

An alternative approach to alternating sign matrices

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Outline

- ASMs: definition, λ -determinant, square ice
- Monotone triangles with prescribed bottom row
- ASMs with general boundary conditions
- Linear relations for refined enumerations of ASMs
- Directed triangles ?

Alternating sign matrices

Square $0, 1, -1$ matrices such that in each row and each column

- the non-zero entries appear with alternating signs and
- the sum of entries is 1, i.e. the first and the last non-zero entry is a 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem (Zeilberger 1996). The number of $n \times n$ alternating sign matrices is

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} =: A_n.$$

The origin of ASMs: λ -determinant

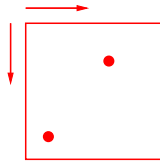
Leibniz formula for the determinant of a matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$:

$$\det(M) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{n,\sigma(n)}$$

or

$$\det(M) = \sum_{B \in \mathcal{P}_n} (-1)^{\mathcal{I}(B)} \prod_{i,j=1}^n m_{i,j}^{B_{i,j}}$$

where \mathcal{P}_n is the set of $n \times n$ permutation matrices and

$$\mathcal{I}(B) = \sum_{\substack{(i_1,j_1),(i_2,j_2) \\ i_1 > i_2, j_1 < j_2}} b_{i_1,j_1} b_{i_2,j_2}$$


is the inversion number of B .

The Desnanot–Jacobi identity...

$$\det \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \times \det \left(\begin{array}{|c|} \hline \text{border} \\ \hline \end{array} \right) = \det \left(\begin{array}{cc} \det \left(\begin{array}{|c|} \hline \text{border} \\ \hline \end{array} \right) & \det \left(\begin{array}{|c|} \hline \text{border} \\ \hline \end{array} \right) \\ \det \left(\begin{array}{|c|} \hline \text{border} \\ \hline \end{array} \right) & \det \left(\begin{array}{|c|} \hline \text{border} \\ \hline \end{array} \right) \end{array} \right)$$

...can be used to see that $n \times n$ determinants are expressible in terms of 2×2 determinants. (Condensation of determinants, Charles L. Dodgson, 1866)

3 × 3 determinants

$$\det \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{pmatrix} = \frac{1}{m_{2,2}}$$
$$\times \det \begin{pmatrix} \det \begin{pmatrix} m_{2,2} & m_{2,3} \\ m_{3,2} & m_{3,3} \end{pmatrix} & \det \begin{pmatrix} m_{2,1} & m_{2,2} \\ m_{3,1} & m_{3,2} \end{pmatrix} \\ \det \begin{pmatrix} m_{1,2} & m_{1,3} \\ m_{2,2} & m_{2,3} \end{pmatrix} & \det \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix} \end{pmatrix}$$

4 × 4 determinants

$$\det \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{pmatrix} = \frac{1}{\det \begin{pmatrix} m_{2,2} & m_{2,3} \\ m_{3,2} & m_{3,3} \end{pmatrix}} \\
 \times \det \left(\begin{array}{c} \det \begin{pmatrix} m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,2} & m_{4,3} & m_{4,4} \end{pmatrix} \\ \det \begin{pmatrix} m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,2} & m_{3,3} & m_{3,4} \end{pmatrix} \end{array} \right) \det \left(\begin{array}{c} \det \begin{pmatrix} m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \\ m_{4,1} & m_{4,2} & m_{4,3} \end{pmatrix} \\ \det \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{pmatrix} \end{array} \right)$$

3 × 3 determinants are expressible in terms of 2 × 2 determinants...and so are 4 × 4 determinants!

Robbins and Rumsey in the 1980s: what happens if we generalize the definition of a 2×2 determinant to

$$\det_{\lambda} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21}$$

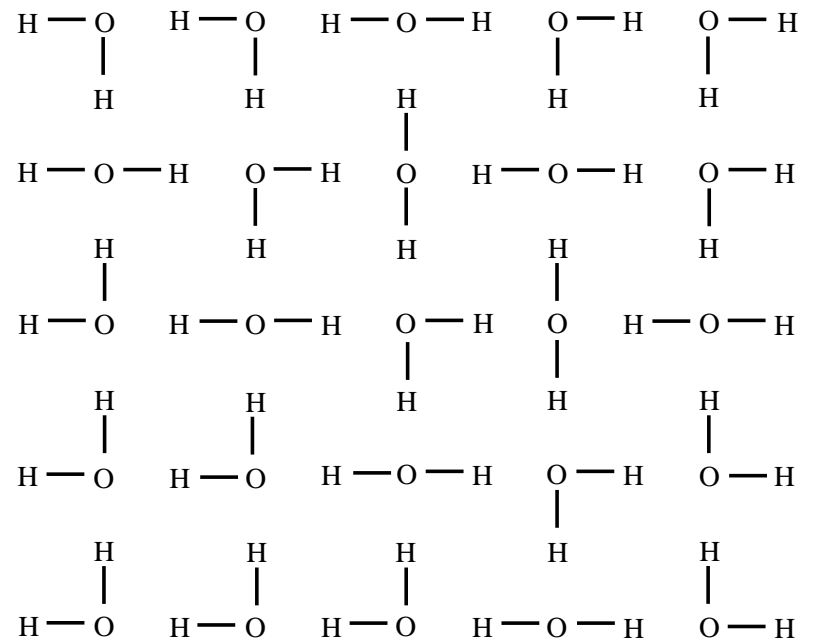
and, furthermore, use the previous observation to generalize the $n \times n$ determinant?

Theorem (Robbins and Rumsey). Let $M = (m_{i,j})$ be an $n \times n$ matrix, \mathcal{A}_n the set of $n \times n$ alternating sign matrices, $\mathcal{I}(B)$ the inversion number of B and $\mathcal{N}(B)$ the number of -1 s in B then

$$\det_{\lambda}(M) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^n m_{i,j}^{B_{i,j}}.$$

ASMs in statistical physics: square ice

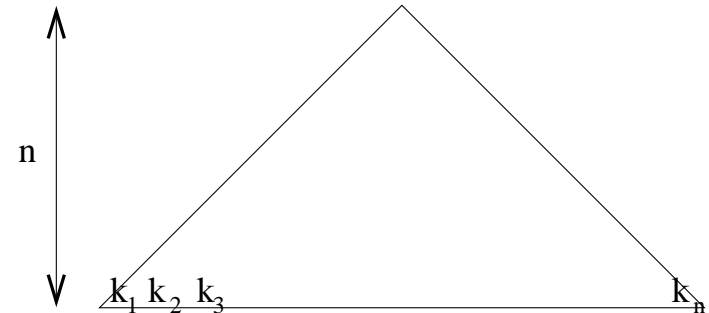
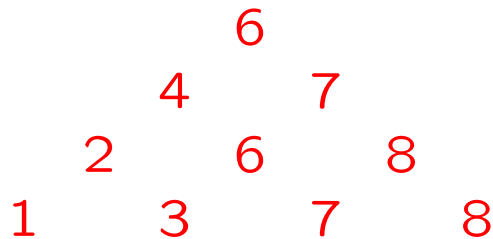
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Outline

