Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order

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Introduction: ASMs and DASASMs

ASMs = Alternating sign matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

Theorem (Zeilberger, 1995).

$$\# \text{ of } n \times n \text{ ASMs} = \frac{1!4!7! \cdots (3n-2)!}{n!(n+1)! \cdots (2n-1)!} = \prod_{i=1}^{n-1} \frac{\binom{3i+1}{i}}{\binom{2i}{i}}$$

DASASMs = Diagonally and antidiagonally symmetric alternating sign matrices of odd order

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Conjecture (Robbins 1980s). The number of $(2n+1) \times (2n+1)$ DASASMs is

$$\prod_{i=1}^{n} \frac{\binom{3i}{i}}{\binom{2i-1}{i}}.$$

This conjecture has recently been settled by Behrend, Konvalinka and myself.

Behavior along diagonals and antidiagonals

An $n \times n$ DASASM $A = (a_{i,j})$ is uniquely determined by its fundamental triangle $\{a_{i,j} | 1 \leq i \leq (n+1)/2, i \leq j \leq n+1-i\}$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$\alpha \in \{-1, 0, 1\}$: $\mathrm{N}_\alpha(A) = \#$ of α's along the diagonal and the antidiagonal in the fundamental triangle.

$$\mathrm{N}_{-1}(A) = 2, \qquad \mathrm{N}_0(A) = 4, \qquad \mathrm{N}_1(A) = 1$$

Bounds for $\mathsf{N}_\alpha(A)$

Proposition. Suppose A is a $(2n+1)\times(2n+1)$ DASASM. Then the statistics $\mathsf{N}_\alpha(A)$ range in the following intervals.

- $n \leq \mathsf{N}_0(A) \leq 2n$
- $0 \leq \mathsf{N}_1(A) \leq n+1$
- $0 \leq \mathsf{N}_{-1}(A) \leq n$

All inequalities are sharp.

Back in Summer 2012: Arvind and I discovered that for four out of the six inequalities, the numbers of DASASMs where equality is attained are equal to numbers that have previously appeared in plane partition or alternating sign matrix counting.

Main results: Number of extreme DASASMs

Upper bounds for $\mathsf{N}_\alpha(A)$ when $\alpha = \pm 1$

of $(2n+1) \times (2n+1)$ DASASMs with $\mathsf{N}_{-1}(A) = n$
$= 1, 1, 2, 7, 42, 429, 7436, 218348, \ldots$

(OEIS): alternating sign matrices of order n

of $(2n+1) \times (2n+1)$ DASASMs with $\mathsf{N}_{1}(A) = n+1$
$= 1, 2, 5, 20, 132, 1452, 26741, 826540, \ldots$

OEIS: cyclically symmetric plane partitions in an $n \times n \times n$ box

Bounds for $\mathrm{N}_0(A)$

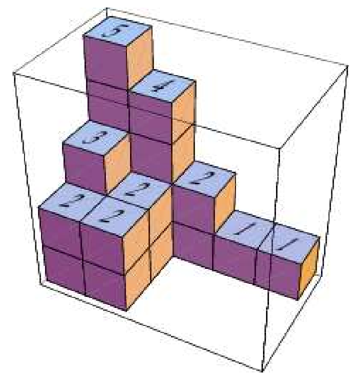
\# of $(2n+1)\times(2n+1)$ DASASMs with $\mathrm{N}_0(A)=n$
$= 1, 2, 7, 42, 429, 7436, 218348, 10850216, \ldots$

OEIS: alternating sign matrices of order $n+1$

\# of $(2n+1)\times(2n+1)$ DASASMs with $\mathrm{N}_0(A)=2n$
$= 1, 1, 2, 6, 33, 286, 4420, 109820, \ldots$

OEIS: vertically and horizontally symmetric alternating sign matrices of order $2n+3$

What is a cyclically symmetric plane partition?

What is a plane partition? 

A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$$

with

$$(i, j, k) \in PP \Rightarrow (i', j', k') \in PP \quad \forall (i', j', k') \leq (i, j, k).$$

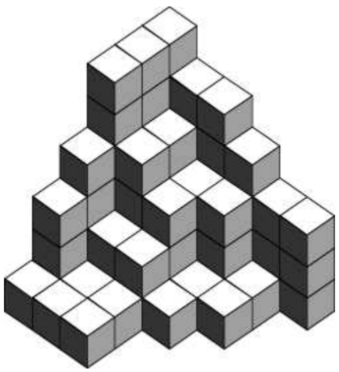
Cyclically symmetric plane partitions=CSPPs

An $n \times n \times n$ PP is cyclically symmetric if

$$(i, j, k) \in PP \Rightarrow (j, k, i) \in PP.$$

In 1979, George Andrews proved that the number of $n \times n \times n$ cyclically symmetric plane partitons is

$$\prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$



Main results

Theorem 1. The number of $(2n+1)\times(2n+1)$ DASASMs with $\mathrm{N}_{-1}(A)=n$ is equal to the total number of $n\times n$ ASMs.

