Refined enumerations of alternating sign matrices

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Central question

Which enumeration problems have a solution in terms of a closed formula that (for instance) only involves the basic arithmetic operations?

Some facts

• Only very few enumeration problems have a nice solution in terms of a closed formula.

• More surprising: combinatorialists can still hardly predict when this rare event occurs.

• One indication to this is that such problems (that admit nice formulas) are often found by chance.

• It is often possible to guess these formula easily by considering small instances of the parameters involved.

• Although these guesses are always correct, many of these formulas (still) require highly non-trivial proofs, which do not provide much insight...

• ...and in this sense these proofs act more as confirmations than as explanations.

An example in this respect: alternating sign matrices

Quadratic $0, 1, -1$ matrices, such that in every row and every column

- the non–zero entries appear with alternating signs and
- the sum of entries is 1, that is the first and the last non-zero entry is a 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The origin of ASMs: λ–determinant

Notation: For a matrix M let $M_{i_1,\ldots,i_m}^{j_1,\ldots,j_n}$ denote the matrix that remains when the rows $i_1, \ldots, i_m$ and the columns $j_1, \ldots, j_n$ of M are deleted.

The Desnanot–Jacobi identity:

$$\det(M)\det(M_{1,n}^{1,n}) = \det \begin{pmatrix} \det(M_1^1) & \det(M_1^n) \\ \det(M_n^1) & \det(M_n^n) \end{pmatrix}$$

Charles L. Dodgson (Lewis Carroll) used this to devise an algorithm for calculating determinants that required only 2×2 determinants. (Condensation of determinants, 1866)

3×3 determinants

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \frac{1}{a_{2,2}}$$

$$\times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \end{pmatrix}$$

4×4 determinants

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} = \frac{1}{\det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix}}$$

$$\times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \end{pmatrix}$$

3×3 determinants are expressible in terms of 2×2 determinants...and so are 4×4 determinants!

Robbins and Rumsey in the 1980s: What happens if we generalize the definition of a 2×2 determinant to

$$\det_\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21}$$

and, furthermore, use the previous observations to generalize the $n \times n$ determinant?

Theorem (Robbins and Rumsey). Let M be an $n \times n$ matrix with entries $a_{i,j}$, $\mathcal{A}_n$ the set of $n \times n$ alternating sign matrices, $\mathcal{I}(B)$ the inversion number of B and $\mathcal{N}(B)$ the number of -1s in B then

$$\det_\lambda(M) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}.$$

Inversion number of a matrix B: Generalizes the inversion number of a permutation(-matrix).

$$\mathcal{I}(B) = \sum_{\substack{(i_1,j_1),(i_2,j_2) \\ i_1>i_2, j_1<j_2}} b_{i_1,j_1} b_{i_2,j_2}$$

Special case $\lambda = -1$ (ordinary determinant):

$$\det(M) = \sum_{B \in \mathcal{A}_n} (-1)^{\mathcal{I}(B)} 0^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}$$

$$= \sum_{B \in \mathcal{A}_n, b_{i,j} \neq -1} (-1)^{\mathcal{I}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}} = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} a_{i,\sigma(i)}$$

This is the well-known Leibniz formula!

How many $n \times n$ alternating sign matrices are there?

$$(1)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Six 3×3 permutation matrices $+ \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

42

Conjecture (Mills, Robbins, Rumsey, early 1980s). The number of $n \times n$ alternating sign matrices is

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Why do these strange objects have such a simple closed enumeration formula, whereas for other objects with a definition of less complexity it is simply impossible to write down just any explicit formula?

How to guess such a formula ?

