subset $I \subseteq [n]$, where $[n] = \{1, 2, \ldots, n\}$, and an integer vector $x \in \mathbb{Z}^n$, we define

$$\mathrm{N}_{A,I}(x) = \#\{z \in \mathbb{Z}^d | (A \cdot z)_i \leq x_i \, \forall i \in I, (A \cdot z)_i = x_i \, \forall i \in [n] \setminus I\}.$$

The quantity $\mathrm{N}_{A,[n]}(x)$ obviously counts the integer points in a generic rational polyhedron. We study the behavior of $\mathrm{N}_{A,I}(x)$ as a function in x. Varying x obviously corresponds to parallel translations of bounding hyperplanes of the polytope; the bounding hyperplanes are of course a subset. Letting

$$\mathrm{Cube}_{n,I} := \{x \in \mathbb{Q}^n | 0 \leq x_i \leq 1 \, \forall i \in I, x_i = 0 \, \forall i \in [n] \setminus I\}$$

and $\langle A \rangle := \{A \cdot z | z \in \mathbb{Q}^d\}$, it is obvious that the function $x \to \mathrm{N}_{A,I}(x)$ vanishes outside the cone $\mathcal{C}_{A,I} := \langle A \rangle + \mathbb{Q}_{\geq 0} \, \mathrm{Cube}_{n,I}$. We assume $\mathrm{N}_{A,I}(x) < \infty$ for all x; a justification is given in Appendix A. The boundedness implies the linear independence of the columns of A and thus we have $n \geq d$.

1.2. Quasipolynomials, walls and a first qualitative result.

Definition 1. *An n–quasipolynomial is an expression of the form*

$$\sum_{(i_1,\ldots,i_n) \in \mathbb{Z}^n_{\geq 0}} c_{i_1,i_2,\ldots,i_n}(x_1, \ldots, x_n) \, x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

where the coefficients $c_{i_1,i_2,\ldots,i_n} : \mathbb{Z}^n \to \mathbb{Q}$ are functions that are periodic in each component and almost all of them vanish.

We let $(e_1, e_2, \ldots, e_n)$ denote the standard basis of $\mathbb{Q}^n$.

Definition 2 (Walls, hyperplanes and chambers of (A, I))**.** *A wall of* (A, I) *is a set of the following form: Let* $J \subseteq I$ *be such that the vectors* $(e_i)_{i \in J}$ *together with* $\langle A \rangle$ *span a hyperplane in* $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$*. Then the cone*

$$\mathrm{W}_{A,J} := \langle A \rangle + \mathbb{Q}_{\geq 0}\,\mathrm{Cube}_{n,J}$$

is said to be a wall of (A, I)*;* $\mathrm{H}_{A,J}$ *denotes the hyperplane containing* $\mathrm{W}_{A,J}$ *and is said to be a hyperplane of* (A, I)*. The connected components of the complement of the union of all walls in* $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$ *are said to be the chambers of* (A, I)*.*

The bounding cones of $\mathcal{C}_{A,I}$ can all be found among the walls of (A, I). We are able to state a first qualitative result.

Theorem 1. *The function* $x \to \mathrm{N}_{A,I}(x)$ *is, for each chamber* C *of* (A, I)*, a quasipolynomial of degree no greater than* $d + |I| - \dim \mathcal{C}_{A,I}$ *on* $(C - \mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$*.*

If there is a minimal superset $I \subseteq I' \subseteq [n]$ *with* $\dim \mathcal{C}_{A,I'} = n$ *such that for all* $J \subseteq I$ *with* $|J| = \dim \mathcal{C}_{A,I} - d$ *we have* $\det(A') \in \{0, 1, -1\}$*, where* A' *is the matrix we obtain from* A *by adding the columns* $(e_j)_{j \in (I' \setminus I) \cup J}$*, then the quasipolynomials can be chosen to be polynomials.*

Observe that $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$ is the union of the sets $C - \mathrm{Cube}_{n,I}$ where C ranges over all chambers of (A, I): Each element on a wall can be shifted to a chamber by adding an appropriate element of $\mathrm{Cube}_{n,I}$.

1.3. **Example.** $n = 3, d = 1, a_1 = (1, -1, -1), I = \{1, 2, 3\}$: The hyperplanes of (A, I) are

$$x_1 + x_2 = 0, \quad x_1 + x_3 = 0, \quad x_2 - x_3 = 0,$$

where the first two of them are the bounding hyperplanes of $\mathcal{C}_{A,I}$. We have $\mathrm{N}_{A,I}(x) = x_1 + x_2 + 1$ on $\{(x_1 + x_2 > -2, x_2 - x_3 < 1\}$ and $\mathrm{N}_{A,I}(x) = x_1 + x_3 + 1$ on $\{(x_1 + x_3 > -2, x_2 - x_3 > -1\}$ as can be checked directly.

1.4. **Relation to existing work and motivation.** The following extensively studied object is a close relative of the function $\mathrm{N}_{A,I}(x)$: Let P be an $n \times d$ integer matrix of rank n, $x \in \mathbb{Z}^n$ and define

$$\Phi_P(x) = \#\{z \in \mathbb{Z}^d_{\geq 0} | P \cdot z = x\}.$$

This function appears in the literature as the *vector partition function* associated with P.

It was presumably first mentioned by Blakley [2] that there is a finite decomposition of $\mathbb{Z}^n$ such that $x \to \Phi_P(x)$ is a quasipolynomial of degree $d - n$ on each piece. Such a decomposition was then explicitly given by Dahmen and Micchelli [3]. However, they did not give the coarsets decomposition possible. Remarkably, they also found a characterisation of the quasipolynomials using certain zeros they could identify. In [5, 4], De Concini, Procesi and Vergne also elaborate on this approach (among many other things). A weaker analog of Theorem 1 for $\Phi_P(x)$ was also given in in [10]. (Sturmfels did not mention the overlaps of the quasipolynomials' domains of validity.) The first appearance of the result in its full strength can probably be found in [11]; see also the above mentioned [5] for a nice proof. Theorem 1 is the analogue of their theorem for our setting. A very elementary, but less general treatment is given in [1].

By moving a given convex rational polytope into the positive orthant and introducing slack variables to convert the inequalities in its hyperplane presentation into equalities, the enumeration of the integer points in the polytope can obviously always be reduced to the evaluation of a certain vector partition function. The quasipolynomiality of $x \to \mathrm{N}_{A,I}(x)$ can in fact be deduced from the quasipolynomiality of $x \to \Phi_P(x)$, however, the decomposition of $\mathbb{Z}^n$ is not as coarse as the one given in Theorem 1 when trying to deduce this theorem from the corresponding theorem for $x \to \Phi_P(x)$.

In a recent preprint, Henk and Linke [8] provide a result similar to Theorem 1: They generalize it to $x \in \mathbb{Q}^n$ and are furthermore able to deduce properties of the coefficients of the quasipolynomials. Interestingly, they have a different description of the chambers C and they do not deal with the overlaps of the domains of validity.

Our interest in this subject arose from the study of concrete enumeration problems, such as plane partition and alternating sign matrix enumeration problems (see for instance [6, 7]), that lead to polynomial enumeration formulas. Theorem 1 provides a uniform explanation of the polynomiality or quasipolynomiality of these formulas because these enumeration problems can usually be formulated as problems of enumerating the integer points in certain rational polytopes. Moreover, we believe that the value of this more general point of view for our concrete enumeration problems goes beyond the first qualitative observation provided by Theorem 1 and we hope to make it through our elementary treatment also more accessible for other combinatorialists.

As an indication for this, we want to draw the readers attention to a byproduct of our proof.

Theorem 2. *Let C_1, C_2 be chambers of (A, I) that are adjacent via the hyperplane $H = \{x \in \mathbb{Q}^n : \langle h, x\rangle = 0\}$, $h \in \mathbb{Z}^n$, and P_{C_1}, P_{C_2} the corresponding quasipolynomials. Let* $\mathrm{pos} = \sum_{i \in I : h_i > 0} h_i$, $\mathrm{neg} = \sum_{i \in I : h_i < 0} h_i$ *and r be an integer with* $\mathrm{neg} < r < \mathrm{pos}$. *Then the difference $P_{C_1}(x) - P_{C_2}(x)$ vanishes on $H^r \cap \mathbb{Z}^n$, where $H^r = \{x \in \mathbb{Q}^n : \langle h, x\rangle + r = 0\}$.*

This shows that the difference of two "adjacent" quasipolynomials vanishes on certain affine hyperplanes, which implies that the difference has certain linear factors. If the adjacency happens to be on the boundary of the support of $\mathrm{N}_{A,I}(x)$ then this explains certain linear factors of the quasipolynomial representing $\mathrm{N}_{A,I}(x)$ on the chamber that lies inside the cone $\mathcal{C}_{A,I}$. In our concrete enumeration problems we frequently come across such linear factors; Theorem 2 might be an explanation. More general, we plan to investigate applications of the theoretical results provided in this paper to concrete enumeration problems in subsequent papers.

1.5. **Plan of the paper.** The rest of the paper is devoted to the proof of Theorem 1 and the exploration of the insights it provides for the actual computation of the quasipolynomials. In Section 2, we start by proving a weak version, which concerns a refined decomposition of $\mathbb{Z}^n$ and does not deal with the overlaps of the domains of validity of the quasipolynomials. We are then able to strengthen the result in Section 3 and conclude the proof of Theorem 1.

In Section 4, we consider the case that the quasipolynomials representing $\mathrm{N}_{A,I}(x)$ are in fact polynomials and we provide tools to effectively compute these polynomials. For this purpose, we introduce certain signed sums of the polynomials representing a given counting function $\mathrm{N}_{A,I}(x)$; it turns out that it is easier to compute these "signed sums" rather than to compute the polynomials themselves "directly". However, we also show that the polynomials representing $\mathrm{N}_{A,I}(x)$ can be expressed as sums of these signed sums. Another indication to the fact that these signed sums are of interest is that we are able to provide a combinatorial interpretation for what the signed sums count (in terms of a signed enumeration) which is valid for all $x \in \mathbb{Z}^n$; for the polynomial representing $\mathrm{N}_{A,I}(x)$ such a combinatorial interpretation does not exist. In Section 5, we apply the results to certain polytopes that are induced by directed graphs; this class of polytopes includes the order polytopes of posets as special cases.

In the appendices, we justify certain assumptions and summarize elementary facts on quasipolynomials. We also provide a proof of the Ehrhart-Macdonald reciprocity within this framework, thus providing a combinatorial interpretation of the quasipolynomial representing $\mathrm{N}_{A,I}(x)$ on the chamber C on its negative $-C$.

1.6. **Idea of the proof of Theorem 1.** It is fairly easy to give an idea of the proof of Theorem 1: We use induction with respect $|I|$. For the induction step we use the following recursion, which is valid for all $j \in J$.

$$\mathrm{N}_{A,I}(x) = \sum_{z=0}^{\infty} \mathrm{N}_{A,I \setminus \{j\}}(x - z \cdot e_j) \tag{1.1}$$

The result then basically follows from the fact that "summing a quasipolynomial" leads to another quasipolynomial, namely in the summation bounds; the precise statement is given in Fact B.1, Appendix B.

The reason why more than one quasipolynomial is needed to represent $x \to \mathrm{N}_{A,I}(x)$ is roughy as follows: It is rather obvious for the base case of the induction, see Lemma 1. It is also evident as $x \to \mathrm{N}_{A,I}(x)$ vanishes on $(\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}) \setminus \mathcal{C}_{A,I}$ and is therefore represented by the zero polynomial on this domain. Concerning the induction step, reconsider (1.1): For fix x and j, the halfline $\{x - z \cdot e_j : z \geq 0\}$ traverses different domains of validity of the quasipolynomials representing $x \to \mathrm{N}_{A,I\setminus\{j\}}(x)$. These domains surely vary as x varies, but coincide in a neighborhood of x; these neighborhoods are the domains of validity of the quasipolynomials representing $x \to \mathrm{N}_{A,I}(x)$; in fact it turns out that there is only a finite number of such neighborhoods.

Another interesting feature about Theorem 1 are the overlaps of the domains of validity of the quasipolynomials, which are caused by the term "$-\,\mathrm{Cube}_{n,I}$". The background is as follows: Throughout this article we use the extended definition of summation, i.e. $\sum_{i=a}^{b} f(i) = -\sum_{b+1}^{a-1} f(i)$ if $a > b$. (Note that this implies $\sum_{i=a}^{a-1} f(i) = 0$.) If $f(i)$ is a polynomial and $g(i)$ is a polynomial with $f(i) = g(i+1) - g(i)$ (it is not hard to see that such a $g(i)$ always exists) then $\sum_{i=a}^{b} f(i) = g(b+1) - g(a)$ for all $a, b \in \mathbb{Z}$; in particular we see that $\sum_{i=a}^{b} f(i)$ is a polynomial in a, b. But then again, a more natural definition of $\sum_{i=a}^{b} f(i)$ for $a > b$ is to set it to zero; the sum $\sum_{i=a}^{b} f(i)$ is of course not a polynomial in a, b when using this definition. However, the two definitions coincide for $b = a - 1$ which is in a sense the explanation for the overlaps; note that this effect is strengthened in the case of nested sums.

2. Proof of a weak version of Theorem 1

This and the following section is devoted to the proof of Theorem 1. We assume $\dim \mathcal{C}_{A,I} = n$; the justification is given in Appendix A. Suppose $\mathcal{H}$ is a finite set of linear hyperplanes[1] in $\mathbb{Q}^n$. The *rooms* of $\mathcal{H}$ are the connected components of the complement of the union of all hyperplanes in $\mathcal{H}$. In this section we show that $x \to \mathrm{N}_{A,I}(x)$ is a quasipolynomial on $\overline{R} \cap \mathbb{Z}^n$ for each room R of the set of hyperplane of (A, I) contained in $\mathcal{C}_{A,I}$. As $\mathcal{H}$ includes the bounding hyperplanes of $\mathcal{C}_{A,I}$, each room not contained in the cone lies entirely in the complement; the function $x \to \mathrm{N}_{A,I}(x)$ vanishes there.

[1] That is they contain the origin.

2.1. **Base case of the induction.** The proof is by induction with respect to $|I|$. The case $|I| = n - d$ is dealt with in the following lemma, actually in full strength.

Lemma 1. *If $|I| = n - d$ then the function $x \to \mathrm{N}_{A,I}(x)$ is periodic on $(\mathcal{C}^{\circ}_{A,I} - \mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$ and takes values in $\{0, 1\}$. In case $\det(A') = \pm 1$, where A' is the matrix obtained from A by appending the columns $(e_i)_{i \in I}$, the function is constant* 1.

Proof. As A' is invertible by the assumption that $\dim \mathcal{C}_{A,I} = 0$ and the vector consisting of the first d components of $A'^{-1}x$ is the only candidate for an element of the respective polytope, we have $\mathrm{N}_{A,I}(x) \in \{0, 1\}$. In fact note that $\mathrm{N}_{A,I}(x) = 1$ if and only if $A^{-1}x \in \mathbb{Z}$ and $(A^{-1}x)_i > -1$ for $i > d$.

Now suppose $x_1, x_2 \in (\mathcal{C}^{\circ}_{A,I} - \mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$ (which is actually equivalent to $(A'^{-1}x_j)_i > -1$ for $i > d$, $j = 1, 2$) and every coordinate of $x_2 - x_1$ is divisible by $\det A'$. Consequently $A'^{-1}(x_2 - x_1) \in \mathbb{Z}^n$. Thus we have $\mathrm{N}_{A,I}(x_1) = 1$ if and only if $A'^{-1}x_1 \in \mathbb{Z}^n$, which is equivalent to $A'^{-1}(x_2 - x_1) + A'^{-1}x_1 = A'^{-1}x_2 \in \mathbb{Z}^n$ and, therefore to $\mathrm{N}_{A,I}(x_2) = 1$. $\square$

2.2. **Induction step.** The next lemma will be fundamental for performing the induction step. We need the following notion: given an element $a \in \mathbb{Q}^n$ and a finite set of (linear) hyperplanes $\mathcal{H}$ in $\mathbb{Q}^n$, we define the *hyperplanes associated with* $(\mathcal{H}, a)$ as the union of $\mathcal{H}$ and the set of all hyperplanes H of the following type: For each pair of distinct hyperplanes $H_1, H_2 \in \mathcal{H}$, let H be the unique hyperplane which contains $H_1 \cap H_2$ and a. (If $H_i = \{x \in \mathbb{Q}^d : \langle h_i, x\rangle = 0\}$ and $\langle h_i, a\rangle = 1$ then $H = \{x \in \mathbb{Q}^d : \langle h_1 - h_2, x\rangle = 0\}$.)

Lemma 2. *Assume $|I| > n - d$ and let $j \in I$ be such that $\langle A\rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I\setminus\{j\}} = \mathbb{Q}^n$. Suppose $\mathcal{H}$ is a finite set of hyperplanes containing the bounding hyperplanes of $\mathcal{C}_{A,I\setminus\{j\}}$ such that the function $x \to \mathrm{N}_{A,I\setminus\{j\}}(x)$ is a quasipolynomial on $\overline{R'} \cap \mathbb{Z}^n$ for every room $R' \subseteq \mathcal{C}_{A,I\setminus\{j\}}$ of $\mathcal{H}$. Then the function $x \to \mathrm{N}_{A,I}(x)$ is a quasipolynomial on $\overline{R} \cap \mathbb{Z}^n$ for every room $R \subseteq \mathcal{C}_{A,I}$ of the set of hyperplanes of $(\mathcal{H}, e_j)$. If $x \to \mathrm{N}_{A,I\setminus\{j\}}(x)$ is a polynomial on each room and, for each $H \in \mathcal{H}$, there exists an $h \in \mathbb{Z}^n$ such that $H = \{q \in \mathbb{Q}^n | \langle h, q\rangle = 0\}$ and $h_j \in \{0, 1\}$ then the quasipolynomials is in fact a polynomials.*

Proof. The proof relies on the recursion (1.1). Let $\mathcal{H} = \{H_1, \dots, H_m\}$ and choose $h_i \in \mathbb{Q}^n$ such that $H_i = \{q \in \mathbb{Q}^n : \langle h_i, q\rangle = 0\}$ and $(h_i)_j \in \{0, 1\}$ for all i. The subset of hyperplanes in $\mathcal{H}$ met by the halfline $\{x - z \cdot e_j : z \geq 0\}$ coincides for all x that lie in the same room of $(\mathcal{H}, e_j)$: This is because an intersection is only possible if $(h_i)_j = 1$ and the point of intersection is characterized by $z = \langle h_i, x\rangle$; thus the halfline meets a hyperplane if and only if $(h_i)_j = 1$ and $\langle h_i, x\rangle > 0$. They are also met in the same order: The halfline $\{x - z \cdot e_j : z \geq 0\}$ first meets H_s and then H_t iff $\langle h_s, x\rangle < \langle h_t, x\rangle$. But this is equivalent to having x on the "positive" side of $\{q \in \mathbb{Q}^n : \langle h_t - h_s, q\rangle = 0\}$, the latter being a hyperplane of $(\mathcal{H}, e_j)$.

