A polynomial method for the enumeration of plane partitions and alternating sign matrices

Habilitationsschrift
zur
Erlangung der Lehrbefugnis (venia docendi)
für Mathematik
an der
Fakultät für Mathematik
der
Universität Wien

von

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Wien, im September 2005

Contents

Preface

This habilitation thesis consists of my four articles [**15, 16, 17, 18**], which can be found in the four chapters of the thesis. The first three articles are published, or are accepted for publication, in refereed journals. The very recent (and thus unpublished) preprint [**18**, resp. Chapter 4] contains an application of the operator formula in [**17**, resp. Chapter 3], providing a new proof of the long-standing *refined alternating sign matrix (ex-)Conjecture* first proved by Zeilberger [**65**]. Thereby, it underlines the significance of this operator formula and should consequently be included in this collection of articles.

There are common themes linking the four parts of the thesis. One connection is quite obvious – the articles deal with enumeration problems on closely related combinatorial objects. In [**15, 16**, resp. Chapter 1 and 2] I have solved two *plane partition* enumeration problems and in [**17, 18**, resp. Chapter 3 and 4] I am considering *alternating sign matrices*. Plane partitions are two-dimensional integer arrays, which are decreasing along rows and columns, thus generalizing ordinary (one-dimensional) partitions. Alternating sign matrices, on the other hand, are equivalent to two-dimensional integer arrays of triangular shape *(monotone triangles)*, which also decrease along rows and columns and, in addition, strictly decrease along diagonals. The significance of these objects is partly due to their close relations to other areas such as *representation theory of classical groups* and *statistical mechanics*. On the other hand, this field is undoubtedly challenging and thus attracting because the enumeration of these objects subject to a variety of different constraints leads to nice product formulas of compelling simplicity, which (still) seem to require highly non-trivial proofs. In the following two paragraphs we give a short account on the history of this area. (An extensive treatment of this topic can be found in Bressoud's book [**7**].)

Plane partitions appeared for the first time in the work of MacMahon [**40**] about a century ago. In the early 1960s, contributions by Carlitz, Cheema, Gordon and Houten [**8, 9, 22, 23, 24, 25**] prove an increasing interest in these objects. Then, in two important papers by Bender and Knuth [**6**] and by Stanley [**55**] respectively, the connection between *symmetric functions* and the enumeration of plane partitions is established and, from that time on, plane partitions are intensively studied objects in enumerative and algebraic combinatorics. The paper by Bender and Knuth also contains the well-known Bender-Knuth (ex-)Conjecture, which was subsequently proved by Andrews [**2**], Gordon [**26**], Macdonald [**38**, Ex. 19, p. 53] [**39**, Ex. 19, p. 86] and Proctor [**46**, Prop. 7.2]. In fact, Andrews showed the equivalence of this conjecture with a conjecture on the generating function of certain symmetric plane partitions, which is already due to MacMahon [**40**] and was proved by Andrews [**1**] much later. Another proof based on the Weyl character formula for type B_n was given by Macdonald in [**38**, Ex. 17, p. 52] [**39**, Ex. 17, p. 84 – 85]. In the following, the interest focused on the enumeration of the nine other *symmetry classes of plane partitions* [**56**], all which, quite remarkably, result in simple product formulas. This task was finally accomplished in 1995 by the work of Andrews [**4**], Kuperberg [**36**], Mills, Robbins and Rumsey [**41, 43**], Proctor [**45, 47**], Stanley [**56**] and Stembridge [**60**]. The only exception is the q-enumeration of totally symmetric plane partitions and a refined enumeration of totally symmetric self-complementary plane partitions [**43**], which are still open.

Alternating sign matrices were introduced by Robbins and Rumsey [**52**] in the course of generalizing Dodgson's [1] condensation of determinants algorithm for evaluating determinants. Out of curiosity, they raised the problem of enumerating alternating sign matrices of fixed size and came up with the appealing conjecture that the number of $n \times n$ alternating sign matrices is given by the simple product formula