A_n = Number of $n \times n$ ASMs $= 1, 2, 7, 42, 429, 7436, 218348,$
$10850216, 911835460, 129534272700, \ldots$

$$A_{n+1}/A_n = 2, \tfrac{7}{2}, 6, \tfrac{143}{14}, \tfrac{52}{3}, \tfrac{323}{11}, \tfrac{646}{13}, \tfrac{2185}{26}, \tfrac{2415}{17}, \tfrac{310155}{1292}, \ldots$$

$$(A_{n+2}/A_{n+1})/(A_{n+1}/A_n) = \tfrac{7}{4}, \tfrac{12}{7}, \tfrac{143}{84}, \tfrac{56}{33}, \tfrac{969}{572}, \tfrac{22}{13}, \tfrac{115}{68}, \tfrac{546}{323}, \tfrac{899}{532}, \tfrac{272}{161}, \ldots$$

Rational interpolation:

$$(A_{n+2}/A_{n+1})/(A_{n+1}/A_n) = \frac{3(3n+2)(3n+4)}{4(2n+1)(2n+3)}$$

Beauty is truth!

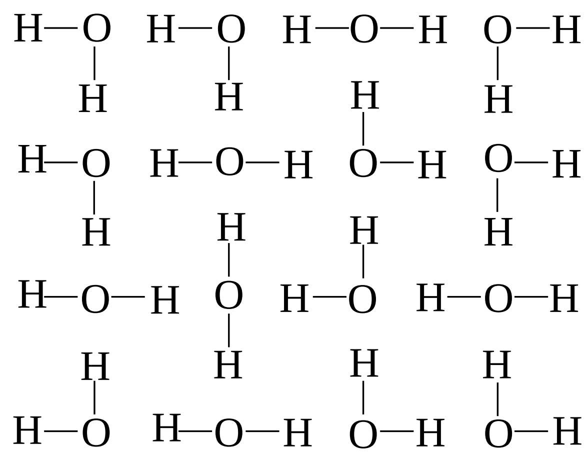
In 1996, Doron Zeilberger finally succeeded in proving the formula! His proof is based in constant term identities and is 84 pages long.

The final breakthrough: Jim Propp realizes that physicists had been studying a model (square ice/six-vertex model) for years, which is equivalent to alternating sign matrices. Greg Kuperberg was able to use this to give another, shorter, proof of the alternating sign matrix theorem.

ASMs in statistical physics

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

ASM



Square ice

The story is not yet over: symmetry classes of ASMs

Vertically symmetric alternating sign matrices (Kuperberg 2001):

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad A^V_{2n+1} = \prod_{j=0}^{n-1} \frac{(3j+2)(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!}$$

Half-turn symmetric alternating sign matrices: Kuperberg 2001 (even order), Razumov/Stroganov 2005 (odd order):

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} A^{HT}_{2n} &= \prod_{j=0}^{n-1} \frac{(3j+2)(3j+1)!^2}{(3j+1)(n+j)!^2} \\ A^{HT}_{2n+1} &= \prod_{j=1}^{n} \frac{4(3j)!^2 j!^2}{3(2j)!^4} \end{aligned}$$

Other cases

Quarter-turn symmetric ASMs. Kuperberg 2001 (even order), Razumov/Stroganov 2005 (odd order).

Vertically and horizontally symmetric ASMs. Okada 2005.

Diagonally and antidiagonally symmetric ASMs of odd order.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \frac{A_{2n+1}^{DA}}{A_{2n-1}^{DA}} = \frac{\binom{3n}{n}}{\binom{2n-1}{n}}$$

Open??

Now: an approach to refined enumerations of ASMs!

joint work with Dan Romik

A (very first) refined enumeration of ASMs

An ASM has a unique “1” in the top row as

- the top row must contain at least one “1” and

- two “1”s would force a “-1” to be in the top row.

Example.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Refined alternating sign matrix theorem

Theorem (Zeilberger 1996). The number of $n \times n$ alternating sign matrices where the unique 1 in the top row is in column k is

$$\binom{n+k-2}{n-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} =: A_{n,k}.$$

The theorem on the total number of $n \times n$ alternating sign matrices follows from this after summing over all k and consulting the appropriate hypergeometric identity.

