

CONSTANT TERM FORMULAS FOR REFINED ENUMERATIONS OF GOG AND MAGOG TRAPEZOIDS

ILSE FISCHER

ABSTRACT. Gog and Magog trapezoids are certain arrays of positive integers that generalize alternating sign matrices (ASMs) and totally symmetric self-complementary plane partitions (TSSCPPs) respectively. Zeilberger used constant term formulas to prove that there is the same number of (n, k) -Gog trapezoids as there is of (n, k) -Magog trapezoids, thereby providing so far the only proof for a weak version of a conjecture by Mills, Robbins and Rumsey from 1986. About 20 years ago, Krattenthaler generalized Gog and Magog trapezoids and formulated an extension of their conjecture, and, recently, Biane and Cheballah generalized Gog trapezoids further and formulated a related conjecture. In this paper, we derive constant term formulas for various refined enumerations of generalized Gog trapezoids including those considered by Krattenthaler and by Biane and Cheballah. For this purpose we employ a result on the enumeration of truncated monotone triangles which is in turn based in the author's operator formula for the number of monotone triangles with prescribed bottom row. As a byproduct, we also generalize the operator formula for monotone triangles by including the inversion number and the complementary inversion number for ASMs. Constant term formulas as well as determinant formulas for the refined Magog trapezoid numbers that appear in Krattenthaler's conjecture are also deduced by using the classical approach based on non-intersecting lattice paths and the Lindström-Gessel-Viennot theorem. Finally, we review and partly extend a few existing tools that may be helpful in relating constant term formulas for Gogs to those for Magogs to eventually prove the above mentioned conjectures.

1. INTRODUCTION

When Robbins and Rumsey [RR86] introduced *alternating sign matrices* in the 1980s, this was exciting for several reasons. On the one hand, they formulated a conjecture together with Mills [MRR82] which states that the number of $n \times n$ ASMs is given by the compelling simple

product formula $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$. This conjecture turned out to be difficult to prove [Zei96, Kup96]

and still deserves to be better understood. On the other hand, it was soon realized that the product formula has appeared before as the counting formula for two classes of plane partitions, namely for *totally symmetric self-complementary plane partitions* in an $2n \times 2n \times 2n$ box [MRR86] and for *descending plane partitions* whose parts do not exceed n [MRR83]. This gave rise to the search for explicit bijections between the three classes of objects to prove these facts directly. However, up to this day, no one was able to provide these bijections, and to find them is for many people the most important open problem in this field.

As part of their attempt to provide a bijection between ASMs and TSSCPPs, Mills, Robbins and Rumsey [MRR86, Conjecture 7] introduced two new classes of objects, one of which generalizes ASMs, while the other generalizes TSSCPPs and they conjectured equinumeracy

The author acknowledges support from the Austrian Science Foundation FWF, START grant Y463 and SFB grant F50.

between the two classes. In his proof of the ASM counting formula, Zeilberger [Zei96] proved their conjecture. Following Zeilberger, the two types of objects are since then called (n, k) -Gog trapezoids and (n, k) -Magog trapezoids n and k being arbitrary non-negative integers. Bijections exist only for the cases $k = 1, 2$, see [CB12, BC13, Bet16]. Already Mills, Robbins and Rumsey introduced a pair of statistics on Gog trapezoids as well as a pair of statistics on Magog trapezoids, and they conjectured that these pairs have the same joint distribution [MRR86, Conjecture 7']. In 1996, Krattenthaler [Kra96, Kra16] generalized Gog trapezoids and Magog trapezoids by introducing a third parameter m and formulated an extension of the (strong version of the) conjecture by Mills, Robbins and Rumsey. Recently, Biane and Cheballah introduced *Gog pentagons* and *GOGAm pentagons*, for which they also conjecture equinumeracy, and they have provided a bijective proof of their conjecture in the special cases when $k = 1, 2$.

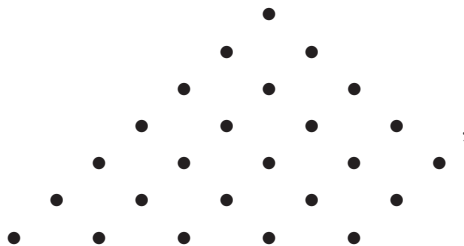
In this article, we deduce constant term formulas for refined enumerations of Gogs including those that appear in the conjectures of Krattenthaler and of Biane and Cheballah. The derivation is based on the main result of [Fis11] which provides a formula for the number of *truncated monotone triangles*. This result is a generalization of the operator formula for the number of monotone triangles with prescribed bottom row [Fis06]. In fact, in order to include as many refinement parameters as possible, we extend the main results from [Fis11] and from [Fis06] by introducing the inversion number as well as the complementary inversion number of ASMs, see Section 2. In order to be able to compare Gogs to Magogs, we also derive constant term formulas for refined enumerations of Magog trapezoids, see Section 4. In the final section, we mention (and partly extend) some known tools that may be helpful to relate constant term formulas for Gogs to those for Magogs.

In general, the purpose of our results is twofold: On the one hand, they provide a framework for computational proofs of the conjectures of Krattenthaler, and of Biane and Cheballah, but, on the other hand, they can also be useful in the (experimental) search for statistics on the two classes of objects that have the same distribution to eventually construct an explicit bijection between Gogs and Magogs. In particular, the presentation aims at illustrating a certain versatility of the result on truncated monotone triangles from [Fis11] and possibly indicates how to include other statistics on Gogs when it turns out to be useful.

Before we present the preliminaries in Section 2, which are then applied in Section 3 to derive the constant term formulas for Gogs, we recall the definition of Gog trapezoids and Magog trapezoids, and state Krattenthaler's extension of [MRR86, Conjecture 7'].

Gog trapezoids, Magog trapezoids and Krattenthaler's conjecture.

Gog trapezoids. In [Kra96, Kra16], Krattenthaler generalizes Zeilberger's Gog trapezoids [Zei96] (which appeared unnamed for the first time in [MRR86, Conjecture 7]) as follows: For m, n, k non-negative integers with $k \leq n$, an (m, n, k) -Gog trapezoid is defined as an arrangement of positive integers of the following form (the bullets correspond to the integers)



where n is the number of rows ($n = 7$ in the example), k is the number of north-east diagonals ($k = 5$ in the example) and the entries in the i -th south-east diagonal ($1 \leq i \leq n$, counted from the left) are bounded from above by $m + i$ such that

- the entries are weakly increasing along NE-diagonals and SE-diagonals, and
- strictly increasing along rows.

(By the strict increase along rows, it suffices to require that the entries in the i -th SE-diagonal are bounded from above by $m + i$ for $k \leq i \leq n$.) A $(3, 7, 5)$ -Gog trapezoid is given next.

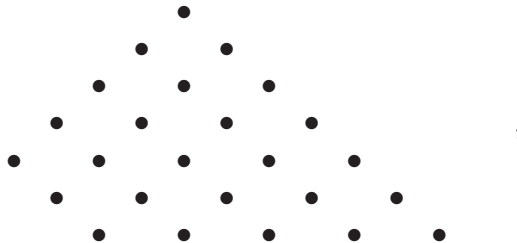
				4		
			3	5		
		2	5	8		
		2	4	7	9	
	1	4	6	7	10	
	1	4	5	8	8	
1	3	5	6	8		

Zeilberger’s original $n \times k$ -Gog trapezoids are just $(0, n, k)$ -Gog trapezoids as defined by Krattenthaler. According to [MRR86, Kra96, Kra16], we define minima and maxima of Gog trapezoids as follows:

- A *minimum* is an entry equal to 1. (The minima are located at the bottom of the leftmost NE-diagonal.)
- A *maximum* is an entry in the k -th NE-diagonal that is equal to the upper bound for the entries in its SE-diagonal.

In the example, we have 3 minima (indicated in red) and 2 maxima (indicated in blue).

Magog trapezoids. In [Kra96, Kra16], Krattenthaler generalized also Zeilberger’s Magog trapezoids [Zei96] (the latter also appeared essentially for the first time in [MRR86, Conjecture 7]): For m, n, k non-negative integers with $k \leq n$, an (m, n, k) -Magog trapezoid is defined as an arrangement of positive integers of the following form



where n is the number of rows ($n = 7$ in the example), k is the number of SE-diagonals ($k = 5$ in the example) and the entries in the i -th NE-diagonal ($1 \leq i \leq n$, counted from the left) are bounded from above by $m + i$ such that the entries are weakly increasing along NE-diagonals

and SE-diagonals. Next we display a $(2, 7, 5)$ -Magog trapezoid.

$$\begin{array}{ccccccc}
 & & & & & & 3 \\
 & & & & 2 & 3 & \\
 & & & 2 & 3 & 4 & \\
 & & 1 & 3 & 4 & 5 & \\
 \mathbf{1} & & 2 & 3 & 5 & 7 & \\
 & \mathbf{1} & 3 & 4 & 6 & 7 & \\
 & 2 & 3 & 4 & 7 & 9 &
 \end{array} \tag{1.1}$$

Zeilberger's $n \times k$ -Magog trapezoids are $(0, n, k)$ -Magog trapezoids as defined by Krattenthaler. According to [MRR86, Kra96, Kra16], we define minima and maxima of Magog trapezoids as follows:

- A *minimum* is an entry equal to 1 that is located in the leftmost SE-diagonal.
- A *maximum* is an entry in the rightmost SE-diagonal that is equal to the upper bound for the entries in its NE-diagonal.

In the example, we have 2 minima (indicated in red) and 3 maxima (indicated in blue).

Krattenthaler [Kra96, Kra16] conjectures the following relation between Gog trapezoids and Magog trapezoids.

Conjecture 1. *The number of (m, n, k) -Gog trapezoids with p minima and q maxima is equal to the number of (m, n, k) -Magog trapezoids with p maxima and q minima.*

The case $m = 0$ is Conjecture 7' in [MRR86]. Conjecture 1 implies in particular that the number of (m, n, k) -Gog trapezoids is equal to the number of (m, n, k) -Magog trapezoids. In [Zei96], Zeiberger proved this for $m = 0$.

2. PRELIMINARIES: MONOTONE TRIANGLES AND TRUNCATED MONOTONE TRIANGLES

2.1. Inversion numbers. The *inversion number* of an ASM $A = (a_{i,j})_{1 \leq i, j \leq n}$, as defined in [RR86, Eq. (18)], is given by

$$\text{inv}(A) = \sum_{1 \leq i' < i \leq n, 1 \leq j' \leq j \leq n} a_{i',j} a_{i,j'}.$$

It generalizes the inversion number of permutations, that is, for a permutation matrix A , $\text{inv}(A)$ is the number of inversions of the permutation $\pi = (\pi_1, \dots, \pi_n)$, where π_i is the column of the unique 1 in row i . Define the *complementary inversion number* $\text{inv}'(A)$ as follows.

$$\text{inv}'(A) = \sum_{1 \leq i' < i \leq n, 1 \leq j \leq j' \leq n} a_{i',j} a_{i,j'}$$

If A is the permutation matrix of $\pi = (\pi_1, \dots, \pi_n)$, then $\text{inv}'(A)$ is clearly the number of inversions of $(\pi_n, \pi_{n-1}, \dots, \pi_1)$, and so we have $\text{inv}(A) + \text{inv}'(A) = \binom{n}{2}$ for permutation matrices.

It is a well-known fact [Bre99] that $n \times n$ alternating sign matrices correspond to *Gelfand-Tsetlin patterns* with strictly increasing rows and bottom row $1, \dots, n$. Recall that a Gelfand-Tsetlin pattern is a triangular array $(m_{i,j})_{1 \leq j \leq i \leq n}$ of integers, where the elements are usually

arranged as follows

$$\begin{array}{cccccccc}
& & & & & & & m_{1,1} \\
& & & & & & & m_{2,1} & m_{2,2} \\
& & & & & \dots & & \dots & \dots \\
& & & & \dots & & \dots & \dots & \dots & m_{n-2,n-2} \\
& & m_{n-2,1} & & \dots & & \dots & & \dots & \\
& m_{n-1,1} & & m_{n-1,2} & & \dots & & \dots & & m_{n-1,n-1} \\
m_{n,1} & & m_{n,2} & & m_{n,3} & & \dots & & \dots & m_{n,n}
\end{array} \tag{2.1}$$

such that there is a weak increase in northeast and southeast direction, i.e., $m_{i+1,j} \leq m_{i,j} \leq m_{i+1,j+1}$ for all i, j with $1 \leq j \leq i < n$. Throughout this paper, we use the indexing of the entries of triangular arrays and partial triangular arrays as given here. A Gelfand-Tsetlin pattern in which each row is strictly increasing except for possibly the bottom row is said to be a *monotone triangle*. (This definition deviates from the standard definition where also the bottom row needs to be strictly increasing.) Monotone triangles of order n where also the bottom row is increasing and the entries are positive integers no greater than $m+n$ are just (m, n, n) -Gog trapezoids.

Suppose $M = (m_{i,j})_{1 \leq j \leq i \leq n}$ is the monotone triangle corresponding to the ASM $A = (a_{i,j})_{1 \leq i, j \leq n}$, then it is not hard to see that

$$\text{inv}(A) = \#\{(i, j) : m_{i,j} = m_{i+1,j+1}\} \quad \text{and} \quad \text{inv}'(A) = \#\{(i, j) : m_{i,j} = m_{i+1,j}\}.$$

Since the -1 's of an $n \times n$ ASM A correspond to the entries $m_{i,j}$ of the associated monotone triangle with $i < n$ and $m_{i+1,j} < m_{i,j} < m_{i+1,j+1}$, it follows that

$$\text{inv}(A) + \text{inv}'(A) = \binom{n}{2} - (\# \text{ of } -1\text{'s in } A). \tag{2.2}$$

We use this to extend the definition of the inversion number and the complementary inversion number to all monotone triangles, that is,

$$\text{inv}(M) = \#\{(i, j) : m_{i,j} = m_{i+1,j+1}\} \quad \text{and} \quad \text{inv}'(M) = \#\{(i, j) : m_{i,j} = m_{i+1,j}\}.$$

In [Fis16, Corollary 3.1], it was shown that the number of monotone triangles with bottom row b_1, \dots, b_n is the constant term of

$$\prod_{i=1}^n (1 + x_i)^{b_i} x_i^{-n+1} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 + x_j + x_i x_j). \tag{2.3}$$

(This derivation was based on [Fis06].) In this section, we aim at generalizing this in order to obtain a constant term expression for the generating function of monotone triangles with respect to the two inversion numbers. For an increasing sequence $\mathbf{b} = (b_1, \dots, b_n)$, we define the generating function

$$\text{MT}_{\mathbf{b}}(u, v) = \sum_M u^{\text{inv}(M)} v^{\text{inv}'(M)},$$

where the sum is over all monotone triangles with bottom row \mathbf{b} .