$$\prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!}.$$

This turned out to be one of the most difficult problems in enumeration that was ever considered. In 1996 Zeilberger [**64**] finally succeeded in proving this conjecture. His proof is based on *constant term identities.* A further breakthrough was possible after Jim Propp's discovery of a surprising relation to *statistical mechanics:* physicists had introduced a model *(six-vertex model)* for "square ice", which turned out to be equivalent to alternating sign matrices. Kuperberg [**34**] was able to use their results on the six-vertex model to give another proof of the alternating sign matrix theorem. As early as in the 1980s and in perfect analogy to the considerations regarding plane partitions, Robbins [**53**] proposed the study of the eight *symmetry classes of alternating sign matrices,* as in six cases the enumeration again seemed to result in simple product formulas. Kuperberg solved some of these problems and introduced new classes of alternating sign matrices in [**35**]. Enumerations of further symmetry classes, as well as refined enumerations, 2-enumerations and 3-enumerations of alternating sign matrices were established by Colomo and Pronko, Eisenkölbl, Hamel and King, Okada, Razumov and Stroganov, Stroganov [**10, 11, 14, 28, 44, 49, 50, 51, 61, 62, 63**]. Now, all conjectures from [**53**] are solved, except for the conjecture on diagonally and antidiagonally symmetric alternating sign matrices of odd order. However, there is another fact which still seeks for an explanation: there are the same numbers of many types of alternating sign matrices as there are of other types of plane partitions. For instance, there are exactly as many alternating sign matrices with no symmetry as there are plane partitions with full symmetry. So far, none of these relations between alternating sign matrices and plane partitions is bijectively explained and to construct such a bijection is currently one of the most challenging problems in this field.

In view of these relations between plane partitions and alternating sign matrices it is natural to aim for a unified approach to solve the respective enumeration problems. This is a central motivation for the research presented in this thesis. In the four articles, I have developed a *"polynomial method"* to solve various problems in this area. Indeed, many enumeration formulas are polynomials in certain parameters if the other parameters are fixed. For instance, the binomial coefficient $\binom{x}{n}$ is a polynomial in x for fixed n. It is a fundamental fact that a polynomial (in one variable) of degree n is determined by $n+1$ "independent" properties. Examples for such properties are zeros or certain symmetries of the polynomial. This fact motivates the following method for proving polynomial enumeration formulas: suppose we consider an enumeration problem where we suspect the enumeration formula to be a polynomial in certain parameters. The method consists in the following three steps.

(1) First we have to prove that the enumeration formula is indeed a polynomial. This is often accomplished by a simple recursion.
(2) Next we have to compute the degree of the polynomial.

[1] Reverend Charles Lutwidge Dodgson is better known as Lewis Carroll, the author of "Alice in wonderland".

(3) Finally we have to deduce enough other properties of the polynomial in order to be able to compute it. What "enough" is, is determined by the degree of the polynomial.

This establishes the basis for my point of view on plane partition and alternating sign matrix enumeration problems here. My method can be seen as an alternative approach *complementing the standard technique* in this field, where one translates[2] the enumeration problem into a (in many cases highly non-trivial) determinant evaluation problem. Next I elaborate on this alternative approach by individually discussing the articles of the habilitation thesis.

In [**15**, resp. Chapter 1] I present a new refinement of the Bender-Knuth (ex-)Conjecture, which easily implies the Bender-Knuth (ex-)Conjecture itself. This is probably the most elementary way currently known to prove this result. In [**6**, p.50] Bender and Knuth conjectured that the generating function of strict plane partitions with at most c columns and parts in $\{1, 2, \ldots, n\}$ is equal to

$$\sum_{\pi} q^{n(\pi)} = \prod_{i=1}^{n} \frac{[c+i;q]_i}{[i;q]_i},$$

where $[n;q] = 1 + q + \cdots + q^{n-1}$, $[a;q]_n = \prod_{i=0}^{n-1} [a+i;q]$ and the norm $n(\pi)$ of a plane partition is defined as the sum of its parts. Surprisingly, my research led me to the observation that the refined enumeration with respect to a new parameter k, which counts the number of parts equal to n, still results in a simple product formula. Namely, the generating function of strict plane partitions with at most c columns, parts in $\{1, 2, \ldots, n\}$ and k parts equal to n is

$$\frac{q^{kn}[k+1;q]_{n-1}[1+c-k;q]_{n-1}}{[1;q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1;q]_{i-1}}{[i;q]_i}.$$

If we set $q = 1$ in this formula and fix n, we obtain a polynomial in k of degree $2n-2$, which gives the number of strict plane partitions with at most c columns, parts in $\{1, 2, \ldots, n\}$ and k parts equal to n. Remarkably, the polynomial factorizes nicely into distinct linear factors over $\mathbb{Z}$. With enumeration polynomials of this type, the properties characterizing the polynomial in Step 3 of our method are typically the zeros together with an additional evaluation of the polynomial. The idea, which we have used in [**15**, resp. Chapter 1] to explain these zeros, is interesting in its own right: we find a "natural" *combinatorial extension* of strict plane partitions with at most c columns, parts in $\{1, 2, \ldots, n\}$ and exactly k parts equal to n to arbitrary integers k and notice that the extension is simply impossible if k is a zero of the enumeration polynomial. This may sound strange at first, however in this case, the combinatorial extension can be defined by using Gelfand-Tsetlin patterns, which are objects equivalent to strict plane partitions. This explains roughly how I prove the Bender-Knuth (ex-)Conjecture refinement in the special case $q = 1$. In order to obtain the result for arbitrary q, I had to extend the method to so-called q-polynomials.