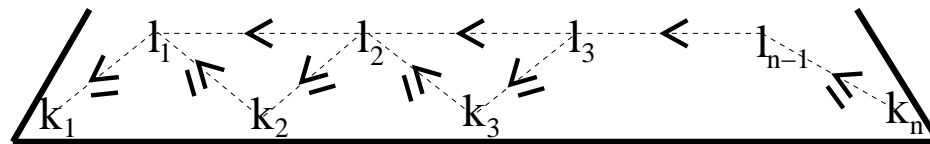
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Monotone triangles with arbitrary strictly increasing bottom row



$\alpha(n; k_1, \dots, k_n) = \#$ of monotone triangles with bottom row (k_1, \dots, k_n)

Advantage: recursive structure!



$$\alpha(n; k_1, k_2, \dots, k_n) = \sum_{\substack{k_1 \leq l_1 \leq k_2 \leq l_2 \leq k_3 \leq \dots \leq l_{n-1} \leq k_n \\ l_i < l_{i+1}}} \alpha(n-1; l_1, \dots, l_{n-1})$$

Operator formula for the number of monotone triangles

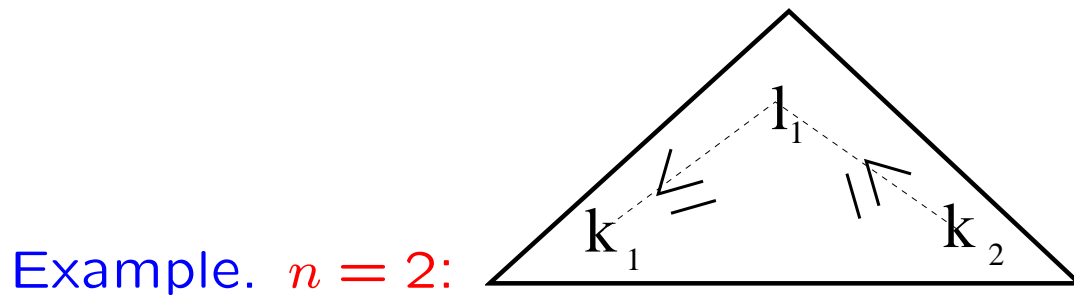
Theorem (2006). The number of monotone triangles with bottom row (k_1, \dots, k_n) is

$$\prod_{1 \leq p < q \leq n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i} =: \alpha(n; k_1, \dots, k_n),$$

where $E_x p(x) = p(x + 1)$.

How to use the formula:

- Treat k_i as a variable.
- Apply the product of operators to the polynomial $\prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}$.
- This gives a polynomial in (k_1, \dots, k_n) in which you may replace k_i by the desired integer.



$$\begin{aligned}
 & (E_{k_1} + E_{k_2}^{-1} - E_{k_1}E_{k_2}^{-1})(k_2 - k_1 + 1) \\
 &= E_{k_1}(k_2 - k_1 + 1) + E_{k_2}^{-1}(k_2 - k_1 + 1) - E_{k_1}E_{k_2}^{-1}(k_2 - k_1 + 1) \\
 &= k_2 - k_1 + 1 = \# \text{ of integers between } k_1 \text{ and } k_2
 \end{aligned}$$

Proof by induction

Need to show:

$$\begin{aligned}
 & \prod_{1 \leq p < q \leq n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i} \\
 &= \sum_{\substack{k_1 \leq l_1 \leq k_2 \leq l_2 \leq k_3 \leq l_3 \leq \dots \leq l_{n-1} \leq k_n \\ l_i < l_{i+1}}} \prod_{1 \leq p < q \leq n-1} (E_{l_p} + E_{l_q}^{-1} - E_{l_p} E_{l_q}^{-1}) \prod_{1 \leq i < j \leq n-1} \frac{l_j - l_i + j - i}{j - i}
 \end{aligned}$$

Summation is the discrete equivalent of integration!

Discrete equivalent of the fundamental theorem of calculus. Let $f : \mathbb{Z} \rightarrow \mathbb{C}$. If $F : \mathbb{Z} \rightarrow \mathbb{C}$ is a function such that $f(i) = \Delta_i F(i) := F(i+1) - F(i)$ then

$$\sum_{i=a}^b f(i) = F(b+1) - F(a).$$

A multivariate discrete equivalent – related to the recursion

Let $a : \mathbb{Z}^{n-1} \rightarrow \mathbb{C}$. If $A : \mathbb{Z}^{n-1} \rightarrow \mathbb{C}$ is a function with

$$a(l_1, \dots, l_{n-1}) = \Delta_{l_1} \cdots \Delta_{l_{n-1}} A(l_1, \dots, l_{n-1})$$

and, for all $i \in \{1, 2, \dots, n-2\}$,

$$\begin{aligned} & A(l_1, \dots, l_{i-1}, l_i - 1, l_i - 1, l_{i+2}, \dots, l_{n-1}) \\ & \quad + A(l_1, \dots, l_{i-1}, l_i, l_i, l_{i+2}, \dots, l_{n-1}) \\ & \quad - A(l_1, \dots, l_{i-1}, l_i - 1, l_i, l_{i+2}, \dots, l_{n-1}) = 0 \end{aligned}$$

then

$$\begin{aligned} \sum_{\substack{k_1 \leq l_1 \leq k_2 \leq l_2 \leq k_3 \leq \dots \leq l_{n-1} \leq k_n \\ l_i < l_{i+1}}} a(l_1, \dots, l_{n-1}) &= A(k_2 + 1, k_3 + 1, \dots, k_n + 1) \\ &- A(k_1, k_3 + 1, \dots, k_n + 1) + A(k_1, k_2, k_4 + 1, \dots, k_n + 1) \\ &\quad - \dots + (-1)^{n-1} A(k_1, k_2, \dots, k_{n-1}). \end{aligned}$$

How to compute the discrete antiderivative of a polynomial

Trick: use the binomial basis $\binom{x}{k}_{k \geq 0}$ instead of $x^k_{k \geq 0}$:

$$p(x) = a_0 \binom{x}{0} + a_1 \binom{x}{1} + a_2 \binom{x}{2} + \dots + a_n \binom{x}{n}$$

Antiderivative of $p(x)$:

$$P(x) = a_0 \binom{x}{1} + a_1 \binom{x}{2} + a_2 \binom{x}{3} + \dots + a_n \binom{x}{n+1}$$