We say that the two non-decreasing sequences $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{a} = (a_1, \dots, a_{n-1})$ are *interlacing* (in symbols: $\mathbf{a} \prec \mathbf{b}$), if

$$b_1 \leq a_1 \leq b_2 \leq a_2 \leq \dots \leq a_{n-1} \leq b_n.$$

Consecutive rows of monotone triangles are obviously interlacing sequences, and we can write down the following recursion for $\text{MT}_{\mathbf{b}}(u, v)$:

$$\text{MT}_{\mathbf{b}}(u, v) = \sum_{\substack{\mathbf{a} < \mathbf{b} \\ \mathbf{a} \text{ strictly increasing}}} \text{MT}_{\mathbf{a}}(u, v) u^{\#\{i: a_i = b_{i+1}\}} v^{\#\{i: a_i = b_i\}}$$

We fix some notation that is needed in the following: We use the *shift operator* E_x , the *forward difference* $\overline{\Delta}_x$ and the *backward difference* $\underline{\Delta}_x$, which are defined as follows.

$$\begin{aligned} E_x p(x) &:= p(x+1) \\ \overline{\Delta}_x &:= E_x - \text{Id} \\ \underline{\Delta}_x &:= \text{Id} - E_x^{-1} \end{aligned}$$

We also need the following operator.

$$\text{Strict}_{x,y} = E_x^{-1} E_y + u \underline{\Delta}_x E_y - v E_x^{-1} \overline{\Delta}_y = u E_y + v E_x^{-1} + (1 - u - v) E_x^{-1} E_y$$

Moreover, we need to work with the following extended definition of the summation

$$\sum_{i=a}^b f(i) = \begin{cases} f(a) + f(a+1) + \dots + f(b), & a \leq b, \\ 0, & b = a - 1, \\ -f(b+1) - f(b+2) - \dots - f(a-1), & b+1 \leq a-1 \end{cases}.$$

The crucial property of the operator is the following: (Here we use the *Iversion bracket*, i.e., $[\text{statement}] = 1$ if the statement is true and $[\text{statement}] = 0$ otherwise.)

$$\sum_{\substack{(a_{i-1}, a_i) \prec (b_{i-1}, b_i, b_{i+1}) \\ a_{i-1} < a_i}} f(a_{i-1}, a_i) u^{[a_{i-1} = b_i]} v^{[a_i = b_i]} = \left[\text{Strict}_{b_i^{(1)}, b_i^{(2)}} \sum_{a_{i-1} = b_{i-1}}^{b_i^{(1)}} \sum_{a_i = b_i^{(2)}}^{b_{i+1}} f(a_{i-1}, a_i) \right] \Big|_{b_i^{(1)} = b_i^{(2)} = b_i},$$

which is true provided that $b_{i-1} < b_i < b_{i+1}$. (Here, we need to work with the following extended definition of the summation

$$\sum_{i=a}^b f(i) = \begin{cases} f(a) + f(a+1) + \dots + f(b), & a \leq b, \\ 0, & b = a - 1, \\ -f(b+1) - f(b+2) - \dots - f(a-1), & b+1 \leq a-1 \end{cases}.$$

Furthermore, we use the *Iversion bracket*, i.e., $[\text{statement}] = 1$ if the statement is true and $[\text{statement}] = 0$ otherwise.) In case $u = v = 1$, it is also true if $b_{i-1} \leq b_i \leq b_{i+1}$. However, note that also

$$\sum_{a_i = b_i}^{b_{i+1}} f(a_i) v^{[a_i = b_i]} = \left[\text{Strict}_{b_i^{(1)}, b_i^{(2)}} \sum_{a_i = b_i^{(2)}}^{b_{i+1}} f(a_i) \right] \Big|_{b_i^{(1)} = b_i^{(2)} = b_i} \quad (2.4)$$

and

$$\sum_{a_i = b_i}^{b_{i+1}} f(a_i) u^{[a_i = b_{i+1}]} = \left[\text{Strict}_{b_{i+1}^{(1)}, b_{i+1}^{(2)}} \sum_{a_i = b_i}^{b_{i+1}^{(1)}} f(a_i) \right] \Big|_{b_{i+1}^{(1)} = b_{i+1}^{(2)} = b_{i+1}} \quad (2.5)$$

whenever $b_i \leq b_{i+1}$. This implies

$$\sum_{\substack{\mathbf{a} \prec \mathbf{b} \\ \mathbf{a} \text{ strictly increasing}}} f(\mathbf{a}) u^{\#\{i: a_i = b_{i+1}\}} v^{\#\{i: a_i = b_i\}}$$

$$= \left[\text{Strict}_{b_1^{(1)}, b_1^{(2)}} \dots \text{Strict}_{b_n^{(1)}, b_n^{(2)}} \sum_{a_1 = b_1^{(2)}}^{b_2^{(1)}} \sum_{a_2 = b_2^{(2)}}^{b_3^{(1)}} \dots \sum_{a_{n-1} = b_{n-1}^{(2)}}^{b_n^{(1)}} f(\mathbf{a}) \right] \Big|_{b_i^{(1)} = b_i^{(2)} = b_i}$$

whenever $b_1 < b_2 < \dots < b_n$, and, under the assumption that $u = v = 1$, it suffices to require $b_1 \leq b_2 \leq \dots \leq b_n$. For $\mathbf{x} = (x_1, \dots, x_n)$, we define

$$\text{GT}_n(\mathbf{x}) = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i + j - i}{j - i} \quad \text{and} \quad M_n(u, v, \mathbf{x}) = \prod_{1 \leq p < q \leq n} \text{Strict}_{x_q, x_p} \text{GT}_n(\mathbf{x}).$$

Theorem 2.1 in [Fis16] can be generalized as follows.

Theorem 2. *Suppose $\mathbf{b} = (b_1, \dots, b_n)$ is a strictly increasing sequence of integers, then the generating function of monotone triangles with bottom row b_1, \dots, b_n w.r.t. the two inversion numbers is the evaluation of the polynomial $M_n(u, v, \mathbf{x})$ at $(x_1, \dots, x_n) = (b_1, \dots, b_n)$. When considering the special case $u = v = 1$, it suffices to require that (b_1, \dots, b_n) is weakly increasing.*

The proof is analogous to the proof of Theorem 2.1 in [Fis16]. Moreover, using the fact that

$$\text{Strict}_{x,y} = E_x^{-1}(\text{Id} + u \bar{\Delta}_x + (1 - v) \bar{\Delta}_y + u \bar{\Delta}_x \bar{\Delta}_y),$$

the following generalization of Corollary 3.1 in [Fis16] can be proved.

Corollary 3. *Suppose $\mathbf{b} = (b_1, \dots, b_n)$ is a strictly increasing sequence of integers, then the generating function of monotone triangles with bottom row b_1, \dots, b_n w.r.t. the two inversion numbers is the constant term in x_1, \dots, x_n of the following Laurent polynomial.*

$$\prod_{i=1}^n (1 + x_i)^{b_i} x_i^{-n+1} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 + (1 - v)x_i + u(x_j + x_i x_j))$$

When considering the special case $u = v = 1$, it suffices to require that (b_1, \dots, b_n) is weakly increasing.

In Appendix A, we elaborate on the case $v = 1 - u$, which is an interesting case since there exist certain combinatorial tools to handle it. Also note that we can recover the generating function with respect to number of -1 in the ASM proved in [Fis10] by setting $u = v = 1/Q$ and multiplying with $Q^{\binom{n}{2}}$. This follows from (2.2).

Remark 2.1. There exist two alternative constant term expressions for the number of monotone triangles with bottom row $\mathbf{b} = (b_1, \dots, b_n)$ that could replace (2.3) in all what follows in the special case $u = v = 1$. Both of them are also based on [Fis06]. First, it was shown in [Fis09, Theorem 3] that the constant term of

$$\prod_{i=1}^n x_i^{-n+1-b_i} (1 - x_i)^{-n} \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 - x_j + x_i x_j) \quad (2.6)$$

is the number of monotone triangles with bottom row $\mathbf{b} = (b_1, \dots, b_n)$. Note that this is not a Laurent polynomial but a rational function, and so we need to clarify how we expand it into a Laurent series. Here and throughout the remainder of the paper, unless stated otherwise, we expand in powers with non-negative exponents, in particular, we have

$$(1 - x_i)^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j} (-1)^j x_i^j$$

in (2.6). In order to present the second alternative formula, we define $\mathcal{AS}_{x_1, \dots, x_n}$ to be the antisymmetrizer w.r.t. x_1, \dots, x_n , that is

$$\mathcal{AS}_{x_1, \dots, x_n} p(x_1, \dots, x_n) := \sum_{\sigma \in \mathcal{S}_n} \text{sgn } \sigma p(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \quad (2.7)$$

In [FR15] it was shown that the number of monotone triangles with bottom row $\mathbf{b} = (b_1, \dots, b_n)$ is given by the constant term of the following expression.

$$\mathcal{AS}_{x_1, \dots, x_n} \left[\prod_{i=1}^n (1 + x_i)^{b_i} \prod_{1 \leq i < j \leq n} (1 + x_j + x_i x_j) \right] \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-1} \quad (2.8)$$

Note that this expression is a polynomial in x_1, \dots, x_n (and not only a Laurent polynomial).

It should be noted that (2.6) and (2.8) could actually be generalized to arbitrary u and v .

2.2. Refined enumeration with respect to the top entry. For a positive integer n , non-negative integers $\min \leq a \leq \max$ and $\mathbf{x} = (x_1, \dots, x_n)$, we define

$$\text{GT}_{n,a,\min,\max}(\mathbf{x}) = \sum_{b=0}^{\max - \min} (-1)^{a+\min+b} \binom{b}{a - \min} \det_{1 \leq i, j \leq n} \left(\binom{x_i + i - \min - 1}{j - 1 + b[j = n]} \right).$$

Suppose $\min \leq b_1 \leq b_2 \leq \dots \leq b_n \leq \max$ is a sequence of integers, then $\text{GT}_{n,a,\min,\max}(b_1, \dots, b_n)$ is the number of Gelfand-Tsetlin patterns with bottom row b_1, \dots, b_n and top row a . This follows, for instance, from a result in [Fis10, Theorem 3, $S = \emptyset, P = Q = 1$], as

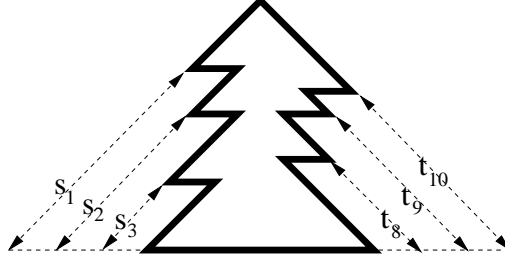
$$\prod_{\min \leq i \leq \max, i \neq a} \frac{x - i}{a - i} = \sum_{b=0}^{\max - \min} (-1)^{a+\min+b} \binom{b}{a - \min} \binom{x - \min}{b}.$$

Further, we define

$$M_{n,a,\min,\max}(u, v, \mathbf{x}) = \prod_{1 \leq p < q \leq n} \text{Strict}_{x_q, x_p} \text{GT}_{n,a,\min,\max}(\mathbf{x}),$$

then it follows from [Fis10, Theorem 3] and the ideas above that the evaluation of $M_{n,a,\min,\max}(u, v, \mathbf{x})$ at $\mathbf{x} = (b_1, \dots, b_n)$ is the generating function of monotone triangles with bottom row b_1, \dots, b_n and top row a , provided that $\min \leq b_1 < b_2 < \dots < b_n \leq \max$. The refined version of Corollary 3 is then the following.

Theorem 4. *Suppose $\mathbf{b} = (b_1, \dots, b_n)$ is a strictly increasing sequence of integers and $\min \leq a \leq \max$ are integers with $\min \leq b_1$ and $b_n \leq \max$. Then the generating function of monotone*

FIGURE 1. Shape of an (\mathbf{s}, \mathbf{t}) -tree.

triangles with bottom row b_1, \dots, b_n and top row a is the constant term of the rational function one obtains by multiplying the Laurent polynomial in Corollary 3 with

$$\sum_{b=0}^{\max - \min} (-1)^{a+\min+b} \binom{b}{a - \min} \prod_{i=1}^n (1 + x_i)^{-\min} \langle z^b \rangle \prod_{i=1}^n \frac{1}{1 - x_i^{-1} z},$$

where $\langle z^b \rangle$ means that we take the coefficient of z^b and $\frac{1}{1-x_i^{-1}z} = \sum_{k=0}^{\infty} z^k x_i^{-k}$. Again, when considering the special case $u = v = 1$, it suffices to require that $\mathbf{b} = (b_1, \dots, b_n)$ is weakly increasing.

2.3. Truncated monotone triangles: (\mathbf{s}, \mathbf{t}) -trees. Next we turn to certain partial monotone triangles. Let l, r, n be non-negative integers with $l + r \leq n$. Suppose $\mathbf{s} = (s_1, s_2, \dots, s_l)$, $\mathbf{t} = (t_{n-r+1}, t_{n-r+2}, \dots, t_n)$ are sequences of non-negative integers, where \mathbf{s} is weakly decreasing, while \mathbf{t} is weakly increasing. An (\mathbf{s}, \mathbf{t}) -tree of order n is an integer array whose shape is obtained from the shape of a monotone triangle with n rows when deleting the bottom s_i entries from the i -th NE-diagonal for $1 \leq i \leq l$ (NE-diagonals are counted from the left) and the bottom t_i entries from the i -th SE-diagonal for $n - r + 1 \leq i \leq n$ (SE-diagonals are also counted from the left), see Figure 1. We assume in the following that there is no interference between the deletion of the entries in the l leftmost NE-diagonals (as prescribed by \mathbf{s}) with the deletion of the entries from the r rightmost SE-diagonals (as prescribed by \mathbf{t}).

In an (\mathbf{s}, \mathbf{t}) -tree, an entry $m_{i,j}$ is said to be *regular* if it has a SW neighbour $m_{i+1,j}$ and a SE neighbour $m_{i+1,j+1}$. We require the following monotonicity properties in an (\mathbf{s}, \mathbf{t}) -tree:

- (1) Each regular entry $m_{i,j}$ has to fulfill $m_{i+1,j} \leq m_{i,j} \leq m_{i+1,j+1}$.
- (2) Two adjacent regular entries $m_{i,j}, m_{i,j+1}$ in the same row have to be distinct.

This extends the notion of monotone triangles, as a monotone triangle of order n is just an (\mathbf{s}, \mathbf{t}) -tree, where l, r are any two numbers with $l + r \leq n$, $\mathbf{s} = (\underbrace{0, \dots, 0}_l)$ and $\mathbf{t} = (\underbrace{0, \dots, 0}_r)$.

We extend the definition of the two inversion numbers as follows. Suppose $M = (m_{i,j})$ (using the indexing indicated in (2.1)) is an (\mathbf{s}, \mathbf{t}) -tree of order n and define

$$\begin{aligned} \text{inv}(M) &= \#\{m_{i,j} \text{ regular} : m_{i,j} = m_{i+1,j+1}\}, \\ \text{inv}'(M) &= \#\{m_{i,j} \text{ regular} : m_{i,j} = m_{i+1,j}\}. \end{aligned}$$

We define further generalizations of the inversion numbers that are useful for our purposes: The generalization of inv depends on a subset $J \subseteq \{j | t_j < t_{j+1}\} \cup \{n\}$ of *exceptional* SE-diagonals

and is defined as

$$\text{inv}_J(M) = \#\{m_{i,j} \text{ regular} : m_{i,j} = m_{i+1,j+1} \\ \text{and } m_{i+1,j+1} \text{ is not the bottom entry of an exceptional SE-diagonal}\}.$$

The generalization of the complementary inversion number inv' depends on a subset $I \subseteq \{1\} \cup \{i | s_{i-1} > s_i\}$ of exceptional NE-diagonals and is defined as

$$\text{inv}'_I(M) = \#\{m_{i,j} \text{ regular} : m_{i,j} = m_{i+1,j} \\ \text{and } m_{i+1,j} \text{ is not the bottom entry of an exceptional NE-diagonal}\}.$$

Next we define generalizations of the forward difference operator and of the backward difference operator, namely the v -forward difference operator $\overline{\Delta}[v]_x$ (w.r.t. the variable x) and the u -backward difference operator $\underline{\Delta}[u]_x$.

$$\overline{\Delta}[v]_x = (1 + (1 - v)\overline{\Delta}_x)^{-1}\overline{\Delta}_x \quad \text{and} \quad \underline{\Delta}[u]_x = (1 + (u - 1)\underline{\Delta}_x)^{-1}\underline{\Delta}_x$$

These operators are well-defined in our context, since

$$(1 + (1 - v)\overline{\Delta}_x)^{-1} = \sum_{i=0}^{\infty} (v - 1)^i \overline{\Delta}_x^i \quad \text{and} \quad (1 + (u - 1)\underline{\Delta}_x)^{-1} = \sum_{i=0}^{\infty} (1 - u)^i \underline{\Delta}_x^i$$

and the sums are finite if applied to polynomials, because for each polynomial $p(x)$ of degree d we have $\overline{\Delta}_x^i p(x) = \underline{\Delta}_x^i p(x) = 0$ for all $i > d$. The following property of $\overline{\Delta}[v]_x$ is crucial:

$$\begin{aligned} & -\overline{\Delta}[v]_{b_1} \left[\text{Strict}_{b_1^{(1)}, b_1^{(2)}} \dots \text{Strict}_{b_n^{(1)}, b_n^{(2)}} \sum_{a_1=b_1^{(2)}}^{b_2^{(1)}} \sum_{a_2=b_2^{(2)}}^{b_3^{(1)}} \dots \sum_{a_{n-1}=b_{n-1}^{(2)}}^{b_n^{(1)}} f(\mathbf{a}) \right] \Bigg|_{b_i^{(1)}=b_i^{(2)}=b_i} \\ &= -\overline{\Delta}_{b_1} \left[\text{Strict}_{b_2^{(1)}, b_2^{(2)}} \dots \text{Strict}_{b_n^{(1)}, b_n^{(2)}} \sum_{a_1=b_1}^{b_2^{(1)}} \sum_{a_2=b_2^{(2)}}^{b_3^{(1)}} \dots \sum_{a_{n-1}=b_{n-1}^{(2)}}^{b_n^{(1)}} f(\mathbf{a}) \right] \Bigg|_{b_i^{(1)}=b_i^{(2)}=b_i} \\ &= \left[\text{Strict}_{b_2^{(1)}, b_2^{(2)}} \dots \text{Strict}_{b_n^{(1)}, b_n^{(2)}} \sum_{a_2=b_2^{(2)}}^{b_3^{(1)}} \dots \sum_{a_{n-1}=b_{n-1}^{(2)}}^{b_n^{(1)}} f(b_1, a_2, \dots, a_{n-1}) \right] \Bigg|_{b_i^{(1)}=b_i^{(2)}=b_i} \end{aligned}$$

A similar identity is true for $\underline{\Delta}[u]_x$. These observations imply almost immediately the following result. The case $u = v = 1$ appeared first in [Fis11], however the proof given there can be generalized easily, see also [Fis16].