Theorem 2. The number of $(2n+1)\times(2n+1)$ DASASMs with $\mathrm{N}_{1}(A)=n+1$ is equal to the number of CSPPs in an $n\times n\times n$ box.

Theorem 3. The number of $(2n+1)\times(2n+1)$ DASASMs with $\mathrm{N}_{0}(A)=n$ is equal to the total number of $(n+1)\times(n+1)$ ASMs.

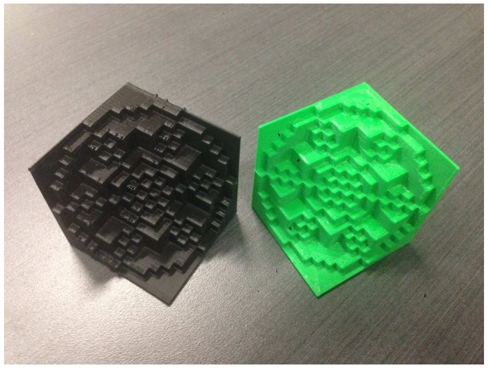
Theorem 4. The number of $(2n+1)\times(2n+1)$ DASASMs with $\mathrm{N}_{0}(A)=2n$ is equal to the number of $(2n+3)\times(2n+3)$ vertically and horizontally symmetric ASMs.

Remarks

• Our proofs are not bijective proofs. Thus each theorem gives rise to a new open problem, namely to find a bijective proof!

• In one case we have been able to establish refinements in the sense that we have identified pairs of statistics that have the same distribution. This should help in finding bijections.

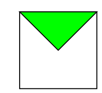
• Plane partitions do not only "appear" in Theorem 2: Totally symmetric self-complementary PPs =TSSCPPs are plane partitions that are invariant under permutation of the three axes and that are equal to their complement. There is the same number of TSSCPPs in a $2n \times 2n \times 2n$ box as there is of $n \times n$ ASMs.

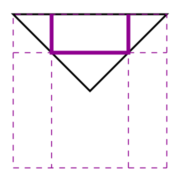


ASTs, QASTs and odd order OOSASMs

DASASM-triangles

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- It suffices to know the entries in the fundamental triangle: 
- Conversely, a triangular array is the fundamental triangle of a DASASM if 1's and -1's alternate and add up to 1 along paths of the following type:



DASASM of order $2n+1 \Rightarrow$ DASASMs-triangle of order n.

ASTs = Alternating sign triangles

An AST of order n is a triangular array with n centered rows of length $2n-1, 2n-3, \ldots, 1$ such that
(1) the non-zero entries alternate in each row and each column,
(2) all row sums are 1, and
(3) the top most entry of each column is 1 (if it exists).

ASTs of order 3:

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{array} \Bigg| \begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{array} \Bigg| \begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{array} \Bigg| \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & \\ & & 1 & & \end{array}$$

$$\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & \\ & & 1 & & \end{array} \Bigg| \begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & \\ & & 1 & & \end{array} \Bigg| \begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ & 1 & -1 & 1 & \\ & & 1 & & \end{array}$$

• Let A be a DASASM-triangle of order n. Then $\mathrm{N}_{-1}(A) = n$, iff, after deleting the boundary elements of each row, we obtain an AST of order n.

• Conversely, given an ASTs of order n, there is a unique way to add boundary elements such that the resulting triangular array A is a DASASM-triangle of order n with $\mathrm{N}_{-1}(A) = n$: place a -1 below each column with sum 1, and a 0 otherwise.

QASTs = Quasi alternating sign triangles

A QAST of order n is a triangular array with n centered rows of length $2n-1, 2n-3, \ldots, 1$ such that
(1) the non-zero entries alternate in each row and each column,
(2) the row sums are 1 for rows $1, 2, \ldots, n-1$, and 0 or 1 for row n, and
(3) the top most entry of each column is 1 (if it exists).

There is a simple bijection between QASTs of order n and DASASMs-triangle A of order n with $\mathrm{N}_1(A) = n + 1$.

$$\begin{array}{ccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 1 & -1 & 1 & 0 & 0 & \\
 & & 0 & 0 & 1 & & \\
 & & & 0 & & &
\end{array}
\quad\Big|\quad
\begin{array}{ccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & -1 & 1 & 0 & 0 & \\
 & & 0 & 0 & 0 & & \\
 & & & -1 & & &
\end{array}
\quad\Big|\quad
\begin{array}{ccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & 0 & 1 & & \\
 & & & 1 & -1 & 1 & & & \\
 & & & & 1 & & & &
\end{array}$$

• Decrease elements by 1 on the left and right boundary whenever it is possible.