Towards another proof: ASMs $\Rightarrow$ Monotone triangles

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 2 & & 3 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Monotone triangles are triangular integer arrays of the following shape

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & a_{n,n} & & & & & & & & \\
 & & & & & & & a_{n-1,n-1} & & a_{n-1,n} & & & & & & & \\
 & & & & & & a_{n-2,n-2} & & a_{n-2,n-1} & & a_{n-2,n} & & & & & & \\
 & & & & & \dots & & \dots & & \dots & & \dots & & & & & \\
 & & & & a_{2,2} & & a_{2,3} & & \dots & & \dots & & a_{2,n} & & & & \\
 & & & a_{1,1} & & a_{1,2} & & a_{1,3} & & \dots & & \dots & & a_{1,n} & & &
\end{array},$$

that are monotone increasing in $\nearrow$ direction and in $\searrow$ direction and strictly increasing along rows.

$n \times n$ ASMs $\leftrightarrow$ MTs with bottom row $1, 2, \dots, n$
$=:$ n–complete MTs

$n \times n$ ASMs where the unique 1 in the top row is in column k $\leftrightarrow$ n–complete MTs with top row k.

Important ingredient: the operator formula

Theorem (F). The number of monotone triangles with n rows and prescribed bottom row $k_1, k_2, \ldots, k_n$ is

$$\left(\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_p} E_{k_q} - E_{k_p}) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} =: \alpha(n; k_1, \ldots, k_n),$$

where $E_x p(x) = p(x+1)$.

A new type of formula: the shift operator is used in addition to the basic arithmetic operations!

Follows from the operator formula: $\alpha(n; k_1, \ldots, k_n)$ is a polynomial in $k_1, k_2, \ldots, k_n$ of degree $\leq n-1$ in every k_i.

We consider the following specialization

$$g_n(x) = \alpha(n; 1, 2, \ldots, n-1, n+x).$$

Since $g_n(x)$ is a polynomial in x of degree $\leq n-1$, it has an expansion in terms of the polynomial basis $\left(\binom{x+k-1}{k-1}\right)_{k \geq 1}$:

$$g_n(x) = \sum_{k=1}^{n} A_{n,k} \binom{x+k-1}{k-1}$$

Surprisingly, the coefficients $A_{n,k}$ are the numbers in the refined alternating sign matrix theorem.

How does the proof proceed?

The following three properties of $\alpha(n;k_1,\ldots,k_n)$

- $\alpha(n;k_1,k_2,k_3,\ldots,k_n) = (-1)^{n-1}\alpha(n;k_2,\ldots,k_n,k_1-n)$
- $\alpha(n;k_1,k_2,\ldots,k_n) = \alpha(n;-k_n,-k_{n-1},\ldots,-k_1)$
- $\alpha(n;k_1,\ldots,k_n) = \alpha(n;k_1+t,\ldots,k_n+t) \quad (t \in \mathbb{Z})$

...imply the following symmetry property of $g_n(x)$:

$$\begin{aligned} g_n(x) &= \alpha(n; 1, 2, \ldots, n-1, n+x) \\ &= (-1)^{n-1}\alpha(n; 2n+x, 1, 2, \ldots, n-1) \\ &= (-1)^{n-1}\alpha(n; -n+1, -n+2, \ldots, -1, -2n-x) \\ &= (-1)^{n-1}\alpha(n; 1, 2, \ldots, n-1, -n-x) \\ &= (-1)^{n-1}g_n(-2n-x) \end{aligned}$$

Therefore...

$$\sum_{k=1}^{n} A_{n,k}\binom{x+k-1}{k-1} = g_n(x)$$

$$= (-1)^{n-1} g_n(-2n-x) = (-1)^{n-1}\sum_{j=1}^{n} A_{n,j}\binom{-x-2n+j-1}{j-1}$$

By the Vandermonde summation the right-hand side is equal to

$$(-1)^{n-1}\sum_{j=1}^{n} A_{n,j}\sum_{k=1}^{j}(-1)^{k-1}\binom{x+k-1}{k-1}\binom{-2n+j}{j-k}$$

and, by comparing coefficients, this implies, for $1 \le k \le n$,

$$A_{n,k} = \sum_{j=k}^{n} A_{n,j}(-1)^{n-k}\binom{-2n+j}{j-k} = \sum_{j=k}^{n} A_{n,j}(-1)^{n+j}\binom{2n-k-1}{j-k}.$$

This is a system of linear equations for the numbers $A_{n,k}$!