In [**16**, resp. Chapter 2] I present a further refinement of the Bender-Knuth (ex-)Conjecture, thereby refining Krattenthaler's refinement [**30**] and my own refinement from [**15**, resp. Chapter 1]. Namely, I have computed the generating function of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns, p rows of odd length and k parts equal to n. The method,

[2] In the case of plane partitions, this translation is usually accomplished by the Lindström-Gessel-Viennot theorem for non-intersecting lattice paths [**21, 37**], whereas with alternating sign matrices the determinantal expression for the partition function of square ice [**34, 35**] can often be used.

which I use for proving the result, is an extension of the method presented in [**15**, resp. Chapter 1]. On the one hand, I had to extend the method to so-called q-quasi-polynomials. More remarkable is that in this case, the zeros together with one extra evaluation of the q-quasi polynomial do not suffice in order to uniquely determine it. Consequently, some additional properties of the enumeration polynomial had to be deduced. The proofs in this paper are rather technical. In the end, they heavily rely on non-trivial transformation and summation formulas for basic hypergeometric series.

In [**17**, resp. Chapter 3] the situation is different. There, I consider monotone triangles with prescribed bottom row $(k_1, \ldots, k_n)$. Monotone triangles with bottom row $(1, 2, \ldots, n)$ are in bijection with $n \times n$ alternating sign matrices. It is not hard to see that the number of monotone triangles with prescribed bottom row $(k_1, k_2, \ldots, k_n)$ is given by a polynomial in $(k_1, k_2, \ldots, k_n)$. Thus, in contrast to the enumeration polynomials in [**15, 16**, resp. Chapter 1 and 2], we consider a *multivariate enumeration polynomial* in this case. Moreover, this polynomial does not factor over $\mathbb{Z}$, and, therefore, we have to work with other properties of the polynomial. In this case, I found operators, which are polynomials in shift operators, whose application to the enumeration polynomial produce certain symmetries of the polynomial. Thereby, I was able to prove that the number of monotone triangles with prescribed bottom row $(k_1, k_2, \ldots, k_n)$ is

$$\left(\prod_{1 \le p < q \le n} \left(\mathrm{id} + E_{k_p} \Delta_{k_q} \right) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

In this formula, E_x denotes the shift operator, defined by $E_x \, p(x) = p(x+1)$, and $\Delta_x := E_x - \mathrm{id}$ denotes the difference operator. This class of enumeration formulas seems to be new and, very likely, it allows elegant presentations for some other enumeration formulas as well. Let us remark that *operator formulas* of this type can often be translated into constant term identities.

Even though the operator formula is a compact way to express the number of monotone triangles with prescribed bottom row, it can still be challenging to apply it as we see in the last chapter [**18**, resp. Chapter 4] of this thesis. Namely, in order to use the operator formula to give another proof of the refined alternating sign matrix theorem, one has to evaluate it at $(k_1, k_2, \ldots, k_n) = (1, 2, \ldots, n)$. I have carried out this (non-trivial) evaluation in [**17**, resp. Chapter 3], where I succeeded in proving a refined version of the *alternating sign matrix theorem* first given by Zeilberger [**65**]. Moreover, in this article, I was able to compute the number of certain 0–1–(-1) matrices generalizing alternating sign matrices, thereby providing a new generalization of the alternating sign matrix theorem.

This need not to be the end of the story. I plan to further apply (and thereby further develop) the polynomial method in the theory of plane partitions and alternating sign matrices and beyond. For example, it would be nice to find other applications of the method for proving polynomial enumeration formulas that factorize over $\mathbb{Z}$ as presented in [**15**, resp. Chapter 1]. Moreover, it would be interesting to find q-versions of the operator formula from [**17**, resp. Chapter 3], thereby providing weighted enumerations of monotone triangles. However, I also believe that there are other (triangular) arrays of non-negative integers, whose enumeration formula can be expressed using operator formulas. An example may be the triangular integer arrays corresponding to totally symmetric self-complementary plane partitions (TSSCPP-triangles) [**43**]. (Zeilberger [**64**] calls generalizations of them "magog trapezoids".) This, and our experience from [**18**, resp. Chapter 4] in evaluating operator formulas, could be helpful in attacking a long-standing conjecture from [**42**] on the refined enumeration of

TSSCPP-triangles. This research may also help in finally constructing a bijection between TSSCPP-triangles and monotone triangles.