• Add elements on the left and right boundary so that 1's and -1's alternate along the special paths.

$\mathrm{N}_0(A) = n$ and $\mathrm{N}_0(A) = 2n$

$\mathrm{N}_0(A) = n$: A DASASM-triangle A of order n satisfies $\mathrm{N}_0(A) = n$ iff we obtain an AST of order n when replacing all -1's on the left and right boundary of each row with a zero. This is a bijection, and so Theorem 1 and Theorem 3 are equivalent.

$\mathrm{N}_0(A) = 2n$: Kuperberg introduced off-diagonally and off-antidiagonally symmetric ASMs (OOSASMs) as DASASMs of order $4n$ that have only zeros on the diagonal and antidiagonal. No formula for the plain enumeration of OOSASMs is currently known (but he derived a formula for the partition function).

Such DASASMs do not exist for other orders!

Odd order DASASMs A with $\mathrm{N}_0(A) = 2n$ are as close as one can get to OOSASMs, since the central entry of an odd order DASASM has to be non-zero. Thus, we call them odd order OOSASMs, and we were able to enumerate them!

Doubly refined enumeration of ASTs

Number of -1's in ASMs and ASTs

Let A be an ASM or an AST. Then we define

$$\mu(A) = \#\text{ of } -1\text{'s in } A.$$

Obviously

$$|\{A \in \mathrm{ASM}(n) \mid \mu(A) = 0\}| = n! = |\{A \in \mathrm{AST}(n) \mid \mu(A) = 0\}|.$$

Generalization of Theorem 1: Let m, n be non-negative integers. Then

$$|\{A \in \mathrm{ASM}(n) \mid \mu(A) = m\}| = |\{A \in \mathrm{AST}(n) \mid \mu(A) = m\}|.$$

Inversion numbers

Let $\pi = (\pi_1, \ldots, \pi_n)$ be a permutation and A be the permutation matrix of π, that is π_i is the column of the unique 1 in row i. Then

$$\mathrm{inv}(A) = \sum_{1 \le i' < i \le n, 1 \le j' \le j \le n} a_{i'j} a_{ij'}$$

is the number of inversions in π. We define the inversion number of an ASMs accordingly.

Let $A = (a_{i,j})_{1 \le i \le n, i \le j \le 2n-i}$ be an AST. We define

$$\mathrm{inv}(A) = \sum_{i' < i, j' \le j} a_{i'j} a_{ij'}.$$

Generalization of the generalization of Theorem 1: Let m, n, i be non-negative integers. Then

$$|\{A \in \mathsf{ASM}(n) \mid \mu(A) = m, \mathsf{inv}(A) = i\}| \\ = |\{A \in \mathsf{AST}(n) \mid \mu(A) = m, \mathsf{inv}(A) = i\}|.$$

Refine even more: What is a statistic on ASTs that corresponds to the column of the unique 1 in the top row of an ASM?

Column of the unique 1 in the top row of an ASM

Refined ASM-Theorem

Observation: There is a unique 1 in the top row of an ASM.

Theorem (Zeilberger, 1996): The number of $n \times n$ ASMs with a 1 in the top row and column r is

$$\binom{n+r-2}{n-1}\frac{(2n-r-1)!}{(n-r)!}\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n+j)!} = A_{n,r}.$$

Find a statistic on ASTs that has the same distribution as the column of the 1 in the first row of an ASM!

Statistic on ASTs

In an AST, the elements of a column add up to 0 or 1. We say that a column is a 1-column if they add up to 1.

Let T be an AST with n rows. Define

$$\rho(T) = (\#\text{1-columns in the left half of } T \text{ that have a 1 at the bottom}) \\ + (\#\text{1-columns in the right half of } T \text{ that have a 0 at the bottom}) + 1.$$

Conjecture (Behrend 2015). The number of ASTs T with n rows and $\rho(T) = r$ is equal to $A_{n,r}$.

A constant term expression

Theorem (Fischer 2016). Define

$$P_n(X_1,\ldots,X_{n-1}) = \sum_{0\le i_1<i_2<\ldots<i_{n-1}\le 2n-3} X_1^{-i_1} X_2^{-i_2} \cdots X_{n-1}^{-i_{n-1}}.$$

The constant term of

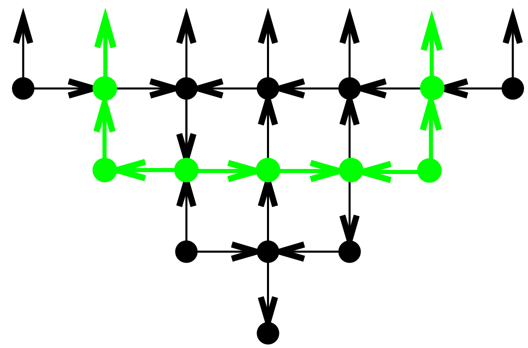
$$P_n(X_1,\ldots,X_{n-1}) t^{-r} \prod_{i=1}^{n-1} (t+X_i) \prod_{1\le i<j\le n-1} (1+X_i+X_iX_j)(X_j-X_i)$$

in the variables $X_1, X_2, \ldots, X_{n-1}, t$ is equal to the number of ASTs T with n rows and $\rho(T) = r$.