Theorem 5. *Let n, l, r be non-negative integers with $l + r \leq n$. Suppose b_1, \dots, b_n is a strictly increasing sequence of integers, and $\mathbf{s} = (s_1, s_2, \dots, s_l)$, $\mathbf{t} = (t_{n-r+1}, t_{n-r+2}, \dots, t_n)$ are a weakly decreasing and a weakly increasing sequence of non-negative integers, respectively. Then the evaluation of the following polynomial*

$$(-\overline{\Delta}[v]_{x_1})^{s_1} \dots (-\overline{\Delta}[v]_{x_l})^{s_l} \underline{\Delta}[u]_{x_{n-r+1}}^{t_{n-r+1}} \dots \underline{\Delta}[u]_{x_n}^{t_n} M_n(u, v, \mathbf{x}) \quad (2.9)$$

at $(x_1, \dots, x_n) = (b_1, \dots, b_n)$ is the generating function with respect to inv and inv' of (\mathbf{s}, \mathbf{t}) -trees of order n with the following properties:

- For $1 \leq i \leq n - r$, the bottom entry of the i -th NE-diagonal is b_i .
- For $n - r + 1 \leq i \leq n$, the bottom entry of the i -th SE-diagonal is b_i .

Furthermore, suppose $I \subseteq \{1\} \cup \{i | s_{i-1} > s_i\}$ and $J \subseteq \{j | t_j < t_{j+1}\} \cup \{n\}$, then the generating function with respect to inv_J and inv'_I of these (\mathbf{s}, \mathbf{t}) -trees is obtained from (2.9) by applying

$$\prod_{i \in I} (1 + (1 - v)\overline{\Delta}_{x_i})^{-1} \prod_{j \in J} (1 + (u - 1)\underline{\Delta}_{x_j})^{-1},$$

and then evaluating at $(x_1, \dots, x_n) = (b_1, \dots, b_n)$. Finally, suppose $\min \leq a \leq \max$ are integers with $\min \leq b_1$ and $b_n \leq \max$, then we obtain the generating function of the above mentioned (\mathbf{s}, \mathbf{t}) -trees that have a in the top row if $M_n(u, v, \mathbf{x})$ is replaced by $M_{n,a,\min,\max}(u, v, \mathbf{x})$.

When considering the special case $u = v = 1$, all results are true also if we only require that b_1, \dots, b_n is weakly increasing.

In order to prove the modification concerning the generalized inversion numbers inv_J and inv'_I , it has to be noted that

$$\text{Strict}_{x,y} = 1 + (1 - v)\overline{\Delta}_y$$

when applied to functions that are independent of x , while

$$\text{Strict}_{x,y} = 1 + (u - 1)\underline{\Delta}_x$$

when applied to functions that are independent of y . Compare also to (2.4) and (2.5).

We translate the formula in the theorem into a constant term expression generalizing the constant term expression of Corollary 3: To this end, we first observe that the application of $\overline{\Delta}_{b_i}$ to the constant term expression in Corollary 3 corresponds to the multiplication with $(1 + x_i) - 1 = x_i$, while the application of $\underline{\Delta}_{b_i}$ corresponds to the multiplication with $1 - 1/(1 + x_i) = x_i/(1 + x_i)$. This implies the following relations:

$$\overline{\Delta}[u]_{b_i} \sim \frac{x_i}{1 + (1 - v)x_i} \quad \text{and} \quad \underline{\Delta}[u]_{b_i} \sim \frac{x_i}{1 + ux_i}.$$

Corollary 6. *The generating function of (\mathbf{s}, \mathbf{t}) -trees as described in the theorem is given by the constant term in x_1, \dots, x_n of the following expression:*

$$\begin{aligned} & \prod_{i=1}^l \left(\frac{x_i}{(v-1)x_i - 1} \right)^{s_i} \prod_{i=n-r+1}^n \left(\frac{x_i}{ux_i + 1} \right)^{t_i} \prod_{i=1}^n (1 + x_i)^{b_i} x_i^{-n+1} \\ & \quad \times \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 + (1 - v)x_i + u(x_j + x_i x_j)) \end{aligned}$$

If I, J are two sets as described in the theorem, then the generating function with respect to $\text{inv}_J, \text{inv}'_I$ is the constant term in x_1, \dots, x_n of the expression one obtains by multiplying the rational function above with

$$\prod_{i \in I} \frac{1}{1 + (1 - v)x_i} \prod_{j \in J} \frac{1 + x_j}{1 + ux_j}.$$

If $\min \leq a \leq \max$ are integers with $\min \leq b_1$ and $b_n \leq \max$, then the generating function of the above mentioned (\mathbf{s}, \mathbf{t}) -trees whose top row is a is the constant term of the rational function which is obtained by multiplying the respective expression by $P_{n,a,\min,\max}(\mathbf{x})$.

3. APPLICATION: CONSTANT TERM FORMULA FOR THE GENERATING FUNCTION OF (m, n, k) -GOG TRAPEZOIDS

Using the terminology of the previous section, (m, n, k) -Gog trapezoids with bottom row b_1, \dots, b_k correspond to (\mathbf{s}, \mathbf{t}) -trees of order n , where $\mathbf{s} = \emptyset$, $\mathbf{t} = (0, 1, \dots, n - k - 1)$, such that

- (1) the bottom entries of the first k NE-diagonals are b_1, \dots, b_k ,
- (2) the bottom entries of the last $n - k$ SE-diagonals are $m + k + 1, \dots, m + n$.

In order to obtain the Gog trapezoid from the (\mathbf{s}, \mathbf{t}) -tree, one has to delete the $(k + 1)$ -st NE-diagonal from the (\mathbf{s}, \mathbf{t}) -tree.

Corollary 6 now implies the following formulas.

Theorem 7. *The number of (m, n, k) -Gog trapezoids with bottom row b_1, \dots, b_k is the constant term of*

$$\prod_{i=1}^k (1 + x_i)^{b_i} x_i^{-n+1} \prod_{i=k+1}^n (1 + x_i)^{m+k+1} x_i^{-n+i-k} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 + x_j + x_i x_j).$$

If $m = 0$, then the bottom row is necessarily $1, 2, \dots, k$, and so the number of $(0, n, k)$ -Gog trapezoids is the constant term of

$$\prod_{i=1}^k (1 + x_i)^i x_i^{-n+1} \prod_{i=k+1}^n (1 + x_i)^{k+1} x_i^{-n+i-k} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 + x_j + x_i x_j).$$

We are interested in the refined enumeration of (m, n, k) -Gog trapezoids with respect to four parameters.

3.1. Inversion numbers. The two inversion numbers are defined in the obvious way: Suppose $M = (m_{i,j})$ is an (m, n, k) -Gog trapezoid, then

$$\begin{aligned} \text{inv}(M) &= \#\{m_{i,j} : m_{i,j} = m_{i+1,j+1}\}, \\ \text{inv}'(M) &= \#\{m_{i,j} : m_{i,j} = m_{i+1,j}\}. \end{aligned}$$

So, in order for the entry $m_{i,j}$ to contribute to $\text{inv}(A)$, it must have a SE neighbor in the (m, n, k) -Gog trapezoid and this neighbor must be equal to $m_{i,j}$; however, $m_{i,j}$ need not to have a SW neighbor. Similar for the complementary inversion number. Corollary 6 implies the following.

Theorem 8. *The generating function of (m, n, k) -Gog trapezoids with bottom row b_1, \dots, b_k w.r.t. inv and inv' is the constant term of*

$$\prod_{i=1}^k (1 + x_i)^{b_i} x_i^{-n+1} \prod_{i=k+1}^n (1 + x_i)^{m+i+1} (1 + u x_i)^{k-i} x_i^{-n+i-k} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 + (1 - v)x_i + u(x_j + x_i x_j)) \quad (3.1)$$

in x_1, \dots, x_n .

The additional factor $\prod_{i=k+1}^n \frac{1+x_i}{1+u x_i}$ is caused by the fact that the u -weight should not take into account the entries in the k -th NE-diagonal that are equal to their upper bound.

3.2. Minima and maxima. In order to involve the number of minima and the number of maxima in the Gog-trapezoids, we have to apply the version of Corollary 6 where the requirement on b_1, b_2, \dots, b_n is only that it is weakly increasing, and thus we cannot involve the inversion numbers at the same time.

An (m, n, k) -Gog trapezoids with p minima can be identified with (\mathbf{s}, \mathbf{t}) -trees of order n , where $\mathbf{s} = (p-1)$, $\mathbf{t} = (0, 1, \dots, n-k-1)$ and where the bottommost entry of the first NE-diagonal is set to 2. In order to see this, delete in the (m, n, k) -Gog trapezoids with p minima the occurrences of 1 except for the topmost which is replaced by 2. In order to obtain a formula for the number we have to multiply (3.1) by $(-x_1)^{p-1}$ and set $b_1 = 2$. We obtain

$$(-1)^{p-1}(1+x_1)^2 x_1^{p-n} \prod_{i=2}^k (1+x_i)^{b_i} x_i^{-n+1} \prod_{i=k+1}^n (1+x_i)^{m+k+1} x_i^{-n+i-k} \times \prod_{1 \leq i < j \leq n} (x_i - x_j)(1+x_j+x_i x_j). \quad (3.2)$$

The case $p = 0$ is not covered by this formula. In that case, we use (3.1) with the appropriate $b_1 > 1$.

Now let $M \subseteq \{k+1, \dots, n\}$ and suppose we require to have a maximum in the i -th SE-diagonal precisely if $i \in M$ (and possibly in the k -th SE-diagonal, in which case we have $b_k = m+k$). Then we need to multiply (3.2) by

$$\prod_{i \in M} \frac{x_i}{1+x_i} \prod_{i \in \{k+1, k+2, \dots, n\} \setminus M} \frac{1}{1+x_i}.$$

If we sum over all subsets M with cardinality q , we obtain

$$\prod_{i=k+1}^n \frac{1}{1+x_i} e_q(x_{k+1}, \dots, x_n),$$

where e_q denotes the q -th elementary symmetric function. As $e_q(x_{k+1}, \dots, x_n)$ is the coefficient of Q^q in $\prod_{i=k+1}^n (1+Qx_i)$, we obtain the following theorem.

Theorem 9. *The generating function of (m, n, k) -Gog trapezoids with bottom row $1, b_2, \dots, b_k$ w.r.t. the weight*

$$P^{\# \text{ of minima}} Q^{\# \text{ of maxima not including } b_k}$$

is the constant term of

$$P(1+Px_1)^{-1}(1+x_1)^2 x_1^{-n+1} \prod_{i=2}^k (1+x_i)^{b_i} x_i^{-n+1} \prod_{i=k+1}^n (1+Qx_i)(1+x_i)^{m+k} x_i^{-n+i-k} \times \prod_{1 \leq i < j \leq n} (x_i - x_j)(1+x_j+x_i x_j) \quad (3.3)$$

in x_1, \dots, x_n , while the generating function of (m, n, k) -Gog trapezoids with bottom row b_1, b_2, \dots, b_k w.r.t. the weight

$$Q^{\# \text{ of maxima not including } b_k}$$

is the constant term of

$$\prod_{i=1}^k (1+x_i)^{b_i} x_i^{-n+1} \prod_{i=k+1}^n (1+Qx_i)(1+x_i)^{m+k} x_i^{-n+i-k} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1+x_j+x_i x_j) \quad (3.4)$$

in x_1, \dots, x_n .

The generating functions in Theorems 8 and 9 can be restricted further to Gog trapezoids with fixed top entry by multiplying with the Laurent series provided in Theorem 4.

In case $m = 0$, there is always a minimum and the bottom row is forced. Also note that the last entry of the bottom row is a maximum in this case.

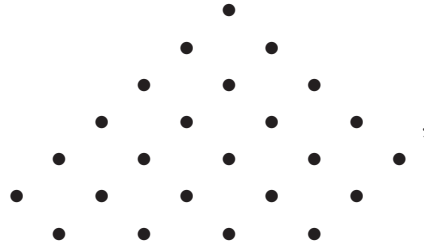
Corollary 10. *The generating function of $(0, n, k)$ -Gog trapezoids w.r.t. the number of minima and maxima is the constant term of*

$$PQ(1+Px_1)^{-1}(1+x_1)^2 x_1^{-n+1} \prod_{i=2}^k (1+x_i)^i x_i^{-n+1} \prod_{i=k+1}^n (1+Qx_i)(1+x_i)^{m+k} x_i^{-n+i-k} \times \prod_{1 \leq i < j \leq n} (x_i - x_j)(1+x_j+x_i x_j).$$

In order to obtain the number of all (m, n, k) -Gog trapezoids with p minima and q maxima also if $m \neq 0$, one has to sum over all possible bottom rows and take into account whether the rightmost entry in the bottom row is a maximum. In the expressions in Theorem 9, the bottom row appears only through the factors $\prod_{i=2}^k (1+x_i)^{b_i}$ and $\prod_{i=1}^k (1+x_i)^{b_i}$, respectively.

3.3. Gog pentagons. In this section, we consider generalizations of Gog trapezoids (so-called *Gog pentagons*) that were recently introduced by Biane and Cheballah [BC16]. In that paper also certain Magog-type objects were introduced (so-called *GOGAm pentagons*) along with the conjecture that there is the same number of Gog pentagons of a given type as there is of GOGAm pentagons of the same type and that this is still true if we restrict to pentagons that have a prescribed bottom entry. In this section, we use Corollary 6 to derive constant term expressions for various refined countings of Gog pentagons including fixing the bottom entry.