Acknowledgements

I thank Christian Krattenthaler for introducing me to the enumeration of plane partitions and alternating sign matrices and for helpful discussions. His “identification of factors” method for evaluating determinants had an important impact on the developement of the polynomial method presented in this thesis. Moreover, I thank my parents and my friends for their continuing support and, especially, for accepting the various side effects, which an overdose of plane partitions and alternating sign matrices can sometimes have.

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CHAPTER 1

A method for proving polynomial enumeration formulas

Abstract. We present an elementary method for proving enumeration formulas which are polynomials in certain parameters if others are fixed and factorize into distinct linear factors over $\mathbb{Z}$. Roughly speaking the idea is to prove such formulas by "explaining" their zeros using an appropriate combinatorial extension of the objects under consideration to negative integer parameters. We apply this method to prove a new refinement of the Bender-Knuth (ex-)Conjecture, which easily implies the Bender-Knuth (ex-)Conjecture itself. This is probably the most elementary way to prove this result currently known. Furthermore we adapt our method to q-polynomials, which allows us to derive generating function results as well. Finally we use this method to give another proof for the enumeration of semistandard tableaux of a fixed shape which differs from our proof of the Bender-Knuth (ex-)Conjecture in that it is a multivariate application of our method.

1. Introduction

1.1. A simple example. Let $F(r,k)$ denote the number of partitions $(\lambda_1,\ldots,\lambda_r)$, i.e. $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r \ge 0$, of length r, with parts in $\{0,1,\ldots,k\}$. It is basic combinatorial knowledge that

$$F(r,k) = \binom{k+r}{r} = \frac{(k+1)\cdot(k+2)\cdot\ldots\cdot(k+r)}{r!}.$$

For fixed r this expression is a polynomial in k with distinct integer zeros. In this paper we present an elementary method for proving polynomial enumeration formulas of that type, together with some non-trivial applications. The underlying idea is to find the appropriate extension of the combinatorial objects under consideration to (typically) negative integer parameters and with this "explain" the zeros of the enumeration polynomial.

To be more concrete let us first demonstrate this 3 step method in terms of our simple example.

(1) In the first step we extend the combinatorial interpretation of $F(r,k)$ to negative integer k's. For $k<0$ we define

$$F(r,k) = (-1)^r [\#(\lambda_1,\ldots,\lambda_r) \in \mathbb{Z}^r \text{ with } k < \lambda_1 < \lambda_2 < \ldots < \lambda_r < 0].$$

This definition seems to appear from nowhere, however, the following step should convince us that it was a good choice.

(2) In this step we show that for fixed r the function $k \to F(r,k)$ can be expressed by a polynomial in k of degree at most r. This is equivalent to $\Delta^{r+1} F(r,k) = 0$, where the differences are taken with respect to the parameter k. In order to show this we use induction with respect to r. The initial step follows from $F(1,k) = k+1$. Assume that $r>1$ and $k \ge 0$. Then

$$\begin{aligned}
\Delta F(r,k) &= F(r,k+1) - F(r,k)\\
&= [\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k+1 \ge \lambda_1 \ge \ldots \ge \lambda_r \ge 0]\\
&\quad - [\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k \ge \lambda_1 \ge \ldots \ge \lambda_r \ge 0]\\
&= [\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k+1 \ge \lambda_1 \ge \ldots \ge \lambda_r \ge 0 \text{ and } \lambda_1 = k+1]\\
&= F(r-1,k+1).
\end{aligned}$$

If $k<0$ we have

$$\begin{aligned}
\Delta F(r,k) &= F(r,k+1) - F(r,k)\\
&= (-1)^r[\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k+1 < \lambda_1 < \ldots < \lambda_r < 0]\\
&\quad - (-1)^r[\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k < \lambda_1 < \ldots < \lambda_r < 0]\\
&= (-1)^{r-1}[\#(\lambda_1,\lambda_2,\ldots,\lambda_r) \text{ with } k < \lambda_1 < \ldots < \lambda_r < 0 \text{ and } \lambda_1 = k+1]\\
&= F(r-1,k+1).
\end{aligned}$$

The induction hypothesis implies $\Delta^r F(r-1,k+1) = 0$ and thus $\Delta^{r+1} F(r,k) = 0$.

(3) In the final step we explore the integer zeros of $F(r,k)$ in k. Consider the definition of $F(r,k)$ for negative k's and observe that $F(r,k) = 0$ for $k = -1,-2,\ldots,-r$. By Step 2 $F(r,k)$ is a polynomial in k and therefore it has the factor $(k+1)_r$, where the Pochhammer symbol $(a)_n$ is defined by $(a)_n = \prod_{i=0}^{n-1}(a+i)$. The degree estimation of Step 2 implies that this factor

determines $F(r,k)$ up to a factor independent of k. Observe that $F(r,0)=1$, and thus this factor is equal to $1/r!$ and the formula is proved.