Triangular six vertex configurations

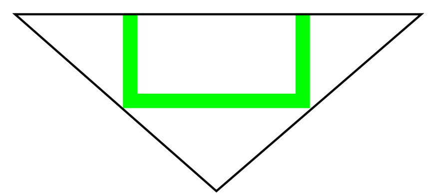
DASASM-triangle → 6-vertex configuration

$$\begin{array}{ccccccccccccc} 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 \\ & & & 1 & & -1 & & 0 & & 1 & & 0 & \\ & & & & 0 & & 1 & & -1 & & & & \\ & & & & & & -1 & & & & & & \end{array}$$



Transformation:

1) Top edges are oriented upward.

2) Work through all paths of type  :

- Along straight lines, change orientation iff ± 1.
- As for turns, change orientation iff 0.

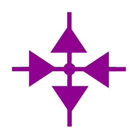
Triangular 6-vertex configuration

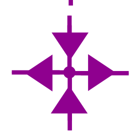
We obtain orientations of triangular graphs s.t.

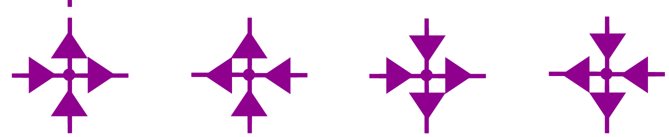
- all degree 4 vertices are “balanced”,
- all top edges are oriented upward.

1-1-correspondence with DASASMs

Dictionary:

$1 \leftrightarrow$ 

$-1 \leftrightarrow$ 

$0 \leftrightarrow$ 

Proofs

Weighted enumeration

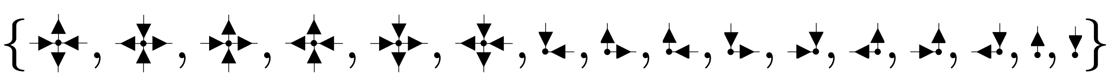
- Assign to each vertex v a weight $\mathsf{W}(v)$.

- Weight $\mathsf{W}(C)$ of a configuration C: $\mathsf{W}(C) = \prod_{v \in C} \mathsf{W}(v)$

- Generating function=partition function:

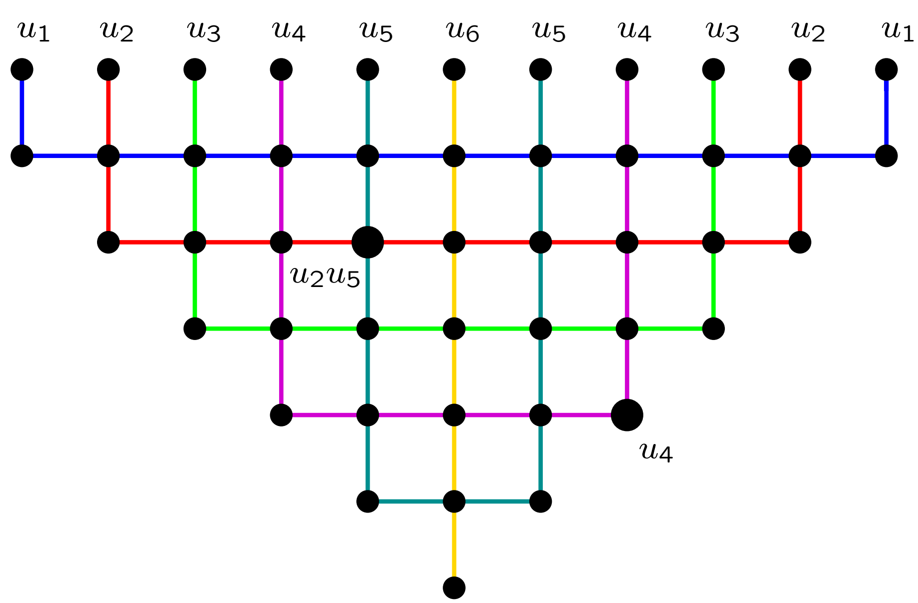
$$Z_n = \sum_{C \text{ 6-vertex configuration}} \mathsf{W}(C)$$

- Specialize parameters to obtain the number of extreme DASASMs.

- The vertex weights $\mathsf{W}(c, u)$ depend on the orientations c of the surrounding edges, i.e. $c \in \{$  $\}$, and the label u of the vertex.

Label of a vertex

Each colored path is assigned a parameter u_i as follows.