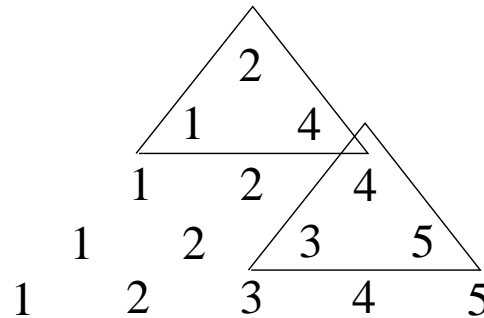
Proof:

$$\Delta_x \binom{x}{k+1} = \binom{x+1}{k+1} - \binom{x}{k+1} = \binom{x}{k}$$

A generalization: the number of -1 s in an ASM

How is a -1 in the ASM reflected in the monotone triangle?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Weight: $Q^{\#}$ of (-1) -triangles

Theorem (2009). The weighted enumeration of monotone triangles with bottom row (k_1, k_2, \dots, k_n) is given by

$$\prod_{1 \leq p < q \leq n} (E_{k_p} + E_{k_q}^{-1} + (Q - 2)E_{k_p}E_{k_q}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}.$$

Operator formula for halved monotone triangles

A halved monotone triangle is a triangular array of integers of shape

$$\begin{array}{ccccccc} & & & & & & a_{1,1} \\ & & & & & a_{2,1} & \\ & & & a_{3,1} & & a_{3,2} & \\ & & a_{4,1} & & a_{4,2} & & \\ & a_{5,1} & & a_{5,2} & & a_{5,3} & \\ a_{6,1} & & a_{6,2} & & a_{6,3} & & \end{array}$$

which is monotone increasing in ↗-direction and ↘-direction and strictly increasing along rows.

Theorem (2007). The number of halved monotone triangles with n rows, bottom row $(k_1, \dots, k_{\lceil n/2 \rceil})$ and where no entry exceeds x is equal to

$$\prod_{1 \leq p < q \leq (n+1)/2} \left(E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1} \right) \left(E_{k_p}^{-1} + E_{k_q}^{-1} - \text{id} \right) \\ \prod_{1 \leq i < j \leq (n+1)/2} \frac{(k_j - k_i + j - i)(2x - 1 - k_i - k_j - i - j)}{(j - i)(j + i - 1)}$$

if n is odd and equal to

$$\prod_{1 \leq p < q \leq n/2} \left(E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1} \right) \left(E_{k_p}^{-1} + E_{k_q}^{-1} - \text{id} \right) \\ \prod_{1 \leq i < j \leq n/2} \frac{(k_j - k_i + j - i)(2x - k_i - k_j - i - j)}{(j - i)(j + i)} \prod_{i=1}^{n/2} \frac{x - k_i - i}{i}$$

if n is even.

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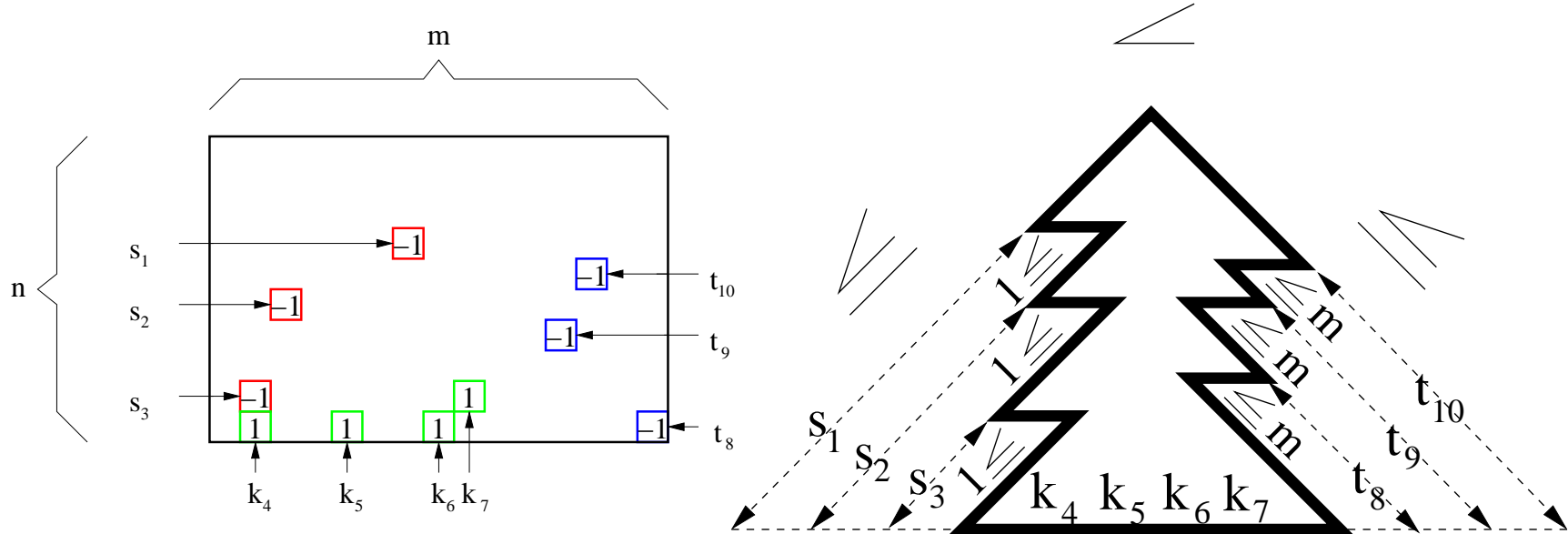
General boundary conditions: left, right and bottom

Loosen the requirement that the first/last non-zero entry in each row/column is 1 ...

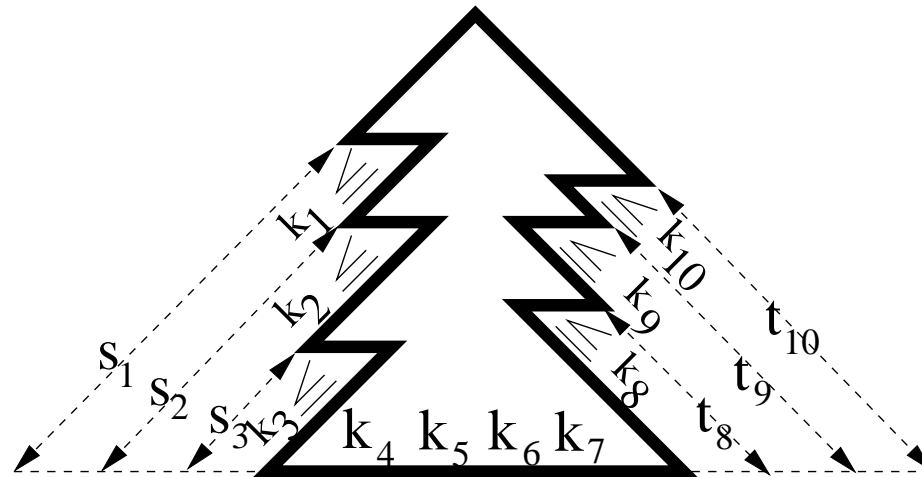
	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
	0	0	0	0	0	0	1	0	0	-1	1	0	0	0	0
	0	0	0	0	0	1	0	0	0	0	-1	0	1	0	0
s_1	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0
	0	0	1	0	0	-1	0	1	-1	0	1	0	-1	0	0
s_2	0	0	-1	1	0	0	0	-1	0	1	-1	1	0	0	0
	0	1	0	-1	0	1	0	0	0	0	0	-1	0	0	0
	0	0	1	0	0	-1	1	0	0	-1	0	0	1	0	0
s_3	0	-1	0	0	0	1	-1	0	1	0	0	0	-1	0	1
	0	1	-1	0	1	-1	0	1	0	0	0	0	0	0	-1
		k_4		k_5		k_6	k_7								
														t_8	
														t_9	
														t_{10}	

... with the exception of the first non-zero entry in the columns.