1.2. The method. We summarize the general strategy in the example above and with this establish our method for proving polynomial enumeration formulas. It applies to the enumeration of combinatorial objects which depend on an integer parameter k and where we suspect the existence of an enumeration formula which is polynomial in k and factorizes into distinct linear factors over $\mathbb{Z}$. The method is divided into the following three steps.

(1) **Extension of the combinatorial interpretation.** Typically the admissible domain of k is a set S of non-negative integers. In the first step of our method we have to find (most likely new) combinatorial objects indexed by an *arbitrary* integer k which are in bijection with the original objects for $k \in S$.
(2) **The extending objects are enumerated by a polynomial.** The extension of the combinatorial interpretation in the previous step has to be chosen so that we are able to prove that the new objects are enumerated by a polynomial in k. In many cases this is done with the help of a recursion. Moreover the degree of this polynomial has to be computed.
(3) **Exploring "natural" linear factors.** Finally one has to find the k's for which there exist none of these objects, i.e. one has to compute the (integer) zeros of the polynomial.[1] Typically these zeros will not lie in S, which made the extension in Step 1 necessary. Moreover one has to find an additional (and thus non-zero) evaluation of the polynomial which is easy to compute. The zeros and the single non-zero evaluation determine the polynomial uniquely and we are finally able to compute it.

The last step shows the limits of this method. Even if one succeeds in the first two steps, it may be that the polynomial has non-integer zeros or multiple zeros and the method as described does not work. On the other hand the enumeration problems which result in polynomials that factorize totally over $\mathbb{Z}$ are exactly the one we are especially interested in and where we are longing for an understanding of the simplicity of the result.

1.3. A refinement of the Bender-Knuth (ex-)Conjecture. Next we explain a plane partition enumeration result we have obtained by using this method. The main purpose of the rest of the paper is the proof of this result. Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ be a partition. A *strict plane partition* of shape λ is an array $\pi_{1\le i\le r, 1\le j\le \lambda_i}$ of non-negative integers such that the rows are weakly decreasing and the columns are strictly decreasing. The *norm* $n(\pi)$ of a strict plane partition is defined as the sum of its parts and π is said to be a strict plane partition of the non-negative integer $n(\pi)$. For instance

$$\begin{array}{cccccc} 7 & 5 & 5 & 4 & 3 & 2 \\ 6 & 4 & 3 & 2 & & \\ 5 & 2 & & & & \\ 3 & 1 & & & & \end{array}$$

[1]In the first step it may have been necessary to introduce a signed enumeration outside of the admissible domain in order to have the same enumeration polynomial for all k's. In this case we have to find the k's for which objects cancel in pairs with respect to the sign.

is a strict plane partition of shape $(6,4,2,2)$ with norm 52. In [**6**, p.50] Bender and Knuth conjectured that the generating function of strict plane partitions with at most c columns, parts in $\{1,2,\ldots,n\}$ and with respect to this norm is equal to

$$\sum q^{n(\pi)} = \prod_{i=1}^{n} \frac{[c+i;q]_i}{[i;q]_i},$$

where $[n;q] = 1 + q + \cdots + q^{n-1}$ and $[a;q]_n = \prod_{i=0}^{n-1}[a+i;q]$. This conjecture was proved by Andrews [**2**], Gordon [**26**], Macdonald [**38**, Ex. 19, p.53] and Proctor [**46**, Prop. 7.2]. For related papers, which mostly include generalizations of the Bender-Knuth (ex-)Conjecture see [**12, 13, 29, 30, 48, 59**].

Using a "q-extension" of our method we have obtained the following new refinement of this result. As an additional parameter k we introduce the number of parts equal to n in the strict plane partition.

Theorem 1.1. *The generating function of strict plane partitions with parts in* $\{1,2,\ldots,n\}$, *at most c columns and k parts equal to n is*

$$\sum q^{n(\pi)} = \frac{q^{kn}[k+1;q]_{n-1}[1+c-k;q]_{n-1}}{[1;q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1;q]_{i-1}}{[i;q]_i}.$$

If we sum this generating function over all k's, $0 \le k \le c$, we easily obtain the Bender-Knuth (ex-)Conjecture. Probably this detour via Theorem 1.1 is the easiest and most elementary way to prove the Bender-Knuth (ex-)Conjecture currently known. In [**32**, Sec. 3] the authors come to the conclusion that all other proofs of the Bender-Knuth (ex-)Conjecture "share more or less explicitly an identity, which relates Schur functions and odd orthogonal characters of the symmetric group of rectangular shape". In our elementary proof this is not the case.