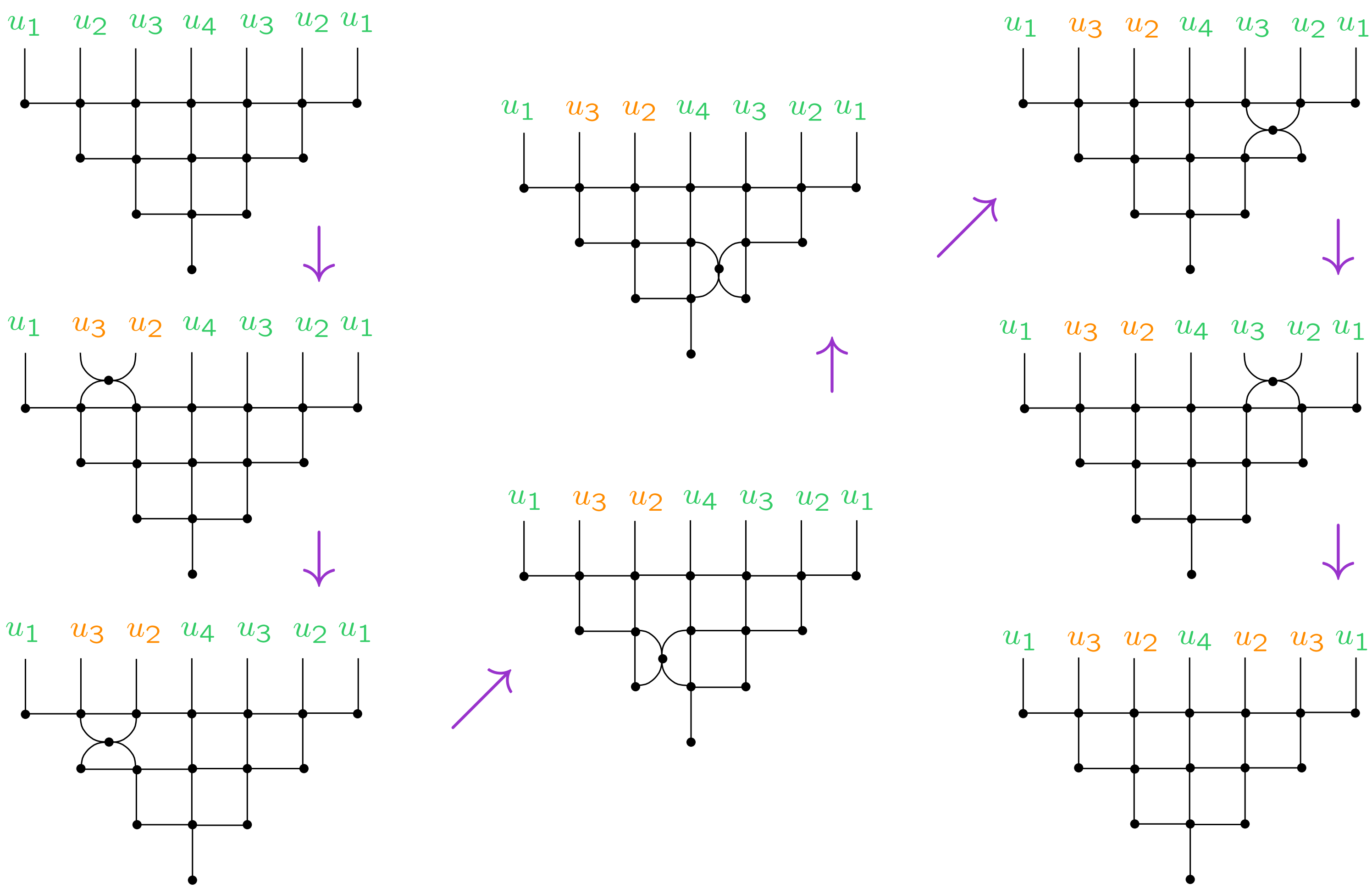


• A degree 4 vertex is contained in two colored paths u_i and u_j $\Rightarrow$ label u_iu_j

• All boundary vertices have a unique path u_i $\Rightarrow$ label u_i

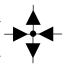
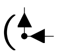
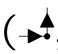
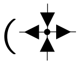
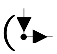
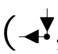
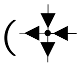
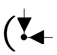
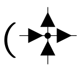
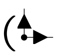
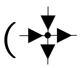
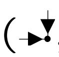
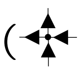
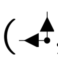
Generating function: $Z_n(u_1, \ldots, u_n; u_{n+1})$.