Let m, n, k_L, k_R be non-negative integers with $n+1 \leq k_L+k_R$. An (m, n, k_L, k_R) -Gog pentagon is an arrangement of positive integers of the following form



where n is the number of rows ($n = 7$ in the example), k_L is the number of NE-diagonals ($k_L = 5$ in the example) and k_R is the number of SE-diagonals ($k_R = 6$ in the example). Moreover, the entries in the i -th NE-diagonal, $1 \leq i \leq k_L$ (counted from the left), are bounded from below by i and the entries in the i -th SE-diagonal, $1 \leq i \leq k_R$ (counted from the right), are

bounded from above by $m + n + 1 - i$. The entries are weakly increasing along NE-diagonals and SE-diagonals, and strictly increasing along rows. (By the strict increase along rows, it suffices to require that the entries in the i -th NE-diagonal are bounded from below by i for $1 \leq i \leq n - k_R + 1$ and that the entries in the i -th SE-diagonal are bounded from above by $m + n + 1 - i$ for $1 \leq i \leq n - k_L + 1$.) A $(3, 7, 5, 6)$ -Gog pentagon is displayed next.

$$\begin{array}{cccccc}
 & & & & 4 & & \\
 & & & & 3 & 5 & \\
 & & & 2 & 5 & 8 & \\
 & & 2 & 4 & 7 & 9 & \\
 & 1 & 4 & 6 & 7 & 10 & \\
 1 & 4 & 5 & 8 & 8 & & \\
 & 3 & 5 & 6 & 8 & &
 \end{array}$$

Obviously, (m, n, k) -Gog trapezoids are (m, n, k, n) -Gog pentagons. Gog pentagons were first defined by Biane and Cheballah in [BC16]; they use different parameters: (m, n, k_L, k_R) -Gog pentagons as defined here correspond to their $(m + n, k_R, k_L, n)$ -Gog pentagons. Also, as we reflected their pentagons along a horizontal axis, the bottom entry in their pentagons is the top entry of the pentagons as defined here.

3.4. Inversion numbers. We observe that (m, n, k_L, k_R) -Gog pentagons with bottom row $b_{n-k_R+1}, b_{n-k_R+2}, \dots, b_{k_L}$ correspond to (\mathbf{s}, \mathbf{t}) -trees of order n , where $\mathbf{s} = (n - k_R - 1, n - k_R - 2, \dots, 0)$ and $\mathbf{t} = (0, 1, \dots, n - k_L - 1)$. This implies that the number of (m, n, k_L, k_R) -Gog pentagons with bottom row $b_{n-k_R+1}, \dots, b_{k_L}$ is equal to

$$\begin{aligned}
 & (-1)^{\binom{n-k_R}{2}} \prod_{i=1}^{n-k_R} (1+x_i)^i x_i^{1-k_R-i} \prod_{i=n-k_R+1}^{k_L} (1+x_i)^{b_i} x_i^{-n+1} \prod_{i=k_L+1}^n (1+x_i)^{m+k_L+1} x_i^{-n+i-k_L} \\
 & \quad \times \prod_{1 \leq i < j \leq n} (x_i - x_j)(1+x_j+x_i x_j).
 \end{aligned}$$

If we define the inversion numbers for Gog pentagons in the obvious way, it follows that the generating function of (m, n, k_L, k_R) -Gog pentagons with bottom row $b_{n-k_R+1}, \dots, b_{k_L}$ and w.r.t. the two inversion numbers are

$$\begin{aligned}
 & (-1)^{\binom{n-k_R}{2}} \prod_{i=1}^{n-k_R} (1+x_i)^i (1+(1-v)x_i)^{-n+k_R+i-1} x_i^{1-k_R-i} \prod_{i=n-k_R+1}^{k_L} (1+x_i)^{b_i} x_i^{-n+1} \\
 & \quad \times \prod_{i=k_L+1}^n (1+x_i)^{m+i+1} (1+ux_i)^{k_L-i} x_i^{-n+i-k_L} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1+(1-v)x_i + u(x_j + x_i x_j)).
 \end{aligned}$$

In order to obtain the generating function of (m, n, k_L, k_R) -Gog pentagons with top row a , one has to multiply the expression with $P_{n,a,\min,\max}(\mathbf{x})$, where $\min \leq a \leq \max$ and $\min \leq 1$ and $\max \geq m + n$.

3.5. Minima and maxima. A *bottom-minimum* is an entry in the leftmost SE-diagonal that is equal to the lower bound of its SE-diagonal, while a *bottom-maximum* is an entry in the rightmost NE-diagonal that is equal to the upper bound of its NE-diagonal. A *top-minimum* is an entry equal to 1 and such entries can only be in the leftmost NE-diagonal. A *top-maximum* is

an entry equal to $m+n$ and such entries can only be in the rightmost SE-diagonal. Observe that minima (resp. maxima) as defined for Gog trapezoids are top-minima (resp. bottom-maxima) as defined for Gog pentagons.

Also in this case, it is not possible to include both, the inversions numbers, and the numbers of maxima and minima because for the numbers of minima and maxima we need to apply Corollary 6 in instances where b_1, \dots, b_n is not necessarily strictly increasing.

Corollary 6 implies in a similar way as for Gog trapezoids that the generating function of (m, n, k_L, k_R) -Gog pentagons with bottom row $b_{n-k_R+1}, \dots, b_{k_L}$ w.r.t. the weight

$$Q_L^{\#\text{of bottom minima not including } b_{n-k_R+1}} Q_R^{\#\text{of bottom maxima not including } b_{k_L}}$$

is the constant term of

$$\begin{aligned} & (-1)^{\binom{n-k_R}{2}} \prod_{i=1}^{n-k_R} \left(1 - Q_L \frac{x_i}{1+x_i}\right) (1+x_i)^{i+1} x_i^{1-k_R-i} \prod_{i=n-k_R+1}^{k_L} (1+x_i)^{b_i} x^{-n+1} \\ & \quad \times \prod_{i=k_L+1}^n (1+Q_R x_i)(1+x_i)^{m+k_L} x_i^{-n+i-k_L} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1+x_j+x_i x_j) \end{aligned}$$

w.r.t. x_1, \dots, x_n . Finally, the generating function of (m, n, k_L, k_R) -Gog pentagons with bottom row $b_{n-k_R+1}, \dots, b_{k_L}$ that have at least one top-minimum and at least one top-maximum w.r.t. the weight

$$P_L^{\#\text{of top minima}} P_R^{\#\text{of top maxima}} Q_L^{\#\text{of bottom minima not including } b_{n-k_R+1}} Q_R^{\#\text{of bottom maxima not including } b_{k_L}}$$

is the constant term of the following expression in x_1, \dots, x_n .

$$\begin{aligned} & (-1)^{\binom{n-k_R}{2}+1} P_L (1+P_L x_1)^{-1} (1+x_1)^2 x_1^{-k_R+1} P_R (1+x_n - P_R x_n)^{-1} x_n^{-k_L+1} (1+x_n)^{k_L+m} \\ & \quad \times Q_L Q_R \prod_{i=2}^{n-k_R} \left(1 - Q_L \frac{x_i}{1+x_i}\right) (1+x_i)^{i+1} x_i^{1-k_R-i} \prod_{i=n-k_R}^{k_L} (1+x_i)^{b_i} x_i^{-n+1} \\ & \quad \times \prod_{i=k_L+1}^{n-1} (1+Q_R x_i)(1+x_i)^{m+k_L} x_i^{-n+i-k_L} \prod_{1 \leq i < j \leq n} (x_i - x_j)(1+x_j+x_i x_j). \end{aligned}$$

In order to obtain the generating functions of Gog pentagons where the top row is fixed, one has to proceed as in the previous subsection.

4. CONSTANT TERM FORMULA FOR THE NUMBER OF (m, n, k) -MAGOG TRAPEZOIDS WITH PRESCRIBED NUMBERS OF MINIMA AND MAXIMA

In this section we derive constant term formulas for the number of (m, n, k) -Magog trapezoids with p maxima and q minima. Together with Theorem 9, where a constant term formula for the number of (m, n, k) -Gog trapezoids with p minima and q maxima is provided, this could serve as a framework to give a computational proof of Conjecture 1.

In special cases such constant term formulas have been derived before: Zeilberger gave a constant term expression for the unrestricted enumeration of $(0, n, k)$ -Magog trapezoids in [Zei96] (see also [Zei94]), Krattenthaler extended this to (m, n, k) -Magog trapezoids in [Kra96], and

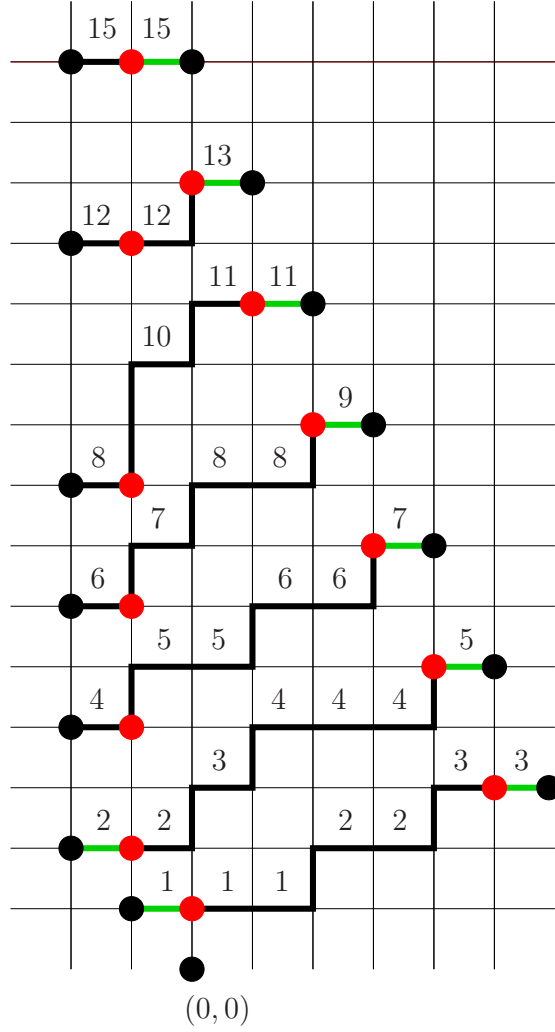


FIGURE 2. Non-intersecting lattice paths

which evaluates to

$$\binom{c+d-a-b-1}{d-b-1} + P \binom{c+d-a-b-1}{d-b}$$

if $Q = 1$. If we use

$$\binom{n}{k} = \begin{cases} \frac{n(n-1)\cdots(n-k+1)}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases},$$

then the formulas for the generating functions are true for any choice of integers a, b, c, d (in particular also when there is no path from (a, b) to (c, d)) if whenever the upper parameter of each binomial coefficient is negative, then also the lower parameter is negative. The Lindström-Gessel-Viennot theorem [Lin73, GV85] now implies that the generating function of (m, n, k) -Magog trapezoids that have bottom row (b_{n-k+1}, \dots, b_n) is

$$\det_{1 \leq i, j \leq n} \left(\begin{cases} \binom{j+k+m-3}{2j+m-i-3} + (P+Q) \binom{j+k+m-3}{2j+m-i-2} + PQ \binom{j+k+m-3}{2j+m-i-1}, & i = 1, \dots, n-k \\ \binom{j+m+n-b_i-i-1}{2j+m-b_i-i-1} + P \binom{j+m+n-b_i-i-1}{2j+m-b_i-i}, & i = n-k+1, \dots, n \end{cases} \right),$$

where the weight is

$$P^{\# \text{ of maxima}} \cdot Q^{\# \text{ of minima not contained in the bottom row}}$$

In case $1 \leq i \leq n - k$, the upper parameter of a binomial coefficient in the matrix is negative only if $j = 1, k = 1, m = 0$; then all lower parameters are also negative, except the one after PQ if $i = 1$ which is 0 and it can be checked that also in this case we have the correct generating function. In case $n - k + 1 \leq i \leq n$, the upper parameters are either no less than the lower parameters or non-negative or the lower parameters are negative, except for the binomial coefficient after P when $j + m + n - b_i - i - 1 = -1$ and $2j + m - b_i - i = 0$, and it can be checked that also in this case we obtain the correct result.

Using $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$, this is equal to

$$(-1)^{(m-1)n + \binom{n+1}{2} + \sum_{i=n-k+1}^n b_i} \times \det_{1 \leq i, j \leq n} \left(\begin{cases} \binom{j-i-k-1}{2j+m-i-3} - (P+Q)\binom{j-i-k}{2j+m-i-2} + PQ\binom{j-i-k+1}{2j+m-i-1}, & i = 1, \dots, n-k \\ \binom{j-n-1}{2j+m-b_i-i-1} - P\binom{j-n}{2j+m-b_i-i}, & i = n-k+1, \dots, n \end{cases} \right).$$

Now, as $\text{CT}_x \frac{(1+x)^n}{x^k} = \binom{n}{k}$, where CT_x denotes the constant term in x , this is equal to

$$\text{CT}_{x_1, \dots, x_n} (-1)^{(m-1)n + \binom{n+1}{2} + \sum_{i=n-k+1}^n b_i} \times \det_{1 \leq i, j \leq n} \left(\begin{cases} \frac{(1+x_i)^{j-i-k-1}}{x_i^{2j+m-i-3}} - (P+Q)\frac{(1+x_i)^{j-i-k}}{x_i^{2j+m-i-2}} + PQ\frac{(1+x_i)^{j-i-k+1}}{x_i^{2j+m-i-1}}, & i = 1, \dots, n-k \\ \frac{(1+x_i)^{j-n-1}}{x_i^{2j+m-b_i-i-1}} - P\frac{(1+x_i)^{j-n}}{x_i^{2j+m-b_i-i}}, & i = n-k+1, \dots, n \end{cases} \right).$$

After pulling out the factor

$$\prod_{i=1}^{n-k} \frac{(1+x_i)^{-i-k}}{x_i^{m-i+1}} (x_i - P(1+x_i)) (x_i - Q(1+x_i)) \prod_{i=n-k+1}^n \frac{(1+x_i)^{-n}}{x_i^{m-b_i-i+2}} (x_i - P(1+x_i))$$

of the determinant, it remains

$$\det_{1 \leq i, j \leq n} ((x_i^{-1} + x_i^{-2})^{j-1}) = \prod_{1 \leq i < j \leq n} (x_j^{-1} + x_j^{-2} - x_i^{-1} - x_i^{-2}) = \prod_{1 \leq i < j \leq n} \frac{(x_i - x_j)(x_i + x_j + x_i x_j)}{x_i^2 x_j^2},$$

which can be computed using the Vandermonde determinant evaluation.

We obtain the following theorem.

Theorem 11. *The generating function of (m, n, k) -Magog trapezoids with bottom row b_{n-k+1}, \dots, b_n and w.r.t. the weight*

$$P^{\# \text{ of maxima}} Q^{\# \text{ of minima not including } b_{n-k+1}}$$

is the constant term of the following expression in x_1, \dots, x_n .

$$(-1)^{(m-1)n + \binom{n+1}{2} + \sum_{i=n-k+1}^n b_i} \prod_{i=1}^{n-k} (1+x_i)^{-i-k} x_i^{-m-2n+i+1} (x_i - P(1+x_i)) (x_i - Q(1+x_i)) \times \prod_{i=n-k+1}^n (1+x_i)^{-n} x_i^{-m-2n+b_i+i} (x_i - P(1+x_i)) \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i + x_j + x_i x_j). \quad (4.1)$$

We set

$$\begin{aligned} M_{m,n,k}(x_1, \dots, x_n) &= (-1)^{(m-1)n + \binom{n-k+1}{2}} \prod_{i=1}^{n-k} (1+x_i)^{-i-k} x_i^{-m-2n+i+1} (x_i - P(1+x_i)) (x_i - Q(1+x_i)) \\ &\quad \times \prod_{i=n-k+1}^n (1+x_i)^{-n} x_i^{-m-2n} (x_i - P(1+x_i)) \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i + x_j + x_i x_j), \end{aligned}$$

so that the generating function is

$$\prod_{i=n-k+1}^n (-x_i)^{b_i+i} M_{m,n,k}(x_1, \dots, x_n).$$

It will be crucial that $M_{m,n,k}(x_1, \dots, x_n)$ is an antisymmetric function in x_{n-k+1}, \dots, x_n . It follows that the generating function of (m, n, k) -Magogs w.r.t. the weight

$$P^{\# \text{ of maxima}} \cdot Q^{\# \text{ of minima}}$$

is the constant term of

$$\begin{aligned} &\left(Q \left(\sum_{1 \leq b_{n-k+2} \leq b_{n-k+3} \leq \dots \leq b_n} y_{n-k+1}^{n-k+2} \prod_{i=n-k+2}^n y_i^{b_i+i} \right) + \left(\sum_{2 \leq b_{n-k+1} \leq b_{n-k+2} \leq \dots \leq b_n} \prod_{i=n-k+1}^n y_i^{b_i+i} \right) \right) \\ &\quad \times M_{m,n,k}(x_1, \dots, x_n) \\ &= \left(Q + \frac{\prod_{i=n-k+1}^n y_i}{1 - \prod_{i=n-k+1}^n y_i} \right) \frac{\prod_{i=n-k+1}^n y_i^{i+1}}{\prod_{i=n-k+2}^n (1 - \prod_{j=i}^n y_j)} M_{m,n,k}(x_1, \dots, x_n), \quad (4.2) \end{aligned}$$

where we set $y_i = -x_i$. We use the following notation: Suppose $F(x_{r+1}, \dots, x_{r+s})$ is a function in x_{r+1}, \dots, x_{r+s} and $\sigma \in \mathcal{S}_s$, then

$$\sigma[F(x_{r+1}, \dots, x_{r+s})] := F(x_{\sigma(r+1)}, \dots, x_{\sigma(r+s)}).$$

Now the constant term of the expression in (4.2) is

$$\begin{aligned} &\frac{1}{(k-1)!} \sum_{\sigma \in \mathcal{S}_{k-1}} Q y_{n-k+1}^{n-k+2} M_{m,n,k}(x_1, \dots, x_n) \operatorname{sgn} \sigma \cdot \sigma \left[\prod_{i=n-k+2}^n \frac{y_i^{i+1}}{1 - \prod_{j=i}^n y_j} \right] \\ &\quad + \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} M_{m,n,k}(x_1, \dots, x_n) \operatorname{sgn} \sigma \cdot \sigma \left[\prod_{i=n-k+1}^n \frac{y_i^{i+2}}{1 - \prod_{j=i}^n y_j} \right]. \quad (4.3) \end{aligned}$$

We need the following lemma from [Zei96, Subsublemma 1.1.3].