General boundary conditions: left, right and bottom



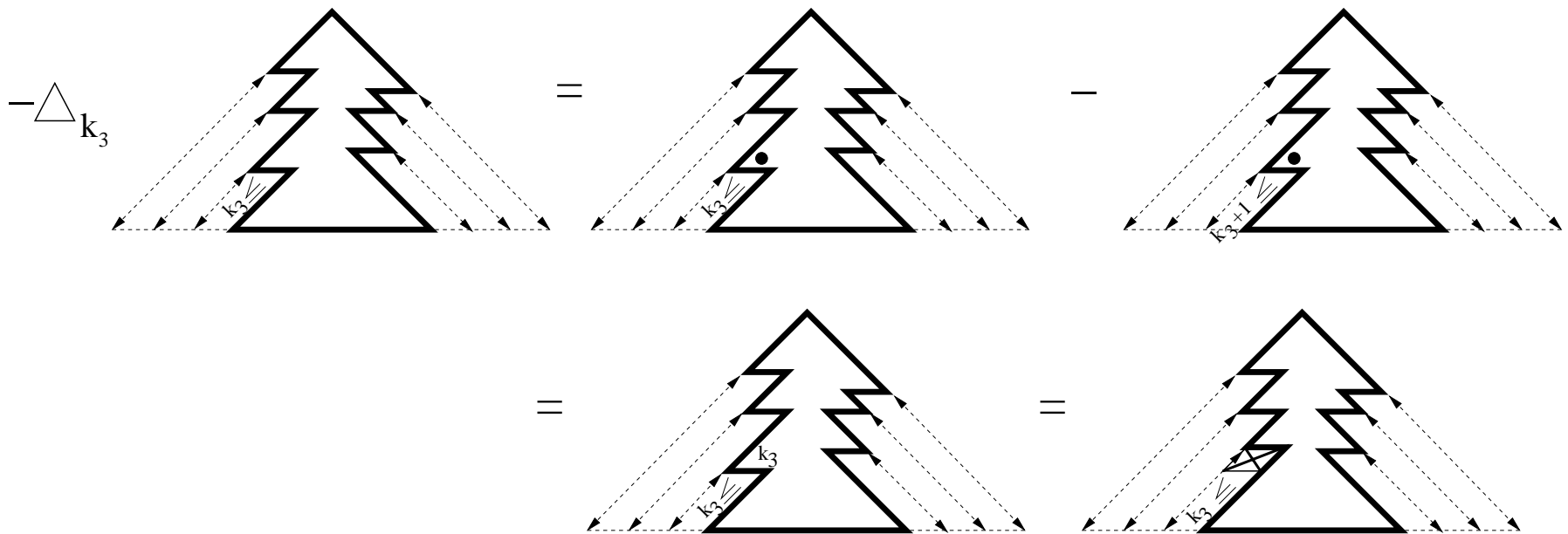
More general partial monotone triangles



Lower bound of the i -th NE-diagonal: $1 \rightarrow k_i$

Upper bound of the i -th SE-diagonal: $m \rightarrow k_i$

The negative forward difference $-\Delta_{k_i}$ truncates the i -th NE-diagonal.



Similar: the backward difference δ_{k_i} , where $\delta_x p(x) = p(x) - p(x - 1)$, truncates the i -th **SE-diagonal**.

To obtain a formula for more general partial monotone triangles:

- Start with a full monotone triangle
- Δ_{k_i} “eats” the NE-diagonals
- δ_{k_i} “eats” the SE-diagonals

Therefore: the number of more general partial monotone triangles is

$$(-\Delta_{k_1})^{s_1-1} (-\Delta_{k_2})^{s_2-1} \dots (-\Delta_{k_l})^{s_l-1} \delta_{k_{n-r+1}}^{t_{n-r+1}-1} \delta_{k_{n-r+2}}^{t_{n-r+2}-1} \dots \delta_{k_n}^{t_n-1} \alpha(n; k_1, \dots, k_n).$$

General boundary conditions: left, right and bottom

Theorem (2010). The number of ASMs with boundary conditions

$$(s_1, \dots, s_l), (k_{l+1}, \dots, k_{n-r}), (t_{n-r+1}, \dots, t_n)$$

is

$$\Delta_{k_1}^{s_1-1} \Delta_{k_2}^{s_2-1} \dots \Delta_{k_l}^{s_l-1} \delta_{k_{n-r+1}}^{t_{n-r+1}-1} \delta_{k_{n-r+2}}^{t_{n-r+2}-1} \dots \delta_{k_n}^{t_n-1} (-1)^{s_1+s_2+\dots+s_l+l} \\ \times \alpha(n; k_1, \dots, k_n) |_{k_1=k_2=\dots=k_l=1, k_{n-r+1}=k_{n-r+2}=\dots=k_n=m}.$$

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Strategy

Use identities for $\alpha(n; k_1, \dots, k_n)$ to deduce identities for enumerations of ASMs with general boundary conditions = refined enumerations of ASMs

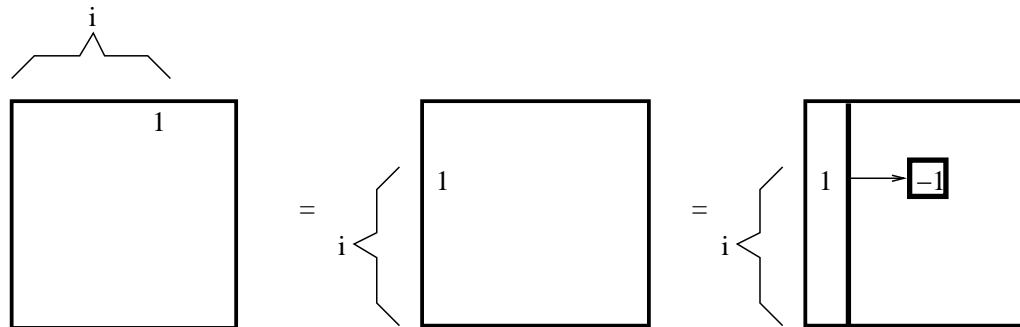
Identity 1.