Symmetry in $u_1, u_2, \ldots, u_n$



Vertex weights

Notation: $x^{-1} = \bar{x}$ and $\sigma(x) = x - \bar{x}$

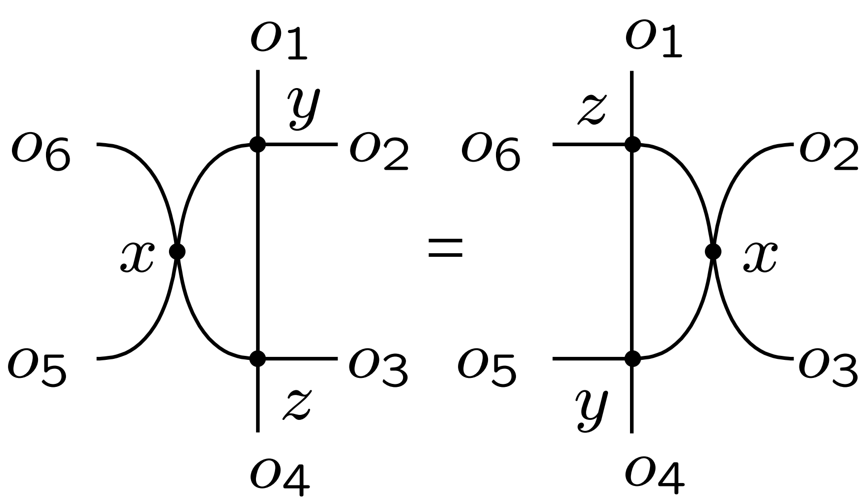
Bulk weights	Left boundary weights	Right boundary weights
W(, u) $= 1$	W(, u) $= \frac{\beta_L\, qu + \gamma_L\, \bar{q}\bar{u}}{\sigma(q^2)}$	W(, u) $= \frac{\beta_R\, q\bar{u} + \gamma_R\, \bar{q}u}{\sigma(q^2)}$
W(, u) $= 1$	W(, u) $= \frac{\gamma_L\, qu + \beta_L\, \bar{q}\bar{u}}{\sigma(q^2)}$	W(, u) $= \frac{\gamma_R\, q\bar{u} + \beta_R\, \bar{q}u}{\sigma(q^2)}$
W(, u) $= \frac{\sigma(q^2 u)}{\sigma(q^4)}$	W(, u) $= \alpha_L \frac{\sigma(q^2 u^2)}{\sigma(q^2)}$	
W(, u) $= \frac{\sigma(q^2 u)}{\sigma(q^4)}$	W(, u) $= \delta_L \frac{\sigma(q^2 u^2)}{\sigma(q^2)}$	
W(, u) $= \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		W(, u) $= \alpha_R \frac{\sigma(q^2 \bar{u}^2)}{\sigma(q^2)}$
W(, u) $= \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		W(, u) $= \delta_R \frac{\sigma(q^2 \bar{u}^2)}{\sigma(q^2)}$

Degree 1 vertices have weight 1.

If $u = 1$ and $q = e^{i\pi/6}$, all bulk weights are 1!

Yang-Baxter equation

Theorem. If $xyz = q^2$ and $o_1, o_2, \ldots, o_6 \in \{\text{in}, \text{out}\}$, then

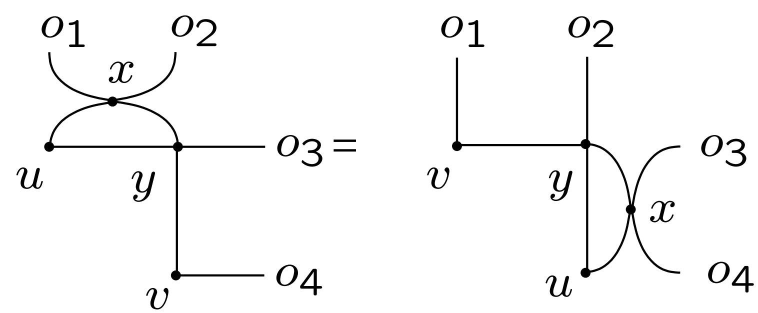


.

A diagram stands for the generating function of all orientations of the graph such that the external edges have the prescribed orientations $o_1, o_2, \ldots, o_6$, degree 4 vertices are balanced, and the vertex weights are as given in the table, where the letter close to a vertex indicates its label (rotate until the label is in the SW corner).

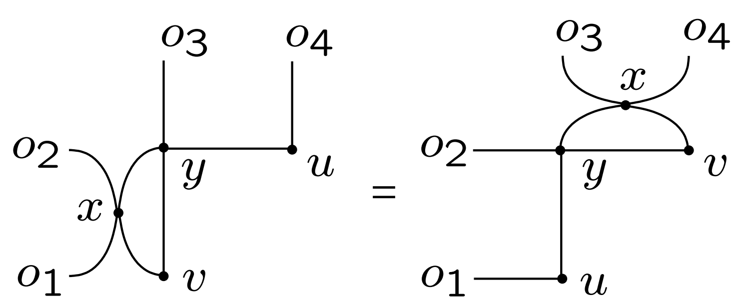
Reflection equations

Theorem. Suppose $o_1, o_2, o_3, o_4 \in \{\text{in}, \text{out}\}$. If $x = q^2\bar{u}v$ and $y = uv$, then



,

and if $x = q^2\bar{u}v$ and $y = \bar{u}\bar{v}$, then

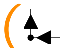
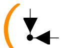
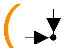


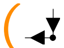
.

