Sequences of labeled trees related to Gelfand-Tsetlin patterns

or

Towards a combinatorial proof of the ASM-Theorem?

ASM=Alternating sign matrix

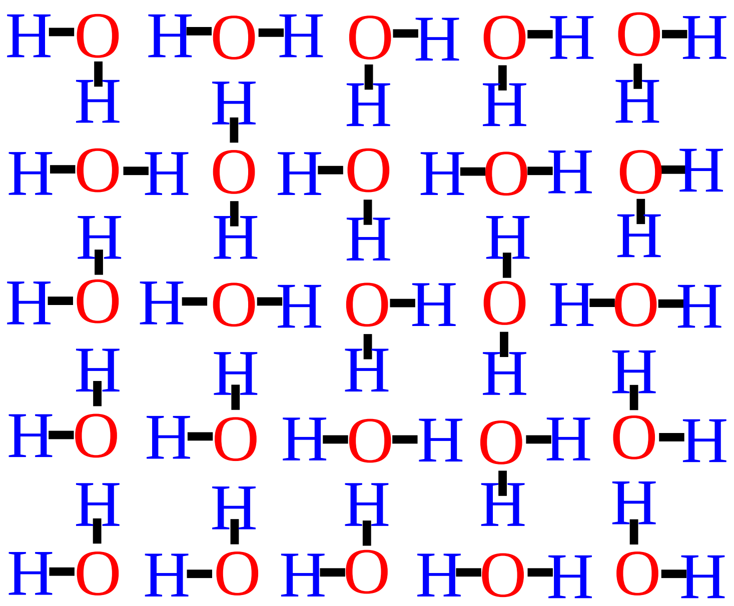
Quadratic $0, 1, -1$ matrix such that in each row and each column

- the non–zero entries appear with alternating signs and
- the sum of entries is 1, that is the first and the last non-zero entry is a 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Equivalent to square ice:

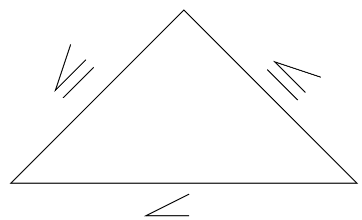
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Monotone triangles

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Triangular arrays of integers with monotonicity requirements:



Monotone triangles with bottom row $1, 2, \ldots, n \Leftrightarrow n \times n$ ASMs

ASM-Theorem

$$\#\text{ of } n \times n \text{ ASMs } = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} =: A_n$$

Refined ASM-Theorem

$\#$ of $n \times n$ ASMs with 1 in position $(1,i)$

$$= \binom{n+i-2}{n-1} \frac{(2n-i-1)!}{(n-i)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} =: A_{n,i}$$

A theorem that implies the refined ASM-Theorem

Fix $n \geq 2$ and assume that $(A_{n-1,i})_{1 \leq i \leq n-1}$ is known. Then the numbers $A_{n,i}$ are uniquely determined by the following system of linear equations:

$$
\begin{aligned}
A_{n,1} &= A_{n-1} = \sum_{i=1}^{n-1} A_{n-1,i} && (1)\\
A_{n,i} &= A_{n,n+1-i} \qquad 1 \leq i \leq n && (2)\\
A_{n,i} &= \sum_{k=1}^{n} \binom{2n-1-i}{k-i} (-1)^{k+n} A_{n,k} \qquad 1 \leq i \leq n && (3)
\end{aligned}
$$

Remarks

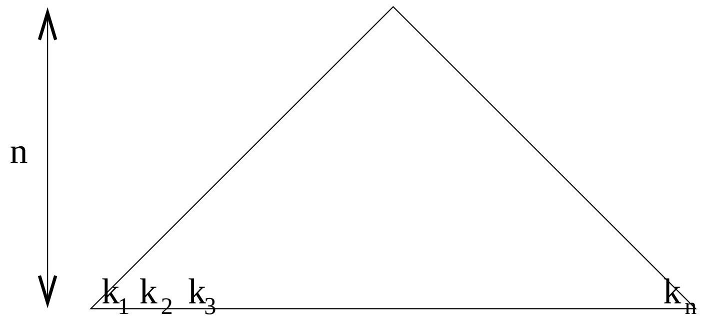
1. It is obvious that the number $A_{n,i}$ satisfy (1) and (2).

2. To show that the system of linear equations has a unique solution is more than a complicated linear algebra exercise – it involves a determinant evaluation of Andrews related to descending plane partitions.