$$\alpha(n; k_1, \dots, k_n) = (-1)^{n-1} \alpha(n; k_2, \dots, k_n, k_1 - n)$$

Refined alternating sign matrix theorem

Theorem (Zeilberger 1996). The number of $n \times n$ ASMs with a 1 in position $(1, i)$ is

$$\binom{n+i-2}{n-1} \frac{(2n-i-1)!}{(n-i)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} =: A_{n,i}.$$



Two expressions for $A_{n,i}$:

$$\begin{aligned} A_{n,i} &= (-1)^{i-1} \Delta_{k_1}^{i-1} \alpha(n; k_1, \dots, k_n) \Big|_{(k_1, \dots, k_n) = (1, 1, 2, \dots, n-1)} \\ &= \delta_{k_n}^{i-1} \alpha(n; k_1, \dots, k_n) \Big|_{(k_1, \dots, k_n) = (1, 2, 3, \dots, n-1, n-1)} \end{aligned}$$

Use Identity 1 to express one in terms of the other:

$$\begin{aligned} A_{n,i} &= (-1)^{i-1} \Delta_{k_1}^{i-1} \alpha(n; k_1, \dots, k_n) \Big|_{(k_1, \dots, k_n) = (1, 1, 2, \dots, n-1)} \\ &= (-1)^{i+n} \Delta_{k_1}^{i-1} \alpha(n; k_2, \dots, k_n, k_1 - n) \Big|_{(k_1, \dots, k_n) = (1, 1, 2, \dots, n-1)} \\ &= (-1)^{i+n} E_{k_1}^{-2n+2} \Delta_{k_1}^{i-1} \alpha(n; k_2, \dots, k_n, k_1) \Big|_{(k_2, \dots, k_n, k_1) = (1, 2, \dots, n-1, n-1)} \end{aligned}$$

Now $\Delta_x = E_x - \text{id} = E_x(\text{id} - E_x^{-1}) = E_x \delta_x$, which gives

$$(-1)^{i+n} E_{k_n}^{-2n+1+i} \delta_{k_n}^{i-1} \alpha(n; k_1, k_2, \dots, k_n) \Big|_{(k_1, \dots, k_{n-1}, k_n) = (1, 2, \dots, n-1, n-1)} .$$

Getting rid of the shift: binomial theorem

$$E_x^{-m} = (\text{id} - (\text{id} - E_x^{-1}))^m = (\text{id} - \delta_x)^m = \sum_{j=0}^m \binom{m}{j} (-1)^j \delta_x^j$$

Therefore

$$\begin{aligned} A_{n,i} &= \sum_{j=0}^{2n-1-i} \binom{2n-1-i}{j} (-1)^{i+j+n} \delta_{k_n}^{i+j-1} \alpha(n; k_1, k_2, \dots, k_n) |_{(k_1, \dots, k_{n-1}, k_n) = (1, 2, \dots, n-1, n-1)} \\ &= \sum_{j=0}^{2n-1-i} \binom{2n-1-i}{j} (-1)^{i+j+n} A_{n,i+j} = \sum_{k=i}^n \binom{2n-1-i}{k-i} (-1)^{k+n} A_{n,k} \\ &= \sum_{k=i}^n \binom{2n-1-i}{k-i} (-1)^{k+n} A_{n,n+1-k} \end{aligned}$$

for $1 \leq i \leq n$.

System of linear equations for $(A_{n,i})_{1 \leq i \leq n}$ – determines the numbers uniquely!

A doubly refined enumeration of ASMs: $\overline{\overline{A}}_{n,i,j}$

Situation for the two top rows of an ASM:

$$\begin{pmatrix} & i & & k & j & & \\ 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 1 & 0 & -1 & 1 & 0 & \\ & & & \vdots & & & \end{pmatrix} \quad \begin{pmatrix} & i=k & & j & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 0 & \\ & & & \vdots & & & \end{pmatrix} \quad \begin{pmatrix} & i & & k=j & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ & & & \vdots & & & \end{pmatrix}$$

$\Rightarrow i \leq k \leq j$; the choice of k has no influence on the rest of the matrix.

Let $\overline{\overline{A}}_{n,i,j}$ denote the number of $n \times n$ ASMs where the two top rows are fixed and i, j are as above.

Theorem (F. and Romik, 2009). There exists an extension of $\overline{\overline{A}}_{n,i,j}$ to all $1 \leq i, j \leq n$ such that

$$\overline{\overline{A}}_{n,i,j} = \sum_{p,q=1}^n (-1)^{p+q} \binom{2n-i-2}{p-i} \binom{2n-j-2}{q-j} \overline{\overline{A}}_{n,n+1-p,n+1-q} + r_{n,i,j},$$

where

$$r_{n,i,j} = A_{n-1}(-1)^n \left([i=1] \left(\binom{2n-2-j}{n-1-j} - (2n-3) \binom{2n-2-j}{n-j} \right) - [i=2] \binom{2n-2-j}{n-j} \right)$$

and $[statement] = 1$ iff the statement is true and zero otherwise.

Conjecture. The system of linear equations together with $\overline{\overline{A}}_{n,i,n} = A_{n-1,i}$ determines the numbers $\overline{\overline{A}}_{n,i,j}$ uniquely.

Extends to refined enumerations with respect to the d top rows and the c bottom rows!

Linear relations between different types of refined enumerations of ASMs

Identity 2. If $j \geq 1$ then

$$e_j(\Delta_{k_1}, \dots, \Delta_{k_n})\alpha(n; k_1, \dots, k_n) = 0,$$

where

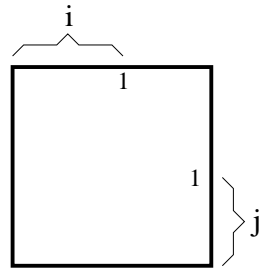
$$e_j(X_1, \dots, X_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n} X_{i_1} X_{i_2} \cdots X_{i_p}$$

is the j -th elementary symmetric function.