Lemma 12. *Let $r \geq 1$ be an integer. Then*

$$\sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn} \sigma \cdot \sigma \left[\prod_{i=1}^r \frac{y_i^{i-1}}{1 - \prod_{j=i}^r y_j} \right] = \prod_{i=1}^r (1 - y_i)^{-1} \prod_{1 \leq i < j \leq r} \frac{y_j - y_i}{1 - y_i y_j}.$$

From the lemma it now follows that the expression in (4.3) is equal to

$$\begin{aligned} & \frac{Q \cdot y_{n-k+1}^{n-k+2} M_{m,n,k}(x_1, \dots, x_n) \prod_{i=n-k+2}^n y_i^{n-k+3} (1-y_i)^{-1}}{(k-1)!} \prod_{n-k+2 \leq i < j \leq n} \frac{y_j - y_i}{1 - y_i y_j} \\ & + \frac{M_{m,n,k}(x_1, \dots, x_n) \prod_{i=n-k+1}^n y_i^{n-k+3} (1-y_i)^{-1}}{k!} \prod_{n-k+1 \leq i < j \leq n} \frac{y_j - y_i}{1 - y_i y_j}. \end{aligned}$$

This can also be written as

$$\begin{aligned} & \frac{Q \cdot y_{n-k+1}^{n-k+2} M_{m,n,k}(x_1, \dots, x_n) \prod_{i=n-k+2}^n (1-y_i)^{-1}}{(k-1)!} \prod_{n-k+2 \leq i < j \leq n} \frac{1}{1 - y_i y_j} \sum_{\sigma \in \mathcal{S}_{k-1}} \text{sgn } \sigma \sigma \left[\prod_{i=n-k+2}^n y_i^{i+1} \right] \\ & + \frac{M_{m,n,k}(x_1, \dots, x_n) \prod_{i=n-k+1}^n (1-y_i)^{-1}}{k!} \prod_{n-k+1 \leq i < j \leq n} \frac{1}{1 - y_i y_j} \sum_{\sigma \in \mathcal{S}_k} \text{sgn } \sigma \sigma \left[\prod_{i=n-k+1}^n y_i^{i+2} \right]. \end{aligned}$$

We again employ the antisymmetry of $M_{m,n,k}(x_1, \dots, x_n)$ to see that this is equal to

$$\begin{aligned} & Q \cdot y_{n-k+1}^{n-k+2} M_{m,n,k}(x_1, \dots, x_n) \prod_{i=n-k+2}^n y_i^{i+1} (1-y_i)^{-1} \prod_{n-k+2 \leq i < j \leq n} \frac{1}{1 - y_i y_j} \\ & + M_{m,n,k}(x_1, \dots, x_n) \prod_{i=n-k+1}^n y_i^{i+2} (1-y_i)^{-1} \prod_{n-k+1 \leq i < j \leq n} \frac{1}{1 - y_i y_j} \\ & = (-1)^{\binom{n+3}{2} + \binom{n-k+3}{2}} M_{m,n,k}(x_1, \dots, x_n) \prod_{i=n-k+1}^n \frac{x_i^{i+2}}{1+x_i} \prod_{n-k+1 \leq i < j \leq n} \frac{1}{1-x_i x_j} \\ & \quad \times \left((-1)^k Q (1+x_{n-k+1}^{-1}) \prod_{j=n-k+2}^n (x_j^{-1} - x_{n-k+1}) + 1 \right). \end{aligned}$$

We summarize our result in the following theorem.

Theorem 13. *The generating function of (m, n, k) -Magog trapezoids w.r.t. the weight*

$$P^{\# \text{ of maxima}} Q^{\# \text{ of minima}}$$

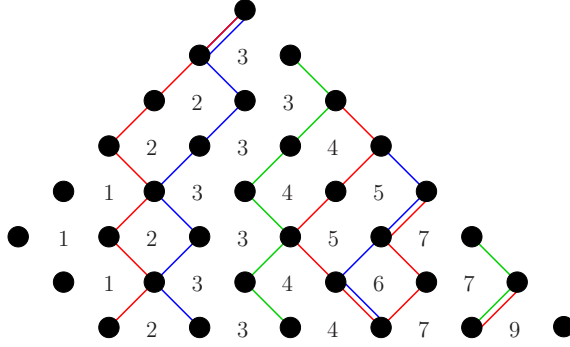


FIGURE 3. An $(2, 7, 5)$ -Magog trapezoids and its separating lattice paths

is the constant term in x_1, \dots, x_n of the following expression.

$$\begin{aligned}
& (-1)^{(m-1)n + \binom{n+1}{2}} \left((-1)^k Q(1 + x_{n-k+1}^{-1}) \prod_{j=n-k+2}^n (x_j^{-1} - x_{n-k+1}) + 1 \right) \\
& \times \prod_{i=1}^{n-k} (1 + x_i)^{-i-k} x_i^{-m-2n+i+1} (x_i - P(1 + x_i)) (x_i - Q(1 + x_i)) \\
& \times \prod_{i=n-k+1}^n (1 + x_i)^{-n-1} x_i^{-m-2n+i+2} (x_i - P(1 + x_i)) \\
& \times \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i + x_j + x_i x_j) \prod_{n-k+1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}
\end{aligned}$$

4.2. Second version. There are other options to encode (m, n, k) -Magog trapezoids as families of non-intersecting lattice paths, see for instance [Kra96]. We derive the constant term expression for a second possibility in this subsection, following an encoding that was used for instance in [FZJ08]. Instead of interpreting NE-diagonals as lattice paths, we now consider the lattice paths that separate the entries that are less than or equal to i in the Magog trapezoid from the entries that are greater than or equal to $i + 1$, $1 \leq i \leq m + n - 1$, see Figure 3. By rotating the picture and shifting the paths appropriately, these paths can be transformed into a family of non-intersecting lattice paths, see Figure 4 (the leftmost separating path in Figure 3, i.e. the one separating the 1's from 2's, corresponds to the bottom path in Figure 4).

It can be worked out that (m, n, k) -Magog trapezoids correspond to families of non-intersecting lattice paths with starting points $(-i + 1, i - 1)$, $i = 1, \dots, m - 1$, and $(-i + 1, 2i - m - 1)$, $m \leq i \leq m + n - 1$, and end points $(k - b_j + 1, n - k + b_j - 1)$, $1 \leq j \leq m + n - 1$, for some $1 \leq b_1 < b_2 < \dots < b_{m+n-1} \leq m + n + k - 1$, with north steps and east steps. The maximums correspond to the lattice paths with starting points $(-i + 1, 2i - m - 1)$, $m \leq i \leq m + n - 1$, such that the first step is horizontal, while the number of minima is just the height of the last horizontal step of the path ending at $(k - b_1 + 1, n - k + b_1 - 1)$ if $b_1 = 1$, otherwise it is $n - k + 1$. It follows that the number of (m, n, k) -Magog trapezoids with q minima is equal to the number of the following family of lattice paths depending on whether or not $q = n - k + 1$.

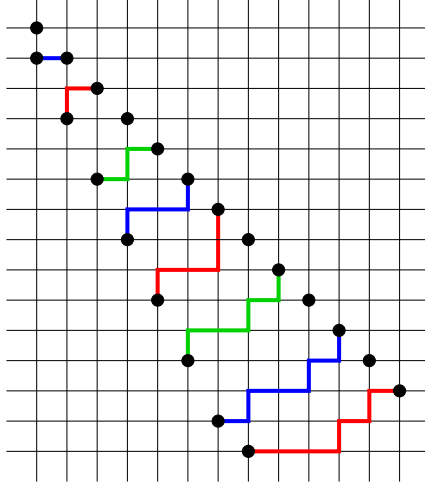


FIGURE 4. Non-intersecting lattice paths for the second version

Case $q \leq n - k$: In this case, we need to count families of non-intersecting lattice paths starting at $(-i + 1, i - 1)$, $i = 1, \dots, m - 1$, and $(-i + 1, 2i - m - 1)$, $m \leq i \leq m + n - 1$, and with ending points $(k - 1, q)$ and $(k - b_j + 1, n - k + b_j - 1)$, $2 \leq j \leq m + n - 1$, for some $2 \leq b_2 < \dots < b_{m+n-1} \leq m + n + k - 1$.

Case $q = n - k + 1$: Here, we need to count families of non-intersecting lattice paths starting at $(-i + 1, i - 1)$, $i = 1, \dots, m - 1$, and $(-i + 1, 2i - m - 1)$, $m \leq i \leq m + n - 1$, and with ending points $(k - b_j + 1, n - k + b_j - 1)$, $1 \leq j \leq m + n - 1$, for some $2 \leq b_1 < \dots < b_{m+n-1} \leq m + n + k - 1$.

First we assume $q \leq n - k$.

The generating function of lattice paths starting at (a, b) and ending at (c, d) with respect to the weight

$$P^{\text{[first step is horizontal]}}$$

allowing steps in north direction and east direction is

$$\begin{aligned} P \cdot \mathcal{N}((a + 1, b) \rightarrow (c, d)) + \mathcal{N}((a, b + 1) \rightarrow (c, d)) \\ = P \binom{c + d - a - b - 1}{c - a - 1} + \binom{c + d - a - b - 1}{c - a}. \end{aligned}$$

The formula is true whenever $c + d - a - b > 0$. It is also true when $c + d - a - b = 0$ and $c - a = 0$. If $c + d - a - b = 0$ and $c - a \neq 0$ or $c + d - a - b < 0$, then the generating function vanishes. It follows that the generating function of (m, n, k) -Magog trapezoids with q minima and w.r.t. the number of maxima is

$$\det_{1 \leq i, j \leq m+n-1} \left(\begin{array}{c|c} \binom{k+q-1}{k+i-2} & \binom{n}{i+k-b_j} \\ \hline \begin{cases} P \binom{k+m+q-i-2}{k+i-3} + \binom{k+m+q-i-2}{k+i-2}, & i \leq k + m + q - 2 \\ 0, & i \geq k + m + q - 1 \end{cases} & P \binom{m+n-i-1}{i+k-b_j-1} + \binom{m+n-i-1}{i+k-b_j} \end{array} \right),$$

where we need to distinguish between $1 \leq i \leq m - 1$ and $m \leq i \leq m + n - 1$ concerning the row, and between $j = 1$ and $2 \leq j \leq m + n - 1$ concerning the columns, and we have to exclude

the case when $k = 1$ and $m + q = 1$. This is equal to the constant term of

$$\det_{1 \leq i, j \leq m+n-1} \left(\begin{array}{c|c} \frac{(1+x_i)^{k+q-1}}{x_i^{i+k-3}} & \frac{(1+x_i)^n}{x_i^{i+k-b_j}} \\ \hline \begin{cases} P \frac{(1+x_i)^{k+m+q-i-2}}{x_i^{i+k-3}} + \frac{(1+x_i)^{k+m+q-i-2}}{x_i^{i+k-2}} & i \leq k+m+q-2 \\ 0 & i \geq k+m+q-1 \end{cases} & P \frac{(1+x_i)^{m+n-i-1}}{x_i^{i+k-b_j-1}} + \frac{(1+x_i)^{m+n-i-1}}{x_i^{i+k-b_j}} \end{array} \right).$$

We pull out the factor

$$P^{[m=0]} \prod_{i=1}^{m+n-1} (1+x_i)^n x_i^{-i-k} \prod_{i=\max(1,m)}^{m+n-1} (1+x_i)^{m-i-1} (Px_i + 1),$$

and we obtain the following determinant

$$\det_{1 \leq i, j \leq m+n-1} \left(\begin{array}{c|c} (1+x_i)^{k+q-n-1} x_i^2, & i \leq k+m+q-2 \\ 0, & i \geq k+m+q-1 \end{array} \middle| x_i^{b_j} \right),$$

where we now only have to distinguish between $j = 1$ and $2 \leq j \leq m+n-1$ concerning the columns, while we have a homogeneous definition in the rows. We expand w.r.t. the first column and obtain

$$\sum_{l=1}^{k+m+q-2} (-1)^{l+1} (1+x_l)^{k+q-n-1} x_l^2 \det_{\substack{2 \leq j \leq m+n-1 \\ 1 \leq i \leq m+n-1, i \neq l}} (x_i^{b_j}). \quad (4.4)$$

Now we aim to sum over all $2 \leq b_2 < \dots < b_{m+n-1} \leq m+n+k-1$. However, since the constant term is zero if $b_{m+n-1} \geq m+n+k$, we can simply sum over all $2 \leq b_2 < \dots < b_{m+n-1}$. Observe that by Lemma 12 we get

$$\begin{aligned} \sum_{b \leq b_1 < b_2 < \dots < b_r} \det_{1 \leq i, j \leq r} (x_i^{b_j}) &= \sum_{b \leq b_1 < b_2 < \dots < b_r} \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn} \sigma \cdot \sigma [x_1^{b_1} \dots x_r^{b_r}] \\ &= \prod_{i=1}^l x_i^b \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn} \sigma \cdot \sigma \left[\sum_{0 \geq b_1 < b_2 < \dots < b_r} x_1^{b_1} \dots x_r^{b_r} \right] = \prod_{i=1}^r x_i^b \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn} \sigma \cdot \sigma \left[\prod_{i=1}^r \frac{x_i^{i-1}}{1 - \prod_{j=i}^r x_j} \right] \\ &= \prod_{i=1}^r x_i^b (1-x_i)^{-1} \prod_{1 \leq i < j \leq r} \frac{x_j - x_i}{1 - x_i x_j}. \quad (4.5) \end{aligned}$$

We sum (4.4) over all b_2, \dots, b_{m+n-1} with $2 \leq b_2 < \dots < b_{m+n-1}$ and obtain

$$\sum_{l=1}^{k+m+q-2} (-1)^{l+1} (1+x_l)^{k+q-n-1} x_l^2 \prod_{\substack{1 \leq i \leq m+n-1 \\ i \neq l}} x_i^2 (1-x_i)^{-1} \prod_{\substack{1 \leq i < j \leq m+n-1 \\ i, j \neq l}} \frac{x_j - x_i}{1 - x_i x_j}$$

It follows that the generating function of (m, n, k) -Magog trapezoids with q Minima w.r.t. the weight $P^\#$ of maxima is the constant term of

$$P^{[m=0]} \prod_{i=1}^{m+n-1} (1+x_i)^n x_i^{-i-k} \prod_{i=\max(m,1)}^{m+n-1} (1+x_i)^{m-i-1} (Px_i+1) \\ \times \sum_{l=1}^{k+m+q-2} (-1)^{l+1} (1+x_l)^{k+q-n-1} x_l^2 \prod_{\substack{1 \leq i \leq m+n-1 \\ i \neq l}}^{m+n-1} x_i^2 (1-x_i)^{-1} \prod_{\substack{1 \leq i < j \leq m+n-1 \\ i, j \neq l}} \frac{x_j - x_i}{1 - x_i x_j}$$

if we assume $q \leq n - k$.