Fix a room R of $(\mathcal{H}, e_j)$ contained in $\mathcal{C}_{A,I}$ and let $x \in R$. The intersection of the halfline $\{x - z \cdot e_j : z \geq 0\}$ with $\mathcal{C}_{A,I\setminus\{j\}}$ is non-empty, connected and bounded, since $x \in \mathcal{C}_{A,I}$, $\mathcal{C}_{A,I\setminus\{j\}}$ is convex and our polytopes are assumed to be finite. Without loss of generality, let $H_1, H_2, \dots, H_r$ be the hyperplanes of $\mathcal{H}$ that the halfline $\{x - z \cdot e_j : z \geq 0\}$ meets in $\mathcal{C}_{A,I\setminus\{j\}}$, in this order.

For all z with $\langle h_{i-1}, x\rangle \leq z \leq \langle h_i, x\rangle$ $(1 < i \leq r)$, the points $x - z \cdot e_j$ lie in the closure of the same room R'_i of $\mathcal{H}$. Moreover, all $x - z \cdot e_j$ with $0 \leq z < \langle h_1, x\rangle$ either lie in the same room R'_1 of $\mathcal{H}$ or outside of $\mathcal{C}_{A,I\setminus\{j\}}$; in the latter case we set $R'_1 := \mathbb{Q}^d \setminus \mathcal{C}_{A,I\setminus\{j\}}$. If $z > \langle h_r, x\rangle$ then $x - z \cdot e_j$ is also not contained in $\mathcal{C}_{A,I\setminus\{j\}}$. All this is also true if $x \in \partial R$: There exists a sequence $(x_n)_{n\geq 0}$ in R converging to x and thus we have $0 \leq \langle h_1, x\rangle \leq \langle h_2, x\rangle \leq \ldots \leq \langle h_r, x\rangle$ and $x - z \cdot e_j \in \overline{R'_i}$ for $\langle h_{i-1}, x\rangle \leq z \leq \langle h_i, x\rangle$ if $1 < i \leq r - 1$. Similarly, $x - z \cdot e_j \in \overline{R'_1}$ if

$0 \le z < \langle h_1, x \rangle$. If however $R_1' = \mathbb{Q}^d \setminus \mathcal{C}_{A,I \setminus \{j\}}$ then $x - z \cdot e_j \in R_1'$ if $z < \langle h_1, x \rangle$: In this case H_1 is a bounding hyperplane of $\mathcal{C}_{A,I\setminus\{j\}}$ and $\langle h_1, y \rangle > 0$ implies $y \notin \mathcal{C}_{A,I\setminus\{j\}}$. Similarly, $x - z \cdot e_j \in \mathbb{Q}^d \setminus \mathcal{C}_{A,I\setminus\{j\}}$ if $z > \langle h_r, x \rangle$.

By the recursion, we have

$$\mathrm{N}_{A,I}(x) = \sum_{z \in [0, \langle h_1, x \rangle)} \mathrm{N}_{A,I\setminus\{j\}}(x - z e_j) + \sum_{i=2}^{r} \sum_{z \in (\langle h_{i-1}, x\rangle, \langle h_i, x \rangle)} \mathrm{N}_{A,I\setminus\{j\}}(x - z e_j) + \sum_{i=1}^{r} [\langle h_i, x \rangle \in \mathbb{Z}] \cdot \mathrm{N}_{A,I\setminus\{j\}}(x - \langle h_i, x \rangle e_j)$$

for all $x \in \overline{R} \cap \mathbb{Z}^d$, where [statement] = 1 if the statement is true and zero otherwise. For a room $R' \subseteq \mathcal{C}_{A,I\setminus\{j\}}$ of $\mathcal{H}$, let $Q_{R'}(x)$ denote the quasipolynomial with $Q_{R'}(x) = \mathrm{N}_{A,I\setminus\{j\}}(x)$ for all $x \in \overline{R'} \cap \mathbb{Z}^d$. Moreover, set $Q_{\mathbb{Q}^d \setminus \mathcal{C}_{A,I\setminus\{j\}}}(x) = 0$. With this notation we have

$$\mathrm{N}_{A,I}(x) = \sum_{z \in [0, \langle h_1, x \rangle)} Q_{R_1'}(x - z e_j) + \sum_{i=2}^{r} \sum_{z \in (\langle h_{i-1}, x\rangle, \langle h_i, x \rangle)} Q_{R_i'}(x - z e_j) + \sum_{i=1}^{r} [\langle h_i, x \rangle \in \mathbb{Z}] \cdot Q_{R'_{i+\epsilon_i}}(x - \langle h_i, x \rangle e_j), \quad (2.1)$$

for all $x \in \overline{R} \cap \mathbb{Z}^d$, where $\epsilon_r = 0$, $\epsilon_1 = 1$ if $R_1' = \mathbb{Q}^d \setminus \mathcal{C}_{A,I\setminus\{j\}}$ and we are free to choose $\epsilon_i \in \{0, 1\}$ otherwise. The assertion follows as the right hand side is a quasipolynomial in x by Fact B.1 for quasipolynomials, see Appendix B. Note that the additional assumption implies $h \in \mathbb{Z}$ and $Q_{R'}(x)$ are polynomials and we are able to conclude that $\mathrm{N}_{A,I}(x)$ is a polynomial in this case.

□

The purpose of next lemma is to show that the subdivision of $\mathbb{Q}^n$ into rooms given by the previous lemma is too fine. This essentially follows from the freedom for the choice of e_j.

Lemma 3. *Suppose $|I| > n - d$. Let $K \subseteq I$ be the set of integers k such that $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{A,I\setminus\{k\}} = \mathbb{Q}^n$, $\mathcal{G}^k$ be the set of hyperplanes associated with $(A, I \setminus \{k\})$ and $\mathcal{H}^k$ be the set of hyperplanes associated with $(\mathcal{G}^k, e_k)$. Then $\bigcap_{k \in K} \mathcal{H}^k$ is the set of hyperplane associated with (A, I).*

Proof. Let H be a hyperplane of (A, I). We show that H lies in the intersection. If $e_k \notin H$ or e_k can be excluded from the generators without lowering the dimension then $H \in \mathcal{G}^k$ and thus $H \in \mathcal{H}^k$. Therefore it can be assumed that excluding e_k from the list of generators of H gives a subspace G of dimension $n-2$. By the assumption that $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{A,I\setminus\{k\}} = \mathbb{Q}^n$, there are $s, t \in I \setminus \{k\}$ such that $H_s := G + \mathbb{Q}e_s$, $H_t := G + \mathbb{Q}e_t$ are two distinct hyperplanes in $\mathcal{G}^k$. Therefore H is the unique hyperplane containing e_k and the intersection $H_s \cap H_t$, which implies that it lies in $\mathcal{H}^k$.

Concerning the other direction, let $k \in K$ and $H_s, H_t \in \mathcal{G}^k$ with $e_k \notin H_s, H_t$. Denote by H the unique hyperplane containing $H_s \cap H_t$ and e_k. We have to show that the fact that H is contained in the intersection displayed in the lemma implies that H is a hyperplane of (A, I). As $H_s + \mathbb{Q}e_k = \mathbb{Q}^n$, there must be a $j \in I \setminus \{k\}$ with $e_j \in H_s$ but $e_j \notin H$, which implies also $e_j \notin H_t$. However, since $H_t + \mathbb{Q}e_k = \mathbb{Q}^n$, this implies $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{A,I\setminus\{j\}} = \mathbb{Q}^n$ and therefore $j \in K$. Thus $H \in \mathcal{H}^j$ and, since $e_j \notin H$, $H \in \mathcal{G}^j$. □

Now we are in the position to show inductively that the function $x \to \mathrm{N}_{A,I}(x)$ is a quasipolynomial on $\overline{R} \cap \mathbb{Z}^n$ for every room $R \subseteq \mathcal{C}_{A,I}$ of the hyperplanes of (A, I): The case $|I| = n - d$ was dealt with in Lemma 1. Suppose $|I| > n - d$. Let $k \in I$ be with $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{A,I\setminus\{k\}} = \mathbb{Q}^n$ and $\mathcal{G}$ denote the set of hyperplanes of $(A, I \setminus \{k\})$. The induction hypothesis and Lemma 2 implies that the function $x \to \mathrm{N}_{A,I}(x)$ is a quasipolynomial on the closure of every room of the hyperplanes associated with $(\mathcal{G}, e_k)$ which is contained in $\mathcal{C}_{A,I}$. (Note that Lemma 3 implies that each room is either contained in $\mathcal{C}_{A,I}$ or in the complement.) By Lemma 3 and Fact B.3, Appendix B, the assertion follows.

2.3. **Polynomiality.** We show the assertion on the polynomiality stated in the theorem: If $|I| = n-d$ then this follows from Lemma 1. Otherwise we use the respective part of the statement of Lemma 2: Suppose H is a hyperplane of $(A, I \setminus \{j\})$ and $a_1, a_2, \ldots, a_d, e_{i_1}, \ldots, e_{i_{n-d-1}}$ are its generators where $i_l \in I \setminus \{j\}$. If $e_j \in H$ then for each normal vector h of H we have $h_j = 0$. Otherwise $\det(A') = \pm 1$ where A' is the square matrix we obtain from A by adding $e_{i_1}, \ldots, e_{i_{n-d-1}}, e_j$, in this order. Thus A'^{-1} is a matrix with integers entries; its bottom row is a normal vector h of H with integer coefficients and $h_j = 1$.

3. Difference of adjacent quasipolynomials and the proof of Theorem 1

3.1. **A formula for the difference of adjacent quasipolynomials.** For a room R of the set of hyperplanes of (A, I) contained in $\mathcal{C}_{A,I}$, let $P_R(x)$ denote the quasipolynomial such that $\mathrm{N}_{A,I}(x) = P_R(x)$ for all $x \in \overline{R} \cap \mathbb{Z}^n$. We set $P_R(x) = 0$ if R is a room not contained in $\mathcal{C}_{A,I}$ or $R = \mathbb{Q}^n \setminus \mathcal{C}_{A,I}$. In this section we fix two adjacent rooms R_1, R_2 of the set of hyperplanes of (A, I) and let H denote the separating hyperplane, i.e. $\overline{R_1} \cap \overline{R_2} \subseteq H$ and $\dim \overline{R_1} \cap \overline{R_2} = n-1$. We aim to derive a formula for the difference $P_{R_1}(x) - P_{R_2}(x)$.

We assume the existence of $j \in I$ with $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I\setminus\{j\}} = \mathbb{Q}^n$ such that e_j is not contained in H. We first consider the case that e_j is on the same side of H as R_1. Observe that there exists an $x \in R_1$ such that H is the first hyperplane of $(A, I \setminus \{j\})$ that the halfline $\{x - ze_j | z \geq 0\}$ meets and enters R_2 after this point of intersection.

Recall that we have derived the following formula for $P_R(x)$ in the proof of Lemma 2: If $j \in I$ and $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I\setminus\{j\}} = \mathbb{Q}^n$ then, using the notation from Lemma 2,

$$P_R(x) = \sum_{z \in [0, \langle h_1, x \rangle)} Q_{R'_1}(x - ze_j) + \sum_{i=2}^{r} \sum_{z \in (\langle h_{i-1}, x \rangle, \langle h_i, x \rangle)} Q_{R'_i}(x - ze_j) + \sum_{i=1}^{r} [\langle h_i, x \rangle \in \mathbb{Z}] \cdot Q_{R'_{i+\epsilon_i}}(x - \langle h_i, x \rangle e_j), \quad (3.1)$$

where $\epsilon_r = 0$, $\epsilon_1 = 1$ if $R'_1 = \mathbb{Q}^d \setminus \mathcal{C}_{A,I\setminus\{j\}}$ and we are free to choose $\epsilon_i \in \{0, 1\}$ otherwise. It is a crucial point of this approach that this gives a formula for the quasipolynomial $P_R(x)$ for all $x \in \mathbb{Z}^n$ if we use the extended definition of the summation provided in Subsection 1.5. We apply this to our situation to see that

$$P_{R_1}(x) = \sum_{z \in [0, \langle h_1, x \rangle)} Q_{R'_1}(x - ze_j) + \sum_{i=2}^{r} \sum_{z \in (\langle h_{i-1}, x \rangle, \langle h_i, x \rangle)} Q_{R'_i}(x - ze_j) + \sum_{i=1}^{r} [\langle h_i, x \rangle \in \mathbb{Z}] \cdot Q_{R'_{i+\epsilon_i}}(x - \langle h_i, x \rangle e_j)$$

and

$$P_{R_2}(x) = \sum_{z\in[0,\langle h_2,x\rangle)} Q_{R'_2}(x-ze_j) + \sum_{i=3}^{r} \sum_{z\in(\langle h_{i-1},x\rangle,\langle h_i,x\rangle)} Q_{R'_i}(x-ze_j) + \sum_{i=2}^{r}[\langle h_i,x\rangle \in \mathbb{Z}] \cdot Q_{R'_{i+\epsilon_i}}(x-\langle h_i,x\rangle e_j).$$

Then, for the difference of the two quasipolynomials, we obtain the following after noting that we have the following identity for the extended summation

$$\sum_{z\in(a,b)} q(z) - \sum_{z\in[c,b)} q(z) = -\sum_{z\in[b,a]} q(z) - \sum_{z\in[c,b)} q(z) = -\sum_{z\in[c,a)} q(z) - [a\in\mathbb{Z}]q(z).$$[2]

$$\begin{aligned} P_{R_1}(x) - P_{R_2}(x) = &\sum_{z\in[0,\langle h_1,x\rangle)} Q_{R'_1}(x-ze_j) + \sum_{z\in(\langle h_1,x\rangle,\langle h_2,x\rangle)} Q_{R'_2}(x-ze_j) + \\ &[\langle h_1,x\rangle\in\mathbb{Z}]Q_{R'_{1+\epsilon_1}}(x-\langle h_1,x\rangle e_j) - \sum_{z\in[0,\langle h_2,x\rangle)} Q_{R'_2}(x-ze_j) = \\ &\sum_{z\in[0,\langle h_1,x\rangle)} Q_{R'_1}(x-ze_j) - Q_{R'_2}(x-ze_j) \\ &+ [\langle h_1,x\rangle\in\mathbb{Z}](Q_{R'_{1+\epsilon_1}}(x-\langle h_1,x\rangle e_j) - Q_{R'_2}(x-\langle h_1,x\rangle e_j)) \end{aligned}$$

We can choose $\epsilon_1 \in \{0,1\}$ arbitrarily except for two cases: We have $\epsilon_1 = 0$ if $r=1$ and $\epsilon_1 = 1$ if $R'_1 = \mathbb{Q}^d \setminus \mathcal{C}_{A,I\setminus\{j\}}$. Thus

$$\begin{aligned} P_{R_1}(x) - P_{R_2}(x) &= \sum_{z\in[0,\langle h_1,x\rangle)} Q_{R'_1}(x-ze_j) - Q_{R'_2}(x-ze_j) \\ &= \sum_{z\in[0,\langle h_1,x\rangle]} Q_{R'_1}(x-ze_j) - Q_{R'_2}(x-ze_j), \end{aligned}$$

where only the first formula is valid if, for all $i \in I\setminus\{j\}$, e_i is either on H or on the opposite side of H as e_j and only the second formula is valid if, for all $i\in I\setminus\{j\}$, e_i is either on H or on the same side of H as e_j. Since the formula is equivalent to the formula we obtain by interchanging the role of R_1 and R_2, it is also vaild if e_j is on the same side of H as R_2. Clearly, the formula is also true if $R_1 \cup R_2 \subseteq \mathbb{Q}^d \setminus \mathcal{C}_{A,I}$. We summarize this in the following lemma.

Lemma 4. *Let R_1, R_2 be two rooms adjacent via $H = \{q\in\mathbb{Q}^d : \langle h,q\rangle = 0\}$ and $j\in I$ such that $e_j \notin H$ and $\langle A\rangle + \mathbb{Q}\,\mathrm{Cube}_{A,I\setminus\{j\}} = \mathbb{Q}^n$. For $i=1,2$ denote by R'_i the room of $(A, I\setminus\{j\})$ containing R_i. Then we have*

$$\begin{aligned} P_{R_1}(x) - P_{R_2}(x) = &\sum_{z\in[0,\langle h,x\rangle/h_j)} Q_{R'_1}(x-ze_j) - Q_{R'_2}(x-ze_j) = \\ &\sum_{z\in[0,\langle h,x\rangle/h_j]} Q_{R'_1}(x-ze_j) - Q_{R'_2}(x-ze_j), \end{aligned}$$

[2]Note that it is a bit awkward with this extended definition of summation that $\sum_{z\in(a,a)} q(z) = \sum_{z=\lfloor a\rfloor+1}^{\lceil a\rceil-1} q(z) = -[a\in\mathbb{Z}]q(a)$.

where only the first formula is valid if $h_i/h_j \leq 0$ for all $i \in I \setminus \{j\}$ and only the second formula is valid if $h_i/h_j \geq 0$ for all $i \in I \setminus \{j\}$.

We turn our attention to the conclusion of the proof of Theorem 1. Two facts remain to be shown: Firstly, we have to show $P_{R_1}(x) = P_{R_2}(x)$ if the rooms R_1 and R_2 are contained in the same chamber of (A, I) and, secondly, we have to show that $\mathrm{N}_{A,I}(x) = P_R(x)$ for all $x \in R - \mathrm{Cube}_{n,I}$, where R is a room of (A, I). We start with the first assertion.