$\Rightarrow$ Symmetry of $Z_n(u_1, \ldots, u_n; u_{n+1})$ in $u_1, \ldots, u_n$.

Specialization of the boundary constants $\alpha_L, \beta_L, \gamma_L, \delta_L, \alpha_R, \beta_R, \gamma_R, \delta_R$ for ASTs

We need to have

W(, 1) = W(, 1) = W(, 1) = W(, 1) = 0,

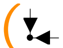
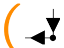
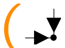
W(, 1) = W(, 1) = W(, 1) = W(, 1) = 1,

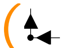
and this is satisfied if we specialize as follows.

α_L	β_L	γ_L	δ_L	α_R	β_R	γ_R	δ_R
0	$-\bar{q}$	q	1	0	$-\bar{q}$	q	1

Specialization of the boundary constants $\alpha_L, \beta_L, \gamma_L, \delta_L, \alpha_R, \beta_R, \gamma_R, \delta_R$ for QASTs

We need to have

W( , 1) = W( , 1) = W( , 1) = W( , 1) = 0,

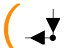
W( , 1) = W( , 1) = W( , 1) = W( , 1) = 1,

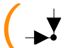
and this is satisfied if we specialize as follows.

α_L	β_L	γ_L	δ_L	α_R	β_R	γ_R	δ_R
0	q	$-\bar{q}$	1	0	q	$-\bar{q}$	1

Specialization of the boundary constants $\alpha_L, \beta_L, \gamma_L, \delta_L, \alpha_R, \beta_R, \gamma_R, \delta_R$ for odd OOSASMs

We need to have

W(, 1) = W(, 1) = W(, 1) = W(, 1) = 0,

W(, 1) = W(, 1) = W(, 1) = W(, 1) = 1,

and this is satisfied if we specialize as follows.

$\alpha_L = \alpha_R$	$\beta_L = \beta_R$	$\gamma_L = \gamma_R$	$\delta_L = \delta_R$
1	0	0	1

AST partition function

The AST partition function of order n is

$$\frac{\prod_{j=1}^{n}\sigma(u_j)\,\prod_{i=1}^{n}\prod_{j=1}^{n+1}\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(q^2)^{2n}\,\sigma(q^4)^{n(n-1)}\,\prod_{i=1}^{n}\sigma(u_i\bar{u}_{n+1})\,\prod_{1\le i<j\le n}\sigma(u_i\bar{u}_j)^2}$$
$$\times\ \det_{1\le i,j\le n+1}\left(\begin{cases}\frac{1}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}, & i\le n\\ 1-\frac{\sigma(\bar{u}_{n+1}u_j)}{\sigma(u_j)}, & i=n+1\end{cases}\right).$$

The specialization at $q=e^{\frac{i\pi}{6}}$ is

$$3^{-\binom{n+1}{2}}\prod_{i=1}^{n}(u_i^2+1+\bar{u}_i^2)\,s_{(n-1,n-1,n-2,n-2,\dots,1,1)}(u_1^2,\bar{u}_1^2,\dots,u_n^2,\bar{u}_n^2).$$

QAST partition function

The QAST partition function of order n is

$$\frac{\prod_{j=1}^{n}\sigma(u_j)\prod_{i=1}^{n}\prod_{j=1}^{n+1}\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(q^2)^{2n}\sigma(q^4)^{n^2}\prod_{i=1}^{n}\sigma(u_i\bar{u}_{n+1})\prod_{1\le i<j\le n}\sigma(u_i\bar{u}_j)^2}$$

$$\times\ \det_{1\le i,j\le n+1}\left(\begin{cases}\frac{1}{\sigma(q^2u_iu_j)}+\frac{1}{\sigma(q^2\bar{u}_i\bar{u}_j)}, & i\le n\\ \frac{\sigma(u_{n+1})}{\sigma(u_j)}, & i=n+1\end{cases}\right).$$

The specialization at $q=e^{\frac{i\pi}{6}}$ is

$$3^{-\binom{n+1}{2}}\prod_{i=1}^{n}(u_i^2+1+\bar{u}_i^2)s_{(n,n-1,n-1,n-2,n-2,\dots,1,1)}(u_1^2,\bar{u}_1^2,\dots,u_n^2,\bar{u}_n^2).$$