3. Combinatorial proof of (3) $\Rightarrow$ combinatorial proof of the refined ASM-Theorem.

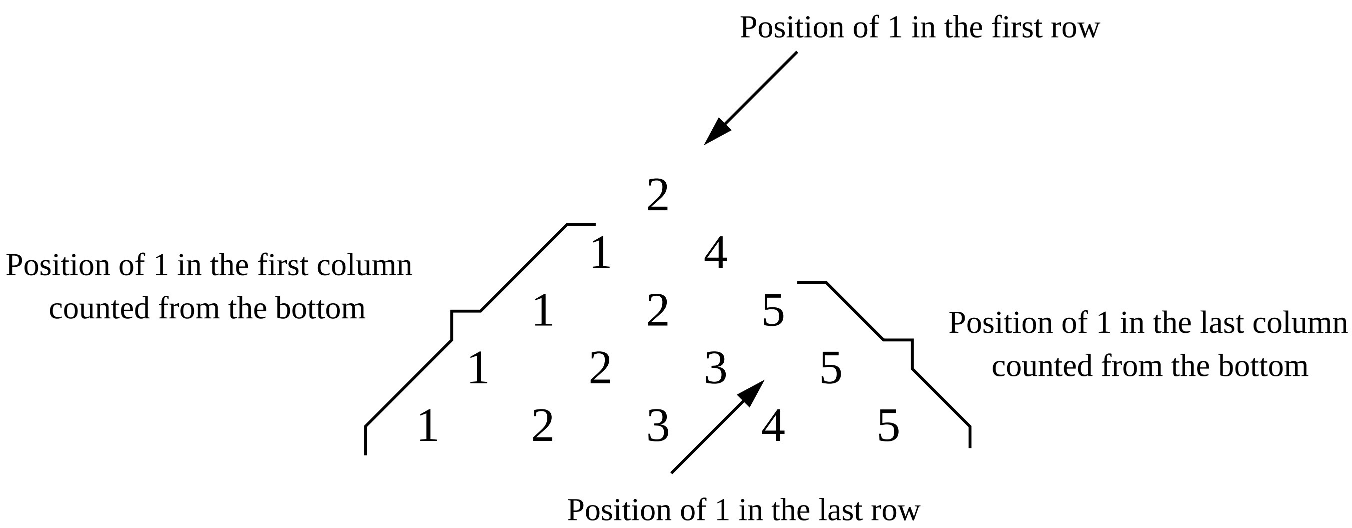
Non-combinatorial proof of (3)

Everything is in terms of monotone triangles:

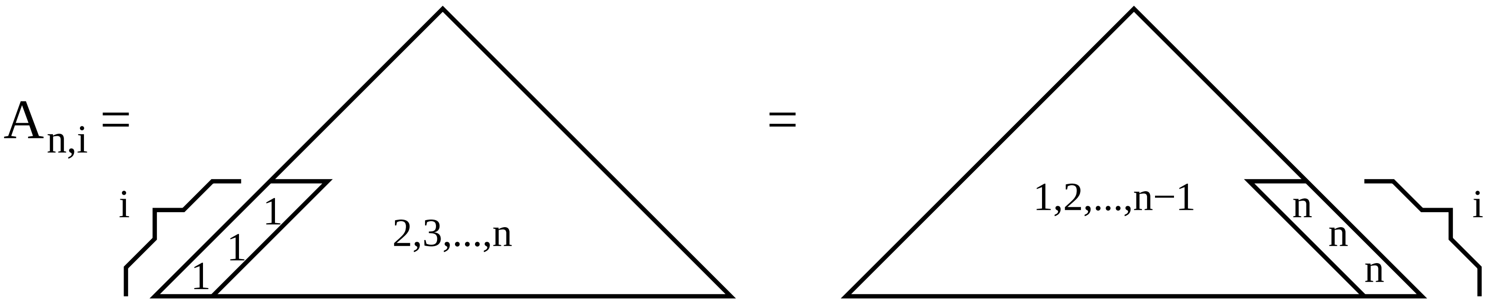


$$\# = \alpha(n; k_1, \ldots, k_n)$$

How is the position of the unique 1 in the first row of an ASM reflected in the corresponding monotone triangle?

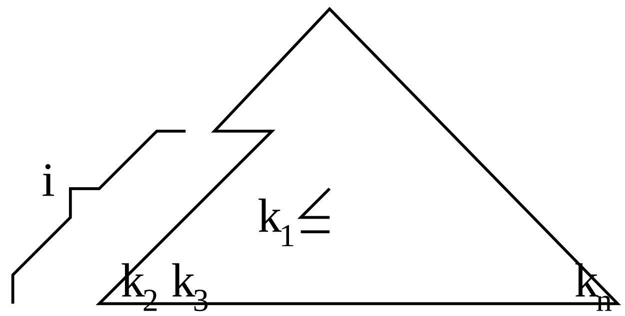


Therefore...



Partial monotone triangles with truncated first ↗-diagonal, respectively truncated last ↘-diagonal!

Lemma. $k_1 \leq k_2 < \ldots < k_n$, $1 \leq i \leq n$

\# of 

$$= (-1)^{i-1} \Delta_{k_1}^{i-1} \alpha(n; k_1, \ldots, k_n)$$

$$\Delta_x p(x) = p(x+1) - p(x)$$

Idea of the proof

Recursion:

$$\alpha(n; k_1, \ldots, k_n) = \sum_{\substack{k_1 \le l_1 \le k_2 \le l_2 \le \cdots \le l_{n-1} \le k_n \\ l_i \ne l_{i+1}}} \alpha(n-1; l_1, \ldots, l_{n-1})$$

Now

$$-\Delta_{k_1} \alpha(n; k_1, \ldots, k_n) = \sum_{\substack{k_1 \le l_1 \le k_2 \le l_2 \le \cdots \le l_{n-1} \le k_n \\ l_i \ne l_{i+1}}} \alpha(n-1; l_1, \ldots, l_{n-1})$$

$$- \sum_{\substack{k_1+1 \le l_1 \le k_2 \le l_2 \le \cdots \le l_{n-1} \le k_n \\ l_i \ne l_{i+1}}} \alpha(n-1; l_1, \ldots, l_{n-1})$$

$$= \sum_{\substack{k_2 \le l_2 \le k_3 \le \cdots \le l_{n-1} \le k_n \\ l_i \ne l_{i+1}}} \alpha(n-1; k_1, l_2, \ldots, l_{n-1}).$$

Ingredients for the proof of (3)

Corollary.

$$A_{n,i} = (-1)^{i-1} \Delta_{k_1}^{i-1} \alpha(n; k_1, \ldots, k_n)\Big|_{(k_1,\ldots,k_n)=(2,2,3,\ldots,n)}$$
$$= \delta_{k_n}^{i-1} \alpha(n; k_1, \ldots, k_n)\Big|_{(k_1,\ldots,k_n)=(1,2,3,\ldots,n-1,n-1)}$$

where $\delta_x p(x) = p(x) - p(x-1)$.