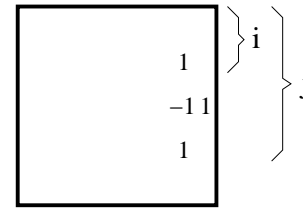
Example. $j = 1, n = 2$:

$$e_1(\Delta_{k_1}, \Delta_{k_2})\alpha(2; k_1, k_2) = (\Delta_{k_1} + \Delta_{k_2})(k_2 - k_1 + 1) = 0$$

Two doubly refined enumerations of ASMs



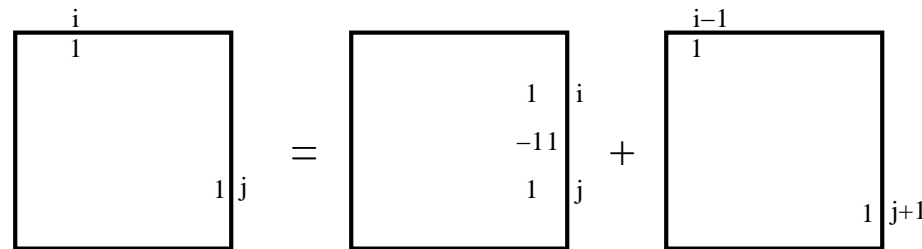
$$\overline{A}|_{n,i,j}$$



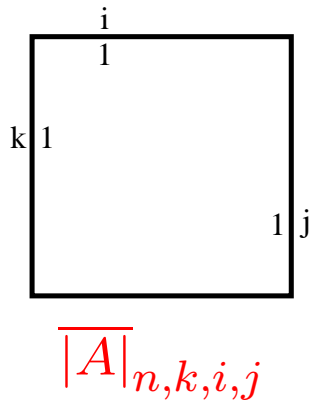
$$A||_{n,i,j} = \overline{\overline{A}}|_{n,i,j}$$

Theorem (2010). If $j \neq 1$ then

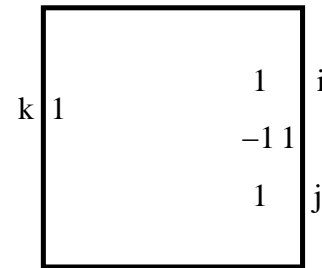
$$\overline{A}|_{n,i,j} = A||_{n,i,j} + \overline{A}|_{n,i-1,j+1}$$



Two triply refined enumerations of ASMs



$$\overline{|A|}_{n,k,i,j}$$



$$|A||_{n,k,i,j}$$

Theorem (2010). If $j, k \neq 1$ then

$$\begin{aligned} & \overline{|A|}_{n,k+1,i,j} + \overline{|A|}_{n,k+1,i,j+1} + \overline{|A|}_{n,k,i+1,j} \\ & = |A||_{n,k,i,j} + \overline{|A|}_{n,k+1,i-1,j+1} + \overline{|A|}_{n,k,i,j+1} + \overline{|A|}_{n,k+1,i+1,j} \end{aligned}$$

Outline

- ASMs: definition, λ -determinant, square ice
- Monotone triangles with prescribed bottom row
- ASMs with general boundary conditions
- Linear relations for refined enumerations of ASMs
- Directed triangles !

Less computations, more combinatorics!

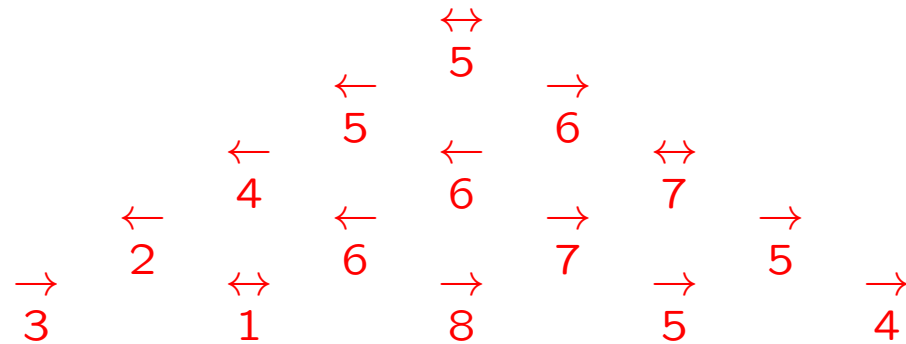
Combinatorial proof of

$$\alpha(n; k_1, \dots, k_n) = (-1)^{n-1} \alpha(n; k_2, \dots, k_n, k_1 - n)?$$

Problem: The right hand side has no combinatorial meaning if $k_1 < k_2 < \dots < k_n$ as $k_2, k_3, \dots, k_n, k_1 - n$ is not strictly increasing in this case.

Combinatorial interpretation of $\alpha(n; k_1, \dots, k_n)$ for all $(k_1, \dots, k_n) \in \mathbb{Z}^n$

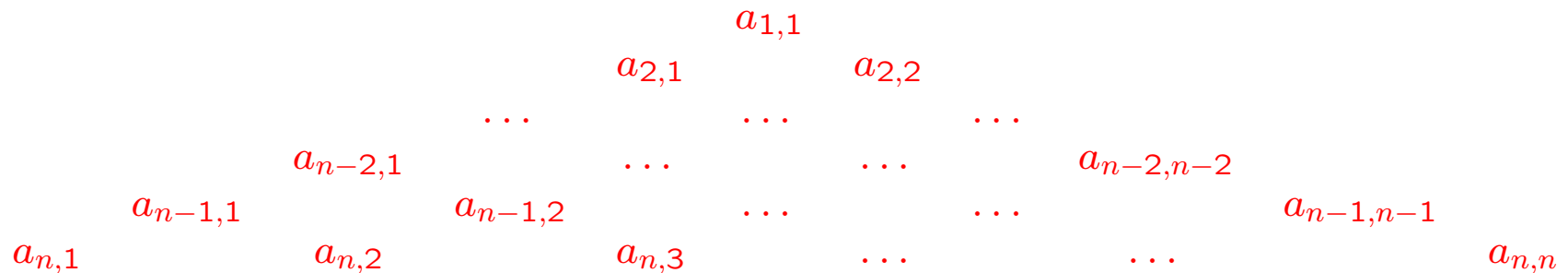
Example.



- $x \overset{y}{\prec} z$: each entry y lies (numerically) in between its SW-neighbor x and its SE-neighbor z .
- “Lying in between” is not precise: the arrows indicate whether the inequalities are strict or not.

Directed triangles

Triangular arrays of integers of the following shape



together with a function $f : \{a_{i,j}\} \rightarrow \{\leftarrow, \rightarrow, \leftrightarrow\}$, such that for all entries y that are not situated in the bottom row the following is fulfilled:

Four cases:

$$1. \quad \begin{array}{c} \leftarrow \\ x \end{array} \quad \begin{array}{c} y \\ \leftarrow, \leftrightarrow \\ z \end{array} : x \leq y < z \text{ or } x > y \geq z$$

$$2. \quad \begin{array}{c} \leftarrow \\ x \end{array} \quad \begin{array}{c} y \\ \rightarrow \\ z \end{array} : x \leq y \leq z \text{ or } x > y > z$$

$$3. \quad \begin{array}{c} \leftrightarrow, \rightarrow \\ x \end{array} \quad \begin{array}{c} y \\ \leftarrow, \leftrightarrow \\ z \end{array} : x < y < z \text{ or } x \geq y \geq z$$

$$4. \quad \begin{array}{c} \leftrightarrow, \rightarrow \\ x \end{array} \quad \begin{array}{c} y \\ \rightarrow \\ z \end{array} : x < y \leq z \text{ or } x \geq y > z$$

Rules:

- if (x, z) are “in order” and an arrow is pointing towards y then the respective inequality is strict.
- if (x, z) are not in order then

$$\leq \leftrightarrow > \text{ and } \geq \leftrightarrow < .$$

Signed enumeration

Whenever (x, z) are not “in order” (blue cases) then y is said to be an **inversion**. The sign of a **directed triangle** is

$$(-1)^{\# \text{ of inversions} + \# \text{ of } \leftrightarrow}.$$

Theorem (2011). The signed enumeration of directed triangles with bottom row k_1, \dots, k_n is

$$\prod_{1 \leq p < q \leq n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}.$$

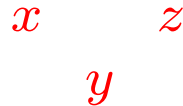
Thank you for your attention!