$n = 4$:

$$\begin{pmatrix} -1 & 6 & -15 & 20 \\ 0 & 1 & -5 & 10 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A_{n,1} \\ A_{n,2} \\ A_{n,3} \\ A_{n,4} \end{pmatrix} = \begin{pmatrix} A_{n,1} \\ A_{n,2} \\ A_{n,3} \\ A_{n,4} \end{pmatrix}$$

Unfortunately, it does not determine the $A_{n,k}$'s uniquely! Not even up to a constant, as the (algebraic and geometric) multiplicity of the eigenvalue 1 is 2.

This can be repaired!

$$A_{n,k} = \sum_{j=k}^{n} A_{n,j} (-1)^{n+j} \binom{2n-k-1}{j-k}$$

Fact. $A_{n,j} = A_{n,n+1-j}$ (reflect ASM along the vertical axis)

Replace j by $n+1-j$ and use the fact to see that

$$A_{n,k} = \sum_{j=1}^{n} A_{n,j} (-1)^{j+1} \binom{2n-k-1}{n-j-k+1} \quad \text{for } 1 \le k \le n.$$

Proposition. This system of linear equations together with $A_{n,1} = A_{n-1} = \sum_{k=1}^{n-1} A_{n-1,k}$ determines the numbers $A_{n,k}$ uniquely (inductively with n).

Proof. It suffices to show that the geometric multiplicity of the eigenvalue 1 of the matrix $\left((-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\right)_{1\le i,j\le n}$ is 1. Equivalently: the rank of

$$\left((-1)^j\binom{2n-i-1}{n-i-j+1}+\delta_{i,j}\right)_{1\le i,j\le n}$$

is $n-1$.

This follows after we have showed that

$$\det_{2\le i,j\le n}\left((-1)^j\binom{2n-i-1}{n-i-j+1}+\delta_{i,j}\right)\neq 0.$$

By conjugating the matrix, this determinant is equal to a determinant which was computed by Andrews to compute the number of descending plane partitions. $\square$

Incidentally, there is the same number of descending plane partitions with largest part less than or equal to n as there is of $n\times n$ alternating sign matrices.

Three years later, Dan Romik observed that this all generalizes to a certian doubly–refined enumeration...

Consider

$$g_n(x,y) = \alpha(n; 1, 2, \ldots, n-2, n-1+x, n+y)$$

and its expansion

$$g_n(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{n,i,j} \binom{x+i-1}{i-1} \binom{y+j}{j-1}.$$

A monotone (d, n)–trapezoid is a monotone triangle with n rows where the first $d-1$ rows were removed. For instance,

$$\begin{array}{ccccccccc} & & 2 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 6 & \\ 1 & & 2 & & 3 & & 5 & & 6 \end{array}$$

is a monotone $(3, 5)$–trapezoid.

Theorem (F., Romik). If $1 \leq i < j \leq n$ then the coefficient $A_{n,i,j}$ is the number of monotone $(2, n)$–trapezoids with bottom row $1, 2, \ldots, n$ and top row i, j.

In terms of ASMs...

The coefficient $A_{n,i,j}$ enumerates $n \times n$ ASMs with respect to the two top rows: k is fixed, lies in $\{i, i+1, \ldots, j-1, j\}$ and is the position of the unique 1 in the top row.

$$\begin{array}{cccccc} & \text{i} & & \text{k} & \text{j} & \\ \end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ & & & \vdots & & \end{pmatrix}$$

$$\begin{array}{cccccc} & \text{i=k} & & & \text{j} & \\ \end{array}$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ & & & \vdots & & \end{pmatrix}$$

$$\begin{array}{cccccc} & \text{i} & & & \text{k=j} & \\ \end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ & & & \vdots & & \end{pmatrix}$$