Lemma.

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n)$$

$$
\begin{aligned}
A_{n,i} &= (-1)^{i-1}\Delta_{k_1}^{i-1}\alpha(n;k_1,\ldots,k_n)\Big|_{(k_1,\ldots,k_n)=(1,1,2,\ldots,n-1)} \\
&= (-1)^{i+n}\Delta_{k_1}^{i-1}\alpha(n;k_2,\ldots,k_n,k_1-n)\Big|_{(k_1,\ldots,k_n)=(1,1,2,\ldots,n-1)} \\
&= (-1)^{i+n}\Delta_{k_1}^{i-1}E_{k_1}^{-2n+2}\alpha(n;k_2,\ldots,k_n,k_1)\Big|_{(k_2,\ldots,k_n,k_1)=(1,2,\ldots,n-1,n-1)} \\
&= (-1)^{i+n}\delta_{k_n}^{i-1}E_{k_n}^{-2n+1+i}\alpha(n;k_1,k_2,\ldots,k_n)\Big|_{(k_1,\ldots,k_{n-1},k_n)=(1,2,\ldots,n-1,n-1)}
\end{aligned}
$$

Binomial Theorem: $E_x^{-m} = (\mathrm{id} - \delta_x)^m = \sum_{j=0}^{m} \binom{m}{j}(-1)^j \delta_x^j$

$$
\begin{aligned}
A_{n,i} &= \sum_{j=0}^{2n-1-i} \binom{2n-1-i}{j}(-1)^{i+j+n}\delta_{k_n}^{i+j-1}\,\alpha(n;k_1,k_2,\ldots,k_n)\Big|_{(k_1,\ldots,k_{n-1},k_n)=(1,2,\ldots,n-1,n-1)} \\
&= \sum_{j=0}^{2n-1-i} \binom{2n-1-i}{j}(-1)^{i+j+n}A_{n,i+j} = \sum_{k=i}^{n} \binom{2n-1-i}{k-i}(-1)^{k+n}A_{n,k}
\end{aligned}
$$

A combinatorial proof of

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n)$$

implies a combinatorial proof of the refined ASM-Theorem.

Problem: The right hand side has no combinatorial meaning if $k_1 < k_2 < \ldots < k_n$.

Formula for α:

$$\alpha(n; k_1, \ldots, k_n) = \prod_{1 \le p < q \le n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}$$

where $E_x p(x) = p(x+1)$.

Combinatorial interpretation for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$

Example.

$$
\begin{array}{ccccccccc}
 & & & & \overset{\leftrightarrow}{5} & & & & \\
 & & & \overset{\leftarrow}{5} & & \overset{\rightarrow}{6} & & & \\
 & & \overset{\leftarrow}{4} & & \overset{\leftarrow}{6} & & \overset{\leftrightarrow}{7} & & \\
 & \overset{\leftarrow}{2} & & \overset{\leftarrow}{6} & & \overset{\rightarrow}{7} & & \overset{\rightarrow}{5} & \\
\overset{\rightarrow}{3} & & \overset{\leftrightarrow}{1} & & \overset{\rightarrow}{8} & & \overset{\rightarrow}{5} & & \overset{\rightarrow}{4}
\end{array}
$$

- Each entry $a_{i,j}$ lies between its SW-neighbor $a_{i+1,j}$ and its SE-neighbor $a_{i+1,j+1}$.

- The arrows indicate whether the inequalities are strict or not.

Arrow triangles (better name?)

Triangular arrays of integers of the following shape

$$\begin{array}{ccccccccccc} & & & & & a_{1,1} & & & & & \\ & & & & a_{2,1} & & a_{2,2} & & & & \\ & & & \dots & & \dots & & \dots & & & \\ & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} & & \\ & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} & \\ a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n} \end{array}$$

together with a function $f : \{a_{i,j}\} \to \{\leftarrow, \rightarrow, \leftrightarrow\}$, such that for all $a_{i,j}$ with $i < n$ the following is fulfilled:

Four cases:

$$\begin{array}{ccc} & a_{i,j} & \\ \leftarrow & & \leftarrow,\leftrightarrow \\ a_{i+1,j} & & a_{i+1,j+1} \end{array} : \; a_{i+1,j} \leq a_{i,j} < a_{i+1,j+1} \text{ or } a_{i+1,j} > a_{i,j} \geq a_{i+1,j+1}$$

$$\begin{array}{ccc} & a_{i,j} & \\ \leftarrow & & \rightarrow \\ a_{i+1,j} & & a_{i+1,j+1} \end{array} : \; a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1} \text{ or } a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$$

$$\begin{array}{ccc} & a_{i,j} & \\ \leftrightarrow,\rightarrow & & \leftarrow,\leftrightarrow \\ a_{i+1,j} & & a_{i+1,j+1} \end{array} : \; a_{i+1,j} < a_{i,j} < a_{i+1,j+1} \text{ or } a_{i+1,j} \geq a_{i,j} \geq a_{i+1,j+1}$$

$$\begin{array}{ccc} & a_{i,j} & \\ \leftrightarrow,\rightarrow & & \rightarrow \\ a_{i+1,j} & & a_{i+1,j+1} \end{array} : \; a_{i+1,j} < a_{i,j} \leq a_{i+1,j+1} \text{ or } a_{i+1,j} \geq a_{i,j} > a_{i+1,j}$$

Signed enumeration

If we are in the second case then $a_{i,j}$ is said to be an inversion. The sign of an "arrow triangle" is

$$(-1)^{\#\text{ of inversions}}(-1)^{\#\text{ of }\leftrightarrow}.$$

Theorem. The signed enumeration of arrow triangles with bottom row $k_1, \ldots, k_n$ is

$$\prod_{1 \le p < q \le n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}.$$

Remark

If $k_1 < k_2 < \ldots < k_n$ then an arrow triangle is not a monotone triangle!