We first prove the special case $q = 1$ of Theorem 1.1, i.e. we compute the number of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n, see Theorem 1.2. (Observe that for $q = 1$ the formula in Theorem 1.1 is a polynomial in k, which factorizes into distinct linear factors over $\mathbb{Z}$.) This result is new as well. Later we will see that the method can be extended to q-polynomials in order to prove the general result.

1.4. Outlook and outline of the paper. We plan to apply this method to other enumeration problems in the future. The most ambitious project in this direction is probably our current effort to give another proof of the Refined Alternating Sign Matrix Theorem. There is some hope for a proof along the lines of the proof of Theorem 1.1, for details see Section 7. Moreover we plan to extend our method to polynomial enumeration formulas that do not factor into distinct linear factors over $\mathbb{Z}$. Hopefully the lack of integer zeros can be compensated by other properties of the polynomial.

The rest of the paper is organized as follows. In Section 2 we introduce our combinatorial extension with respect to k of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n as proposed in *Step 1* of our method above. In Section 3 we show that these objects are enumerated by a polynomial in k which is of degree at most $2n-2$ (*Step 2*) and in Section 4 we show that the polynomial has the predicted zeros (*Step 3*). This concludes the proof of Theorem 1.1 for $q = 1$. In Section 5 we apply the method to give another proof of the formula for the number of semistandard tableaux of a fixed shape. This application of our method is of interest since in this case we have to work with more than just one polynomial parameter. Finally we extend our method to what we call "q-polynomials" and prove Theorem 1.1 in its full strength in Section 6. In Section 7 a connection of our result to the Refined Alternating Sign Matrix Theorem is presented.

Throughout the whole article we use the extended definition of the summation symbol, namely,

$$\sum_{i=a}^{b} f(i) = \begin{cases} f(a) + f(a+1) + \cdots + f(b) & \text{if } a \leq b \\ 0 & \text{if } b = a-1 \\ -f(b+1) - f(b+2) - \cdots - f(a-1) & \text{if } b+1 \leq a-1 \end{cases}. \tag{1.1}$$

This assures that for any polynomial $p(X)$ over an arbitrary integral domain I containing $\mathbb{Q}$ there exists a unique polynomial $q(X)$ over I such that $\sum_{x=0}^{y} p(x) = q(y)$ for all integers y. We usually write $\sum_{x=0}^{y} p(x)$ for $q(y)$. We use the analogous extended definition of the product symbol. In particular this extends the definition of the Pochhammer symbol, for instance $(a)_{-1} = 1/(a-1)$.

2. From strict plane partitions to generalized $(n-1, n, c)$ Gelfand-Tsetlin patterns

Let n be a positive integer. A Gelfand-Tsetlin pattern with n rows is a triangular array of integers, say

$$\begin{array}{ccccccccccccc}
 & & & & & & a_{n,n} & & & & & & \\
 & & & & & a_{n-1,n-1} & & a_{n-1,n} & & & & & \\
 & & & & \dots & & \dots & & \dots & & & & \\
 & & & a_{3,3} & & \dots & & \dots & & a_{3,n} & & & \\
 & & a_{2,2} & & a_{2,3} & & \dots & & \dots & & a_{2,n} & & \\
 & a_{1,1} & & a_{1,2} & & a_{1,3} & & \dots & & \dots & & a_{1,n} &
\end{array},$$

such that $a_{i,j} \leq a_{i-1,j}$ for $1 < i \leq j \leq n$ and $a_{i,j} \leq a_{i+1,j+1}$ for $1 \leq i \leq j < n$, see [**58**, p. 313] or [**20**, (3)] for the original reference. An example of a Gelfand-Tsetlin pattern with 7 rows is given below.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 3 & & & & \\
 & & & 0 & & 1 & & 2 & & 4 & & & \\
 & & 0 & & 1 & & 1 & & 3 & & 5 & & \\
 & 0 & & 0 & & 1 & & 2 & & 4 & & 6 & \\
0 & & 0 & & 0 & & 2 & & 2 & & 4 & & 6
\end{array}$$

The following correspondence between Gelfand-Tsetlin patterns and strict plane partitions is crucial for our paper.

LEMMA 1.1. *There is a bijection between Gelfand-Tsetlin patterns $(a_{i,j})$ with n rows, parts in $\{0, 1, \dots, c\}$ and fixed $a_{n,n} = k$, and strict plane partitions with parts in $\{1, 2, \dots, n\}$, at most c columns and k parts equal to n. In this bijection $(a_{1,n}, a_{1,n-1}, \dots, a_{1,1})$ is the shape of the strict plane partition.*

Proof. Given such a Gelfand-Tsetlin pattern, the corresponding strict plane partition is such that the shape filled by entries greater than i corresponds to the partition given by the