3.2. **From rooms to chambers.** Suppose $H = \langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,J}$, where $J \subseteq I$ and $|J| = n-d-1$. Choose $k \in I$ with $H + \mathbb{Q}e_k = \mathbb{Q}^n$. For $j \in J$, set $H^j = \langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,J\setminus\{j\}\cup\{k\}}$ and choose $h_j \in \mathbb{Q}^d$ with $H^j = \{q \in \mathbb{Q}^d : \langle h_j, q \rangle = 0\}$. For a fixed j, we have either $\langle h_j, q \rangle \geq 0$ for all $q \in \overline{R_1} \cup \overline{R_2}$ or $\langle h_j, q \rangle \leq 0$ for all $q \in \overline{R_1} \cup \overline{R_2}$. This implies the existence of a unique $\epsilon \in \{0, 1\}^J$ such that

$$\overline{R_1} \cap \overline{R_2} \subseteq \langle A \rangle + \mathrm{cone}((-1)^{\epsilon_j} e_j | j \in J).$$

The following remains to be shown: If for all choices of generators $(e_j)_{j\in J}$ of H we have $\epsilon \neq 0$ then $P_{R_1}(x) = P_{R_2}(x)$. This is done by induction with respect to $|I|$. The case $|I| = n - d$ is dealt with in Lemma 1. Thus we assume $|I| > n - d$.

Case 1: $k \in I$ can be chosen such that $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I\setminus\{k\}} = \mathbb{Q}^n$. Let R_1', R_2' be the rooms of the set of hyperplanes of $(A, I \setminus \{k\})$ containing R_1, R_2 respectively. Moreover, for generators $(e_j)_{j\in J}$, let $\epsilon \in \{0, 1\}^J$ be such that

$$\overline{R_1'} \cap \overline{R_2'} \subseteq \langle A \rangle + \mathrm{cone}((-1)^{\epsilon_j} e_j | j \in J).$$

Since $\overline{R_1} \cap \overline{R_2} \subseteq \overline{R_1'} \cap \overline{R_2'}$ we can conclude that $\epsilon \neq 0$, and, by the induction hypothesis, $Q_{R_1'} = Q_{R_2'}$. The assertion follows from Lemma 4.

Case 2: There exists a unique $k \in I$ with $H + \mathbb{Q}e_k = \mathbb{Q}^n$. The case $R_1, R_2 \subseteq \mathcal{C}_{A,I}$ is impossible under this assumption and $R_1, R_2 \subseteq \mathbb{Q}^n \setminus \mathcal{C}_{A,I}$ trivial. If $R_1 \subseteq \mathcal{C}_{A,I}$ and $R_2 \subseteq \mathbb{Q}^n \setminus \mathcal{C}_{A,I}$ then $\overline{R_1} \cap \overline{R_2}$ is contained in the unique bounding $(n-1)$–dimensional cone of $\mathcal{C}_{A,I}$ not containing e_k, i.e. $\langle A \rangle + \mathbb{Q}_{\geq 0}\,\mathrm{Cube}_{n,I\setminus\{k\}}$ and there is nothing to prove.

3.3. **Overlaps.** In order to prove the last open assertion of Theorem 1, the following is crucial.

Lemma 5. *Let $x \in \mathbb{Z}^n, w \in \mathrm{Cube}_{n,I}$ be with $\frac{\langle h,x \rangle}{\langle h,x+w \rangle} < 0$. Then $P_{R_1}(x) = P_{R_2}(x)$.*

Proof. We choose h such that $\langle h, x \rangle < 0$ and $\langle h, x+w \rangle > 0$. Note that this implies $\langle h, w \rangle > 0$. We prove the assertion by induction with respect to the number of $j \in I$ such that $e_j \notin H$.

Base case of the induction: there is a unique $j \in I$ with $e_j \notin H$. W.l.o.g. $R_2 \subseteq \mathbb{Q}^n \setminus \mathcal{C}_{A,I}$ and thus it suffices to consider the case $R_1 \subseteq \mathcal{C}_{A,I}$. This implies $P_{R_2} \equiv 0$ and we have to show $P_{R_1}(x) = 0$. By (1.1),

$$P_{R_1}(y) = [\langle h, y \rangle / h_j \in \mathbb{Z}]\, \mathrm{N}_{A,I\setminus\{j\}}\,(y - e_j \langle h, y \rangle / h_j)$$

for all $y \in \overline{R}_1 \cap \mathbb{Z}^n$. This implies that

$$[\langle h, y \rangle / h_j \in \mathbb{Z}]\, P_{R_1}(y)$$

evaluates to $\mathrm{N}_{A,I}(y)$ for all $y \in \overline{R}_1 \cap \mathbb{Z}^n$. This expression is obviously a quasipolynomial and since there is a unique quasipolynomial $P_{R_1}(y)$ that evaluates to $\mathrm{N}_{A,I}$ on $\overline{R}_1 \cap \mathbb{Z}^n$ we have

$$P_{R_1}(y) = [\langle h, y \rangle / h_j \in \mathbb{Z}]\, P_{R_1}(y).$$

Thus, it suffices to show

$$[\langle h, x \rangle / h_j \in \mathbb{Z}] = 0.$$

Let $\epsilon \in [0,1]$ be with $w - \epsilon e_j \in H$. As $0 < \langle h, w \rangle = \langle h, \epsilon e_j \rangle = \epsilon h_j$ we conclude $h_j > 0$. If the fraction $\langle h, x \rangle / h_j$ were an integer then it could not be greater than -1 since $\langle h, x \rangle / h_j < 0$ by assumption. As $\epsilon \in [0,1]$ we have

$$\langle h, x + w \rangle / h_j = \langle h, x \rangle / h_j + \epsilon \leq 0,$$

which is a contradiction to our assumption that $\langle h, x + w \rangle > 0$.

We have to distinguish between two cases for the induction step.

Case A: there exist $i, j \in I$ with $h_i > 0$ and $h_j < 0$. Since $H + \mathbb{Q} e_i = \mathbb{Q}^d$, we have $\langle A \rangle + \mathbb{Q} \operatorname{Cube}_{n, I \setminus \{j\}} = \mathbb{Q}^n$. By Lemma 4,

$$P_{R_1}(x) - P_{R_2}(x) = \sum_{z \in [0, \langle h, x \rangle / h_j)} Q_{R'_1}(x - z e_j) - Q_{R'_2}(x - z e_j), \tag{3.2}$$

where R'_i is the room of the set of hyperplanes of $(A, I \setminus \{j\})$ containing R_i. By our assumptions $\langle h, x \rangle / h_j > 0$. Moreover, $\langle h, x - z e_j \rangle < 0$ and

$$\langle h, x - z e_j + w - \langle w, e_j \rangle e_j \rangle = \langle h, x + w \rangle - (z + w_j) h_j \geq \langle h, x + w \rangle > 0$$

for all $z \in [0, \langle h, x \rangle / h_j)$ as $w_j \geq 0$ and $h_j < 0$. The induction hypothesis (note that $w - \langle w, e_j \rangle e_j \in \operatorname{Cube}_{n, I \setminus \{j\}}$) implies

$$Q_{R'_1}(x - z e_j) = Q_{R'_2}(x - z e_j)$$

for all $z \in [0, \langle h, x \rangle / h_j)$ and the assertion follows in this case.

Case B: there exist at least two $j \in I$ with $e_j \notin H$ and all of them lie on the same side of H. Let j be with this property. Clearly, $\langle A \rangle + \mathbb{Q} \operatorname{Cube}_{n, I \setminus \{j\}} = \mathbb{Q}^n$. By Lemma 4 and the extended definition of the summation over an interval we have

$$P_{R_1}(x) - P_{R_2}(x) = \sum_{z \in (\langle h, x \rangle / h_j, 0)} Q_{R'_2}(x - z e_j) - Q_{R'_1}(x - z e_j)$$

where R'_i is the room of the hyperplanes of $(A, I \setminus \{j\})$ containing R_i. By $\langle h, w \rangle > 0$ we have $h_j > 0$ and thus $\langle h, x \rangle / h_j < 0$. Moreover, $\langle h, x - z e_j \rangle < 0$ and $\langle h, x - z e_j + w - \langle w, e_j \rangle e_j \rangle > 0$ for all $z \in (\langle h, x \rangle / h_j, 0) \cap \mathbb{Z}$. (The second assertion follows since $z \leq -1$ and $w_j \in [0,1]$.) The assertion follows as in the previous case. □

Remark 1. *Identity 3.2 actually shows the following: If there exist $i, j \in I$ with $h_i > 0$ and $h_j < 0$ (Case A) then the assumption $\langle h, x \rangle = 0$ also implies $P_{R_1}(x) = P_{R_2}(x)$.*

Now we are in the position to prove Theorem 2.

Proof of Theorem 2. If $r = 0$ then this follows from the remark. Let $r \in \{1, 2, \ldots, \operatorname{pos} - 1\}$ and $x \in H^r$. Then

$$\frac{\langle h, x \rangle}{\langle h, x + \sum_{i \in I : h_i > 0} e_i \rangle} = \frac{-r}{-r + \operatorname{pos}} < 0$$

and $P_{C_1}(x) - P_{C_2}(x) = 0$ by Lemma 5. The argument is similar if $r \in \{\operatorname{neg} + 1, \operatorname{neg} + 2, \ldots, -1\}$. □

We conclude the proof of Theorem 1: Let R be a room of the hyperplanes of (A, I) and $x \in R, w \in \operatorname{Cube}_{n,I}$ with $x - w \in \mathbb{Z}^n$. We have to show $\mathrm{N}_{A,I}(x - w) = P_R(x - w)$. Let $H_1, \ldots, H_m$ be the hyperplanes of (A, I) and choose $h_i \in \mathbb{Q}^d$ such that $H_i = \{q \in \mathbb{Q}^d : \langle h_i, q \rangle = 0\}$ and $R = \bigcap_{i=1}^{m} \{q \in \mathbb{Q}^d : \langle h_i, q \rangle > 0\}$. Moreover, we assume w.l.o.g. $\langle h_i, x - w \rangle > 0$ for $1 \leq i \leq r$,

$\langle h_i, x-w\rangle = 0$ for $r+1 \le i \le s$ and $\langle h_i, x-w\rangle < 0$ for $s+1 \le i \le m$. Observe that

$$S = \bigcap_{i=1}^{s} \{q \in \mathbb{Q}^d : \langle h_i, q\rangle > 0\} \cap \bigcap_{i=s+1}^{m} \{q \in \mathbb{Q}^d : \langle h_i, q\rangle < 0\}$$

is a non-empty room of the hyperplanes of (A, I): There exists $\epsilon > 0$ such that $x-w+\epsilon x \in S$. We have $x - w \in \overline{S}$.

This implies $P_S(x-w) = \mathrm{N}_{A,I}(x-w)$: In order to see this we have to show $S \subseteq \mathcal{C}_{A,I}$ if $x-w \in \partial(\mathcal{C}_{A,I})$. However, if $x-w \in \partial(\mathcal{C}_{A,I})$ then $x \in \mathcal{C}_{A,I}$ and thus $R \subseteq \mathcal{C}_{A,I}$. Therefore, all bounding hyperplanes of $\mathcal{C}_{A,I}$ must be in $\{H_1, \ldots, H_s\}$. (Otherwise $x-w \in \mathbb{Q}^n \setminus \mathcal{C}_{A,I}$.) Thus, the room S lies on the same side of the bounding hyperplanes of $\mathcal{C}_{A,I}$ as R and is therefore contained in $\mathcal{C}_{A,I}$.

Consequently, it suffices to show

$$P_R(x-w) = P_S(x-w).$$

We prove the assertion by induction with respect to $m-s$. If $m = s$ then $R = S$. If $m > s$ then the set $\{H_{s+1}, \ldots, H_m\}$ contains a bounding hyperplane H' of R with $\dim \overline{R} \cap H' = n-1$ (otherwise S would be on the same side of the bounding hyperplanes of R as R which implies $R = S$); w.l.o.g. let this be H_{s+1}. Let R_1 be the room which is adjacent to R via H_{s+1}. By Lemma 5, we have

$$P_R(x-w) = P_{R_1}(x-w)$$

and, by the induction hypothesis,

$$P_{R_1}(x-w) = P_S(x-w).$$

4. Tools for the computation of the polynomials representing $x \to \mathrm{N}_{A,I}(x)$

4.1. **Iterative application of Lemma 4.** In this section, we assume that (A, I) has the property given in Theorem 1 that implies that the quasipolynomials representing $x \to \mathrm{N}_{A,I}(x)$ are in fact polynomials. For the moment we also stick to our assumption that $\dim \mathcal{C}_{A,I} = n$. At the end of Section 2 it was shown that in this case the property implying the polynomiality is equivalent with the following property (Property (P)): *For all hyperplanes H of (A, I), there is a normal vector $h \in \mathbb{Z}^n$ of H with $h_j \in \{0, 1, -1\}$ for all j.* We say that h is a *reduced normal vector* of H.

This implies the following for the difference of the polynomials representing $x \to \mathrm{N}_{A,I}(x)$ on two adjacent rooms of (A, I): Suppose R_1, R_2 are two rooms adjacent via the hyperplane H. Let P_{R_1}, P_{R_2} denote the respective polynomials, choose $j \in I$ with $e_j \notin H$ and a reduced normal vector $h \in \mathbb{Z}^n$ of H. We have to distinguish between two cases.

Case 1: $\langle A\rangle + \mathbb{Q}\,\mathrm{Cube}_{A,I\setminus\{j\}} \neq \mathbb{Q}^n$. Let R' be the room of $(A, I \setminus \{j\})$ in H containing the inner points of $\overline{R_1} \cap \overline{R_2}$ and $Q_{R'}(x)$ be a quasipolynomial with $\mathrm{N}_{A,I\setminus\{j\}}(x) = Q_{R'}(x)$ on $(R - \mathrm{Cube}_{n,I\setminus\{j\}}) \cap \mathbb{Z}^n$. Then

$$P_{R_1}(x) - P_{R_2}(x) = \pm Q_{R'}(x - \langle h, x\rangle / h_j e_j),$$

where the plus sign is valid if $R_1 \subseteq \mathcal{C}_{A,I}$. This follows essentially from the recursion (1.1) and by choosing x close enough to R'.

Case 2: $\langle A\rangle + \mathbb{Q}\,\mathrm{Cube}_{A,I\setminus\{j\}} = \mathbb{Q}^n$. For $i = 1, 2$, let R'_i denote the chamber of $(A, I \setminus \{j\})$ containing R_i, and let $Q_{R'_i}(x)$ be the respective polynomial. Then, as $\langle h, x\rangle / h_j$ is an integer if $x \in \mathbb{Z}^n$ and by Lemma 4,

$$P_{R_1}(x) - P_{R_2}(x) = \sum_{z=0}^{\langle h,x\rangle/h_j - 1} Q_{R'_1}(x - ze_j) - Q_{R'_2}(x - ze_j)$$
$$= \sum_{z=0}^{\langle h,x\rangle/h_j} Q_{R'_1}(x - ze_j) - Q_{R'_2}(x - ze_j),$$

where only the first formula is valid if $h_i/h_j \in \{0,-1\}$ for all $i \in I \setminus \{j\}$ and only the second formula is valid if $h_i/h_j \in \{0,1\}$ for all $i \in I \setminus \{j\}$.

We apply this formula recursively: Let $Q = \{i_1, i_2, \ldots, i_q\}$ be the set of indices with $i_j \in I$ and e_{i_j} not contained in H and p minimal such that $e_{i_{p+1}}, e_{i_{p+2}}, \ldots, e_{i_q}$ all lie on the same side of H. Then

$$P_{R_1}(x) - P_{R_2}(x) = \pm \sum_{z_1=0}^{\langle h,x\rangle/h_{i_1} - 1} \sum_{z_2=0}^{\langle h,x-z_1e_{i_1}\rangle/h_{i_2}-1} \ldots \sum_{z_p=0}^{\langle h,x-z_1e_{i_1}-\ldots-z_{p-1}e_{i_{p-1}}\rangle/h_{i_p}-1}$$
$$\sum_{z_{p+1}=0}^{\langle h,x-z_1e_{i_1}-\ldots-z_pe_{i_p}\rangle/h_{i_{p+1}}} \ldots \sum_{z_{q-1}=0}^{\langle h,x-z_1e_{i_1}-\ldots-z_{q-2}e_{i_{q-2}}\rangle/h_{i_{q-1}}}$$
$$Q_{R'}(x - z_1e_{i_1} - \ldots - z_{q-1}e_{i_{q-1}} - \langle h, x - z_1e_{i_1} - \ldots - z_{q-1}e_{i_{q-1}}\rangle/h_{i_q}e_{i_q}), \quad (4.1)$$

where the plus sign is valid if R_1 lies on the same side as e_{i_q} and R' is the room of $(A, I \setminus Q)$ in H containing the inner points of $\overline{R_1} \cap \overline{R_2}$. (Note that the upper bounds of summations can also be increased by 1 in the first $p-1$ sums individually.) This can also be phrased differently as follows.