But it is obvious that the signed enumeration of arrow triangles gives the number of monotone triangles in this case:

- All rows are strictly increasing in this case.
- Situation for $a_{i,j}$:

$$\begin{array}{ccc} a_{i-1,j-1} & & a_{i-1,j} \\ & a_{i,j} & \end{array}$$

If $f(a_{i,j}) = \leftarrow$ then $a_{i-1,j-1} < a_{i,j} \leq a_{i-1,j}$.
If $f(a_{i,j}) = \rightarrow$ then $a_{i-1,j-1} \leq a_{i,j} < a_{i-1,j}$.
If $f(a_{i,j}) = \leftrightarrow$ then $a_{i-1,j-1} < a_{i,j} < a_{i-1,j}$.

Try first for easier objects: Gelfand-Tsetlin patterns

Triangular array of integers of the following shape

$$\begin{array}{ccccccccccc} & & & & & a_{1,1} & & & & & \\ & & & & a_{2,1} & & a_{2,2} & & & & \\ & & & \dots & & \dots & & \dots & & & \\ & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} & & \\ & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} & \\ a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n} \end{array},$$

with weak increase in $\nearrow$-direction and in $\searrow$-direction, i.e. $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$ for all i, j.

Semistandard tableaux of fixed shape

$$\begin{array}{ccccccccccc} & & & & & 2 & & & & & \\ & & & & 2 & & 2 & & & & \\ & & & 1 & & 2 & & 4 & & & \\ & & 1 & & 1 & & 3 & & 4 & & \\ & 0 & & 1 & & 3 & & 3 & & 5 & \\ 0 & & 0 & & 2 & & 3 & & 5 & & 6 \end{array}$$

$$\begin{array}{cccccc} 1 & 1 & 3 & 3 & 5 & 6 \\ 2 & 2 & 4 & 6 & 6 & \\ 3 & 5 & 5 & & & \\ 4 & 6 & & & & \end{array}$$

If $(k_1, k_2, \ldots, k_n)$ is the bottom row of the Gelfand-Tsetlin pattern then $(k_n, k_{n-1}, \ldots, k_1)$ is the shape of the respective semistandard tableau.

Enumeration

of semistandard tableau of shape $(k_n, k_{n-1}, \dots, k_1)$

$$= \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i} =: \beta(n; k_1, \dots, k_n)$$

Proof. Rewrite Stanley's hook-content formula.

We also have

$$\beta(n; k_1, \dots, k_n) = (-1)^{n-1} \beta(n; k_2, \dots, k_n, k_1 - n).$$

In fact, this follows from

$$\beta(n; k_1, \dots, k_n) = -\beta(n; k_1, \dots, k_{i-1}, k_{i+1} + 1, k_i - 1, k_{i+2}, \dots, k_n),$$

which is valid for all $i \in \{1, 2, \dots, n-1\}$. (Shift-antisymmetry)

Give a combinatorial proof of the shift-antisymmetry!

What is $\beta(n; k_1, \dots, k_n)$ if $(k_1, \dots, k_n)$ is not weakly increasing?

An extended Gelfand-Tsetlin pattern is a triangular array of integers of the following shape

$$\begin{array}{ccccccccccc} & & & & & a_{1,1} & & & & & \\ & & & & a_{2,1} & & a_{2,2} & & & & \\ & & & \dots & & \dots & & \dots & & & \\ & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} & & \\ & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} & \\ a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n} \end{array},$$

with $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$ or $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$ for all $a_{i,j}$. In the latter case, $a_{i,j}$ is said to be an inversion and the sign of an extended Gelfand-Tsetlin pattern is

$$(-1)^{\#\text{ of inversions}}.$$

Then $\beta(n; k_1, \dots, k_n)$ is the signed enumeration of Gelfand-Tsetlin patterns with bottom row $(k_1, \dots, k_n)$.

- Using this extension it is possible to give a combinatorial proof of the shift-antisymmetry.

- But everything is much nicer when using auxiliary objects: Gelfand-Tsetlin tree sequences.

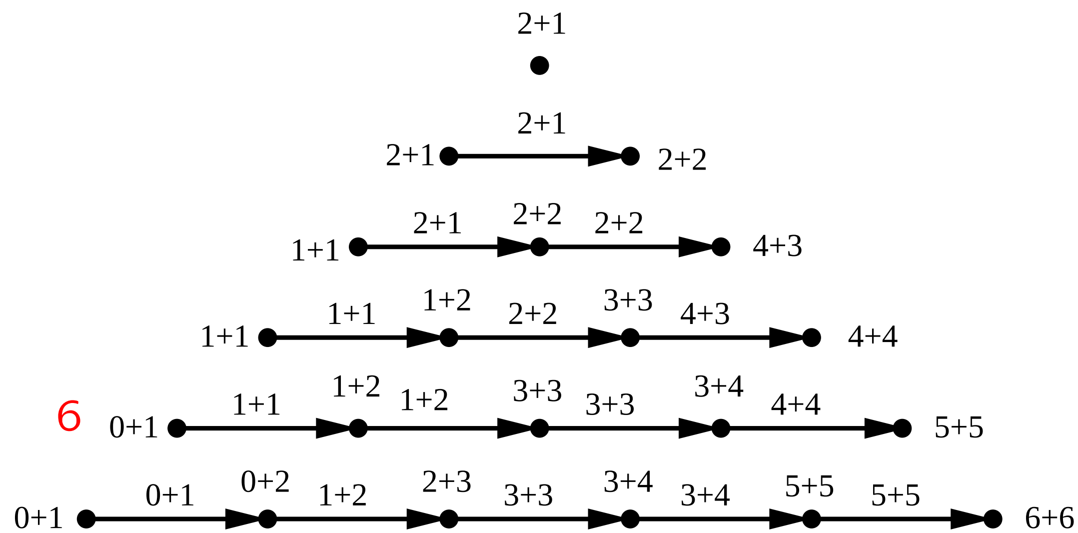
- They provide a family of set of objects; the signed enumeration of each member of this family is given by