$(n-i)$-th row of the Gelfand-Tsetlin pattern, counting from the top. As an example, consider the strict plane partition in the introduction. If we choose $n=7$ and $c=6$ then this strict plane partition corresponds to the Gelfand-Tsetlin pattern above. $\square$

Therefore it suffices to enumerate Gelfand-Tsetlin patterns $(a_{i,j})$ with n rows, parts in $\{0,1,\ldots,c\}$ and fixed $a_{n,n}=k$. Why should this be easier than enumerating the corresponding strict plane partitions? Recall that k is the polynomial parameter in our refinement of the Bender-Knuth (ex-)Conjecture which we want to make use of when applying our method. In order to accomplish Step 1 of the method we have to find a "natural" extended definition of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n, where k is an *arbitrary* integer which does not necessarily lie in $\{0,1,\ldots,c\}$. (Parts equal to n may only appear in the first row of a (column-)strict plane partition with parts in $\{1,2,\ldots,n\}$ and thus, when considering strict plane partitions, we must have $k\in\{0,1,\ldots,c\}$.) "Natural" stands for the fact that the extension has to be chosen such that the extending objects are enumerated by a polynomial in k. In order to find this extension it seems easier to work with Gelfand-Tsetlin patterns rather than with strict plane partitions. Next we define generalized Gelfand-Tsetlin patterns which turn out to be the right extension.

Let r,n,c be integers, r non-negative and n positive. In this paper a *generalized* (r,n,c) *Gelfand-Tsetlin pattern* (for short: (r,n,c)-pattern) is an array $(a_{i,j})_{1\le i\le r+1, i-1\le j\le n+1}$ of integers with

(1) $a_{i,i-1}=0$ and $a_{i,n+1}=c$,
(2) if $a_{i,j}\le a_{i,j+1}$ then $a_{i,j}\le a_{i-1,j}\le a_{i,j+1}$
(3) if $a_{i,j}>a_{i,j+1}$ then $a_{i,j}>a_{i-1,j}>a_{i,j+1}$.

A $(3,6,c)$-pattern for example is of the form

$$\begin{array}{ccccccccccccccc}
 & & & 0 & & a_{4,4} & & a_{4,5} & & a_{4,6} & & c & & & \\
 & & 0 & & a_{3,3} & & a_{3,4} & & a_{3,5} & & a_{3,6} & & c & & \\
 & 0 & & a_{2,2} & & a_{2,3} & & a_{2,4} & & a_{2,5} & & a_{2,6} & & c & \\
0 & & a_{1,1} & & a_{1,2} & & a_{1,3} & & a_{1,4} & & a_{1,5} & & a_{1,6} & & c,
\end{array}$$

such that every entry not in the top row is between its northwest neighbour w and its northeast neighbour e, if $w\le e$ then weakly between, otherwise strictly between. Thus

$$\begin{array}{ccccccccccccccc}
 & & & 0 & & 3 & & -5 & & 10 & & 4 & & & \\
 & & 0 & & 2 & & -2 & & 3 & & 8 & & 4 & & \\
 & 0 & & 2 & & -1 & & 2 & & 4 & & 7 & & 4 & \\
0 & & 0 & & 0 & & 1 & & 2 & & 5 & & 6 & & 4
\end{array}$$

is an example of an $(3,6,4)$-pattern. Note that a generalized $(n-1,n,c)$ Gelfand-Tsetlin pattern $(a_{i,j})$ with $0\le a_{n,n}\le c$ is a Gelfand-Tsetlin pattern with n rows and parts in $\{0,1,\ldots,c\}$ as defined at the beginning of this section. This is because $0\le a_{n,n}\le c$ implies that the third possibility in the definition of a generalized Gelfand-Tsetlin pattern never occurs.

Next we introduce the sign of an (r,n,c)-pattern $a=(a_{i,j})$, since we actually have to work with a signed enumeration if $a_{n,n}\notin\{0,1,\ldots,c\}$. A pair $(a_{i,j},a_{i,j+1})$ with $a_{i,j}>a_{i,j+1}$ and $i\ne 1$ is called an *inversion* of the (r,n,c)-pattern and $(-1)^{\#\text{ of inversions}}$ is said to be the *sign* of the pattern, denoted by $\mathrm{sgn}(a)$. The $(3,6,4)$-pattern in the example above has 6 inversions altogether and thus its sign is 1. We define the following expression

$$F(r,n,c;k_1,k_2,\ldots,k_{n-r}) = \sum_a \mathrm{sgn}(a),$$

where the sum is over all (r,n,c)-patterns $a=(a_{i,j})$ with top row defined by $a_{r+1,r+i}=k_i$ for $i=1,\ldots,n-r$. Now it is important to observe that for $0 \le k \le c$ the number of $(n-1,n,c)$-patterns with $a_{n,n}=k$ is given by $F(n-1,n,c;k)$. This is because an $(n-1,n,c)$-pattern with $0 \le a_{n,n} \le c$ has no inversions. Thus $F(n-1,n,c;k)$ is the quantity we want to compute. It has the advantage that it is defined for all integers k, whereas our original enumeration problem was only defined for $0 \le k \le c$.