Lemma 6. *We assume* $\dim \mathcal{C}_{A,I} = n$ *and that* (A,I) *has Property (P). Suppose* R_1, R_2 *are two rooms of* (A,I) *that are adjacent via the hyperplane* H*, let* $Q := \{i_1,\ldots,i_q\} = \{i \in I | e_i \notin H\}$*, choose a reduced normal vector* h *of* H *and let* R' *denote the room of* $(A, I\setminus Q)$ *in* H *that contains the inner points of* $\overline{R_1} \cap \overline{R_2}$*. Let* $P_{R_1}(x), P_{R_2}(x)$ *be the polynomials that coincide with* $\mathrm{N}_{A,I}(x)$ *on* $(R_1 - \mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$ *and* $(R_2 - \mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$*, respectively, and* $Q_{R'}(x)$ *any polynomial that coincides with* $\mathrm{N}_{A,I\setminus Q}(x)$ *on* $(R' - \mathrm{Cube}_{n,I\setminus Q}) \cap \mathbb{Z}^n$*. Then*

$$P_{R_1}(x) - P_{R_2}(x) = \pm \sum_{(z_1,\ldots,z_q)\in\mathbb{Z}^q} (-1)^{\# i: z_i < 0} Q_{R'}(x - z_1e_{i_1} - \ldots - z_qe_{i_q}),$$

where the sum is over all $z_1, \ldots, z_q$ *with* $z_1h_{i_1} + z_2h_{i_2} + \ldots + z_qh_{i_q} = \langle h, x\rangle$ *such that* $z_j \geq 0$ *if* $\langle h,x\rangle/h_{i_j} \geq 0$ *and* $z_j < 0$ *otherwise, and the plus sign has to be chosen if* x *is on the same side of* H *as* R_1*.*

Proof. By possibly rearranging $(i_1, i_2, \ldots, i_q)$, we may assume $h_{i_1} = h_{i_2} = \ldots = h_{i_p} = -1$ and $h_{i_{p+1}} = h_{i_{p+2}} = \ldots = h_{i_q} = 1$ in (4.1). We consider the case $\langle h,x\rangle \geq 0$ and rewrite (4.1) using the extended definition of the summation as follows.

$$P_{R_1}(x) - P_{R_2}(x) = \pm(-1)^p \sum_{z_1=-\langle h,x\rangle}^{-1} \sum_{z_2=-\langle h,x\rangle-z_1}^{-1} \cdots \sum_{z_p=-\langle h,x\rangle-z_1-z_2-\ldots-z_{p-1}}^{-1}$$
$$\sum_{z_{p+1}=0}^{\langle h,x\rangle+z_1+\ldots+z_p} \sum_{z_{p+2}=0}^{\langle h,x\rangle+z_1+\ldots+z_p-z_{p+1}} \cdots \sum_{z_{q-1}=0}^{\langle h,x\rangle+z_1+\ldots+z_p-z_{p+1}-\ldots-z_{q-2}}$$
$$Q_{R'}(x - z_1 e_{i_1} - \ldots - z_{q-1} e_{i_{q-1}} - (\langle h,x\rangle + z_1 + \ldots + z_p - z_{p+1} - \ldots - z_{q-1})\, e_{i_q})$$

Now the lower bounds of the p outer summations are no greater than 0 and we can rewrite these as the sum over all $(z_1,\ldots,z_p) \in \mathbb{Z}^p$ with $-\langle h,x\rangle \le z_1+\ldots+z_p$ and $z_i < 0$. In the $q-p-1$ inner sums, the upper bounds of the summations are non-negative and we can rewrite them as the sum over all $(z_{p+1},\ldots,z_{q-1}) \in \mathbb{Z}^{q-1-p}$ with $z_{p+1}+\ldots+z_{q-1} \le \langle h,x\rangle + z_1+\ldots+z_p$ and $z_i \ge 0$. We set $z_q = \langle h,x\rangle + z_1 + \ldots + z_p - z_{p+1} - \ldots z_{q-1}$ and the assertion follows in this case. Concerning the sign observe that x is on the same side of H as e_{i_q}. The case $\langle h,x\rangle < 0$ is similar. □

From now on, we deviate from our assumption that $\dim \mathcal{C}_{A,I} = n$. At this point it is important to note that the quasipolynomials representing $x \to \mathrm{N}_{A,I}(x)$ are uniquely determined if $\dim \mathcal{C}_{A,I} = n$ (see also Appendix B.2). In case $\dim \mathcal{C}_{A,I} < n$, there are more choices, however, all possibilities coincide on $\langle A\rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$.

Remark 2. *Lemma 6 can be extended to the case when* $\dim \mathcal{C}_{A,I} < n$ *as follows (see also Appendix A): choose a minimal superset* $I \subseteq I' \subseteq [n]$ *with* $\dim \mathcal{C}_{A,I'} = n$. *We assume that* (A,I') *has the property that implies that the quasipolynomials representing* $x \to \mathrm{N}_{A,I'}(x)$ *are in fact polynomials. As in Lemma 6, suppose* R_1, R_2 *are two rooms of* (A,I) *that are adjacent via the hyperplane* H, *let* $Q := \{i_1,\ldots,i_q\} = \{i \in I | e_i \notin H\}$ *and* R' *denote the room of* $(A, I \setminus Q)$ *in* H *that contains the inner points of* $\overline{R_1} \cap \overline{R_2}$. *Then, for* $i = 1,2$, $S_i := R_i + \sum_{i\in I'\setminus I} \mathbb{Q}_{>0} e_i$ *is a room of* (A,I'), *and* S_1, S_2 *are adjacent via the hyperplane* $G := H + \sum_{i\in I'\setminus I} \mathbb{Q} e_i$. *Note that* $Q = \{i \in I' | e_i \notin G\}$. *Let, for* $i = 1,2$, $P_{S_i}(x)$ *denote the polynomial representing* $\mathrm{N}_{A,I'}(x)$ *on* $(S_i - \mathrm{Cube}_{n,I'}) \cap \mathbb{Z}^n$ *and observe that* $P_{S_i}(x) = \mathrm{N}_{A,I}(x)$ *if* $x \in (R_i - \mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$. *Moreover,* $S' = R' + \sum_{i\in I'\setminus I} \mathbb{Q}_{>0} e_i$ *is a room of* $(A, I' \setminus Q)$ *in* G; S' *contains the inner points of* $\overline{S_1} \cap \overline{S_2}$. *Let* $Q_{S'}(x)$ *be a polynomial representing* $\mathrm{N}_{A,I'\setminus Q}(x)$ *on* $(S' - \mathrm{Cube}_{n,I'\setminus Q}) \cap \mathbb{Z}^n$ *and note that* $Q_{S'}(x) = \mathrm{N}_{A,I\setminus Q}(x)$ *if* $x \in (R' - \mathrm{Cube}_{n,I\setminus Q}) \cap \mathbb{Z}^n$. *Choose a reduced normal vector* h *of* G. *Then, by Lemma 6,*

$$P_{S_1}(x) - P_{S_2}(x) = \pm \sum_{(z_1,\ldots,z_q)\in\mathbb{Z}^q} (-1)^{\# i: z_i<0} Q_{S'}(x - z_1 e_{i_1} - \ldots - z_q e_{i_q}),$$

where the sum is over all $z_1,\ldots,z_q$ *with* $z_1 h_{i_1} + z_2 h_{i_2} + \ldots + z_q h_{i_q} = \langle h,x\rangle$ *such that* $z_j \ge 0$ *if* $\langle h,x\rangle / h_{i_j} \ge 0$ *and* $z_j < 0$ *otherwise, and the plus sign has to be chosen if* x *is on the same side of* H *than* S_1. *This shows that Lemma 6 also holds in case that* $\dim \mathcal{C}_{A,I} < n$ *if the polynomials* $P_{R_1}(x), P_{R_2}(x), Q_{R'}(x)$ *and the normal vector* h *are chosen properly.*

4.2. **Neighborhood of rooms and a signed sum of polynomials associated with it.** We want to apply also the formula provided in Lemma 6 recursively. For this purpose, the notion of m-neighborhoods of rooms of (A,I) is needed. The set of 01-words of length m is denoted by $\mathcal{W}_m$.

Definition 3 (m-neighborhoods of (A,I))**.** *Let m be a non-negative integer. A 0-neighborhood is a single room of (A,I). If $m \ge 1$, it is a collection $(R_w)_{w \in \mathcal{W}_m}$ of rooms of (A,I) such that there exists a hyperplane H of (A,I) with the following properties:*

1. *For all $w \in \mathcal{W}_{m-1}$, the room R_{w0} is adjacent to R_{w1} via H; let R'_w denote the room of (A,I') in H where $I' = \{i \in I | e_i \in H\}$ that contains the inner points of $\overline{R}_{w0} \cap \overline{R}_{w1}$.*
2. *The collection $(R'_w)_{w \in \mathcal{W}_{m-1}}$ is an $(m-1)$-neighborhood of (A,I').*
3. *All rooms R_{w0}, $w \in \mathcal{W}_{m-1}$, are on the same side of H, and this side also contains at least one element e_i with $i \in I$.*

An m-neighborhood induces naturally a sequence of m nested subspaces:

Definition 4 (Sequence of nested subspaces associated with a neighborhood of (A,I))**.** *Let $\mathcal{R} = (R_w)_{w \in \mathcal{W}_m}$ be an m-neighborhood of (A,I). If $m \ge 1$, choose H and $\mathcal{R}' := (R'_w)_{w \in \mathcal{W}_{m-1}}$ as in Definition 3. Then the sequence of m nested subspaces, say $(H_1, H_2, \ldots, H_m)$, associated with $\mathcal{R}$ is defined as follows: $H_1 = H$ and $(H_2, \ldots, H_m)$ is the sequence of nested subspaces of $\mathcal{R}'$. The basement of $\mathcal{R}$ is R_- if $m = 0$; otherwise it is the basement of $\mathcal{R}'$.*

The following proposition characterizes sequences of nested subspace that are associated with neighborhoods of (A,I); in the following we also refer to these sequences as the *sequences of nested subspaces of* (A,I). The proof is easy and left to the reader.

Proposition 1. *Let $(H_1, H_2, \ldots, H_m)$ be a sequence of nested subspaces of $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$ and R be a room of (A,I') with $I' = \{i \in I | e_i \in H_m\}$. Then the sequence is associated with an m-neighborhood of (A,I) with basement R if and only if each subspace H_i is generated by $\langle A \rangle$ and certain e_i, $i \in I$, and $\dim H_i = \dim(\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}) - i$.*

The following neighborhoods are of special interest.

Definition 5 (Complete m-neighborhoods of (A,I))**.** *Let $\mathcal{R}$ be an m-neighborhood of (A,I) and $(H_1, H_2, \ldots, H_m)$ the sequence of nested subspaces associated with it. It is said to be complete if $\{a_i | 1 \le i \le d\} \cup \{e_i | i \in I, e_i \in H_m\}$ is a basis of H_m.*

Complete neighborhoods are relevant since $\mathrm{N}_{A,I'}(x) = 1$ if $x \in \mathcal{C}_{A,I'} \cap \mathbb{Z}^n$ and zero elsewhere, where I' is as in Proposition 1. Observe that for $m = \dim(\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}) - d$ all m-neighborhoods associated with (A,I) are complete: this is because the boundedness of our polytopes implies $\langle A \rangle \cap \{e_i | i \in I\} = \emptyset$.

We will see that the following signed sums associated with neighborhoods $(R_w)_{w \in \mathcal{W}_m} =: \mathcal{R}$ are of interest.

Definition 6 (Signed sums of neighborhoods)**.** *Let $\mathcal{R} = (R_w)_{w \in \mathcal{W}_m}$ be an m-neighborhood of (A,I). The signed sum of $\mathcal{R}$ is defined as the polynomial function*

$$\sum_{w \in \mathcal{W}_m} \operatorname{sgn} w \, P_{R_w}(x) =: \mathcal{S}_{\mathcal{R}}(x)$$

on $\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$, where $\operatorname{sgn} w = (-1)^{\#\text{ of 1s in }w}$ and $P_{R_w}(x)$ is a polynomial representing $\mathrm{N}_{A,I}(x)$ on $(R_w - \mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$.

In particular, we will show the following:

- Each polynomial representing $x \to \mathrm{N}_{A,I}(x)$ can be written as a sum of $\mathcal{S}_{\mathcal{R}}(x)$'s over certain m-neighborhoods $\mathcal{R}$ of (A,I), m fixed, see Proposition 2.
- For complete m-neighborhoods, we give a combinatorial interpretation of $\mathcal{S}_{\mathcal{R}}(x)$ for all $x \in (\langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$. It will be in terms of a signed enumeration, see Section 4.4. (Note that we do not have such an interpretation for the polynomials representing $x \to \mathrm{N}_{A,I}(x)$.)

- We provide tools to compute $\mathcal{S}_{\mathcal{R}}(x)$ in the case of completeness and use them to solve the case $m = 1, 2$, see Section 4.6.
- The signed sum $\mathcal{S}_{\mathcal{R}}(x)$ of a neighborhood (complete or not) depends only on its associated sequence of nested subspaces and the basement.

Remark 3. *Let $\mathcal{R} = (R_w)_{w\in\mathcal{W}_m}$ be an m-neighborhood with associated sequence of nested subspaces $(H_1, H_2, \ldots, H_m)$, and set $H_0 = \langle A\rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$. Suppose that, for all $i \in \{1, 2, \ldots, m\}$, H_i is a supporting hyperplane of the cone $\mathcal{C}_{A,I\cap\{j|j\in H_{i-1}\}}$ in H_{i-1}. Then $R_{00\ldots0}$ is the sole room of $\mathcal{R}$ that is contained in $\mathcal{C}_{A,I}$. In this case $\mathcal{S}_{\mathcal{R}}(x) = P_{R_{00\ldots0}}(x)$.*

4.3. $P_R(x) = \sum_{\mathcal{R}} \mathcal{S}_{\mathcal{R}}(x)$.

Proposition 2. *Fix $m \in \{0, 1, \ldots, \dim(\langle A\rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}) - d\}$ and a room R of (A, I), and let $P_R(x)$ denote a polynomial representing $\mathrm{N}_{A,I}(x)$ on $(R - \mathrm{Cube}_{n,I}) \cap \mathbb{Z}^n$. Then there exists a sequence $(\mathcal{R}_j)_{1\le j\le s}$ of m-neighborhoods of (A, I) such that*

$$P_R(x) = \sum_{j=1}^{s} \mathcal{S}_{\mathcal{R}_j}(x)$$

on $\langle A\rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$.

Proof. If $\dim \mathcal{C}_{A,I} < n$ then choose all polynomials representing $x \to \mathrm{N}_{A,I}(x)$ that appear in this proof with respect to the same minimal superset $I \subseteq I' \subseteq [n]$ such that $\dim \mathcal{C}_{A,I'} = n$; see Remark 2. Also choose the reduced normal vectors of the hyperplanes of (A, I) as indicated there.

We use induction with respect to m; there is nothing to do for $m = 0$. Otherwise there exists a sequence of rooms $R_0, R_1, \ldots, R_t$ with $R_0 = R$ and $R_t \subseteq \mathbb{Q}^n \setminus \mathcal{C}_{A,I}$ such that R_{s-1}, R_s are adjacent for all s. Thus,

$$P_R(x) = P_{R_t}(x) + \sum_{s=1}^{t}(P_{R_{s-1}}(x) - P_{R_s}(x)) = \sum_{s=1}^{t}(P_{R_{s-1}}(x) - P_{R_s}(x)).$$

It suffices to show that the differences $P_{R_{s-1}}(x) - P_{R_s}(x)$ are expressible as sums of signed sums associated with m-neighborhoods. Fix s, suppose R_{s-1}, R_s are adjacent via the hyperplane H of (A, I) and let $Q = \{j \in I | e_j \notin H\}$. Furthermore, let R' be the room of $(A, I \setminus Q)$ containing the inner points of $\overline{R_{s-1}} \cap \overline{R_s}$. By the induction hypothesis, there exists a sequence $\mathcal{R}'_j = (R'^j_w)_{w\in\mathcal{W}_{m-1}}$, $j = 1, 2, \ldots, r$, of $(m-1)$-neighborhoods associated with $(A, I \setminus Q)$ such that

$$Q_{R'}(x) = \sum_{j=1}^{r} \mathcal{S}_{\mathcal{R}'_j}(x).$$

(In the case of $\dim \mathcal{C}_{A,I} < n$, we use a minimal superset $I' \setminus Q \subseteq I'' \subseteq [n]$ with $\dim \mathcal{C}_{A,I''} = n$ to find the polynomials representing $x \to \mathrm{N}_{A,I\setminus Q}$.) For R'^j_w, let R^j_{w0}, R^j_{w1} be rooms of (A, I) that are adjacent via H such that R'^j_w contains the inner points of $\overline{R^j_{w0}} \cap \overline{R^j_{w1}}$ and all rooms R^i_{w0} are on the same side of H as R_{s-1}. By Lemma 6 and Remark 2,

$$P_{R^j_{w0}}(x) - P_{R^j_{w1}}(x) = \pm \sum_{(z_1,\ldots,z_q)\in\mathbb{Z}^q} (-1)^{\#i:z_i<0} Q_{R'^j_w}(x - z_1 e_{i_1} - \ldots - z_q e_{i_q}),$$

where the sum is over all $z_1, \ldots, z_q$ as described there; this is also true when replacing $R^j_{w0}, R^j_{w1}, R'^j_w$ by R_{s-1}, R_s, R', respectively. This implies

$$\begin{aligned}\sum_{j=1}^{r} \sum_{w \in \mathcal{W}_m} \operatorname{sgn} w \, P_{R^j_w}(x) &= \sum_{j=1}^{r} \sum_{w \in \mathcal{W}_{m-1}} \operatorname{sgn} w \, \left(P_{R^j_{w0}}(x) - P_{R^j_{w1}}(x) \right) \\ &= \pm \sum_{(z_1,\ldots,z_q) \in \mathbb{Z}^q} (-1)^{\# i: z_i < 0} \sum_{j=1}^{r} \sum_{w \in \mathcal{W}_{m-1}} \operatorname{sgn} w \, Q_{R'^j_w}(x - z_1 e_{i_1} - \ldots - z_q e_{i_q}) \\ &= \pm \sum_{(z_1,\ldots,z_q) \in \mathbb{Z}^q} (-1)^{\# i: z_i < 0} Q_{R'}(x - z_1 e_{i_1} - \ldots - z_q e_{i_q}) = P_{R_{s-1}}(x) - P_{R_s}(x).\end{aligned}$$

□

4.4. **Combinatorial interpretation of $\mathcal{S}_{\mathcal{R}}(x)$ for all $x \in \mathbb{Z}^n$ if $\mathcal{R}$ is complete.** In the remainder of this section, we assume again $\dim \mathcal{C}_{A,I} = n$. We fix an m-neighborhood $\mathcal{R}$ of (A, I) and let $(H_1, H_2, \ldots, H_m)$ be the sequence of nested subspaces associated with it. In the following proposition we identify nice sequences of vectors $(h_1, h_2, \ldots, h_m)$ with $H_k = \{x \in \mathbb{Q}^n | \langle h_i, x \rangle = 0 \, \forall i \le k\}$ for all k.