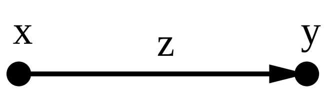
$$\prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}.$$

- Gelfand-Tsetlin patterns are one special member of this family.

Where is the tree sequence in a Gelfand-Tsetlin pattern?

2
2 2
1 2 4
1 1 3 4
0 1 3 3 5
0 0 2 3 5 6

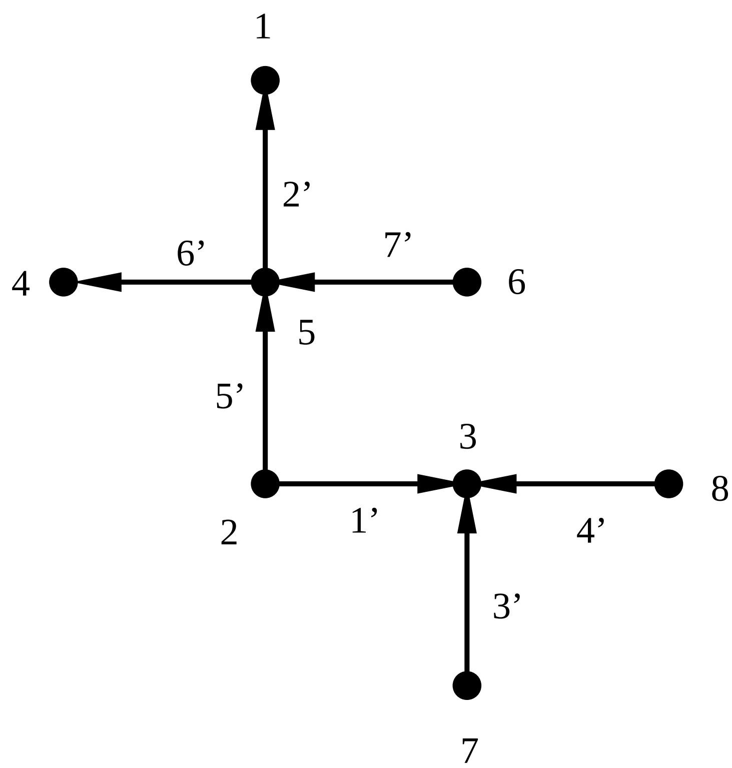


Rule. 

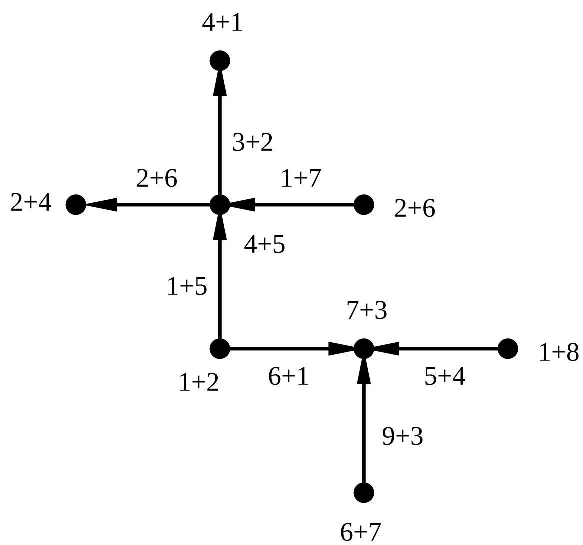
- $\min(x,y) \leq z < \max(x,y)$
- The edge is an inversion if the edge is directed from the maximum vertex label to the minimum vertex label.

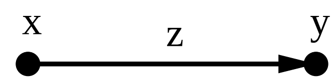
Paths can be replaced by trees!

n-tree: directed tree on n vertices. Identify vertices with elements in $\{1, 2, ..., n\}$ and edges with elements $\{1', 2', ..., (n-1)'\}$.



Admissible labeling



Rule. 

- $\min(x, y) \leq z < \max(x, y)$
- The edge is an inversion if its sink has the label $\min(x, y)$.
- The second summand of the labeling is always the "name" of the vertex/edge.