System of linear equations for the numbers $A_{n,i,j}$

A symmetry property for $g_n(x,y)$

$$\begin{aligned} g_n(x,y) &= \alpha(n;1,2,\dots,n-2,n-1+x,n+y)\\ &= \alpha(n;-n-y,-n+1-x,-n+2,-n+3,\dots,-1)\\ &= (-1)^{n-1}\alpha(n;-n+1-x,-n+2,-n+3,\dots,-1,-2n-y)\\ &= \alpha(n;-n+2,-n+3,\dots,-1,-2n-y,-2n+1-x)\\ &= \alpha(n;1,2,\dots,n-2,n-1+(-2n-y),n+(-2n-x))\\ &= g_n(-2n-y,-2n-x) \end{aligned}$$

leads to the following system of linear equations

$$A_{n,i,j}=\sum_{p=i}^{n}\sum_{q=j}^{n}(-1)^{p+q}\binom{2n-i-2}{p-i}\binom{2n-j-2}{q-j}A_{n,q,p}$$

for $1\le i,j\le n$.

Near–symmetry of the numbers $A_{n,i,j}$

Theorem (F., Romik). The numbers $A_{n,i,j}$ satisfy the symmetry property

$$A_{n,i,j} = A_{n,n+1-j,n+1-i}$$

for all $1 \le i, j \le n$, except when $(i,j) = (n-1, 1)$ or $(i,j) = (n,2)$, in which case we have

$$A_{n,n-1,1} = A_{n,n,2} + A_{n-1}.$$

!! In this case the proof does not follow from the combinatorial interpretation of $A_{n,i,j}$ as we do not have one if $i \ge j$!! (Is there any?)

The sufficiency conjecture

Conjecture (F., Romik). The linear equations given above together with the near symmetry and the special values $A_{n,i,n} = A_{n-1,i}$ determine the numbers $A_{n,i,j}$ uniquely.

One strategy for proving this conjecture is to compute the determinant of a reduced system of linear equations and to show that it is non–zero. Surprisingly, the determinant can conjecturally be represented by a simple product formula (this is due to Krattenthaler). In principal, the latter conjecture could be attacked by Krattenthaler's "identifications of factors" method for computing determinants.

A conjectural formula for $A_{n,i,j}$

Conjecture.

$$A_{n,i,j} = \frac{A_{n-1}(2n-2-i)!(2n-2-j)!(n+i-3)!(n+j-3)!}{(3n-5)!(n-2)!(i-1)!(j-1)!(n-i)!(n-j)!}$$
$$\Big(n+j-i-1+(2+2i+i^2-3j-ij+j^2-2n-2in+jn+n^2)$$
$$\times \lim_{j'\to j} \sum_{k=0}^{\infty} \left(\frac{\binom{3k-3j'+4}{k}\binom{2j'+i-2k-5}{i-1-k}}{\binom{k-j'+i}{i-1}(k-j'+3-n)} - \frac{\binom{3k-3i+4}{k}\binom{2i+j'-2k-5}{i-1-k}(i-1-k)}{\binom{k-i+j'}{i}(k-i+3-n)i} \right)\Bigg)$$

The sufficiency conjecture reduces the proof of this conjecture to certain hypergeometric identities.

A proved formula for $A_{n,i,j}$

Karklinsky and Romik used the six-vertex model approach and Stroganov's formula for the number of $n \times n$ ASMs with given top and bottom rows to show that

$$A_{n,i,j} = \sum_{p=0}^{n-j} \sum_{q=0}^{p} (-1)^q \binom{p}{q} X_n(i+q, j+p),$$

where

$$X_n(s,t) = \frac{1}{A_{n-1}} \Big(A_{n-1,t}(A_{n,s+1} - A_{n,s}) - A_{n-1,s}(A_{n,t+1} - A_{n,t}) \Big).$$

Much more general...