Proposition 3. *Let $\mathcal{R}$ be an m-neighborhood of (A, I) and $(H_1, H_2, \ldots, H_m)$ be the sequence of nested subspaces associated with it. Set $Q(k) := \{i \in I | e_i \notin H_k\}$ and $Q(0) := \emptyset$.*

(1) *Choose, for all j, k with $1 \le j < k \le m$, integers $i(j,k) \in Q(j) \setminus Q(j-1)$. Then there exists a sequence of reduced normal vectors $(h_1, h_2, \ldots, h_m)$ of hyperplanes of (A, I) with $H_k = \{x \in \mathbb{Q}^n | \langle h_i, x \rangle = 0 \, \forall i \le k\}$ and $(h_k)_{i(j,k)} = 0$ for all j, k with $1 \le j < k \le m$. Moreover $(h_k)_i \ne 0$ for all $i \in Q(k) \setminus Q(k-1)$ and all $k \in \{1, 2, \ldots, m\}$.*

(2) *Choose, for all j with $1 \le j < m$, integers $i(j) \in Q(j) \setminus Q(j-1)$ and reduced normal vectors $(h_1, h_2, \ldots, h_m)$ of hyperplanes of (A, I) with $H_k = \{x \in \mathbb{Q}^n | \langle h_i, x \rangle = 0 \, \forall i \le k\}$ and $(h_k)_{i(j)} = 0$ for all $1 \le j < k \le m$. Then, for each pair (s, t) with $1 \le s < t \le m$, one of the following is true:*
   (a) *$(h_t)_i = 0$ for all $i \in Q(s) \setminus Q(s-1)$*
   (b) *there is no $i \in Q(s)$ with $(h_s)_i = (h_t)_i \ne 0$*
   (c) *there is no $i \in Q(s)$ with $(h_s)_i = -(h_t)_i \ne 0$*

Proof. We start with the first assertion: We set $H_0 = \mathbb{Q}^n$ and use induction with respect to k. There is nothing to prove for $k = 0$. Let $k \in \{1, 2, \ldots, m\}$. Since H_k is generated by $\langle A \rangle$ and some e_i, $i \in I$, and $\dim H_k = n - k$, there exists a subset $K \subseteq I$ with $|K| = n - k - d$ such that $\{a_i | 1 \le i \le d\} \cup \{e_i | i \in K\}$ is a basis of H_k. Choose any $i(k,k) \in Q(k) \setminus Q(k-1)$. Then $\{a_i | 1 \le i \le d\} \cup \{e_i | i \in K\} \cup \{e_{i(j,k)} | 1 \le j \le k\}$ is a basis of $\mathbb{Q}^n$. Let h_k be a reduced normal vector of the hyperplane of (A, I) which is generated by the following set of vectors: $\{a_i | 1 \le i \le d\} \cup \{e_i | i \in K\} \cup \{e_{i(j,k)} | 1 \le j < k\}$. Note that $(h_k)_i \ne 0$ for any $Q(k) \setminus Q(k-1)$. Furthermore, it is obvious that

$$H_k \subseteq H_{k-1} \cap \{x \in \mathbb{Q}^n | \langle h_k, x \rangle = 0\} \subseteq H_{k-1}.$$

We do not have $H_{k-1} \cap \{x \in \mathbb{Q}^n | \langle h_k, x \rangle = 0\} = H_{k-1}$ as $e_{i(k,k)} \in H_{k-1}$ but $(h_k)_{i(k,k)} \ne 0$. For reasons of dimension, this implies $H_k = H_{k-1} \cap \{x \in \mathbb{Q}^n | \langle h_k, x \rangle = 0\}$ which completes the proof of the first assertion.

For the second assertion we first note the following: suppose $(h'_1, h'_2, \ldots, h'_m)$ is a sequence of vectors with

$$H_k = \{x \in \mathbb{Q}^n | \langle h'_i, x \rangle = 0 \, \forall i \le k\} \tag{4.2}$$

for all k. Then there exist rational numbers $(\lambda_{j,k})_{1\le j\le k\le m}$ such that $h'_k = \sum_{j=1}^{k} \lambda_{j,k}h_j$ and $\lambda_{k,k} \ne 0$ for all k. Choose $i'(s) \in Q(s)\setminus Q(s-1)$ with $(h_t)_{i'(s)} \ne 0$. If this is not possible then there is nothing to prove. Let $(h'_1, h'_2, \dots, h'_m)$ be a sequence of reduced normal vectors of hyperplanes of (A, I) such that (4.2) is fulfilled and with $(h'_k)_{i(j)} = 0$ for all $1 \le j < k \le m$, except that $(h'_t)_{i(s)} = 0$ is replaced by $(h'_t)_{i'(s)} = 0$. Let $\lambda_1, \lambda_2, \dots, \lambda_t \in \mathbb{Q}$ be with $h'_t = \sum_{k=1}^{t} \lambda_k h_k$ and $\lambda_t \ne 0$. We can assume $\lambda_t = 1$ w.l.o.g. since $|(h_t)_i| = |(h'_t)_i| = 1$ and $(h_k)_i = 0$ for all $i \in Q(t) \setminus Q(t-1)$ and all $k < t$. Moreover $\lambda_j = 0$ for $j \in \{1, 2, \dots, t-1\} \setminus \{s\}$: This is because $(h_k)_{i(j)} = 0$ for $k \in \{1, 2, \dots, t\} \setminus \{j\}$, $(h'_t)_{i(j)} = 0$ and $(h_j)_{i(j)} \ne 0$. Thus $h'_t = \lambda_s h_s + h_t$. We show $\lambda_s = \pm 1$: This follows from $(h'_t)_{i'(s)} = 0$, $(h_s)_{i'(s)} \ne 0$ and $(h_t)_{i'(s)} \ne 0$. If $(h_s)_{i'(s)} = (h_t)_{i'(s)} = \pm 1$ then $\lambda_s = -1$, otherwise $\lambda_s = 1$. Thus $h'_t = h_t \pm h_s$, and the assertion follows since the coordinates of h_s, h_t, h'_t are in $\{0, 1, -1\}$. □

Some notation is needed.

- Let $Q(k) := \{i_1, \dots, i_{q(k)}\} = \{i \in I | e_i \notin H_k\}$ and observe that $0 =: q(0) < q(1) < q(2) < \dots < q(m)$.
- The m-neighborhood $\mathcal{R} = (R_w)_{w\in\mathcal{W}_m} =: (R^0_w)_{w\in\mathcal{W}_m}$ induces an $(m-k)$-neighborhood $(R^k_w)_{w\in\mathcal{W}_{m-k}}$ of rooms of $(A, I \setminus Q(k))$ in H_k via $R^k_w = (R^{k-1}_w)'$.
- Use Proposition 3, to choose, for all $k \in \{1, 2, \dots, m\}$, reduced normal vectors $h_k \in \mathbb{Z}^n \setminus \{0\}$ of hyperplanes of (A, I) such that R^{k-1}_{w0} is on its positive side and $H_k = \{x \in \mathbb{Q}^n | \langle h_i, x\rangle = 0\, \forall i \le k\}$.

The fact that R^{k-1}_{w0} is on the positive side of h_k for all k is equivalent to $R_{0\dots0} \subseteq \{x \in \mathbb{Q}^n | \langle h_k, x\rangle > 0\, \forall k\}$.

Then, by Lemma 6,

$$\begin{aligned}\mathcal{S}_{\mathcal{R}}(x) &= \sum_{w\in\mathcal{W}_m} \operatorname{sgn} w\, P_{R_w}(x) = \sum_{w\in\mathcal{W}_{m-1}} \operatorname{sgn} w\, \left(P_{R_{w0}}(x) - P_{R_{w1}}(x)\right)\\ &= \pm \sum_{z_1,\dots,z_{q(1)}} (-1)^{\# i: z_i<0} \sum_{w\in\mathcal{W}_{m-1}} \operatorname{sgn} w\, Q_{R'_w}(x - z_1 e_{i_1} - \dots - z_{q(1)} e_{i_{q(1)}})\\ &= \pm \sum_{z_1,\dots,z_{q(1)}} (-1)^{\# i: z_i<0} \mathcal{S}_{\mathcal{R}'}(x - z_1 e_{i_1} - \dots - z_{q(1)} e_{i_{q(1)}}),\end{aligned}$$

where $\mathcal{R}' = (R'_w)_{w\in\mathcal{W}_{m-1}}$ and the sum is over all integers $z_1, \dots, z_{q(1)}$ with $z_1(h_1)_{i_1} + \dots + z_{q(1)}(h_1)_{i_{q(1)}} = \langle h_1, x\rangle$ such that $z_j \ge 0$ if $\langle h_1, x\rangle/(h_1)_{i_j} \ge 0$ and $z_j < 0$ otherwise, and the plus sign is chosen if $\langle h_1, x\rangle \ge 0$. We iterate this to see that

$$\begin{aligned}\mathcal{S}_{\mathcal{R}}(x) = \sum_{z_1,\dots,z_{q(m)}} &(-1)^{\# k: \langle h_k, x - z_1 e_{i_1} - \dots - z_{q(k-1)} e_{i_{q(k-1)}}\rangle < 0} (-1)^{\# i: z_i<0}\\ &\times Q_{R^m_-}(x - z_1 e_{i_1} - \dots z_{q(m)} e_{i_{q(m)}}) \quad (4.3)\end{aligned}$$

where the sum is over all integers $z_1, \dots, z_{q(m)}$ with

$$z_{q(k-1)+1}(h_k)_{i_{q(k-1)+1}} + \dots + z_{q(k)}(h_k)_{i_{q(k)}} = \langle h_k, x - z_1 e_{i_1} - \dots z_{q(k-1)} e_{i_{q(k-1)}}\rangle$$

for $k = 1, 2, \dots, m$ and, for $q(k-1) < j \le q(k)$, $z_j \ge 0$ if

$$\langle h_k, x - z_1 e_{i_1} - \dots - z_{q(k-1)} e_{i_{q(k-1)}}\rangle/(h_k)_{i_j} \ge 0$$

and $z_j < 0$ otherwise. After noting that the completeness of $(R_w)_{w \in \mathcal{W}_m}$ would imply $Q_{R^m_-} \equiv 1$ if $R^m_- = \mathcal{C}^\circ_{A,I \setminus Q(m)}$ and $Q_{R^m_-} \equiv 0$ else, we obtain the following.

Theorem 3. *We assume* $\dim \mathcal{C}_{A,I} = n$. *Suppose* $\mathcal{R}$ *is a complete* m*-neighborhood of* (A,I)*, let* $(H_1, H_2, \ldots, H_m)$ *be the sequence of nested subspaces associated with it and* R *be the basement of the neighborhood. Furthermore, let* $Q(k) = \{i \in I | e_i \notin H_k\}$, $Q(0) = \emptyset$ *and, for* $1 \le k \le m$, *choose reduced normal vectors* $h_k \in \mathbb{Z}^n$ *of hyperplanes of* (A,I) *with* $H_k = \{x \in \mathbb{Q}^n | \langle h_i, x \rangle = 0 \, \forall i \le k\}$ *and* $R_{0\ldots0} \subseteq \{x \in \mathbb{Q}^n | \langle h_k, x \rangle > 0 \, \forall k\}$. *Then* $\mathcal{S}_{\mathcal{R}}(x) = 0$ *if* $R \subseteq H_m \setminus \mathcal{C}_{A,I \setminus Q(m)}$. *Otherwise* $\mathcal{S}_{\mathcal{R}}(x)$ *is the signed enumeration of all* $z \in \mathbb{Z}^n$ *with the following properties*

- $\langle h_k, z \rangle = \langle h_k, x \rangle$ *for* $k = 1, 2, \ldots, m$,
- $i \in [n] \setminus Q(m)$ *implies* $z_i = 0$, *and*
- $i \in Q(k) \setminus Q(k-1)$ *implies* $z_i \ge 0$ *if* $\langle h_k, x - z_{Q(k-1)} \rangle / (h_k)_i \ge 0$ *and* $z_i < 0$ *otherwise; here* $z_{Q(k-1)}$ *denotes the vector which coincides with* z *on the set of indices in* $Q(k-1)$ *and is zero otherwise,*

where the sign of the solution z *is* $(-1)^{(\#i: z_i < 0) + (\#k: \langle h_k, x - z_{Q(k-1)} \rangle < 0)}$.

The next proposition follows from (4.3). Note that the neighborhood does not need to be complete in this case.

Proposition 4. *The signed sum* $S_{\mathcal{R}}(x)$ *of a neighborhood* $\mathcal{R}$ *only depends on the sequence of nested subspaces associated with the neighborhood and its basement.*

4.5. **Tools to compute** $\mathcal{S}_{\mathcal{R}}(x)$**.** It is convenient to introduce the following operators.

Definition 7 (The operators $S_{h,i}, S_{h,\mathbf{i}}, T_{h,j}, T_{h,\mathbf{j}}, ST_{h,\mathbf{i},\mathbf{j}}$)**.** *Suppose* $f : \mathbb{Z}^n \to \mathbb{Q}$, $h, x \in \mathbb{Z}^n$, $i \in \{1, 2, \ldots, n\}$, $\mathbf{i} = (i_1, \ldots, i_p) \in \{1, 2, \ldots, n\}^p$ *and* $\mathbf{j} \in \{1, 2, \ldots, n\}^q$. *We define*

$$
\begin{aligned}
(S_{h,i} f)(x) &:= \sum_{z=0}^{-\langle h,x \rangle - 1} f(x - z e_i), \\
(S_{h,\mathbf{i}} f)(x) &:= (S_{h,i_1} S_{h,i_2} \ldots S_{h,i_p} f)(x), \\
(T_{h,i} f)(x) &:= \sum_{z=0}^{\langle h,x \rangle} f(x - z e_i), \\
(T_{h,\mathbf{i}} f)(x) &:= (T_{h,i_1} T_{h,i_2} \ldots T_{h,i_p} f)(x) \quad \textit{and} \\
(ST_{h,\mathbf{i},\mathbf{j}} f)(x) &:= (S_{h,\mathbf{i}} T_{h,\mathbf{j}} f)(x).
\end{aligned}
$$

The following recursion was deduced in Section 4.4, compare also the formula in Lemma 6 with (4.1),

$$
\begin{aligned}
\mathcal{S}_{\mathcal{R}}(x) = & \sum_{z_1=0}^{\langle h_1,x \rangle/(h_1)_{i_1} - 1} \sum_{z_2=0}^{\langle h_1, x - z_1 e_{i_1} \rangle/(h_1)_{i_2} - 1} \cdots \sum_{z_p=0}^{\langle h_1, x - z_1 e_{i_1} - \ldots - z_{k-1} e_{i_{p-1}} \rangle/(h_1)_{i_p} - 1} \\
& \sum_{z_{p+1}=0}^{\langle h_1, x - z_1 e_{i_1} - \ldots - z_k e_{i_p} \rangle} \cdots \sum_{z_{q-1}=0}^{\langle h_1, x - z_1 e_{i_1} - \ldots - z_{q-2} e_{i_{q-2}} \rangle} \\
& \mathcal{S}_{\mathcal{R}'}(x - z_1 e_{i_1} - \ldots - z_{q-1} e_{i_{q-1}} - \langle h_1, x - z_1 e_{i_1} - \ldots - z_{q-1} e_{i_{q-1}} \rangle e_{i_q}), \quad (4.4)
\end{aligned}
$$

where $q = q(1)$, R_{w0} lies on the same side of H_1 as e_{i_q} and, w.l.o.g. $(h_1)_{i_1} = (h_1)_{i_2} = \ldots = (h_1)_{i_p} = -1$ and $(h_1)_{i_{p+1}} = (h_1)_{i_{p+2}} = \ldots = (h_1)_{i_q} = 1$. (Note that $p < q$ by the definition a neighborhood. Moreover, the formula is also true if $(h_1)_{i_j} \in \{1, -1\}$ for $j \in \{1, 2, \ldots, p-1\}$

and we may (individually) increase the upper bounds in the first $p-1$ summations by 1.) Using the operators introduced above and setting $\mathbf{i} = (i_1, i_2, \ldots, i_p)$ and $\mathbf{j} = (i_{p+1}, i_{p+1}, \ldots, i_{q-1})$, this can be written in a more compact way as follows.

$$S_{\mathcal{R}}(x) = ST_{h_1,\mathbf{i},\mathbf{j}} S_{\mathcal{R}'}(x - \langle h_1, x\rangle e_{i_q}) \tag{4.5}$$

The rules for the application of the operators $S_{h,\mathbf{i}}$ and $T_{h,\mathbf{j}}$ provided in the following lemma will be convenient.

Lemma 7. *Suppose $g, g_0 \in \mathbb{Q}^n$, $h, x \in \mathbb{Z}^n$, $m \in \mathbb{Z}_{\geq 0}$, $\mathbf{i} = (i_1, i_2, \ldots, i_p) \in \{1, 2, \ldots, n\}^p$ and $c = d + \langle g_0, x\rangle$ with $d \in \mathbb{Q}$ and $(g_0)_{i_j} = 0$ for all j.*

- *If $(g)_{i_j} = 1$ and $(h)_{i_j} = -1$ for $j = 1, 2, \ldots, p$ then*

$$S_{h,\mathbf{i}} \binom{\langle g, x\rangle + c}{m} = \sum_{k=0}^{m} (-1)^{m+p} \binom{-\langle g+h, x\rangle + m - c - 1}{m-k} \binom{\langle h, x\rangle}{k+p}. \qquad \textbf{(S-Rule)}$$

  Assuming $h = -g$ and $-1 \leq c \leq m-1$, the right-hand side simplifies to $\binom{-\langle h,x\rangle + c + p}{m+p}$.

- *If $(g)_{i_j} = -1$ and $(h)_{i_j} = 1$ for $j = 1, 2, \ldots, p$ then*

$$T_{h,\mathbf{i}} \binom{\langle g, x\rangle + c}{m} = \sum_{k=0}^{m} (-1)^{p} \binom{\langle g+h, x\rangle + c + 1}{m-k} \binom{-\langle h, x\rangle - 1}{k+p}. \qquad \textbf{(T-Rule)}$$

  Assuming $h = -g$ and $-1 \leq c \leq m-1$, the right-hand side simplifies to $(-1)^p \binom{c - \langle h,x\rangle}{m+p}$.

Proof. For the first identities in each case, we use induction with respect to p. The case $p = 0$ follows from the Chu-Vandermonde summation and the induction step from

$$\sum_{z=a}^{b} \binom{z+d}{n} = \sum_{z=a}^{b} \left(\binom{z+d+1}{n+1} - \binom{z+d}{n+1} \right) = \binom{b+d+1}{n+1} - \binom{a+d}{n+1}.$$

The simplifications in the special cases follow from the Chu-Vandermonde summation. □

Lemma 8. *Let $m_1, m_2 \in \mathbb{Z}_{\geq 0}, c_1, c_2 \in \mathbb{Q}$. Then*

$$\binom{X + c_1}{m_1} \binom{X + c_2}{m_2} = \sum_{k=0}^{m_1} \binom{c_1 - c_2 + m_2}{m_1 - k} \binom{k + m_2}{k} \binom{X + c_2}{k + m_2}. \qquad \textbf{(U-Rule)}$$

Proof. We apply the Chu-Vandermonde summation.

$$\begin{aligned} \binom{X + c_1}{m_1} \binom{X + c_2}{m_2} &= \sum_{k=0}^{m_1} \binom{c_1 - c_2 + m_2}{m_1 - k} \binom{X + c_2 - m_2}{k} \binom{X + c_2}{m_2} \\ &= \sum_{k=0}^{m_1} \binom{c_1 - c_2 + m_2}{m_1 - k} \binom{k + m_2}{k} \binom{X + c_2}{k + m_2} \end{aligned}$$

□

The **I-Rule** will refer to the following elementary transformation of the binomial coefficient.

$$\binom{n}{k} = (-1)^k \binom{k - n - 1}{k}$$

In the next subsection we demonstrate how these rule suffice to compute $S_{\mathcal{R}}(x)$ for complete $\mathcal{R}$ if $m = 1, 2$. In fact these rules are also sufficient for the computations if $m > 2$, though the computations get more and more complicated.