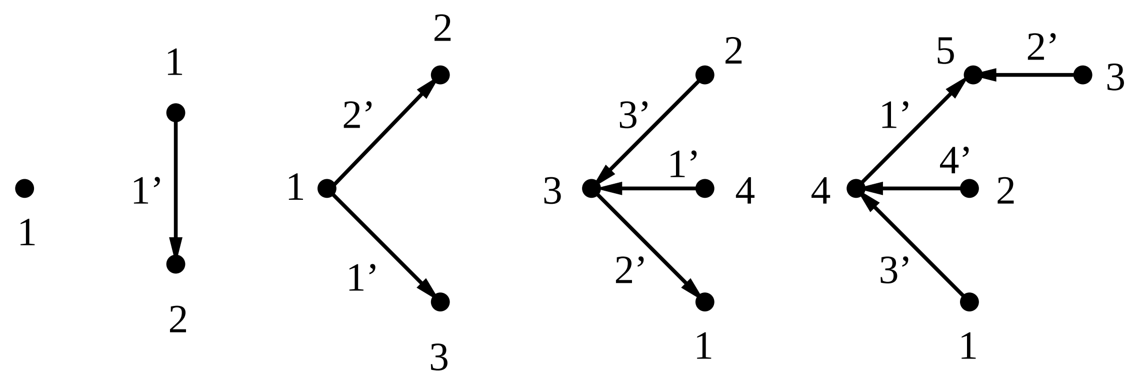
- Collect the first summands in a vector:

vertices $\mathbf{k} := (4, 1, 7, 2, 4, 2, 6, 1)$; edges $\mathbf{l} := (6, 3, 9, 5, 1, 2, 1)$.

- The vector $\mathbf{l}$ is said to be admissible for the pair $(T, \mathbf{k})$; the sign is $(-1)^{\#\text{ of inversions}}$.

Tree sequence of order n

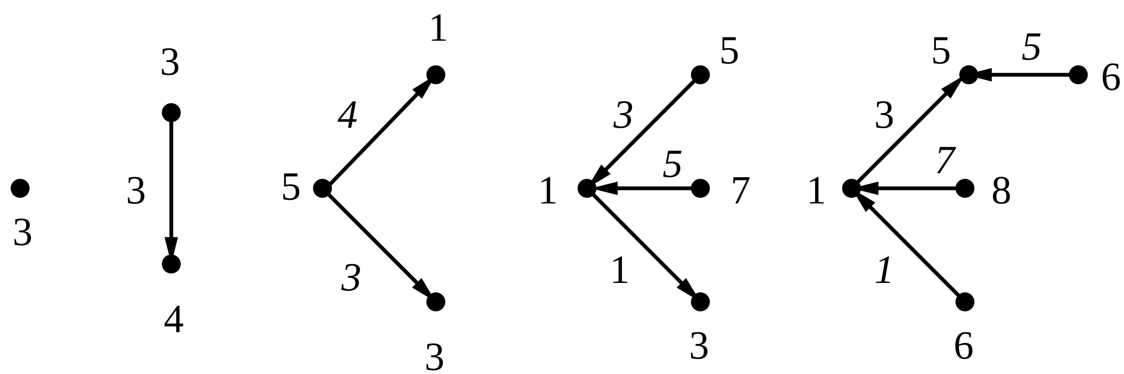
$\mathcal{T} = (T_1, \ldots, T_n)$, where T_i is an i-tree for $1 \le i \le n$.



Gelfand-Tsetlin tree sequence: definition

Given: tree sequence $\mathcal{T} = (T_1, \ldots, T_n)$ and a shifted labeling $\mathbf{k} \in \mathbb{Z}^n$ of the vertices of T_n.

A sequence $(\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_n)$ of vectors $\mathbf{l}_i \in \mathbb{Z}^i$ with $\mathbf{l}_n = \mathbf{k}$ such that $\mathbf{l}_{i-1}$ is admissible for the pair $(T_i, \mathbf{l}_i)$ is called a Gelfand-Tsetlin tree sequence associated with $\mathcal{T}$ and $\mathbf{k}$.



Signed enumeration of GT tree sequences

Theorem. Let $\mathcal{T} = (T_1, \dots, T_n)$ and $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$. Then the signed enumeration of GT tree sequences associated with $\mathcal{T}$ and $\mathbf{k}$ is

$$\prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i} \, \mathrm{sgn}(\mathcal{T}),$$

where $\mathrm{sgn}(\mathcal{T})$ is a certain sign associated with $\mathcal{T}$.

Idea of the proof.

$$L_n(\mathcal{T},\mathbf{k}) := \text{sgn}(\mathcal{T}) \times \sum_{\text{GT tree sequences associated with } (\mathcal{T},\mathbf{k})} (-1)^{\#\text{ of inversions}}$$

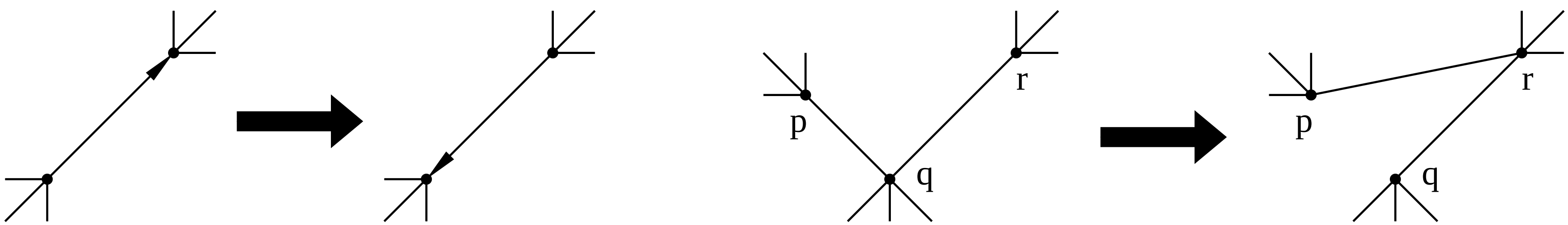
(1) $\Delta_{k_i}^{n} L_n(\mathcal{T}, k_1, \ldots, k_n) = 0$: find a combinatorial interpretation for $\Delta_{k_i}^{j} L_n(\mathcal{T}, k_1, \ldots, k_n)$ if $j \in \{0, 1, \ldots, n-1\}$ and show that $\Delta_{k_i}^{n-1} L_n(\mathcal{T}, k_1, \ldots, k_n)$ does not depend on k_i.