For non–negative integers c, d with $c + d \leq n$, we consider the following expansion:

$$\alpha(n; k_1, \dots, k_c, c+1, c+2, \dots, n-d, k_{n-d+1}, k_{n-d+2}, \dots, k_n)$$
$$= \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \dots \sum_{s_c=1}^{n} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \dots \sum_{i_d=1}^{n} A(n; s_1, s_2, \dots, s_c; i_1, \dots, i_d)$$
$$\times (-1)^{s_1+s_2+\dots+s_c+c} \binom{k_1 - c - 1}{s_c - 1} \binom{k_2 - c - 1}{s_{c-1} - 1} \dots \binom{k_c - c - 1}{s_1 - 1}$$
$$\times \binom{k_{n-d+1} - n + d - 2 + i_1}{i_1 - 1} \binom{k_{n-d+2} - n + d - 2 + i_2}{i_2 - 1} \dots \binom{k_n - n + d - 2 + i_d}{i_d - 1}$$

Theorem (F). For $s_1 < s_2 < \cdots < s_c$ and $i_1 < i_2 < \cdots < i_d$ the coefficient $A(n; s_1, \ldots, s_c; i_1, \ldots, i_d)$ is the number of monotone $(d, n-c)$–trapezoids with $(i_1, \ldots, i_d)$ as top row and whose bottom row consists of the numbers in $\{1, 2, \ldots, n\} \setminus \{s_1, s_2, \ldots, s_c\}$, arranged in increasing order.

Example:

$$A(9; 2, 5, 8; 1, 4, 5, 8) = \# \begin{array}{ccccccccccc} & & 1 & & 4 & & 5 & & 8 & & \\ & ? & & ? & & ? & & ? & & ? & \\ 1 & & 3 & & 4 & & 6 & & 7 & & 9 \end{array}$$

In terms of ASMs...

A (t, n)–partial ASM is a $t \times n$ matrix with entries in $\{0, 1, -1\}$ such that the non–zero entries alternate in each row and column and the row sums are 1.

Example. $\begin{pmatrix} 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$

That is: the restriction on the columnsums has been removed – $1, -1, 0$ are possible columnsums in a (t, n)–partial ASM.

Now: $A(n; s_1, \ldots, s_c; i_1, \ldots, i_d)$ is the number of $(n-c-d, n)$–partial ASMs such that the j–th columnsum is

- 1 iff $j \notin \{i_1, \ldots, i_d, s_1, \ldots, s_c\}$,

- -1 iff $j \in \{i_1, \ldots, i_d\} \cap \{s_1, \ldots, s_c\}$,

- 0 and the first non–zero entry in column j is -1 iff $j \in \{i_1, \ldots, i_d\}$ and

- 0 and the first non–zero entry in column j is 1 iff $j \in \{s_1, \ldots, s_c\}$.

Towards another proof of a formula for $A_{n,i,j}$

The identity

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n)$$

implies

$$A(n; s_1, \ldots, s_c; i_1, \ldots, i_d) = \sum_{i_{d+1}=s_c}^{n} \sum_{i_{d+2}=s_{c-1}}^{n} \cdots \sum_{i_{d+t}=s_{c-t+1}}^{n} A(n; s_1, \ldots, s_{c-t}; i_1, \ldots, i_{d+t})$$
$$\times (-1)^{s_{c-t+1}+\ldots+s_c+tn} \binom{-2n+d-1+i_{d+1}+c}{i_{d+1}-s_c} \cdots \binom{-2n+d-1+i_{d+t}+c}{i_{d+t}-s_{c-t+1}}$$

for $1 \le i_1, i_2, \ldots, i_d \le n$.

Special case $t = 1, c = 2, d = 0$:

$$A(n; s_1, s_2; -) = \sum_{i_1 = s_2}^{n} (-1)^{n+i_1} \binom{2n-2-s_2}{i_1 - s_2} A(n; s_1; i_1)$$

On the one hand: $A_{n,i,j} = A(n; i, j; -) = A(n; -; i, j)$

On the other hand: $A(n; i; j) =$ Number of alternating sign matrices $M = (m_{p,q})_{1 \le p,q \le n}$ with $m_{1,i} = 1$ and $m_{n,j} = 1$.