4.6. **Computation of $S_{\mathcal{R}}(x)$ for complete $\mathcal{R}$ if $m=1,2$.** In the first theorem, we consider the case $m=1$.

Theorem 4. *Suppose $\mathcal{R}=(R_0,R_1)$ is a complete 1-neighborhood of (A,I). Let H be the associated hyperplane of (A,I) and $h \in \mathbb{Z}^n$ be a reduced normal vector of H such that R_0 is on the positive side of h. Let $q=|\{i \in I | e_i \notin H\}|$ and $p=|\{i \in I | h_i < 0\}$. Then $\mathcal{S}_{\mathcal{R}}(x)=0$ or*

$$\mathcal{S}_{\mathcal{R}}(x) = (-1)^p \binom{\langle h,x\rangle + q - p - 1}{q-1}.$$

Proof. We use the notation from (4.4). If $\mathcal{S}_{\mathcal{R}}(x) \neq 0$ then, by (4.5) and the completness,

$$S_{\mathcal{R}}(x) = S T_{h,\mathbf{i},\mathbf{j}} 1.$$

We apply the **T**-Rule to see that

$$T_{h,\mathbf{j}} 1 = (-1)^{q-p-1} \binom{-\langle h,x\rangle - 1}{q-p-1}.$$

The **S**-Rule implies

$$S_{h,\mathbf{i}} T_{h,\mathbf{j}} 1 = (-1)^p \binom{\langle h,x\rangle + q - p - 1}{q-1}.$$

□

Now we deal with the case $m=2$.

Theorem 5. *Let $\mathcal{R}$ be a complete 2-neighborhood, (H_1,H_2) the sequence of nested hyperplanes associated with the neighborhood and $h_1,h_2 \in \mathbb{Z}^n$ two reduced normal vectors of hyperplanes of (A,I) with*

$$H_1 = \{x \in \mathbb{Q}^n | \langle h_1, x\rangle = 0\}$$

as well as

$$H_2 = \{x \in \mathbb{Q}^n | \langle h_1, x\rangle = 0 \text{ and } \langle h_2, x\rangle = 0\},$$

and $R_{00} \subseteq \{x \in \mathbb{Q}^n | \langle h_1,x\rangle > 0 \text{ and } \langle h_2,x\rangle > 0\}$. Choose $i \in I$ with $(h_1)_i = 1$ and, by Proposition 3, we may assume $(h_2)_i = 0$ and that either of the following is true.

- *Case A: there is no $j \in I$ with $0 \neq (h_1)_j = (h_2)_j$.*
- *Case B: there is no $j \in I$ with $0 \neq (h_1)_j = -(h_2)_j$.*

Choose integers $0 \leq p_1 \leq p_2 \leq p_3 < q$ with the following properties

$$\begin{aligned} p_1 &= |\{j \in I | (h_1)_j = -1 \text{ and } (h_2)_j = 1\}| \\ p_2 - p_1 &= |\{j \in I | (h_1)_j = 1 \text{ and } (h_2)_j = -1\}| \\ p_3 - p_2 &= |\{j \in I | (h_1)_j = -1 \text{ and } (h_2)_j = 0\}| \\ q - p_3 &= |\{j \in I | (h_1)_j = 1 \text{ and } (h_2)_j = 0\}| \end{aligned}$$

if we are in Case A, and with

$$\begin{aligned} p_1 &= |\{j \in I | (h_1)_j = -1 \text{ and } (h_2)_j = -1\}| \\ p_2 - p_1 &= |\{j \in I | (h_1)_j = 1 \text{ and } (h_2)_j = 1\}| \\ p_3 - p_2 &= |\{j \in I | (h_1)_j = -1 \text{ and } (h_2)_j = 0\}| \\ q - p_3 &= |\{j \in I | (h_1)_j = 1 \text{ and } (h_2)_j = 0\}| \end{aligned}$$

if we are in Case B. Moreover, let $q(2) = |\{j \in I | (h_1)_j = 0$ *and* $(h_2)_j \neq 0\}|$ *and* $p(2) = |\{j \in I | (h_1)_j = 0$ *and* $(h_2)_j < 0\}$. *In Case A, we have* $\mathcal{S}_{\mathcal{R}}(x) = 0$ *or*

$$\mathcal{S}_{\mathcal{R}}(x) = \sum_{k=0}^{q(2)-1} (-1)^{p(2)+q-1-p_3+p_2-p_1} \binom{\langle h_1 + h_2, x\rangle + q(2) - p(2) - 1 - p_3 + q}{q(2) - 1 - k} \times \binom{k + q - 1 - p_2}{k} \binom{-\langle h_1, x\rangle - 1 + p_3 - p_2 + p_1}{k + q - 1},$$

whereas in Case B, we have $\mathcal{S}_{\mathcal{R}}(x) = 0$ *or*

$$\mathcal{S}_{\mathcal{R}}(x) = \sum_{k=0}^{q(2)-1} (-1)^{p(2)+q(2)+q-p_3+p_2-p_1} \binom{\langle h_1 - h_2, x\rangle + p(2) - 1 + q - p_3}{q(2) - 1 - k} \times \binom{k + q - 1 - p_2}{k} \binom{-\langle h_1, x\rangle - 1 + p_3 - p_2 + p_1}{k + q - 1}.$$

Proof. We use the notation from (4.4) and consider Case A first. Without loss of generality we assume

- $(h_1)_{i_1} = \ldots = (h_1)_{i_{p_1}} = -1$ and $(h_2)_{i_1} = \ldots = (h_2)_{i_{p_1}} = 1$
- $(h_1)_{i_{p_1+1}} = \ldots = (h_1)_{i_{p_2}} = 1$ and $(h_2)_{i_{p_1+1}} = \ldots = (h_2)_{i_{p_2}} = -1$
- $(h_1)_{i_{p_2+1}} = \ldots = (h_1)_{i_{p_3}} = -1$ and $(h_2)_{i_{p_2+1}} = \ldots = (h_2)_{i_{p_3}} = 0$
- $(h_1)_{i_{p_3+1}} = \ldots = (h_1)_{i_q} = 1$ and $(h_2)_{i_{p_3+1}} = \ldots = (h_2)_{i_q} = 0.$

Set

$$\mathbf{i}_L := (i_1, \ldots, i_{p_1}), \mathbf{i}_R := (i_{p_2+1}, \ldots, i_{p_3})$$

and

$$\mathbf{j}_L := (i_{p_1+1}, \ldots, i_{p_2}), \mathbf{j}_R := (i_{p_3+1}, \ldots, i_{q-1}).$$

The recursion (4.4) and Theorem 4 imply

$$S_{\mathcal{R}}(x) = S_{h_1,\mathbf{i}_L} T_{h_1,\mathbf{j}_L} S_{h_1,\mathbf{i}_R} T_{h_1,\mathbf{j}_R} (-1)^{p(2)} \binom{\langle h_2, x\rangle + q(2) - p(2) - 1}{q(2) - 1}.$$

By the **T-Rule**,

$$T_{h_1,\mathbf{j}_R}(-1)^{p(2)} \binom{\langle h_2, x\rangle + q(2) - p(2) - 1}{q(2) - 1} = (-1)^{p(2)+q-1-p_3} \binom{-\langle h_1, x\rangle - 1}{q - 1 - p_3} \binom{\langle h_2, x\rangle + q(2) - p(2) - 1}{q(2) - 1}.$$

The **S-Rule** implies

$$S_{h_1,\mathbf{i}_R} T_{h_1,\mathbf{j}_R}(-1)^{p(2)} \binom{\langle h_2, x\rangle + q(2) - p(2) - 1}{q(2) - 1} = (-1)^{p(2)+q-1-p_3} \binom{\langle h_2, x\rangle + q(2) - p(2) - 1}{q(2) - 1} \binom{-\langle h_1, x\rangle - 1 + p_3 - p_2}{q - 1 - p_2}.$$

We apply the **U-Rule** with $X=-\langle h_1,x\rangle$ and obtain

$$\sum_{k=0}^{q(2)-1}(-1)^{p(2)+q-1-p_3}\binom{\langle h_1+h_2,x\rangle+q(2)-p(2)-1-p_3+q}{q(2)-1-k}\\ \times\binom{k+q-1-p_2}{k}\binom{-\langle h_1,x\rangle-1+p_3-p_2}{k+q-1-p_2}.$$

The **T-Rule** leads us to

$$T_{h_1,\mathbf{j}_L}S_{h_1,\mathbf{i}_R}T_{h_1,\mathbf{j}_R}(-1)^{p(2)}\binom{\langle h_2,x\rangle+q(2)-p(2)-1}{q(2)-1}\\ =\sum_{k=0}^{q(2)-1}(-1)^{p(2)+q-1-p_3+p_2-p_1}\binom{\langle h_1+h_2,x\rangle+q(2)-p(2)-1-p_3+q}{q(2)-1-k}\\ \times\binom{k+q-1-p_2}{k}\binom{-\langle h_1,x\rangle-1+p_3-p_2}{k+q-1-p_1}.$$

By the **S-Rule**, we finally obtain

$$\mathcal{S}_{\mathcal{R}}(x)=S_{h_1,\mathbf{i}_L}T_{h_1,\mathbf{j}_L}S_{h_1,\mathbf{i}_R}T_{h_1,\mathbf{j}_R}(-1)^{p(2)}\binom{\langle h_2,x\rangle+q(2)-p(2)-1}{q(2)-1}\\ =\sum_{k=0}^{q(2)-1}(-1)^{p(2)+q-1-p_3+p_2-p_1}\binom{\langle h_1+h_2,x\rangle+q(2)-p(2)-1-p_3+q}{q(2)-1-k}\\ \times\binom{k+q-1-p_2}{k}\binom{-\langle h_1,x\rangle-1+p_3-p_2+p_1}{k+q-1}.$$

We consider Case B and assume

- $(h_1)_{i_1}=\ldots=(h_1)_{i_{p_1}}=-1$ and $(h_2)_{i_1}=\ldots=(h_2)_{i_{p_1}}=-1$
- $(h_1)_{i_{p_1+1}}=\ldots=(h_1)_{i_{p_2}}=1$ and $(h_2)_{i_{p_1+1}}=\ldots=(h_2)_{i_{p_2}}=1$
- $(h_1)_{i_{p_2+1}}=\ldots=(h_1)_{i_{p_3}}=-1$ and $(h_2)_{i_{p_2+1}}=\ldots=(h_2)_{i_{p_3}}=0$
- $(h_1)_{i_{p_3+1}}=\ldots=(h_1)_{i_q}=1$ and $(h_2)_{i_{p_3+1}}=\ldots=(h_2)_{i_q}=0$.

As in Case A, we have

$$S_{h_1,\mathbf{i}_R}T_{h_1,\mathbf{j}_R}(-1)^{p(2)}\binom{\langle h_2,x\rangle+q(2)-p(2)-1}{q(2)-1}\\ =(-1)^{p(2)+q-1-p_3}\binom{\langle h_2,x\rangle+q(2)-p(2)-1}{q(2)-1}\binom{-\langle h_1,x\rangle-1+p_3-p_2}{q-1-p_2}.$$

By the **I-Rule**,

$$(-1)^{p(2)+q(2)+q-p_3}\binom{-\langle h_2,x\rangle+p(2)-1}{q(2)-1}\binom{-\langle h_1,x\rangle-1+p_3-p_2}{q-1-p_2}.$$

We apply the **U-Rule** with $X=-\langle h_1,x\rangle$ and obtain

$$\sum_{k=0}^{q(2)-1}(-1)^{p(2)+q(2)+q-p_3}\binom{\langle h_1-h_2,x\rangle+p(2)-1+q-p_3}{q(2)-1-k}\\ \times\binom{k+q-1-p_2}{k}\binom{-\langle h_1,x\rangle-1+p_3-p_2}{k+q-1-p_2}.$$

Then, by the **T-Rule**,

$$T_{h_1,\mathbf{j}_L} S_{h_1,\mathbf{i}_R} T_{h_1,\mathbf{j}_R} (-1)^{p(2)} \binom{\langle h_2, x\rangle + q(2) - p(2) - 1}{q(2)-1}$$
$$= \sum_{k=0}^{q(2)-1} (-1)^{p(2)+q(2)+q-p_3+p_2-p_1} \binom{\langle h_1 - h_2, x\rangle + p(2) - 1 + q - p_3}{q(2)-1-k}$$
$$\times \binom{k+q-1-p_2}{k} \binom{-\langle h_1, x\rangle - 1 + p_3 - p_2}{k+q-1-p_1}.$$

The **S-Rule** finally implies

$$S_{h_1,\mathbf{i}_L} T_{h_1,\mathbf{j}_L} S_{h_1,\mathbf{i}_R} T_{h_1,\mathbf{j}_R} (-1)^{p(2)} \binom{\langle h_2, x\rangle + q(2) - p(2) - 1}{q(2)-1}$$
$$= \sum_{k=0}^{q(2)-1} (-1)^{p(2)+q(2)+q-p_3+p_2-p_1} \binom{\langle h_1 - h_2, x\rangle + p(2) - 1 + q - p_3}{q(2)-1-k}$$
$$\times \binom{k+q-1-p_2}{k} \binom{-\langle h_1, x\rangle - 1 + p_3 - p_2 + p_1}{k+q-1}.$$

□

5. Example: a directed graph polytope

5.1. The polytope.

Definition 8 (Directed Graph Polytope)**.** *Let $G = (V, E)$ be a finite directed graph, and let Q and S denote the set of sources and sinks, respectively. For a subset $I \subseteq E \cup Q \cup S$ and a function $x : E \cup Q \cup S \to \mathbb{Q}$, define the polytope $\mathcal{P}_{G,I}(x)$ to be the set of all functions $z : V \to \mathbb{Q}$ with the following properties:*

1. $z(w) - z(v) \le x(e) \quad \forall e = (v, w) \in E$*; we have equality if* $e \in E \setminus I$
2. $z(q) \le x(q) \quad \forall q \in Q$*; we have equality if* $q \in Q \setminus I$
3. $-z(s) \le x(s) \quad \forall s \in S$*; we have equality if* $s \in S \setminus I$

Throughout this section, let $G = (V, E)$ be a finite directed graph, Q and S its set of sources and sinks, respectively, and $I \subseteq E \cup Q \cup S$. We define a matrix $\mathcal{A}_G$ (it is a slight variant of the incidence matrix): its rows are indexed by $E \cup Q \cup S$ and its columns by V. Hence, the column associated with a vertex $v \in V$ is a function $c_v : E \cup S \cup Q \to \mathbb{Q}$ and we define it as follows.

$$c_v(f) = \begin{cases} 1 & \text{if } v \text{ is the head of edge } f \text{ or } v = f \in Q \\ -1 & \text{if } v \text{ is the tail of edge } f \text{ or } v = f \in S \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$\mathcal{P}_{G,I}(x) = \left\{ z \in \mathbb{Q}^V | (\mathcal{A}_G \cdot z)_f \le x(f) \ \forall f \in I, (\mathcal{A}_G \cdot z)_f = x(f) \ \forall f \in (E \cup Q \cup S) \setminus I \right\}.$$

We are interested in the following counting function, which is special case of the function dealt with in Theorem 1.

$$\mathrm{N}_{\mathcal{A}_G,I}(x) = |\mathcal{P}_{G,I}(x) \cap \mathbb{Z}^V| =: \mathrm{N}_{G,I}(x)$$

In particular, we determine the hyperplanes of $(\mathcal{A}_G, I)$ as well as the sequences of nested subspaces of $(\mathcal{A}_G, I)$.

5.2. **Special case: Order Polytope.** Stanley's order polytope of a poset [9], i.e. the set of all order preserving maps from the poset into the intervall $[0, 1]$, is a special case of $\mathcal{P}_{G,I}(x)$. We need to specialize as follows.

- The graph G is the reversed Hasse diagram of the poset.
- $I = E \cup Q \cup S$
- $x(e) = 0$ for all $e \in E$
- $x(q) = 1$ for all $q \in Q$
- $x(s) = 0$ for all $s \in S$

We obtain the order polynomial of the poset (which gives the number of all integer valued order preserving maps from the poset into the interval $[1, t]$) from $\mathrm{N}_{G,I}(x)$ if we set

- The graph G is the reversed Hasse diagram of the poset.
- $I = E \cup Q \cup S$
- $x(e) = 0$ for all $e \in E$,
- $x(q) = t$ for all $q \in Q$, and
- $x(s) = -1$ for all $s \in S$.

5.3. **Boundedness.** Recall that if $\mathrm{N}_{G,I}(x_0) = \infty$ for an x_0 then $\mathrm{N}_{G,I}(x) \in \{0, \infty\}$ for all x. In this case we say that $\mathrm{N}_{G,I}(x)$ is unbounded. For a given graph G and $I \subseteq E \cup Q \cup S$, it is convenient to define the graph $G(I)$, which is obtained from G by adding reversed versions of the edges in $E \setminus I$ to the edge set.

$$G(I) := (V, E \cup \{(v, w) | (w, v) \in E \setminus I\}).$$

We are able to characterize the pairs (G, I) for which $\mathrm{N}_{G,I}(x)$ is bounded.

Proposition 5. *The counting function $\mathrm{N}_{G,I}(x)$ is bounded if and only if for each vertex v there is a directed path in $G(I)$ from a vertex in $Q \cup (S \setminus I)$ to v as well as a directed path from v to a vertex in $S \cup (Q \setminus I)$.*

Proof. First observe that there always exists an $x : E \cup Q \cup S \to \mathbb{Z}$ with $\mathrm{N}_{G,I}(x) \neq 0$: choose a function $z : V \to \mathbb{Z}$ and set $x(v, w) = z(w) - z(v)$ for all $(v, w) \in E$, $x(q) = z(q)$ for all $q \in Q$ and $x(s) = -z(s)$ for all $s \in S$. Then $z \in \mathcal{P}_{G,I}(x) \cap \mathbb{Z}^V$.