Remark

If $k_1 < k_2 < \dots < k_n$ then a directed triangle is not a monotone triangle!

But it is obvious that the signed enumeration of directed triangles gives the number of monotone triangles in this case:

- All rows are **strictly increasing** in this case.
- Situation for y :



If $f(y) = \leftarrow$ then $x < y \leq z$.

If $f(y) = \rightarrow$ then $x \leq y < z$.

If $f(y) = \leftrightarrow$ then $x < y < z$.

Precise definition of generalized ASMs

Boundary conditions:

- $S = \{s_1, s_2, \dots, s_l\} \subseteq \{1, 2, \dots, n\}$, strictly decreasing
- $K = \{k_{l+1}, \dots, k_{n-r}\} \subseteq \{1, 2, \dots, m\}$, strictly increasing
- $T = \{t_{n-r+1}, \dots, t_n\} \subseteq \{1, 2, \dots, n\}$, strictly increasing

rows in S are of type $-1, 1, -1, \dots$; the others are of type $1, -1, 1, \dots$

rows in T are of type $\dots, -1, 1, -1$; the others are of type $\dots, 1, -1, 1$

columns in K are of type $1, -1, \dots, -1, 1$; the others are of type $1, -1, \dots, 1, -1$

$n \times m$ matrices with entries in $\{0, 1, -1\}$, non-zero entries in rows and columns alternate in sign s.t.

- the first non-zero entry in each column is 1
- column sums are 1 precisely for the columns in K
- row sums are 1 for the rows in $\{1, 2, \dots, n\} \setminus S \cup T$
- row sums are -1 for the rows in $S \cap T$
- row sums are 0 and the first non-zero entry is -1 for the rows in S

Open problem

Bijjective proof of

$$\alpha(n; k_1, \dots, k_n) = \prod_{1 \leq p < q \leq n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}$$

if $k_1 < k_2 < \dots < k_n$.

Gelfand-Tsetlin pattern: triangular array of integers of the following shape

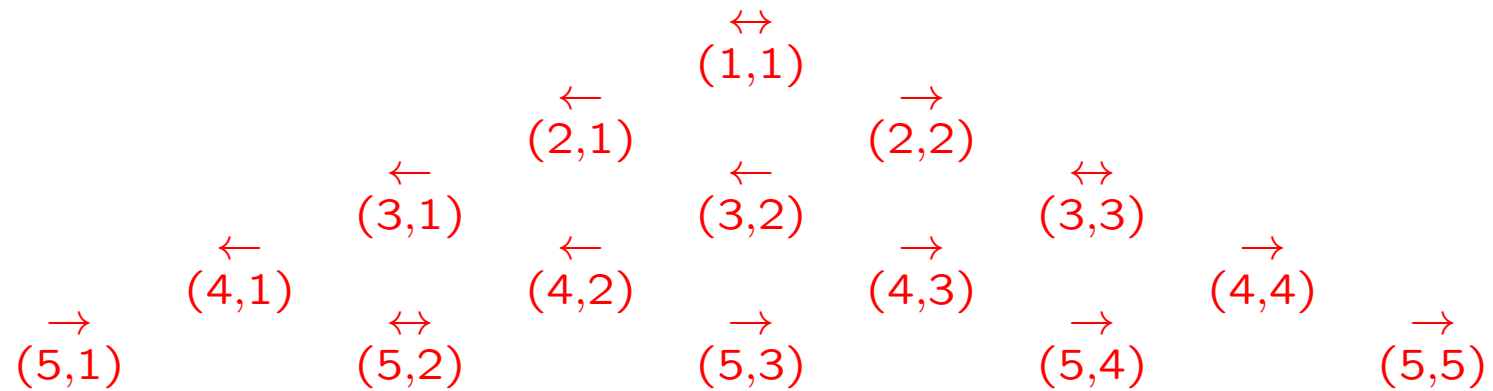
$$\begin{array}{cccccccc}
 & & & & & & & a_{1,1} \\
 & & & & & & a_{2,1} & & a_{2,2} \\
 & & & \dots & & \dots & & \dots & \\
 & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} \\
 & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} \\
 a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n}
 \end{array}$$

with weak increase in \nearrow -direction and in \searrow -direction, i.e. $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$ for all i, j .

of Gelfand-Tsetlin patterns with bottom row (k_1, k_2, \dots, k_n)

$$= \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}$$

Direction pattern: $p : \{(i, j) | 1 \leq j \leq i \leq n\} \rightarrow \{\leftarrow, \rightarrow, \leftrightarrow\}$



$$\text{sgn}(p) := (-1)^{\# \text{ of } \leftrightarrow}$$

Left hand side

Gelfand-Tsetlin pattern respecting a given direction pattern:

$$1. \quad \begin{array}{c} \leftarrow \\ a_{i+1,j} \end{array} \quad \begin{array}{c} a_{i,j} \\ \leftarrow, \leftrightarrow \\ a_{i+1,j+1} \end{array} \quad : \quad a_{i+1,j} \leq a_{i,j} < a_{i+1,j+1}$$

$$2. \quad \begin{array}{c} \leftarrow \\ a_{i+1,j} \end{array} \quad \begin{array}{c} a_{i,j} \\ \rightarrow \\ a_{i+1,j+1} \end{array} \quad : \quad a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$$

$$3. \quad \begin{array}{c} \leftrightarrow, \rightarrow \\ a_{i+1,j} \end{array} \quad \begin{array}{c} a_{i,j} \\ \leftarrow, \leftrightarrow \\ a_{i+1,j+1} \end{array} \quad : \quad a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$$

$$4. \quad \begin{array}{c} \leftrightarrow, \rightarrow \\ a_{i+1,j} \end{array} \quad \begin{array}{c} a_{i,j} \\ \rightarrow \\ a_{i+1,j+1} \end{array} \quad : \quad a_{i+1,j} < a_{i,j} \leq a_{i+1,j+1}$$

$$\text{LHS} = \sum_{p \text{ direction pattern of order } n} \text{sgn}(p)$$

× (# of GTPs with bottom row (k_1, \dots, k_n) respecting direction pattern p)