In case $q = n - k + 1$, the number of Magog trapezoids is the constant term of

$$\det_{1 \leq i, j \leq m+n-1} \left(\begin{cases} \frac{(1+x_i)^n}{x_i^{i+k-b_j}}, & i = 1, \dots, m-1 \\ P \frac{(1+x_i)^{m+n-i-1}}{x_i^{i+k-b_j-1}} + \frac{(1+x_i)^{m+n-i-1}}{x_i^{i+k-b_j}}, & i = m, \dots, m+n-1 \end{cases} \right).$$

Again we pull out the factor

$$P^{[m=0]} \prod_{i=1}^{m+n-1} (1+x_i)^n x_i^{-i-k} \prod_{i=\max(m,1)}^{m+n-1} (1+x_i)^{m-i-1} (Px_i+1),$$

and, in this case, we obtain the following simple determinant.

$$\det_{1 \leq i, j \leq m+n-1} \left(x_i^{b_j} \right),$$

Finally, we sum over all $2 \leq b_1 < b_2 < \dots < b_{m+n-1}$, and, by (4.5), we obtain

$$P^{[m=0]} \prod_{i=1}^{m+n-1} (1+x_i)^n x_i^{-i-k+2} (1-x_i)^{-1} \prod_{i=\max(m,1)}^{m+n-1} (1+x_i)^{m-i-1} (Px_i+1) \prod_{1 \leq i < j \leq m+n-1} \frac{x_j - x_i}{1 - x_i x_j}.$$

We summarize our results in the following theorem.

Theorem 14. *Suppose $k \neq 1$ or $m + q \neq 1$. The generating function of (m, n, k) -Magog trapezoids w.r.t. the weight $P^\#$ of maxima and with q minima is equal to the constant term of*

$$P^{[m=0]} \prod_{i=1}^{m+n-1} (1+x_i)^n x_i^{-i-k+2} (1-x_i)^{-1} \prod_{i=\max(m,1)}^{m+n-1} (1+x_i)^{m-i-1} (Px_i+1) \prod_{1 \leq i < j \leq m+n-1} \frac{x_j - x_i}{1 - x_i x_j} \\ \times \begin{cases} \sum_{i=1}^{k+m+q-2} (1+x_i)^{k+q-n-1} (1-x_i) \prod_{\substack{1 \leq j \leq m+n-1 \\ j \neq i}} \frac{1-x_i x_j}{x_j - x_i}, & \text{if } q \leq n - k \\ 1, & \text{if } q = n - k + 1. \end{cases}$$

5. CONNECTING GOGS AND MAGOGS

In summary, we have seen that the constant term expressions for the number of *Gog-type objects* are all of the following form

$$s \cdot \prod_{i=1}^n f_i(x_i) \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 + x_j + x_i x_j),$$

where $f_i(x)$ are certain (simple) rational function—often of the form

$$f_i(x) = x^{a_i}(1+x)^{b_i},$$

a_i, b_i integers—and s is a sign. These functions usually have (somewhat) homogeneous definitions for certain ranges of i : for instance, when considering (m, n, k) -Gog trapezoids, the definition is homogeneous for $1 \leq i \leq k$ and also for $k+1 \leq i \leq n$, see Theorem 7. In this case, a factor $1+x$ in $f_i(x)$ is replaced by $1+Qx_i$ for $k+1 \leq i \leq n$ when considering the number of maxima, and $f_1(x)$ has an exceptional definition when considering the number of minima. When considering the generating function w.r.t. the two inversion numbers, the term $1+x_j+x_ix_j$ is replaced by $1+(1-v)x_i+u(x_j+x_ix_j)$ and some $f_i(x)$ have an additional factor $\frac{1}{(1+(1-v)x)^{c_i}}$, while others have an additional factor $\frac{1}{(1+ux)^{d_i}}$, c_i, d_i non-negative integers.

Concerning constant term expressions for the number of *Magog-type objects* we need to distinguish between the two versions. In the first case (Theorem 13), we have a sum of two expressions of the form

$$s \cdot \prod_{i=1}^n f_i(x_i) \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i + x_j + x_ix_j) \prod_{n-k+1 \leq i < j \leq n} \frac{1}{1 - x_ix_j},$$

while in the second case (Theorem 14), we have an expression of the form

$$s \cdot \prod_{i=1}^n f_i(x_i) \prod_{1 \leq i < j \leq m+n-1} \frac{x_j - x_i}{1 - x_ix_j}$$

if $q = n - k + 1$ and a slightly more complicated expression otherwise.

In [FZJ08] it was shown that the number of $n \times n$ ASMs (which is the same as the number of (n, n, n) -Gog trapezoids) is the constant term of the following expression

$$\prod_{i=1}^{n-1} (1+x_i)^2 x_i^{-2i+1} \prod_{1 \leq i < j \leq n-1} (x_j - x_i)(1+x_j+x_ix_j). \quad (5.1)$$

This is a Gog-type constant term expression as described above. Interestingly, this formula was derived using the six-vertex model approach, which is different from the approach that was used here. On the other hand, it was shown that the number of TSSCPPs in an $2n \times 2n \times 2n$ box (which is the same as the number of (n, n, n) -Magogs) is the constant term of

$$\prod_{i=1}^n (1+x_i)^i x_i^{-2i+2} (1-x_i^2)^{-1} \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{1 - x_ix_j}, \quad (5.2)$$

and this is a Magog-type constant term expression. In [FZJ08], Fonseca and Zinn-Justin succeeded in showing that the constant terms of the two expressions are equal. It can essentially be deduced from the following theorem, which was conjectured by Di Francesco and Zinn-Justin [ZJDF08] and first proven by Zeilberger [Zei07].

Theorem 15. *Let $S(x_1, \dots, x_n)$ be a power series in x_1, \dots, x_n that is symmetric in x_1, \dots, x_n . Then*

$$\begin{aligned} \text{CT}_{x_1, \dots, x_n} S(x_1, \dots, x_n) \prod_{i=1}^n x_i^{-2i+1} \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 + tx_j + x_i x_j) \\ = \text{CT}_{x_1, \dots, x_n} S(x_1, \dots, x_n) \prod_{i=1}^n \frac{(1 + tx_i)^{i-1} x_i^{-2i+1}}{1 - x_i^2} \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{1 - x_i x_j}. \end{aligned}$$

Fonseca and Zinn-Justin [FZJ08] gave another proof of the result, which is discussed next.

The general strategy (which also appeared in [Zei96]) is to compute *the symmetrizers* of (5.1) and of (5.2). (The symmetrizer is defined as the antisymmetrizer except for the sign $\text{sgn } \sigma$ which is omitted, see (2.7).) Clearly, in order to show that the constant terms of the original expressions are equal, it suffices to show this for the symmetrized expressions.

The computation of the symmetrizer is usually much easier for the Magog-type constant term expressions, as $\prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{1 - x_i x_j}$ is antisymmetric and therefore it suffices to compute the *antisymmetrizer* of $\prod_{i=1}^n f_i(x_i)$ (with $f_i(x) = (1 + x_i)^i x_i^{-2i+2} (1 - x_i^2)^{-1}$ in this case), which can often be accomplished using the Vandermonde determinant evaluation.

As for the Gog-type constant term expressions, the following result of Fonseca and Zinn-Justin [FZJ08] can be used to compute the symmetrizer of (5.1) (and also to prove Theorem 15). The symmetrization of the Gog-type expression (5.1) causes the “core” $\prod_{1 \leq i < j \leq n-1} (x_j - x_i)(1 + x_j + x_i x_j)$ to be transformed into $\prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{1 - x_i x_j}$ (which is the “core” of the Magog-type expression (5.1)), and so the expressions are very similar (but not equal) after symmetrization.¹

Theorem 16. *Let $h_q(w, y) = (qw - q^{-1}y)(qwy - q^{-1})$. Then the antisymmetrizer of*

$$\frac{\prod_{1 \leq i < j \leq n} (qw_i - q^{-1}w_j)}{\prod_{1 \leq i \leq j \leq n} h_1(w_j, y_i) \prod_{1 \leq j \leq i \leq n} h_q(w_j, y_i)} \quad (5.3)$$

w.r.t. w_1, \dots, w_n is

$$\frac{q^{\binom{n}{2}} \det_{1 \leq i, j \leq n} \left(\frac{1}{h_1(w_i, y_j) h_q(w_i, y_j)} \right)}{\prod_{1 \leq i < j \leq n} h_1(y_i, y_j) (1 - q^2 w_i w_j)}.$$

¹The following identity is often helpful for further manipulations (the proof is left to the reader as we will not use it here): Suppose $P(x_1, \dots, x_n), S(x_1, \dots, x_n)$ are two formal power series and $S(x_1, \dots, x_n)$ is symmetric. Then the constant term of

$$\frac{S(x_1, \dots, x_n) P(x_1, \dots, x_n)}{(x_1 \cdots x_n)^{n-1}} \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

agrees with the constant term of

$$\frac{S(0, \dots, 0) P(x_1, \dots, x_n)}{(x_1 \cdots x_n)^{n-1}} \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Now in order to compute the constant term of (5.1) using Theorem 16, one sets $w_i = (x_i q^{-1} - 1)/(x_i q - 1)$ in (5.3) and obtains

$$(q - q^{-1})^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} (1 - (q + q^{-1})x_j + x_i x_j) \prod_{i=1}^n (q x_i - 1)^{n+5} \\ \times \left(\prod_{j=1}^i (-1 + y_j + x_i (q - q^{-1} y_j)) (1 - y_j + x_i (-q^{-1} + q y_j)) \right. \\ \left. \times \prod_{j=i}^n (-q^{-1} + q y_j + x_i (1 - y_j)) (q - q^{-1} y_j + x_i (-1 + y_j)) \right)^{-1}. \quad (5.4)$$

We need to compute the antisymmetrizer of

$$\prod_{i=1}^n x_i^{-2i+1} \prod_{1 \leq i < j \leq n} (1 + x_j + x_i x_j) \quad (5.5)$$

After setting $y_j = 1$ and $q = e^{2i\pi/3}$, (5.4) is up to a factor that is symmetric in x_1, \dots, x_n equal to (5.5).

5.1. A variant of Theorem 16 and its application to (2.3) and to (2.6) when $b_i = i$. Next we state a theorem that is similar to Theorem 16 and that can be proven analogously.

Theorem 17. *Let $h_q(w, y) = (qw - q^{-1}y)$. Then the antisymmetrizer of*

$$\frac{\prod_{1 \leq i < j \leq n} h_q(w_i, w_j)}{\prod_{1 \leq i \leq j \leq n} h_1(w_j, y_i) \prod_{1 \leq j \leq i \leq n} h_q(w_j, y_i)} \quad (5.6)$$

w.r.t. w_1, \dots, w_n is

$$\frac{\det_{1 \leq i, j \leq n} \left(\frac{1}{h_1(w_i, y_j) h_q(w_i, y_j)} \right)}{\prod_{1 \leq i < j \leq n} h_1(y_j, y_i)}.$$

Proof. The proof is by induction w.r.t. n . The case $n = 1$ is easy to check.

Let $A(w_1, \dots, w_n; y_1, \dots, y_n)$ denote the antisymmetrizer of (5.6). We have the following recursion for $A(w_1, \dots, w_n; y_1, \dots, y_n)$.

$$A(w_1, \dots, w_n; y_1, \dots, y_n) = \sum_{k=1}^n (-1)^{k+n} \frac{\prod_{1 \leq i \leq n, i \neq k} h_q(w_i, w_k)}{\prod_{i=1}^n h_1(w_k, y_i) h_q(w_i, y_n)} A(w_1, \dots, \widehat{w}_k, \dots, w_n; y_1, \dots, y_{n-1})$$

By the induction hypothesis, the right-hand side is equal to

$$\sum_{k=1}^n (-1)^{k+n} \det_{\substack{1 \leq i \leq n, i \neq k \\ 1 \leq j \leq n-1}} \left(\frac{1}{h_1(w_i, y_j) h_q(w_i, y_j)} \right) \frac{\prod_{1 \leq i \leq n, i \neq k} h_q(w_i, w_k) \prod_{1 \leq i \leq n-1} h_1(y_n, y_i)}{\prod_{i=1}^n h_1(w_k, y_i) h_q(w_i, y_n) \prod_{1 \leq i < j \leq n} h_1(y_j, y_n)}. \quad (5.7)$$

We define

$$g_l(w_1, \dots, w_n; y_1, \dots, y_n) = \prod_{1 \leq i \leq n-1, i \neq l} \frac{h_1(y_i, y_n)}{h_1(y_l, y_i)} \prod_{i=1}^n \frac{h_q(w_i, y_l)}{h_q(w_i, y_n)}$$

and claim that

$$\sum_{l=1}^n g_l(w_1, \dots, w_n; y_1, \dots, y_n) \frac{1}{h_1(w_k, y_l) h_q(w_k, y_l)} = \frac{\prod_{1 \leq i \leq n, i \neq k} h_q(w_i, w_k) \prod_{1 \leq i \leq n-1} h_1(y_i, y_n)}{\prod_{1 \leq i \leq n} h_1(w_k, y_i) h_q(w_i, y_n)}. \quad (5.8)$$

Assuming (5.8) is true, we can replace in (5.7) the expression on the right-hand side of (5.8) by the left-hand side of (5.8). We change the order of summation and obtain

$$(-1)^{n-1} \prod_{1 \leq i < j \leq n} h_1(y_j, y_i)^{-1} \sum_{l=1}^n g_l \det_{1 \leq i, j \leq n} \left(\frac{1}{h_1(w_i, y_j) h_q(w_i, y_j)} \right) \Big|_{y_n = y_l}$$

and the last expression is equal to the expression in the theorem. In order to prove (5.8), we rearrange the identity and obtain

$$\sum_{l=1}^n \prod_{\substack{1 \leq i \leq n-1 \\ i \neq l}} \frac{h_1(y_i, y_n)}{h_1(y_l, y_i)} \prod_{\substack{1 \leq i \leq n \\ i \neq k}} h_q(w_i, y_l) \prod_{\substack{1 \leq i \leq n \\ i \neq l}} h_1(w_k, y_i) = \prod_{1 \leq i \leq n, i \neq k} h_q(w_i, w_k) \prod_{1 \leq i \leq n-1} h_1(y_i, y_n).$$

We consider both sides as polynomials in w_k . The degree is in both cases not greater than $n-1$ and so it suffices to show that they agree at the evaluations $w_k = y_p$, $1 \leq p \leq n$. In this case, each summand on the left-hand side vanishes except for the summand corresponding to $l = p$. \square

Note that the statement of this theorems differs from the statement of Theorem 16 only in so far that we set $h_q(w, y) = (qw - q^{-1}y)$ instead of $h_q(w, y) = (qw - q^{-1}y)(qwy - q^{-1})$.