This implies that $L_n(\mathcal{T}, k_1, \ldots, k_n)$ is a polynomial in k_i of degree no greater than $n-1$.

(2)

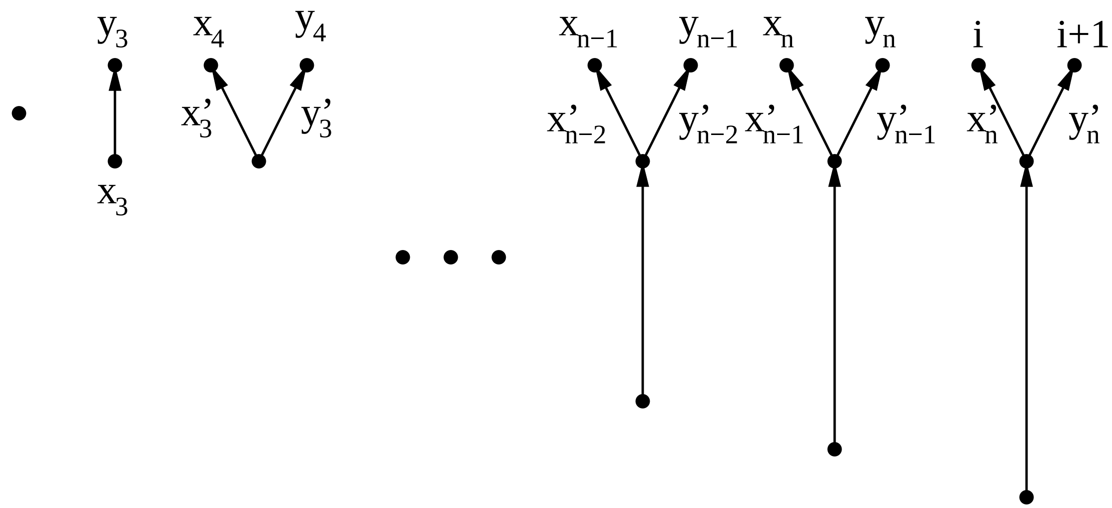
$$L_n(\mathcal{T}, k_1, \ldots, k_n) = -L_n(\mathcal{T}, k_1, \ldots, k_{i-1}, k_{i+1}+1, k_i-1, k_{i+2}, \ldots, k_n)$$

(A) $L_n(\mathcal{T}, \mathbf{k})$ does not depend on $\mathcal{T}$: the signed enumeration is invariant under the following two tree operations (reversing the orientation of an edge; sliding an edge along an adjacent edge):



The result follows as every i-tree can be obtained from every other by means of these operations.

(B) Find a tree sequence for which the shift-antisymmetry is obvious:



(2) $\Rightarrow L_n(\mathcal{T}, k_1, \ldots, k_n) = \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i} P(k_1, \ldots, k_n)$

(1) $\Rightarrow P(k_1, \ldots, k_n) = C$.

There is a unique Gelfand-Tsetlin pattern with bottom row $(1, 1, \ldots, 1)$

$\Rightarrow C = 1$. □

Open problem

Bijective proof of

$$\alpha(n;k_1,\ldots,k_n) = \prod_{1\le p<q\le n} (E_{k_p}+E_{k_q}^{-1}-E_{k_p}E_{k_q}^{-1}) \prod_{1\le i<j\le n} \frac{k_j-k_i+j-i}{j-i}$$

if $k_1 < k_2 < \ldots < k_n$.

Arrow pattern: Function $p : \{(i,j) | 1 \le j \le i \le n\} \to \{\leftarrow, \rightarrow, \leftrightarrow\}$

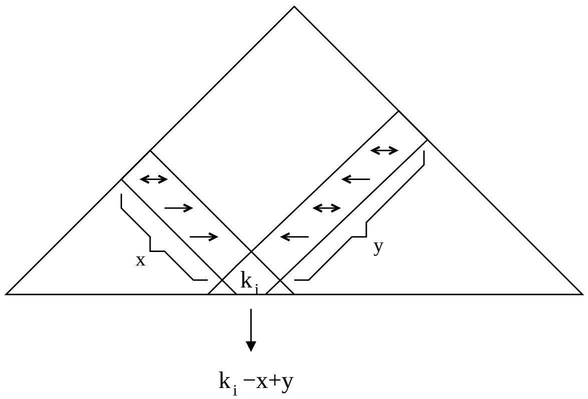
$$\begin{array}{ccccccccc}
 & & & & \overset{\leftrightarrow}{a_{1,1}} & & & & \\
 & & & \overset{\leftarrow}{a_{2,1}} & & \overset{\rightarrow}{a_{2,2}} & & & \\
 & & \overset{\leftarrow}{a_{3,1}} & & \overset{\leftarrow}{a_{3,2}} & & \overset{\leftrightarrow}{a_{3,3}} & & \\
 & \overset{\leftarrow}{a_{4,1}} & & \overset{\leftarrow}{a_{4,2}} & & \overset{\rightarrow}{a_{4,3}} & & \overset{\rightarrow}{a_{4,4}} & \\
\overset{\rightarrow}{a_{5,1}} & & \overset{\leftrightarrow}{a_{5,2}} & & \overset{\rightarrow}{a_{5,3}} & & \overset{\rightarrow}{a_{5,4}} & & \overset{\rightarrow}{a_{5,5}}
\end{array}$$