Writing the recursion in terms of “simple” summations

$n = 3$:

$$\begin{aligned}
 & \sum_{\substack{k_1 \leq l_1 \leq k_2 \leq l_2 \leq k_3 \\ l_1 < l_2}} a(l_1, l_2) \\
 &= \sum_{k_1 \leq l_1 < k_2 \leq l_2 \leq k_3} a(l_1, l_2) + \sum_{k_1 \leq l_1 \leq k_2 < l_2 \leq k_3} a(l_1, l_2) - \sum_{k_1 \leq l_1 < k_2 < l_2 \leq k_3} a(l_1, l_2) \\
 &= \sum_{l_1=k_1}^{k_2-1} \sum_{l_2=k_2}^{k_3} a(l_1, l_2) + \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2+1}^{k_3} a(l_1, l_2) - \sum_{l_1=k_1}^{k_2-1} \sum_{l_2=k_2+1}^{k_3} a(l_1, l_2)
 \end{aligned}$$

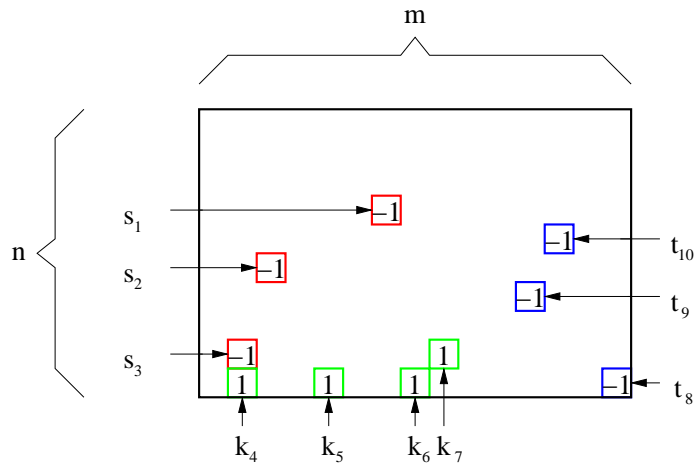
Letting $V_{x,y} = E_x^{-1} + E_y - E_x^{-1}E_y$, this is equal to

$$\left(V_{k_2, k'_2} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k'_2}^{k_3} a(l_1, l_2) \right) \Big|_{k_2=k'_2} .$$

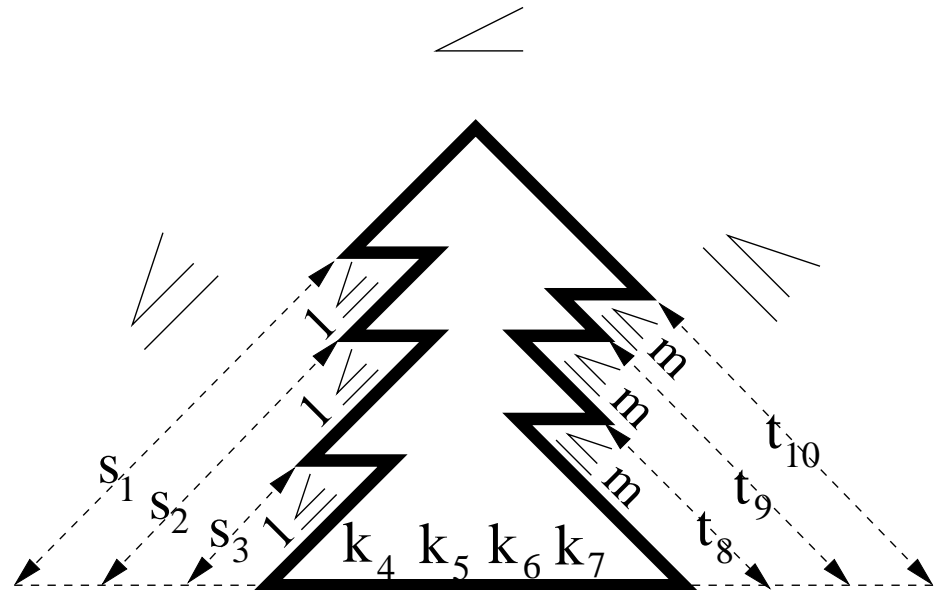
The recursion in terms of simple summations for arbitrary n

$$\begin{aligned}
 & \sum_{\substack{k_1 \leq l_1 \leq k_2 \leq l_2 \leq k_3 \leq \dots \leq l_{n-1} \leq k_n \\ l_i \neq l_{i+1}}} a(l_1, l_2, \dots, l_{n-1}) \\
 = & \left(V_{k_2, k'_2} V_{k_3, k'_3} \cdots V_{k_{n-1}, k'_{n-1}} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k'_2}^{k_3} \sum_{l_3=k'_3}^{k_4} \cdots \sum_{l_{n-1}=k'_{n-1}}^{k_n} a(l_1, \dots, l_{n-1}) \right) \Bigg|_{(k'_2, \dots, k'_{n-1}) = (k_2, \dots, k_{n-1})} \\
 = & \sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)} a(l_1, l_2, \dots, l_{n-1})
 \end{aligned}$$

General boundary conditions: left, right and bottom



- the first non-zero entry is -1 precisely in the rows s_1, s_2, \dots, s_l
- the last non-zero entry is -1 precisely in the rows t_{n-r+1}, \dots, t_n
- the first non-zero entry is 1 in each column
- the column sums are 1 precisely for the columns $k_{l+1}, k_{l+2}, \dots, k_{n-r}$



- Partial monotone triangles with entries in $\{1, \dots, m\}$ s. t.
- the i -th **NE-diagonal** is truncated by s_i elements for $1 \leq i \leq l$
 - the i -th **SE-diagonal** is truncated by t_i elements for $n - r + 1 \leq i \leq n$ and
 - the **bottom row** is $(k_{l+1}, k_{l+2}, \dots, k_{n-r})$.

Application: proof of the operator formula

$$\begin{aligned}
 a(l_1, \dots, l_{n-1}) &= \prod_{1 \leq p < q \leq n-1} (E_{l_p} + E_{l_q}^{-1} - E_{l_p} E_{l_q}^{-1}) \prod_{1 \leq i < j \leq n-1} \frac{l_j - l_i + j - i}{j - i} \\
 &= \prod_{1 \leq p < q \leq n-1} (E_{l_p} + E_{l_q}^{-1} - E_{l_p} E_{l_q}^{-1}) \det_{1 \leq i, j \leq n-1} \left(\binom{l_i + i}{j - 1} \right)
 \end{aligned}$$

(Vandermonde determinant)

Then

$$A(l_1, \dots, l_{n-1}) = \prod_{1 \leq p < q \leq n-1} (E_{l_p} + E_{l_q}^{-1} - E_{l_p} E_{l_q}^{-1}) \det_{1 \leq i, j \leq n-1} \left(\binom{l_i + i}{j} \right)$$

is the discrete antiderivative of $a(l_1, \dots, l_{n-1})$ with the desired property.