Let Q^c denote the set of vertices for which there does not exist a directed path in $G(I)$ starting in $Q \cup (S \setminus I)$ and terminating at the vertices; this implies in particular that the cut $(Q^c, V \setminus Q^c)$ (the set of edges with one endpoint in Q^c and the other in $V \setminus Q^c$) only contains edges that are directed from Q^c to $V \setminus Q^c$. Then, for each non-negative integer n and each $z \in \mathcal{P}_{G,I}(x)$, we have $z + n\mathbf{1}_{Q^c} \in \mathcal{P}_{G,I}(x)$, where $\mathbf{1}_{Q^c} : V \to \mathbb{Q}$ with $\mathbf{1}_{Q^c}(x) = 1$ if $x \in Q^c$ and zero otherwise. This implies that Q^c must be empty if $\mathrm{N}_{G,I}(x)$ was bounded. Similar for the set of vertices for which there does not exist a directed path starting at the vertices and terminating in $S \cup (Q \setminus I)$.

Now suppose that $v_0 e_1 v_1 e_2 v_2 \cdots e_m v_m$ is a directed path, $e_i = (v_{i-1}, v_i)$ in $G(I)$, and $v_m \in S \cup (Q \setminus I)$. For all $z \in \mathcal{P}_{G,I}(x)$,

$$\begin{aligned} z(v_0) &= (z(v_0) - z(v_1)) + (z(v_1) - z(v_2)) + \cdots + (z(v_{m-1}) - z(v_m)) + z(v_m) \\ &\geq -x(e_1) - x(e_2) - \cdots - x(e_m) \mp x(v_m), \end{aligned}$$

where the minus sign has to be chosen if $v_m \in S$ and the plus sign if $v_m \in Q \setminus I$. Hence $z(v_0)$ is bounded from below. The proof that $z(v_0)$ is also bounded from above is analogous. □

In the following we assume that (G, I) has the property given in Proposition 5.

5.4. **The columnspace of $\mathcal{A}_G$ and its orthogonal complement.** We are interested in the columnspace $\langle \mathcal{A}_G \rangle = \sum_{v \in V} c_v \mathbb{Q}$ and its orthogonal complement. Recall that a *flow* on G is an element of $f \in \mathbb{Q}^{E \cup Q \cup S}$ that satisfies Kirchhoff's law, that is:

Definition 9 (Flow)**.** *A flow is a function $f : E \cup Q \cup S \to \mathbb{Q}$ that satisfies the following conditions.*

(1) $\forall v \in V \setminus (Q \cup S) : \sum_{u:(u,v) \in E} f(u, v) = \sum_{w:(v,w) \in E} f(v, w)$,
(2) $\forall v \in Q : f(v) = \sum_{w:(v,w) \in E} f(v, w)$, *and*
(3) $\forall v \in S : f(v) = \sum_{u:(u,v) \in E} f(u, v)$.

However, these are exactly the conditions for a function $f \in \mathbb{Q}^{E \cup Q \cup S}$ to be orthogonal on each column of $\mathcal{A}_G$, thus to be in the orthogonal complement of $\langle \mathcal{A}_G \rangle$. The following class of flows will be of special interest for us.

Definition 10 (Simple flow)**.** *A non-zero flow $f : E \cup Q \cup S \to \mathbb{Q}$ is said to be simple if the support $\{e \in E \cup Q \cup S | f(e) \neq 0\} =: \mathrm{Supp}(f)$ is minimal, i.e. there exists no flow g with $\emptyset \neq \mathrm{Supp}(g) \subsetneq \mathrm{Supp}(f)$. Equivalently, f is a simple flow if the subgraph P_f of G that is induced by $\mathrm{Supp}(f)$ is either a circuit and $\mathrm{Supp}(f) \cap (Q \cup S) = \emptyset$, or a path that connects two elements r_1, r_2 of $Q \cup S$ and $\mathrm{Supp}(f) \cap (Q \cup S) = \{r_1, r_2\}$.*

In the following, circuits that are disjoint from $Q \cup S$ and paths that connect two vertices in $Q \cup S$ are addressed as the *loops* of G. Up to a multiplicative constant loops correspond to simple flows. Clearly, $|f(e)|$ has the same value for all $e \in \mathrm{Supp}(f)$ if f is a simple flow. Simple flows obviously generate the vector space of all flows. To be more precise, we have the following.

Proposition 6. *For each non-zero flow $f : E \cup Q \cup S \to \mathbb{Q}$, there exist simple flows $(f_i)_{1 \leq i \leq m}$ with $\mathrm{Supp}(f_i) \subseteq \mathrm{Supp}(f)$ and $f = \sum_{i=1}^{m} f_i$.*

Proof. Induction on the cardinality of $\mathrm{Supp}(f)$. There is nothing to prove if f is simple. Otherwise choose a simple flow f_1 with $\mathrm{Supp}(f_1) \subseteq \mathrm{Supp}(f)$ and $f_1(e) = f(e)$ for at least one edge in P_{f_1}. Apply the induction hypothesis to $f - f_1$. □

This gives us the following criteria for an element $c : E \cup Q \cup S \to \mathbb{Q}$ to be in $\langle \mathcal{A}_G \rangle$: for each loop P of G, we have $\langle f, c \rangle = 0$, where f is a simple flow with $P_f = P$.

5.5. **The hyperplanes and sequences of nested subspaces of (G, I).** For $x \in E \cup Q \cup S$, let $\mathbf{1}_x \in \mathbb{Q}^{E \cup Q \cup S}$ with $\mathbf{1}_x(y) = \delta_{x,y}$. We say that a hyperplane in $\langle \mathcal{A}_G \rangle + \sum_{x \in I} \mathbb{Q}\mathbf{1}_x =: \langle \mathcal{A}_G \rangle_I$ is a hyperplane of (G, I), if it is generated by $\langle \mathcal{A}_G \rangle$ and some $\mathbf{1}_x$, $x \in I$. Note that these are precisely the hyperplanes of $(\mathcal{A}_G, I)$. For each hyperplane F of $\langle \mathcal{A}_G \rangle_I$, there exists a normal vector f with

$$F = \{x \in \langle \mathcal{A}_G \rangle_I | \langle f, x \rangle = 0\}.$$

Assuming that $\langle \mathcal{A}_G \rangle \subseteq F$, this implies that f is a flow and $\mathrm{Supp}(f) \cap I \neq \emptyset$.

Theorem 6. *The hyperplanes of (G, I) are the orthogonal complements of simple flows f of G such that $\mathrm{Supp}(f) \cap I$ is non-empty but minimal, i.e. there exists no simple flow g of G with $\emptyset \neq \mathrm{Supp}(g) \cap I \subsetneq \mathrm{Supp}(f) \cap I$.*

A hyperplane of (G, I) is a bounding hyperplane of the cone $\langle \mathcal{A}_G \rangle + \sum_{x \in I} \mathbb{Q}_{\geq 0} \mathbf{1}_x$ if for each normal vector f of the hyperplane it is possible to reorient edges in $E \setminus I$ such that P_f is directed.

Proof. Suppose F is a hyperplane of (G, I) and f is a normal vector of F. This implies automatically that f is a flow and $\mathrm{Supp}(f) \cap I \neq \emptyset$. Note that any other flow g is the normal vector of the same hyperplane if and only if $\mathrm{Supp}(f) \cap I = \mathrm{Supp}(g) \cap I$.

We use Proposition 6 to write f as a sum of simple flows $(f_i)_{1 \le i \le m}$ such that $\mathrm{Supp}(f_i) \subseteq \mathrm{Supp}(f)$ for all i. We may exclude the f_i with $\mathrm{Supp}(f_i) \cap I = \emptyset$ from the sum and still obtain a normal vector of the same hyperplane. We show that, after this modification,

$$\mathrm{Supp}(f_1) \cap I = \mathrm{Supp}(f_2) \cap I = \ldots = \mathrm{Supp}(f_m) \cap I.$$

Assume the contrary, i.e. there exist $i, j \in \{1, 2, \ldots m\}$ such that $(\mathrm{Supp}(f_i) \cap I) \setminus (\mathrm{Supp}(f_j) \cap I) \neq \emptyset$ and let e be an element of this complement. As $e \in \mathrm{Supp}(f) \cap I$ and thus $\mathbf{1}_e \notin F$, this implies

$$\langle \mathcal{A}_G \rangle + \sum_{x \in I : f(x)=0} \mathbb{Q}\mathbf{1}_x + \mathbb{Q}\mathbf{1}_e = F + \mathbb{Q}\mathbf{1}_e = \langle \mathcal{A}_G \rangle_I.$$

However, this subspace (i.e. $\langle \mathcal{A}_G \rangle + \sum\limits_{x \in I : f(x)=0} \mathbb{Q}\mathbf{1}_x + \mathbb{Q}\mathbf{1}_e$) also lies in the orthogonal complement of f_j in $\langle \mathcal{A}_G \rangle_I$, the latter being (as $\mathrm{Supp}(f_j) \cap I \neq \emptyset$) a proper subspace of $\langle \mathcal{A}_G \rangle_I$, a contradiction. This implies $\mathrm{Supp}(f_1) \cap I = \mathrm{Supp}(f) \cap I$ and f_1 is a normal vector of F that is also a simple flow.

If there existed a simple flow g with $\emptyset \neq \mathrm{Supp}(g) \cap I \subsetneq \mathrm{Supp}(f) \cap I$ then the orthogonal complement of g in $\langle \mathcal{A}_G \rangle_I$ must lie strictly between F and $\langle \mathcal{A}_G \rangle_I$, which is impossible for reasons of dimension.

Conversely, let f be a simple flow such that $\mathrm{Supp}(f) \cap I$ is non-empty but minimal and F_0 be a hyperplane of (G, I) that contains all $\mathbf{1}_x$, $x \in I$, with $\langle f, \mathbf{1}_x \rangle = f(x) = 0$. Let f_0 be a simple flow such that F_0 is the orthogonal complement of f_0. Hence $\emptyset \neq \mathrm{Supp}(f_0) \cap I \subseteq \mathrm{Supp}(f) \cap I$. By the minimality of f, $\mathrm{Supp}(f_0) \cap I = \mathrm{Supp}(f) \cap I$, which implies that the orthogonal complements of f and f_0 in $\langle \mathcal{A}_G \rangle_I$ coincide.

For the second assertion observe that, in the general theory, the bounding hyperplanes are characterized by the property that $\langle h, e_i \rangle$ has the same sign for all $i \in I$, where h denotes the normal vector of the hyperplane. □

Corollary 1. *The quasipolynomials representing* $\mathrm{N}_{G,I}(x)$ *on the chambers of* $(\mathcal{A}_G, I)$ *are in fact polynomials.*

Proof. This follows from Theorem 6 and Theorem 1 as simple flows f have the property that there exists a constant $c \in \mathbb{Q}$ with $f(x) \in \{0, \pm c\}$ for all $x \in E \cup Q \cup S$. □

From now on we assume without loss of generality that $\langle \mathcal{A}_G \rangle_I = \mathbb{Q}^{E \cup Q \cup S}$. Suppose that $(H_1, H_2, \ldots, H_m)$ is a sequence of nested subspaces of $(\mathcal{A}_G, I)$. Then there exist normal vectors $h_1, h_2, \ldots, h_m$ of hyperplanes of $(\mathcal{A}_G, I)$ such that

$$H_i = \{x \in \mathbb{Q}^{E \cup Q \cup S} | \langle h_j, x \rangle = 0 \, \forall j \in \{1, 2, \ldots, i\}\}.$$

We use Theorem 6 to find such sequences of normal vectors for the directed graph polytope.

Corollary 2. *Suppose* $\langle \mathcal{A}_G \rangle_I = \mathbb{Q}^{E \cup Q \cup S}$. *A sequence* $(F_1, \ldots, F_m)$ *of subspaces of* $\mathbb{Q}^{E \cup Q \cup S}$ *is a sequences of nested subspaces of* (G, I) *if and only if there exist simples flow* $f_1, f_2, \ldots, f_m$ *with*

$$F_i = \{x \in \mathbb{Q}^n | \langle f_j, x \rangle = 0 \, \forall j \in \{1, 2, \ldots, i\}\}$$

such that, for all $i < m$, $\mathrm{Supp}(f_{i+1}) \cap (I \setminus (\mathrm{Supp}(f_1) \cup \cdots \cup \mathrm{Supp}(f_i))) \neq \emptyset$ *and the intersection is minimal for* f_{i+1} *under all simple flows* f *such that* $\mathrm{Supp}(f)$ *has a non-empty intersection with* $I \setminus (\mathrm{Supp}(f_1) \cup \cdots \cup \mathrm{Supp}(f_i))$.

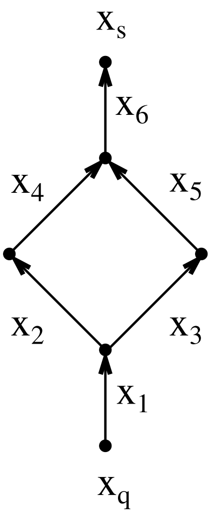

FIGURE 1.

By Proposition 1, this essentially characterizes m-neighborhoods. In the following proposition, we characterize the situation when the union of the set $\{\mathbf{1}_x | x \in I\}$ and the columns of $\mathcal{A}_G$ are linearly independent. This property implies that the polynomials representing $\mathrm{N}_{A,I}(x)$ are constant. The proposition can be used to test whether the neighborhood is complete: in this case the condition in the proposition has to be satisfied by $I \setminus (\mathrm{Supp}(f_1) \cup \cdots \cup \mathrm{Supp}(f_m))$ instead of I.

Proposition 7. *The set $\{c_v | v \in V\} \cup \{\mathbf{1}_x | x \in I\}$ is linearly independent if and only if there is no non-empty subset V' of vertices of G such that I contains all sources and sinks of V' as well as all edges that connect a vertex in V' to a vertex in $V \setminus V'$.*

Proof. It is easy to see that if there exists a subset $V' \subseteq V$ with the property given in the proposition then $\{c_v | v \in V\} \cup \{\mathbf{1}_x | x \in I\}$ is linearly dependent: in this case, $\sum_{v \in V'} c_v$ can be expressed as a linear combination of $\mathbf{1}_x$, $x \in I$.

Conversely, assume there exist subsets $V' \subseteq V, I' \subseteq I$ such that $V' \cup I'$ is non-empty, and, for all $v \in V'$ and $x \in I'$, non-zero rational numbers λ_v and λ_x, respectively, such that $\sum_{v \in V'} \lambda_v c_v + \sum_{x \in I'} \lambda_x \mathbf{1}_x = 0$. As $\{\mathbf{1}_x | x \in I\}$ is linear independent, the set V' contains at least one element. Let v be such an element, suppose there is an edge incident with v that is not contained in I and let w be the other endpoint of the edge. Then $w \in V'$ and $\lambda_v = \lambda_w$. Similarly, if v was a source or a sink then $\mathbf{1}_v$ must take part as well in the linear combination and is therefore included in I. Hence, V' is a subset with the required property. □

5.6. **Example.** We consider the directed graph G displayed in Figure 1 and let $I = E \cup Q \cup S$. By Theorem 6, the hyperplanes of (G, I) are $x_q + x_1 + x_2 + x_4 + x_6 + x_s = 0$, $x_q + x_1 + x_3 + x_5 + x_6 + x_s = 0$ and $x_2 - x_3 + x_4 - x_5 = 0$; the first two of them are the bounding hyperplanes. There are two chambers inside the bounding cone:

$$C_1 = \{(x_q, x_1, \ldots, x_6, x_s) \in \mathbb{Q}^8 | x_q + x_1 + x_2 + x_4 + x_6 + x_s > 0, -x_2 + x_3 - x_4 + x_5 > 0\}$$

$$C_2 = \{(x_q, x_1, \ldots, x_6, x_s) \in \mathbb{Q}^8 | x_q + x_1 + x_3 + x_5 + x_6 + x_s > 0, x_2 - x_3 + x_4 - x_5 > 0\}$$

In this subsection, we present two possibilities to compute the polynomials $P_{C_1}(\mathbf{x})$, $P_{C_2}(\mathbf{x})$ that represent $\mathbf{x} \to \mathrm{N}_{G,I}(\mathbf{x})$ on C_1 and C_2, respectively, where $\mathbf{x} = (x_q, x_1, x_2, x_3, x_4, x_5, x_6, x_s)$.

5.6.1. *First option: Theorem 2.* By Theorem 1, $\deg P_{C_1}(\mathbf{x}) \leq 6$ and $\deg P_{C_2}(\mathbf{x}) \leq 6$. Theorem 2 implies

$$(x_q + x_1 + x_2 + x_4 + x_6 + x_s + 1)_5 | P_{C_1}(\mathbf{x})$$
$$(x_q + x_1 + x_3 + x_5 + x_6 + x_s + 1)_5 | P_{C_2}(\mathbf{x}),$$

where $(a)_n = a(a+1)\ldots(a+n-1)$. Thus

$$P_{C_1}(\mathbf{x}) = (x_q + x_1 + x_2 + x_4 + x_6 + x_s + 1)_5 \\ \times (Ax_q + Bx_1 + Cx_2 + Dx_3 + Ex_4 + Fx_5 + Gx_6 + Hx_s + I)$$

for constants $A, B, C, D, E, F, G, H, I \in \mathbb{Q}$. Note that for each column $\mathbf{c}$ of $\mathcal{A}_G$ and $i = 1, 2$, we have $P_{C_i}(\mathbf{x} + \mathbf{c}) = P_{C_i}(\mathbf{x})$. The columns are

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

This implies $A = B$, $B = C + D$, $C = E$,$D = F$, $E + F = G$ and $G = H$, i.e.

$$P_{C_1}(\mathbf{x}) = (x_q + x_1 + x_2 + x_4 + x_6 + x_s + 1)_5 \\ \times (A(x_q + x_1 + x_3 + x_5 + x_6 + x_s) + C(x_2 - x_3 + x_4 - x_5) + I).$$

By symmetry,

$$P_{C_2}(\mathbf{x}) = (x_q + x_1 + x_3 + x_5 + x_6 + x_s + 1)_5 \\ \times (A(x_q + x_1 + x_2 + x_4 + x_6 + x_s) - C(x_2 - x_3 + x_4 - x_5) + I).$$

Theorem 2 implies $P_{C_1}(\mathbf{x}) = P_{C_2}(\mathbf{x})$ if $x_2 - x_3 + x_4 - x_5 \in \{-1, 0, 1\}$.