We compute the symmetrizer of (2.3): We set $w_i = (x_i + 1 + q^{-1})/(x_i + 1 + q)$ in (5.6) and obtain

$$(q - q^{-1})^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} (q^{-1} + 2 + q + (q^{-1} + 1 + q)x_i + x_j + x_i x_j) \prod_{i=1}^n (1 + q + x_i)^2 \\ \times \left(\prod_{i=1}^n \prod_{j=1}^i (x_i(1 - y_j) + (1 + q^{-1})(1 - qy_j)) \prod_{j=i}^n (x_i(q - q^{-1}y_j) + (1 + q)(1 - y_j q^{-1})) \right)^{-1}.$$

By setting $y_i = 1$, $q = e^{2i\pi/3}$ and applying Theorem 17, we obtain

$$\mathcal{AS}_{x_1, \dots, x_n} \left[\prod_{1 \leq i < j \leq n} (1 + x_j + x_i x_j) \prod_{i=1}^n (1 + x_i)^i x_i^{-n+1} \right] \\ = (-1)^{\binom{n+1}{2}} q^n (q - q^{-1})^{(n+3)n/2} \prod_{i=1}^n (1 + x_i)^{n+1} x_i^{-n+1} (1 + q + x_i)^{-2} \\ \times \lim_{(y_1, \dots, y_n) \rightarrow 1} \det_{1 \leq i, j \leq n} \left(\frac{1}{\left(y_j - \frac{x_i+1+q^{-1}}{x_i+1+q} \right) \left(y_j - q^2 \frac{x_i+1+q^{-1}}{x_i+1+q} \right)} \right) \prod_{1 \leq i < j \leq n} (y_j - y_i)^{-1}. \quad (5.9)$$

Next we aim at using the Cauchy determinant

$$\det_{1 \leq i, j \leq n} \left(\frac{1}{x_i + y_j} \right) = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)}{\prod_{i, j=1}^n (x_i + y_j)}$$

to modify the right-hand side. We use the following partial fraction decomposition

$$\frac{1}{(y-a)(y-b)} = \frac{1}{a-b} \left(\frac{1}{y-a} - \frac{1}{y-b} \right)$$

to rewrite the matrix entry of the determinant. We obtain

$$\begin{aligned} & (-1)^{\binom{n}{2}} (q - q^{-1})^{\binom{n+1}{2}} \prod_{i=1}^n (1 + x_i)^{n+1} x_i^{-n+1} (x_i + 1 + q)^{-1} (x_i + 1 + q^{-1})^{-1} \\ & \times \lim_{(y_1, \dots, y_n) \rightarrow 1} \det_{1 \leq i, j \leq n} \left(\frac{1}{\left(y_j - \frac{x_i+1+q^{-1}}{x_i+1+q}\right)} - \frac{1}{\left(y_j - q^2 \frac{x_i+1+q^{-1}}{x_i+1+q}\right)} \right) \prod_{1 \leq i < j \leq n} (y_j - y_i)^{-1} \end{aligned}$$

for the right-hand side. The determinant can be written as a sum of 2^n Cauchy determinants.

$$\det_{1 \leq i, j \leq n} \left(\frac{1}{\left(y_j - \frac{x_i+1+q^{-1}}{x_i+1+q}\right)} - \frac{1}{\left(y_j - q^2 \frac{x_i+1+q^{-1}}{x_i+1+q}\right)} \right) = \sum_{(s_1, \dots, s_n) \in \{0,1\}^n} \det_{1 \leq i, j \leq n} \left((-1)^{s_i} \frac{1}{\left(y_j - q^{2s_i} \frac{x_i+1+q^{-1}}{x_i+1+q}\right)} \right)$$

In a similar situation, namely the proof of the equality of the constant terms of (5.1) and (5.2), only one of these Cauchy determinants contributes to the constant term. This is not true here, which is mainly due to the factor $\prod_{i=1}^n x_i^{-n+1}$ in (2.3). However, it is still possible to get rid of the extra set of variables y_1, \dots, y_n and it follows that the right-hand side is equal to

$$\begin{aligned} & \prod_{i=1}^n (1+x_i)^{n+1} x_i^{-n+1} (x_i+1+q^{-1})^{-1} \sum_{(s_1, \dots, s_n) \in \{0,1\}^n} (-1)^{(n-1)(s_1+\dots+s_n)+\binom{n}{2}} q^{-s_1-\dots-s_n} \prod_{i=1}^n (1+x_i)^{-ns_i} \\ & \times \prod_{1 \leq i < j \leq n} \left((1+x_i x_j)(s_i - s_j) + x_i(1-s_i-2s_j+3s_i s_j) + x_j(-1+s_j+2s_i-3s_i s_j) \right), \end{aligned}$$

and this can also be written as follows.

$$\prod_{i=1}^n (1+x_i)^{n+1} x_i^{-n+1} (x_i+1+q^{-1})^{-1} \det_{1 \leq i, j \leq n} (x_i^{j-1} - (-1-x_i)^{-j} q^{-1})$$

As for computing the symmetrizer of (2.6) when $b_i = i$, we set $w_i = (x_i q^{-1} - 1)/(x_i q - 1)$ in (5.6) and obtain

$$\begin{aligned} & (q - q^{-1})^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} (1 - (q + q^{-1})x_j + x_i x_j) \prod_{i=1}^n (q x_i - 1)^2 \\ & \times \left(\prod_{j=1}^i (-1 + y_j + x_i(q^{-1} - q y_j)) \prod_{j=i}^n (-q + y_j q^{-1} + x_i(1 - y_j)) \right)^{-1} \end{aligned} \quad (5.10)$$

Furthermore, we set $y_i = 1$ and $q = e^{i\pi/3}$, and Theorem 17 now implies in a similar way as above

$$\begin{aligned} \mathcal{A}S_{x_1, \dots, x_n} & \left[\prod_{1 \leq i < j \leq n} (1 - x_j + x_i x_j) \prod_{i=1}^n (1 - x_i)^{-n} x_i^{-n+1-i} \right] = \\ & \prod_{i=1}^n (1 - x_i)^{-n} x_i^{-n+1} (q^{-1} x_i - 1)^{-1} \sum_{(s_1, \dots, s_n) \in \{0, 1\}^n} (-q)^{-s_1 - \dots - s_n} (-1)^{\binom{s_1 + \dots + s_n}{2} + \binom{n - s_1 - \dots - s_n}{2} + n} \prod_{i=1}^n x_i^{n(s_i - 1)} \\ & \times \prod_{1 \leq i < j \leq n} ((1 + x_i x_j)(s_i - s_j) + x_i(-1 + s_i + 2s_j - 3s_i s_j) + x_j(1 - s_j - 2s_i + 3s_i s_j)). \end{aligned}$$

This can also be written as

$$\prod_{i=1}^n (1 - x_i)^{-n} x_i^{-n+1} (q^{-1} x_i - 1)^{-1} \det_{1 \leq i, j \leq n} (-x_i^{-j} + (1 - x_i)^{j-1} q^{-1}).$$

5.2. The application of Theorem 17 to (2.8) when $b_i = i$. The expression in (2.8) is more complicated at first glance as it already involves the antisymmetrizer operator. However, as our approach involves the computation of the antisymmetrizer anyway, this is no disadvantage.

By (5.9), (2.8) is equal to

$$\begin{aligned} & (-1)^{\binom{n+1}{2}} q^n (q - q^{-1})^{(n+3)n/2} \prod_{i=1}^n (1 + x_i)^{n+1} (1 + q + x_i)^{-2} \\ & \times \lim_{(y_1, \dots, y_n) \rightarrow 1} \det_{1 \leq i, j \leq n} \left(\frac{1}{\left(y_j - \frac{x_i + 1 + q^{-1}}{x_i + 1 + q}\right) \left(y_j - q^2 \frac{x_i + 1 + q^{-1}}{x_i + 1 + q}\right)} \right) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-1} (y_j - y_i)^{-1}, \end{aligned} \quad (5.11)$$

where $q = e^{2\pi/3}$. The same procedure as above can be used to show that this is equal to

$$\begin{aligned} & \prod_{i=1}^n (1 + x_i)^{n+1} (x_i + 1 + q^{-1})^{-1} \sum_{(s_1, \dots, s_n) \in \{0, 1\}^n} (-1)^{(n-1)(s_1 + \dots + s_n) + \binom{n}{2}} q^{-s_1 - \dots - s_n} \prod_{i=1}^n (1 + x_i)^{-ns_i} \\ & \times \prod_{1 \leq i < j \leq n} \frac{(1 + x_i x_j)(s_i - s_j) + x_i(1 - s_i - 2s_j + 3s_i s_j) + x_j(-1 + s_j + 2s_i - 3s_i s_j)}{x_j - x_i} \\ & = \prod_{i=1}^n (1 + x_i)^{n+1} (x_i + 1 + q^{-1})^{-1} \det_{1 \leq i, j \leq n} (x_i^{j-1} - (-1 - x_i)^{-j} q^{-1}) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-1}. \end{aligned} \quad (5.12)$$

On the other hand, the advantage of (2.8) is that it is actually a polynomial. This enables us to use a formula that appeared in [BFZJ12, Eq (43)-(47)] to express the constant term of (5.11) in terms of a ‘‘binomial determinant’’. For convenience, we repeat the procedure here. Suppose $f(x, y)$ is a power series in x and y , then

$$\frac{\det_{1 \leq i, j \leq n} (f(x_i, y_j))}{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)} = \det_{1 \leq i, j \leq n} (f[x_1, \dots, x_i][y_1, \dots, y_j]),$$

where

$$f[x_1, \dots, x_i][y_1, \dots, y_j] = \sum_{k,l=0}^{\infty} c_{i+k-1,j+l-1} h_k(x_1, \dots, x_i) h_l(y_1, \dots, y_j),$$

$c_{i,j}$ is the coefficient of $x^i y^j$ in $f(x, y)$ and $h_p(x_1, \dots, x_q)$ is the p -th complete symmetric function in x_1, \dots, x_q . It follows that

$$\lim_{\substack{(x_1, \dots, x_n) \rightarrow (x, \dots, x) \\ (y_1, \dots, y_n) \rightarrow (y, \dots, y)}} \frac{\det_{1 \leq i, j \leq n} (f(x_i, y_j))}{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)} = \det_{0 \leq i, j \leq n-1} \left(\sum_{k,l=0}^{\infty} c_{i+k,j+l} \binom{i+k-1}{k-1} \binom{j+l-1}{l-1} x^k y^l \right),$$

in particular, if $x = y = 0$, we simply obtain $\det_{0 \leq i, j \leq n-1} (c_{i,j})$. Now, by (5.11), it follows that the number of $n \times n$ ASMs is

$$\begin{aligned} & \lim_{\substack{(x_1, \dots, x_n) \rightarrow 0 \\ (y_1, \dots, y_n) \rightarrow 0}} (-1)^n (q^{-1} - q)^{\binom{n+1}{2}} \prod_{i=1}^n (1 + x_i)^{n+1} (1 + q^{-1} + x_i)^{-1} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-1} (y_j - y_i)^{-1} \\ & \times \det_{1 \leq i, j \leq n} \left(\frac{1}{(y_j + 1)(x_i + 1 + q) - x_i - 1 - q^{-1}} - \frac{1}{(y_j + 1)(x_i + 1 + q) - q^2(x_i + 1 + q^{-1})} \right). \end{aligned} \quad (5.13)$$

The coefficient of $x^i y^j$ in

$$\frac{1}{(y+1)(x+1+q) - x - 1 - q^{-1}} - \frac{1}{(y+1)(x+1+q) - q^2(x+1+q^{-1})}$$

when considering this expression as a power series in x, y is

$$\binom{j}{i} (1+q)^{-i-1} (q-1)^{-j-1} (-1)^j q^{j+1} - \sum_{k=i}^{i+j} \binom{k}{j} \binom{j}{i-k+j} \frac{(1+q)^{k-i-j-1}}{(1-q)^{j+1}} (-1)^k.$$

It follows that the number of $n \times n$ ASMs is given by

$$(1+q)^{-n} (-q)^{-\binom{n}{2}} \det_{0 \leq i, j \leq n-1} \left(\binom{j}{i} (-1)^j q^{j+1} + \sum_{k=0}^{n-1} \binom{k+i}{j} \binom{j}{k} (-1-q)^{k+i-j} \right).$$

Using basis properties of the binomial coefficient and the Chu-Vandermonde summation, it can be shown that

$$\begin{aligned} & \binom{j}{i} (-1)^j q^{j+1} + \sum_{k=0}^{n-1} \binom{k+i}{j} \binom{j}{k} (-1-q)^{k+i-j} \\ & = \sum_{k=0}^{n-1} (-1)^i \binom{i}{k} \binom{-k-1}{j} (q^{j+1} (-1)^k + q^k (-1)^j), \end{aligned}$$

and it follows that the number of $n \times n$ ASMs is given by

$$\det \left[\left(\binom{i}{j} (-1)^{i+j} \right)_{0 \leq i, j \leq n-1} \cdot \left(\binom{i+j}{j} \frac{1 - (-q)^{j+1-i}}{1+q} \right)_{0 \leq i, j \leq n-1} \right] \\ = \det_{0 \leq i, j \leq n-1} \left(\binom{i+j}{j} \frac{1 - (-q)^{j+1-i}}{1+q} \right),$$

provided that $q = e^{2i\pi/3}$. This seems to be a new determinant for the ASM numbers.

In a forthcoming paper, we will compute the more general determinant

$$\det_{0 \leq i, j \leq n-1} \left(\binom{x+i+j}{j} \frac{1 - (-q)^{j+1-i}}{1+q} \right) \quad (5.14)$$

for all sixth roots of unity q not equal to 1, and study consequences of this for the enumeration of Gogs. In these cases, the determinants have only integer zeros as a polynomial in x . Moreover, the determinant is related to the following determinant

$$\det_{0 \leq i, j \leq n-1} \left(\binom{x+i+j}{j} + q \delta_{i,j} \right)$$

that was considered by Ciucu, Eisenkölbl, Krattenthaler and Zare [CEKZ01], and computed for all sixth roots of unity q . To be more precise, the quotient of the first and the second determinant is $(-q)^n$ if $q = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and it is $(-q)^{-n}$ if $q = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Other curious observations such as the following will be studied: For general q , there seems to be a sequence of functions $p_n(x, q)$ that are polynomials in x of degree no greater than $\binom{\lceil (n+1)/2 \rceil}{2}$ and Laurent polynomials in q with highest exponent $\binom{\lfloor (n+1)/2 \rfloor}{2}$ and lowest exponent $-\binom{\lfloor (n+1)/2 \rfloor}{2}$ over \mathbb{Q} such that the determinant in (5.14) is equal to $p_{n-1}(x, q)p_n(x, q)$.

Finally, we point out that we could also have worked with (5.12) and have used the following variant of the formula by Behrend, Di Francesco and Zinn-Justin [BFZJ12, Eq (43)-(47)]: Suppose $f_i(x)$ is for $i \in \{1, \dots, n\}$ a power series in x , then

$$\frac{\det_{1 \leq i, j \leq n} (f_i(x_j))}{\prod_{1 \leq i < j \leq n} (x_j - x_i)} = \det_{1 \leq i, j \leq n} (f_i[x_1, \dots, x_j]),$$

where

$$f_i[x_1, \dots, x_j] = \sum_{l=0}^{\infty} c_{i,j+l} h_l(x_1, \dots, x_j),$$

and $c_{i,j}$ is the coefficient of x^j in $f_i(x)$. It follows that

$$\lim_{(x_1, \dots, x_n) \rightarrow (x, \dots, x)} \frac{\det_{1 \leq i, j \leq n} (f_i(x_j))}{\prod_{1 \leq i < j \leq n} (x_j - x_i)} = \det_{1 \leq i, j \leq n-1} \left(\sum_{l=0}^{\infty} c_{i,j+l-1} \binom{j+l-1}{l} x^l \right).$$

APPENDIX A. THE CASE $v = 1 - u$

A.1. **The cases $(u, v) = (0, 1)$ and $(u, v) = (1, 0)$.** The generating function of monotone triangles with bottom row b_1, \dots, b_n evaluated at $(u, v) = (0, 1)$ is equal to the number of monotone triangles with that bottom row, where SE-diagonals are strictly increasing, while the evaluation at $(u, v) = (1, 0)$ is the number of monotone triangles that are strictly increasing along NE-diagonals. The number is in both cases $\prod_{1 \leq i < j \leq n} \frac{b_j - b_i}{j - i}$ as both sets are in bijective

correspondence with Gelfand-Tsetlin patterns with bottom row $b_1 - 1, b_2 - 2, \dots, b_n - n$ (in the first case, this follows by subtracting i from the i -th NE-diagonal for all i , counted from the left, while in the second case, this follows by subtracting i from the i -th SE-diagonal for all i , also counted from the left), and the number of Gelfand-Tsetlin patterns with bottom row b_1, \dots, b_n is $\prod_{1 \leq i < j \leq n} \frac{b_j - b_i + j - i}{j - i}$, see [GT50] or [Sta99, Corollary 7.21.4 and Lemma 7.21.1].