Odd OOSASM partition function

Let

$$P_m(u_1,\ldots,u_{2m}) = \sigma(q^4)^{-(m-1)2m} \prod_{1\le i<j\le 2m} \frac{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(u_i\bar{u}_j)} \times \operatorname*{Pf}_{1\le i<j\le 2m}\left(\frac{\sigma(u_i\bar{u}_j)}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}\right)$$

and

$$Q_m(u_1,\ldots,u_{2m-1}) = \sigma(q^4)^{-(m-1)(2m-1)} \prod_{1\le i<j\le 2m-1} \frac{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(u_i\bar{u}_j)} \times \operatorname*{Pf}_{1\le i<j\le 2m}\left(\begin{cases}\frac{\sigma(u_i\bar{u}_j)}{\sigma(q^2u_iu_j)} + \frac{\sigma(u_i\bar{u}_j)}{\sigma(q^2\bar{u}_i\bar{u}_j)}, & j<2m\\ 1, & j=2m\end{cases}\right).$$

The odd OOSASM partition function of order $2n+1$ is

$$P_{\lceil \frac{n}{2} \rceil}(u_1, \ldots, u_{2\lceil \frac{n}{2} \rceil})\, Q_{\lceil \frac{n+1}{2} \rceil}(u_1, \ldots, u_{2\lceil \frac{n+1}{2} \rceil - 1}).$$

The specialization at $q = e^{\frac{i\pi}{6}}$ can be expressed in terms of symplectic characters.

Curious fact: $P_n(u_1, \ldots, u_{2n})$ is the partition function of $2n \times 2n$ off-diagonally symmetric ASMs. Moreover, there is the same number of $2n \times 2n$ off-diagonally symmetric ASMs as there is of $(2n+1) \times (2n+1)$ vertically symmetric ASMs.

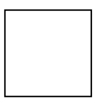
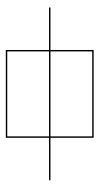
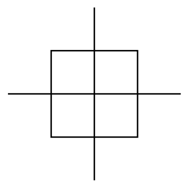
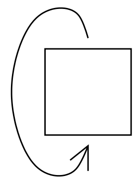
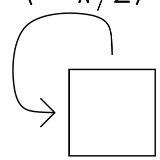
How does one transform a rectangular alternating sign array into a triangular alternating sign array ?

Symmetry classes of ASMs

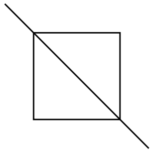
Stanley, Robbins, 1980s: Symmetry group of the square

$$\mathcal{D}_4 = \{\mathcal{I}, \underbrace{\mathcal{V}, \mathcal{H}, \mathcal{D}, \mathcal{A}}_{\text{reflections}}, \underbrace{\mathcal{R}_{\pi/2}, \mathcal{R}_{\pi}, \mathcal{R}_{-\pi/2}}_{\text{rotations}}\}$$

10 subgroups, 8 conjugacy classes

$\{\mathcal{I}\}$	$\langle\mathcal{V}\rangle \sim \langle\mathcal{H}\rangle$	$\langle\mathcal{V}, \mathcal{H}\rangle$	$\langle\mathcal{R}_{\pi}\rangle$	$\langle\mathcal{R}_{\pi/2}\rangle$
				
Zeilberger 1995	Kuperberg 2002	Okada 2004	n even: Kuperberg 2002 n odd: Razumov & Stroganov 2005	n even: Kuperberg 2002 n odd: Razumov & Stroganov 2005

$\langle\mathcal{D}\rangle \sim \langle\mathcal{A}\rangle$

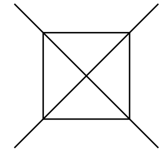


n	#
1	1
2	2
3	5
4	2^4
5	67
6	$2^4 \cdot 23$
7	$2 \cdot 5 \cdot 263$
8	$2^3 \cdot 11 \cdot 277$

$\mathcal{D}_4$

n	#
1	1
2	0
3	1
4	0
5	1
6	0
7	2
8	0
9	2^2
10	0
11	13
12	0
13	$2 \cdot 23$
14	0
15	$2^3 \cdot 31$
16	0
17	$2^2 \cdot 379$

$\langle\mathcal{D}, \mathcal{A}\rangle$



	n	#
$\rightarrow$	1	1
	2	2
$\rightarrow$	3	3
	4	2^3
$\rightarrow$	5	$3 \cdot 5$
	6	$2^2 \cdot 13$
$\rightarrow$	7	$2 \cdot 3^2 \cdot 7$
	8	$2^3 \cdot 71$
$\rightarrow$	9	$2 \cdot 3^4 \cdot 11$
	10	$2^2 \cdot 2609$
$\rightarrow$	11	$3^3 \cdot 11^2 \cdot 13$
	12	$2^3 \cdot 31 \cdot 1303$
$\rightarrow$	13	$2 \cdot 3^2 \cdot 11 \cdot 13^2 \cdot 17$