If $x_3 + x_5 = x_2 + x_4 + 1$ then

$$P_{C_1}(\mathbf{x}) = (x_q + x_1 + x_2 + x_4 + x_6 + x_s + 1)_5 \\ \times (A(x_q + x_1 + x_2 + x_4 + x_6 + x_s + 1) - C + I).$$

and

$$P_{C_2}(\mathbf{x}) = (x_q + x_1 + x_2 + x_4 + x_6 + x_s + 2)_5 \\ \times (A(x_q + x_1 + x_2 + x_4 + x_6 + x_s) + C + I).$$

This implies $A = C + I$ and $A - C + I = 6A$, and, therefore, $C = -2A, I = 3A$. Thus

$$P_{C_1}(\mathbf{x}) = A(x_q + x_1 + x_2 + x_4 + x_6 + x_s + 1)_5 \\ \times (x_q + x_1 - 2x_2 + 3x_3 - 2x_4 + 3x_5 + x_6 + x_s + 3)$$
$$P_{C_2}(\mathbf{x}) = A(x_q + x_1 + x_3 + x_5 + x_6 + x_s + 1)_5 \\ \times (x_q + x_1 + 3x_2 - 2x_3 + 3x_4 - 2x_5 + x_6 + x_s + 3).$$

Finally note that, in general, if we assume that $\mathrm{N}_{A,I}(x)$ is bounded then $\mathrm{N}_{A,I}(\mathbf{0}) = 1$: if there existed a $z \in \mathbb{Z}^d \setminus \{0\}$ with $Az \leq 0$ then $A(nz) \leq 0$ for all $n \in \mathbb{Z}_{\geq 0}$, and thus $\mathrm{N}_{A,I}(\mathbf{0}) = \infty$. This implies $A = \frac{1}{5!3}$.

5.6.2. *Second option: Theorem 5.* Let

$$H_1 = \{\mathbf{x} \in \mathbb{Q}^8 | x_q + x_1 + x_2 + x_4 + x_6 + x_s = 0\}$$
$$H_2 = \{\mathbf{x} \in \mathbb{Q}^8 | x_q + x_1 + x_3 + x_5 + x_6 + x_s = 0\},$$

and $\mathcal{R}_1$ and $\mathcal{R}_2$ be the complete 2-neighborhoods with $(H_1, H_1 \cap H_2)$ and $(H_2, H_1 \cap H_2)$ as associated sequences of nested subspaces, respectively. It is not hard to see that $\mathcal{S}_{\mathcal{R}_1}(\mathbf{x}) = P_{C_1}(\mathbf{x})$ and $\mathcal{S}_{\mathcal{R}_2}(\mathbf{x}) = P_{C_2}(\mathbf{x})$. Observe that we are in Case B of Theorem 5 with the parameters

$$q(2) = 2, p(2) = 0, p_1 = 0, p_2 = 4, p_3 = 4, q = 6$$

for both neighborhoods. This implies

$$\begin{aligned} P_{C_1}(\mathbf{x}) &= \sum_{k=0}^{1} \binom{x_2 - x_3 + x_4 - x_5 + 1}{1-k} \binom{k+1}{k} \binom{-x_q - x_1 - x_2 - x_4 - x_6 - x_s - 1}{k+5} \\ &= \frac{1}{360}(x_q + x_1 + x_2 + x_4 + x_6 + x_s + 1)_5 (x_q + x_1 - 2x_2 + 3x_3 - 2x_4 + 3x_5 + x_6 + x_s + 3) \end{aligned}$$

and

$$\begin{aligned} P_{C_2}(\mathbf{x}) &= \sum_{k=0}^{1} \binom{-x_2 + x_3 - x_4 + x_5 + 1}{1-k} \binom{k+1}{k} \binom{-x_q - x_1 - x_3 - x_5 - x_6 - x_s - 1}{k+5} \\ &= \frac{1}{360}(x_q + x_1 + x_3 + x_5 + x_6 + x_s + 1)_5 (x_q + x_1 + 3x_2 - 2x_3 + 3x_4 - 2x_5 + x_6 + x_s + 3). \end{aligned}$$

Appendix A. Non-generic cases

A.1. **The case** $\dim \mathcal{C}_{A,I} < n$**.** Let $I' \subseteq [n]$ be minimal with $I \subseteq I'$ and $\dim \mathcal{C}_{A,I'} = n$. Then $\mathrm{N}_{A,I'}(x) = \mathrm{N}_{A,I}(x)$ for all $x \in \langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$: Suppose $x \in \langle A \rangle + \mathbb{Q}\,\mathrm{Cube}_{n,I}$ and

$$z^* \in \{z \in \mathbb{Z}^d | (A \cdot z)_i \leq x_i \, \forall i \in I', (A \cdot z)_i = x_i \, \forall i \in [n] \setminus I'\}.$$

Consequently, there exist, for all $i \in I'$, $\alpha_i \in \mathbb{Q}_{\geq 0}$, with $x = \sum_{i=1}^{d} z_i^* a_i + \sum_{i \in I'} \alpha_i e_i$. By the minimality of I', this implies $\alpha_i = 0$ if $i \in I' \setminus I$ and thus

$$z^* \in \{z \in \mathbb{Z}^d | (A \cdot z)_i \leq x_i \, \forall i \in I, (A \cdot z)_i = x_i \, \forall i \in [n] \setminus I\}.$$

A.2. **The infinite case.** We show that the existence of an x_0 with $\mathrm{N}_{A,I}(x_0) = \infty$ implies $\mathrm{N}_{A,I}(x) \in \{0, \infty\}$ for all x: It suffices to show $\mathrm{N}_{A,I}(0) = \infty$. This is obviously the case if $\mathrm{rank}(A) < d$ as there exists a $z \in \mathbb{Z}^d \setminus \{0\}$ with $A \cdot z = 0$ under this assumption. Thus we may assume $\mathrm{rank}(A) = d$. Now suppose that there is an x with $\mathrm{N}_{A,I}(x) = \infty$: let $(z_j)_{j \geq 0}$ be an infinite sequence of distinct elements in $\mathbb{Z}^d$ with $(A \cdot z_j)_i \leq x_i$ for all $i \in I$ and $(A \cdot z_j)_i = x_i$ for all $i \in [n] \setminus I$. By the assumption that $\mathrm{rank}(A) = d$, $(x - A \cdot z_j)_{j \geq 0}$ is then a sequence of distinct elements in $\mathbb{Q}_{\geq 0}\,\mathrm{Cube}_{n,I} \cap \mathbb{Z}^n$. Being a sequence in $\mathbb{Z}^n_{\geq 0}$, it has an increasing subsequence $(x - A \cdot z_{j_k})_{j \geq 0}$, which implies that $(A \cdot (z_{j_k} - z_{j_0}))_{k \geq 0}$ is a decreasing sequence of distinct elements in $\mathbb{Z}^n_{\leq 0}$; for all $i \in [n] \setminus I$, the i-th coordinate of each element is zero. This shows $\mathrm{N}_{A,I}(0) = \infty$. We exclude this case in the following. This implies in particular that our polytopes are bounded: An unbounded rational polytope contains infinitely many integer points or none; in the latter case appropriate parallel shifts of the bounding hyperplanes cause integer points to lie in the polytope. Also note that these assumptions imply that $\langle A \rangle$ contains no element of $\mathbb{Q}_{\geq 0}\,\mathrm{Cube}_{n,I} \setminus \{0\}$ and $n \geq d$.

Appendix B. Three (almost trivial) facts on quasipolynomials

B.1. **Summing quasipolynomials.** *Suppose $\phi(z, x_1, \ldots, x_d)$ is a quasipolynomial in z as well as in $(x_1, \ldots, x_d)$ and let $g, h \in \mathbb{Q}^d$. Then*

$$\sum_{z \in (\langle g,x \rangle, \langle h,x \rangle)} \phi(z, x_1, \ldots, x_d) \tag{B.1}$$

is representable by a quasipolynomial in $x = (x_1, \ldots, x_d)$. This is because

$$\sum_{\substack{z \in (\langle g,x \rangle, \langle h,x \rangle) \\ z \equiv j \mod m}} \binom{\frac{z-j}{m}}{l} = \binom{\lceil \frac{\langle h,x \rangle - j}{m} \rceil}{l+1} - \binom{\lfloor \frac{\langle g,x \rangle - j}{m} \rfloor + 1}{l+1}$$

and every polynomial in z can be written as a linear combination of $\binom{(z-j)/m}{l}$'s (j, m fixed). Note that this also holds for intervals of the other types:

$$[\langle g,x \rangle, \langle h,x \rangle], (\langle g,x \rangle, \langle h,x \rangle], [\langle g,x \rangle, \langle h,x \rangle)$$

B.2. **Quasipolynomials are determined by its values on a room.** *Let $R \subseteq \mathbb{Q}^n$ be a non-empty room of a finite set of linear hyperplanes in $\mathbb{Q}^n$. Then a quasipolynomial is determined by its values on $R \cap \mathbb{Z}^n$:* there exist $b_1, \ldots, b_d \in \mathbb{Z}^n$ that generate $\mathbb{Q}^n$ with $R = \operatorname{cone}(b_1, \ldots, b_d)^\circ$. Suppose Q_1, Q_2 are two quasipolynomials that coincide on $R \cap \mathbb{Z}^n$. W.l.o.g. we may assume $n = d$. We first consider the case $|\det B| = 1$ where $B := (b_1, \ldots, b_n)$. By assumption $x \to Q_1(Bx)$ and $x \to Q_2(Bx)$ coincide on $\mathbb{Z}^n_{>0}$, and, therefore, the two quasipolynomials $Q_1(Bx)$ and $Q_2(Bx)$ must be equal. Consequently, $Q_1(BB^{-1}x)$ and $Q_2(BB^{-1}x)$ must be equal as well. If $|\det B| \neq 1$ then there exists an $z \in [0,1)^n \setminus \{0\}$ such that $Bz \in \mathbb{Z}^n$. Suppose that $z_i \neq 0$. Let B' denote the matrix we obtain from B by replacing the i-th column by Bz. Then $\operatorname{cone} B' \subseteq \operatorname{cone} B$ and $|\det B'| = z_i |\det B| < |\det B|$. We iterate this process until we finally obtain an integer matrix with determinant ± 1.

B.3. **Enlarging rooms.** *Let $\mathcal{G}, H$ be two finite sets of linear hyperplanes in $\mathbb{Q}^n$, both containing the bounding hyperplanes of $\mathcal{C}_{A,I}$ and $\mathrm{N} : \mathbb{Z}^n \to \mathbb{Q}$ a function, which is a quasipolynomial on $\overline{G} \cap \mathbb{Z}^d$ for every room G of $\mathcal{G}$ with $G \subseteq \mathcal{C}_{A,I}$ as well as on $\overline{H} \cap \mathbb{Z}^d$ for every room H of $\mathcal{H}$ with $H \subseteq \mathcal{C}_{A,I}$. Then it is a quasipolynomial on $\overline{R} \cap \mathbb{Z}^n$ for every room R of $\mathcal{G} \cap \mathcal{H}$ with $R \subseteq \mathcal{C}_{A,I}$:* indeed, every room R of $\mathcal{G} \cap \mathcal{H}$ is subdivided into smaller pieces by the hyperplanes in $(\mathcal{G} \cup \mathcal{H}) \setminus (\mathcal{G} \cap \mathcal{H}) =: \mathcal{G} \ominus \mathcal{H}$. Let R and S be two of these rooms. Then there exists a sequence $R_1, \ldots, R_m$ of rooms of $\mathcal{G} \cup \mathcal{H}$ such that $R_1 = R$ and $R_m = S$ and $\overline{R_i} \cap \overline{R_{i+1}}$is a subset of a hyperplane in $\mathcal{G} \ominus \mathcal{H}$ for $i = 1, \ldots, m-1$. Consequently, N is a quasipolynomial on $\overline{(R_i \cup R_{i+1})} \cap \mathbb{Z}^n$. By B.2, N is given by the same quasipolynomial on $R \cap \mathbb{Z}^n$ and on $S \cap \mathbb{Z}^n$.

Appendix C. Proof of the Ehrhart-Macdonald reciprocity

Theorem 7. *Let $x \in \mathbb{Z}^n$, C be a chamber of (A, I) with $x \in \overline{C}$ and $x - \sum_{i \in I} e_i \in C - \mathrm{Cube}_{n,I}$, and $P_C(y)$ be the quasipolynomial with $\mathrm{N}_{A,I}(y) = P_C(y)$ for all $y \in C$. Then*

$$P_C(-x) = (-1)^{n+d+|I|} \, \mathrm{N}_{A,I}(x - \sum_{i \in I} e_i).$$

Proof. The case $C = \mathbb{Q}^d \setminus \mathcal{C}_{A,I}$ is obvious as $P_C \equiv 0$ and $x - \sum_{i \in I} e_i \in C$, since the latter implies $\mathrm{N}_{A,I}(x - \sum_{i \in I} e_i) = 0$. Thus we assume $C \subseteq \mathcal{C}_{A,I}$. We show by induction with respect to $|I|$ that $P_C(-x) = (-1)^{n+d+|I|} P_C(x - \sum_{i \in I} e_i)$. The case $|I| = n - d$ follows from Lemma 1. By (2.1),

$$P_C(-x) = \sum_{z \in [0, \langle h_1, -x \rangle)} Q_{R'_1}(-x - ze_j) + \sum_{i=2}^{r} \sum_{z \in [\langle h_{i-1}, -x \rangle, \langle h_i, -x \rangle)} Q_{R'_i}(-x - ze_j) + [\langle h_r, -x \rangle \in \mathbb{Z}] \cdot Q_{R'_r}(-x - \langle h_r, -x \rangle e_j),$$

where we use the notation from Lemma 2. By the extended definition of the summation, this is equal to

$$\begin{aligned} & - \sum_{z \in [\langle h_1, -x \rangle, 0)} Q_{R'_1}(-x - ze_j) - \sum_{i=2}^{r} \sum_{z \in [\langle h_i, -x \rangle, \langle h_{i-1}, -x \rangle)} Q_{R'_i}(-x - ze_j) \\ & \qquad + [\langle h_r, x \rangle \in \mathbb{Z}] \cdot Q_{R'_r}(-x + \langle h_r, x \rangle e_j) \\ & = - \sum_{z \in (0, \langle h_1, x \rangle]} Q_{R'_1}(-x + ze_j) - \sum_{i=2}^{r} \sum_{z \in (\langle h_{i-1}, x \rangle, \langle h_i, x \rangle]} Q_{R'_i}(-x + ze_j) \\ & \qquad + [\langle h_r, x \rangle \in \mathbb{Z}] \cdot Q_{R'_r}(-x + \langle h_r, x \rangle e_j). \end{aligned}$$

By the induction hypothesis, this is equal to

$$\begin{aligned} & (-1)^{n+d+|I|} \left(\sum_{z \in (0, \langle h_1, x \rangle]} Q_{R'_1}(x - ze_j - E + e_j) + \sum_{i=2}^{r} \sum_{z \in (\langle h_{i-1}, x \rangle, \langle h_i, x \rangle]} Q_{R'_i}(x - ze_j - E + e_j) \right. \\ & \qquad \left. - [\langle h_r, x \rangle \in \mathbb{Z}] \cdot Q_{R'_r}(x - \langle h_r, x \rangle e_j - E + e_j) \right) \\ & = (-1)^{n+d+|I|} \left(\sum_{z \in [0, \langle h_1, x \rangle - 1]} Q_{R'_1}(x - ze_j - E) + \sum_{i=2}^{r} \sum_{z \in (\langle h_{i-1}, x \rangle - 1, \langle h_i, x \rangle - 1]} Q_{R'_i}(x - ze_j - E) \right. \\ & \qquad \left. - [\langle h_r, x \rangle \in \mathbb{Z}] \cdot Q_{R'_r}(x - (\langle h_r, x \rangle - 1)e_j - E) \right), \end{aligned}$$

where $E := \sum\limits_{i \in I} e_i$. By the extended definition of the summation and as $\langle h_1, e_j \rangle = 1$, this is equal to

$$\begin{aligned} & (-1)^{n+d+|I|} \left(\sum_{z \in [0, \langle h_1, x-E \rangle)} Q_{R'_1}(x - ze_j - E) + \sum_{z \in [\langle h_1, x-E \rangle, \langle h_1, x-e_j \rangle]} Q_{R'_1}(x - ze_j - E) \right. \\ & + \sum_{i=2}^{r} \left(\sum_{z \in (\langle h_{i-1}, x-e_j \rangle, \langle h_i, x-E \rangle)} Q_{R'_i}(x - ze_j - E) + \sum_{z \in [\langle h_i, x-E \rangle, \langle h_i, x-e_j \rangle]} Q_{R'_i}(x - ze_j - E) \right) \\ & \qquad \left. - [\langle h_r, x \rangle \in \mathbb{Z}] \cdot Q_{R'_r}(x - (\langle h_r, x \rangle - 1)e_j - E) \right). \qquad \text{(C.1)} \end{aligned}$$

Theorem 2 now implies that, for $i = 1, 2, \dots, r-1$,

$$\sum_{z \in [\langle h_i, x-E \rangle, \langle h_i, x-e_j \rangle]} Q_{R'_i}(x - ze_j - E) = \sum_{z \in [\langle h_i, x-E \rangle, \langle h_i, x-e_j \rangle]} Q_{R'_{i+1}}(x - ze_j - E),$$

while

$$\sum_{z\in(\langle h_i,x-E\rangle,\langle h_i,x-e_j\rangle)} Q_{R'_r}(x-ze_j-E) = \sum_{z\in(\langle h_i,x-E\rangle,\langle h_i,x-e_j\rangle)} Q_{R'_{r+1}}(x-ze_j-E) = 0$$

and, therefore,

$$(-1)^{n+d+|I|}\left(\sum_{z\in[0,\langle h_1,x-E\rangle)} Q_{R'_1}(x-ze_j-E)+\right.$$
$$+\sum_{i=2}^{r}\left(\sum_{z\in[\langle h_{i-1},x-E\rangle,\langle h_{i-1},x-e_j\rangle]} Q_{R'_i}(x-ze_j-E)+\sum_{z\in(\langle h_{i-1},x-e_j\rangle,\langle h_i,x-E\rangle)} Q_{R'_i}(x-ze_j-E)\right)$$
$$\left.+[\langle h_r,x-E\rangle\in\mathbb{Z}]\cdot Q_{R'_r}(x-E-(\langle h_r,x-E\rangle)e_j)\right)$$
$$=(-1)^{n+d+|I|}\left(\sum_{z\in[0,\langle h_1,x-E\rangle)} Q_{R'_1}(x-ze_j-E)++\sum_{i=2}^{r}\sum_{z\in[\langle h_{i-1},x-E\rangle,\langle h_i,x-E\rangle)} Q_{R'_i}(x-ze_j-E)\right.$$
$$\left.+[\langle h_r,x-E\rangle\in\mathbb{Z}]\cdot Q_{R'_r}(x-E-(\langle h_r,x-E\rangle)e_j)\right).$$

The assertion follows from (2.1). □

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