A.2. The case $(u, v) = (\frac{1}{2}, \frac{1}{2})$. On the other hand, the generating function of monotone triangles with bottom row b_1, \dots, b_n evaluated at $(u, v) = (\frac{1}{2}, \frac{1}{2})$ is—up to the factor $2^{\binom{n}{2}}$ —equal to the 2-enumeration of monotone triangles with respect to the number of entries that lie strictly between their SW-neighbors and and their SE-neighbors. By [MRR83, Theorem 2], this number is $2^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} \frac{b_j - b_i}{j - i}$. If $(b_1, \dots, b_n) = (1, \dots, n)$, this corresponds to the 2-enumeration of ASMs with respect to the number of -1 's, or, equivalently, to the enumeration of domino tilings of the Aztec diamond of order n , see [EKLP92] and also [Ciu97, Remark 4.3], where in order to obtain the number of the latter one has to multiply by 2^n .

A.3. The general case. The results mentioned in the previous two paragraphs can also be deduced from Theorem 2 as follows. In all cases, $\prod_{1 \leq p < q \leq n} E_{x_q} \text{Strict}_{x_q, x_p}$ is a symmetric polynomial in $\bar{\Delta}_{x_1}, \dots, \bar{\Delta}_{x_n}$ with constant term 1, and the enumeration formulas follow from the fact that the application of an “operator” polynomial with these properties to $\prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{j - i}$ leaves the polynomial invariant as was shown, e.g., in [Fis16, Lemma 2.5]. In fact, a common generalization follows if we assume that $v = 1 - u$, because in this case the same argument applies and we can conclude that the generating function is $\prod_{1 \leq i < j \leq n} \frac{b_j - b_i}{j - i}$. (This result could also be derived using known proofs for the 2-enumeration of ASMs or the enumeration of perfect matchings of the Aztec diamonds, see, e.g., [MRR83, EKLP92].) In particular, it follows that the generating function is independent of u .

A.3.1. Perfect matchings. The specialization $v = 1 - u$ of the generating function is also the weighted enumeration of perfect matchings of a certain portion of the square grid: First observe that, by extending the bijection between monotone triangles with bottom row $1, 2, \dots, n$ and $n \times n$ ASMs to monotone triangles with arbitrary increasing bottom row of positive² integers, say, b_1, \dots, b_n , we see that the latter are in bijection to $n \times m$ matrices, where m is any positive integer with $m \geq b_n$ that have the same properties as ASMs with the exception that column sums do not have to be 1, but it is still required that the topmost non-zero entry of each column is 1, and, in addition, the column sums have to be 1 precisely for the columns b_1, \dots, b_n .

Now let $\text{AR}^{n,m}$ denote the rectangular graph that consists of n rows of sequences of m consecutive cells of the form \diamond , see Figure 5 (left) for $\text{AR}^{4,7}$. It is a well-known fact, see [Ciu96], that the perfect matchings of $\text{AR}^{n,n}$ can be partitioned into classes that are indexed by $n \times n$ ASMs, such that, for a given ASM, the number of perfect matchings in the associated class is a power of 2, where the exponent is just the number of 1's in the ASMs. This can be extended to the “rectangular alternating sign matrices” described in the previous paragraph as follows. For a given strictly increasing sequence of positive integers $\mathbf{b} = (b_1, \dots, b_n)$ of length n and $m \geq b_n$, let $\text{AR}_{\mathbf{b}}^{n,m}$ denote the graph that is obtained from $\text{AR}^{n,m}$ by adding vertical

²It is no restriction to confine our considerations to monotone triangles that contain only positive integers in the bottom row, because every monotone triangle that contains non-positive integers can be transformed to one that has only positive integers by adding the same positive integer to every entry.

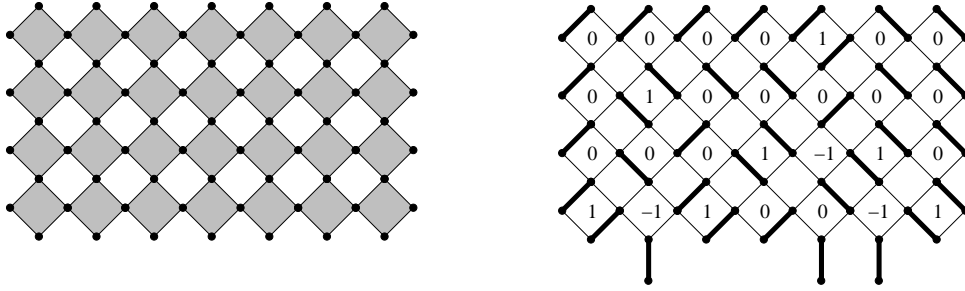


FIGURE 5. $AR^{4,7}$ (left) and an example (right)

edges incident with the bottom vertices of the cells in the bottom row of $AR^{n,m}$ except for the vertices in columns b_1, \dots, b_n . For instance, the underlying graph in Figure 5 (right) is $AR^{4,7}_{1,3,4,7}$. Then the perfect matchings of $AR^{n,m}_{\mathbf{b}}$ can be partitioned such that the classes are indexed by monotone triangles with bottom row \mathbf{b} and the number of elements in each class is a power of 2, where the exponent is the number of 1's in the corresponding rectangular alternating sign matrix. The perfect matching in Figure 5 (right) is a perfect matching that lies in the class of the following monotone triangle.

$$\begin{array}{cccc}
 & & 5 & \\
 & & 2 & 5 \\
 & 2 & 4 & 6 \\
 1 & 3 & 4 & 7
 \end{array}$$

The corresponding rectangular alternating sign matrix can be obtained by counting, for each cell, the number of edges that are part of the perfect matching and subtract 1, see Figure 5 (right), where we have put the numbers into the appropriate cells. The other perfect matchings that are in the class of the given monotone triangle are obtained by “rotating” the perfect matching edges of those cells that contain two matching edges independently.

Next we introduce edge weights such that the weighted enumeration of the perfect matchings of $AR^{n,m}_{\mathbf{b}}$ associated with a fixed monotone triangle M is just $u^{\text{inv}(M)}(1-u)^{\text{inv}'(M)}$: in each cell, we assign the weights $1, 1, u, 1-u$ to the edges, where we start at the NW edge and go around the cell clockwise, see Figure 6 right. As a side remark note that this shows that the special cases $(u, v) = (1, 0)$ and $(u, v) = (0, 1)$ in the generating function amounts to compute the number of perfect matchings of a *hexagonal grid* (since $(u, v) = (1, 0)$ corresponds to the deletion of the SW edge of each cell, while $(u, v) = (0, 1)$ corresponds to the deletion of the SE edge of each cell), which is no surprise, because counting perfect matchings of hexagonal grids corresponds to *lozenge tiling enumeration*, which in turn is related to the enumeration of *semistandard tableaux* and thus of Gelfand-Tsetlin patterns.

A.3.2. *Independence of u .* We sketch an argument that shows that the weighted enumeration of our perfect matchings is independent of u , and thus the generating function of monotone triangles (after setting $v = 1 - u$): This is done with the help of the local graph operation *urban renewal* which was introduced by Kuperberg and Propp, see Figure 6. If we replace a cell with edge weights a, b, c, d by the configuration indicated in Figure 6 (right), the generating function of perfect matchings of the original graph is obtained from the generating function of

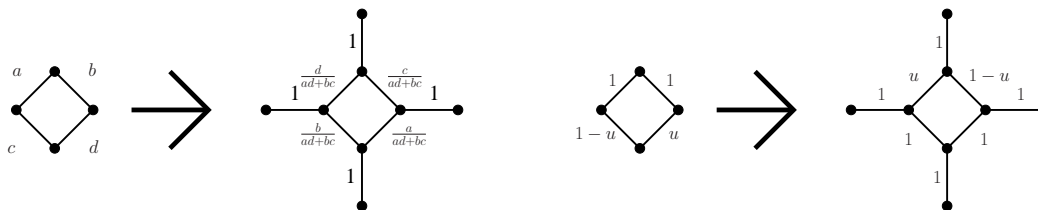


FIGURE 6. Urban renewal

the modified graph by multiplication of $ad + bc$. We perform this operation to every cell of, say, $\text{AR}_{1,3,4,7}^{4,7}$.

In the graph that is obtained this way, there are two trivial simplifications that can be made at several places: If the two edges that are incident with a vertex of degree 2 have weight 1, they can be contracted without changing the generating function, and each edge with weight 1 that is incident with a vertex of degree 1 can be deleted along with all edges incident with the other vertex of the edge. This implies that the weighted enumeration of the perfect matchings of $\text{AR}_{1,3,4,7}^{4,7}$ is equal to the sum of the weighted enumeration of the perfect matchings of the following graphs: $\text{AR}_{2,3,6}^{3,6}$, $\text{AR}_{1,3,6}^{3,6}$, $\text{AR}_{2,3,5}^{3,6}$, $\text{AR}_{1,3,5}^{3,6}$, $\text{AR}_{2,3,4}^{3,6}$, $\text{AR}_{1,3,4}^{3,6}$. (The sequences arise as follows: The complement of $\{1, 3, 4, 7\}$ in $\{1, \dots, 7\}$ is $\{2, 5, 6\}$, and now we allow for each element i in the complement that either the element itself or $i - 1$ is an element of the new complements. If we take complements in $\{1, \dots, 6\}$, then we obtain our sequences.) We can assume by induction with respect to n that the generating function of the perfect matchings of each of these graphs is independent of u .

REFERENCES

- [BC13] P. Biane and H. Cheballah. Gog, Magog and Schützenberger II: left trapezoids. In *25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013)*, Discrete Math. Theor. Comput. Sci. Proc., AS, pages 349–360. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2013.
- [BC16] P. Biane and H. Cheballah. Gog and GOGAm pentagons. *J. Combin. Theory Ser. A*, 138:133–154, 2016.
- [Bet16] J. Bettinelli. A simple explicit bijection between $(n, 2)$ -Gog and Magog-trapezoids. *Sém. Lothar. Combin.*, 75:Article B75e, 2016.
- [BFZJ12] R.E. Behrend, P. Di Francesco, and P. Zinn-Justin. On the weighted enumeration of alternating sign matrices and descending plane partitions. *J. Comb. Theory Ser. A*, 119:331–363, 2012.
- [Bre99] D. Bressoud. *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*. Cambridge: Mathematical Association of America/Cambridge University Press, 1999.
- [CB12] H. Cheballah and P. Biane. Gog and Magog triangles, and the Schützenberger involution. *Sém. Lothar. Combin.*, 66:Art. B66d, 20, 2011/12.
- [CEKZ01] M. Ciucu, T. Eisenkölbl, C. Krattenthaler, and D. Zare. Enumeration of lozenge tilings of hexagons with a central triangular hole. *J. Combin. Theory Ser. A*, 95(2):251–334, 2001.
- [Ciu96] M. Ciucu. Perfect matchings of cellular graphs. *J. Algebraic Combin.*, 5(2):87–103, 1996.
- [Ciu97] M. Ciucu. Enumeration of perfect matchings in graphs with reflective symmetry. *J. Combin. Theory Ser. A*, 77(1):67–97, 1997.
- [EKL92] N.D. Elkies, G. Kuperberg, M. Larsen, and J. Propp. Alternating-sign matrices and domino tilings. I. *J. Algebraic Combin.*, 1(2):111–132, 1992.
- [Fis06] I. Fischer. The number of monotone triangles with prescribed bottom row. *Adv. Appl. Math.*, 37:249–267, 2006.

- [Fis09] I. Fischer. An operator formula for the number of halved monotone triangles with prescribed bottom row. *J. Comb. Theory Ser. A.*, 116:515–538, 2009.
- [Fis10] I. Fischer. The operator formula for monotone triangles - simplified proof and three generalizations. *J. Comb. Theory Ser. A*, 119:1143, 2010.
- [Fis11] I. Fischer. Refined enumerations of alternating sign matrices: monotone (d, m) -trapezoids with prescribed top and bottom row. *J. Alg. Combin.*, 33:239 – 257, 2011.
- [Fis16] I. Fischer. Short proof of the ASM theorem avoiding the six-vertex model. *J. Comb. Theory Ser. A*, 144:139–156, 2016.
- [FR15] I. Fischer and L. Riegler. Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity. *Elect. J. Combin.*, 22:P. 1.5, 32 pp. (electronic), 2015.
- [FZJ08] T. Fonseca and P. Zinn-Justin. On the doubly refined enumeration of alternating sign matrices and totally symmetric self-complementary plane partitions. *Elect. J. Combin.*, 15:35pp, 2008.
- [GT50] I.M. Gelfand and M.L. Tsetlin. Finite-dimensional representations of the group of unimodular matrices (in russian). *Doklady Akad. Nauk. SSSR (N.S.)*, 71:825–828, 1950.
- [GV85] I. Gessel and G. X. Viennot. Binomial determinants, paths, and hook length formulae. *Adv. Math.*, 58(3):300–321, 1985.
- [Ish07] M. Ishikawa. Refined enumerations of totally symmetric self-complementary plane partitions and constant term identities. *Proc. FPSAC'07*, 2007.
- [Kra96] C. Krattenthaler. A Gog-Magog Conjecture. <http://www.mat.univie.ac.at/~kratt/artikel/magog.html>, 1996.
- [Kra16] C. Krattenthaler. Plane partitions in the work of Richard Stanley and his school. In *The mathematical legacy of Richard P. Stanley*, pages 231–261. Amer. Math. Soc., Providence, RI, 2016.
- [Kup96] G. Kuperberg. Another proof of the alternating sign matrix conjecture. *Int. Math. Res. Notices*, 3:139–150, 1996.
- [Lin73] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, 5:85–90, 1973.
- [MRR82] W.H. Mills, D.P. Robbins, and H.C. Rumsey, Jr. Proof of the Macdonald conjecture. *Invent. Math.*, 66(1):73–87, 1982.
- [MRR83] W. H. Mills, D.P. Robbins, and H.C. Rumsey, Jr. Alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 34(3):340–359, 1983.
- [MRR86] W.H. Mills, D.P. Robbins, and H.C. Rumsey, Jr. Self-complementary totally symmetric plane partitions. *J. Combin. Theory Ser. A*, 42(2):277–292, 1986.
- [RR86] D.P. Robbins and H.C. Rumsey, Jr. Determinants and alternating sign matrices. *Adv. Math.*, 62(2):169–184, 1986.
- [Sta99] R. P. Stanley. *Enumerative Combinatorics, Volume 2*. Cambridge University Press, 1999.
- [Zei94] D. Zeilberger. A constant term identity featuring the ubiquitous (and mysterious) Andrews-Mills-Robbins-Rumsey numbers 1, 2, 7, 42, 429, \dots . *J. Combin. Theory Ser. A*, 66(1):17–27, 1994.
- [Zei96] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electron. J. Comb.*, 3:1–84, 1996.
- [Zei07] D. Zeilberger. Proof of a conjecture of Philippe Di Francesco and Paul Zinn-Justin related to the qKZ equations to Dave Robbins' two favorite combinatorial objects. <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/diFrancesco.html>, 2007.
- [ZJDF08] P. Zinn-Justin and P. Di Francesco. The quantum Knizhnik-Zamolodchikov equation, completely symmetric self-complementary plane partitions, and alternating-sign matrices. *Teoret. Mat. Fiz.*, 154(3):387–408, 2008.

ILSE FISCHER, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA

E-mail address: `ilse.fischer@univie.ac.at`