Stroganov (2002) has derived an explicit formula for the latter. Therefore, we have a formula for the first!

Another proved formula for $A_{n,i,j}$

The number of monotone $(2,n)$–trapezoids with bottom row $1,2,\ldots,n$ and top row i,j is given by

$$A_{n,i,j} = \frac{1}{A_{n-1}} \sum_{k=j}^{n} \sum_{l=1}^{i} (-1)^{n+k} \binom{2n-2-j}{k-j} \times \Big(A_{n-1,l-1} A_{n,k-i+l} - A_{n-1,l-1} A_{n,k-i+l-1} + A_{n-1,k-i+l-1} A_{n,l} - A_{n-1,k-i+l-1} A_{n,l-1} \Big)$$

where $A_{n,k}$ is the number of $n \times n$ alternating sign matrices that have a 1 in the first row and k-th column.

Perspectives:
where to go from here?

Further project: Formulas for $A(n; s_1, \ldots, s_c; i_1, \ldots, i_d)$ – at least when c and d are small.

• It suffices to consider the case $c = 0$.

• System of linear equations: for all $i_1, i_2, \ldots, i_d \in \{1, 2, \ldots, n\}$ we have

$$A(n; -, i_1, \ldots, i_d)$$
$$= \sum_{j_1=i_1, j_2=i_2, \ldots, j_d=i_d}^{n} (-1)^{dn+j_1+j_2+\ldots+j_d} A(n; -; j_d, j_{d-1}, \ldots, j_1) \prod_{l=1}^{d} \binom{2n - i_l - d}{j_l - i_l}.$$

• The system is again dependent! Does the near-symmetry generalize and does it provide a characterization of the coefficients $A(n; -; i_1, \ldots, i_d)$? If yes: Cramer's rule implies a determinantal formula.

Obviously: as is there only one 1 in the top row, there is also only one 1 in the first column, in the bottom row and in the last column!

What is about the enumeration of ASMs with respect to

• the first row and first column: seems to be expressible in terms of $A(n; -; i_1, i_2)$.

• the first row, first column and bottom row: is likely to be expressible in terms of $A(n; -; i_1, i_2, i_3)$.

• the first row, first column, bottom row and last column: is possibly expressible in terms of $A(n; -; i_1, i_2, i_3, i_4)$.

Does this extend to symmetry classes?

Vertically symmetric alternating sign matrices: operator formula for "halved monotone triangles":

$$\left(\prod_{1 \le p < q \le n/2} E_{k_p} (\mathrm{id} - E_{k_q} + E_{k_q} E_{k_p}^{-1})(\mathrm{id} - E_{k_q}^{-1} + E_{k_q}^{-1} E_{k_p}^{-1}) \right)$$

$$\prod_{1 \le i < j \le n/2} \frac{(k_j - k_i)(2x + 2 - k_i - k_j)}{(j-i)(j+i)} \prod_{i=1}^{n/2} \frac{(x + 1 - k_i)}{i}$$

Conjecture (F). The number of $(2n+1) \times (2n+1)$ vertically symmetric alternating sign matrices, where the first 1 in the second row is in the i-th column is

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i-2)!} \left(\prod_{j=1}^{n} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!} \right).$$

Essential to this approach: Translate properties of $\alpha(n;k_1,\dots,k_n)$ to identities for the refined enumeration numbers.

Examples.

$$A_{n,k}=\sum_{j=k}^{n} A_{n,j}(-1)^{n+j}\binom{2n-k-1}{j-k}$$

$$\begin{aligned}A_{2n,1,2}+A_{2n,3,4}+\dots+A_{2n,2n-1,2n}&\\ =A_{2n,2,3}+A_{2n,4,5}+\dots+A_{2n,2n-2,2n-1}&\end{aligned}$$

Bijective explanations of such identities would teach us something about the inner structure of ASMs!

Call for applications...

...for one doctoral position and two postdoctoral positions in combinatorics at the Faculty of Mathematics of the University of Vienna.

Anyone interested should consult my webpage:

http://www.mat.univie.ac.at/~ ifischer/