Left hand side

$$\sum_{p:\{(i,j)\}\to\{\leftarrow,\to,\leftrightarrow\}} (-1)^{\#\text{ of }\leftrightarrow}$$
$$\times(\#\text{of Gelfand-Tsetlin patterns with arrow pattern } p \text{ and bottom row } (k_1,\ldots,k_n))$$

Right hand side

Given an arrow pattern $p : \{(i,j)\} \to \{\leftarrow, \rightarrow, \leftrightarrow\}$, associate a deformation of the bottom row $(k_1, \ldots, k_n)$ as follows:



$$\sum_{p:\{(i,j)\}\to\{\leftarrow,\rightarrow,\leftrightarrow\}} (-1)^{\#\text{ of }\leftrightarrow}$$
$$\times\ (\#\text{of Gelfand-Tsetlin patterns with bottom row associated to } p)$$

Non-combinatorial proof of

$$\alpha(n;k_1,\ldots,k_n) = (-1)^{n-1}\alpha(n;k_2,\ldots,k_n,k_1-n)$$

Lemma. Let $V_{x,y} = E_x^{-1} + E_y - E_x^{-1}E_y$ and $i \in \{1,2,\ldots,n-1\}$. Then

$$\alpha(n;k_1,\ldots,k_{i-1},k_{i+1}+1,k_i-1,k_{i+2},\ldots,k_n)$$
$$= -V_{k_i,k_{i+1}}V^{-1}_{k_{i+1},k_i}\alpha(n;k_1,\ldots,k_n).$$

Remark. $V_{x,y}$ is invertible as $V_{x,y} = \mathrm{id} + \delta_x\Delta_y$ and

$$V^{-1}_{x,y} = \sum_{i=0}^{\infty}(-1)^i\delta_x^i\Delta_y^i.$$

The lemma implies

$$
\begin{aligned}
&(-1)^{n-1}\alpha(n; k_2, \ldots, k_n, k_1 - n) \\
&\qquad = (-1)^{n-1}\alpha(n; k_2 + 1, \ldots, k_n + 1, k_1 - n + 1) \\
&\qquad\qquad = \prod_{i=2}^{n} V_{k_1,k_i} V_{k_i,k_1}^{-1} \alpha(n; k_1, \ldots, k_n).
\end{aligned}
$$

It suffices to show that

$$
\left(\prod_{i=2}^{n} V_{k_1,k_i} - \prod_{i=2}^{n} V_{k_i,k_1} \right) \alpha(n; k_1, \ldots, k_n) = 0.
$$

Lemma. Let

$$e_p(X_1, \dots, X_n) = \sum_{1 \le i_1 < i_2 < \dots i_p \le n} X_{i_1} X_{i_2} \cdots X_{i_p}$$

denote the p-th elementary symmetric function. Then, for $p \geq 1$,

$$e_p(\Delta_{k_1}, \dots, \Delta_{k_n}) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i} = 0$$

and

$$e_p(\delta_{k_1}, \dots, \delta_{k_n}) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i} = 0.$$

Corollary. For $p \geq 1$

$$e_p(\Delta_{k_1}, \ldots, \Delta_{k_n})\alpha(n; k_1, \ldots, k_n) = 0.$$

and

$$e_p(\delta_{k_1}, \ldots, \delta_{k_n})\alpha(n; k_1, \ldots, k_n) = 0.$$

Now

$$\prod_{i=2}^{n} V_{k_1,k_i} - \prod_{i=2}^{n} V_{k_i,k_1} = \prod_{i=2}^{n} (\mathrm{id} + \delta_{k_1}\Delta_{k_i}) - \prod_{i=2}^{n} (\mathrm{id} + \Delta_{k_1}\delta_{k_i})$$
$$= \sum_{r=0}^{n-1} \delta_{k_1}^r e_r(\Delta_{k_2}, \ldots, \Delta_{k_n}) - \sum_{r=0}^{n-1} \Delta_{k_1}^r e_r(\delta_{k_2}, \ldots, \delta_{k_n})$$

$$\sum_{r=0}^{n-1} \left(\delta_{k_1}^r \left(e_r(\Delta_{k_1}, \ldots, \Delta_{k_n}) - \Delta_{k_1} e_{r-1}(\Delta_{k_2}, \ldots, \Delta_{k_n})\right)\right.$$

$$\left. - \Delta_{k_1}^r \left(e_r(\delta_{k_1}, \ldots, \delta_{k_n}) - \delta_{k_1} e_{r-1}(\delta_{k_2}, \ldots, \delta_{k_n})\right)\right)$$

$$= \sum_{r=0}^{n-1} \left(\delta_{k_1}^r e_r(\Delta_{k_1}, \ldots, \Delta_{k_n}) - \Delta_{k_1}^r e_r(\delta_{k_1}, \ldots, \delta_{k_n})\right)$$

$$- \sum_{r=1}^{n-1} \left(\delta_{k_1}^r \Delta_{k_1} e_{r-1}(\Delta_{k_2}, \ldots, \Delta_{k_n}) - \Delta_{k_1}^r \delta_{k_1} e_{r-1}(\delta_{k_2}, \ldots, \delta_{k_n})\right) = \ldots$$

$$= \sum_{s=1}^{n} \sum_{r=1}^{n-s} (-1)^s \left(\Delta_{k_1}^{r+s-1} \delta_{k_1}^{s-1} e_r(\delta_{k_1}, \ldots, \delta_{k_n})\right.$$

$$\left. - \delta_{k_1}^{r+s-1} \Delta_{k_1}^{s-1} e_r(\Delta_{k_1}, \ldots, \Delta_{k_n})\right).$$