3. $F(n-1,n,c;k)$ is a polynomial in k of degree at most $2n-2$

In this section we establish Step 2 of the method above for our refinement of the Bender-Knuth (ex-)Conjecture. The following recursion for $F(r,n,c;k_1,k_2,\ldots,k_{n-r})$ is fundamental.

$$F(r,n,c;k_1,k_2,\ldots,k_{n-r}) = \sum_{l_1=0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3}\cdots\sum_{l_{n-r}=k_{n-r-1}}^{k_{n-r}}\sum_{l_{n-r+1}=k_{n-r}}^{c} F(r-1,n,c;l_1,l_2,\ldots,l_{n-r+1}). \quad (1.2)$$

It is obvious for $(k_1,k_2,\ldots,k_{n-r})$ with $0 \le k_1 \le k_2 \le \ldots \le k_{n-r} \le c$. After recalling the extended definition of the summation symbol (1.1) one observes that the generalized (r,n,c) Gelfand-Tsetlin patterns and $F(r,n,c;k_1,\ldots,k_{n-r})$ were simply defined in such a way that this recursion holds for arbitrary integer tuples $(k_1,\ldots,k_{n-r})$. This recursion together with the initial condition

$$F(0,n,c;k_1,k_2,\ldots,k_n)=1$$

implies the following lemma.

Lemma 1.2. *Let r,n be integers, r non-negative and n positive. Then $F(r,n,c;k_1,\ldots,k_{n-r})$ can be expressed by a polynomial in the k_i's and in c.*

In the following $F(r,n,c;k_1,\ldots,k_{n-r})$ is identified with this polynomial. In particular $F(n-1,n,c;k)$ is a polynomial in k and with this we have established the first half of Step 2 in our method. Next we aim to show that $F(r,n,c;k_1,\ldots,k_{n-r})$ is of degree at most $2r$ in every k_i. This will imply that $F(n-1,n,c;k)$ is of degree at most $2n-2$ in k and completing Step 2. However, this degree estimation is complicated and takes Lemmas 1.3–1.6.

The degree of $F(r,n,c;k_1,\ldots,k_{n-r})$ in k_i is the degree of

$$\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}} F(r-1,n,c;l_1,\ldots,l_{n-r+1}), \quad (1.3)$$

in k_i, where $k_0=0$ and $k_{n-r+1}=c$. Let us assume by induction with respect to r that the degree of $F(r-1,n,c;l_1,\ldots,l_{n-r+1})$ is at most $2r-2$ in each of l_i and l_{i+1}. By (1.3) we can easily conclude that the degree of $F(r,n,c;k_1,\ldots,k_{n-r})$ in k_i is at most $4r-2$; however, we want to establish that the degree is at most $2r$. The following lemma shows how to obtain a sharper degree estimation in summations of our type.

In order to state this lemma we have to define an operator D_i which turns out to be crucial for the analysis of the recursion in (1.2). Let $G(k_1, k_2, \ldots, k_m)$ be a function in m variables and $1 \le i \le m-1$. We set

$$D_i G(k_1, \ldots, k_m) := \\ G(k_1, \ldots, k_{i-1}, k_i, k_{i+1}, k_{i+2}, \ldots, k_m) + G(k_1, \ldots, k_{i-1}, k_{i+1}+1, k_i - 1, k_{i+2}, \ldots, k_m).$$

Lemma 1.3. *Let $F(x_1, x_2)$ be a polynomial in x_1 and x_2 which is of degree at most R in each of x_1 and x_2. Moreover assume that $D_1 F(x_1, x_2)$ is of degree at most R as a polynomial in x_1 and x_2, i.e. a linear combination of monomials $x_1^m x_2^n$ with $m+n \le R$. Then $\sum_{x_1=a}^{y} \sum_{x_2=y}^{b} F(x_1, x_2)$ is of degree at most $R+2$ in y. If $D_1 F(x_1, x_2) = 0$ then $\sum_{x_1=a}^{y} \sum_{x_2=y}^{b} F(x_1, x_2)$ is of degree at most $R+1$ in y.*

Proof. Set $F_1(x_1, x_2) = D_1 F(x_1, x_2)/2$ and $F_2(x_1, x_2) = (F(x_1, x_2) - F(x_2+1, x_1-1))/2$. Clearly $F(x_1, x_2) = F_1(x_1, x_2) + F_2(x_1, x_2)$. Observe that $F_2(x_2+1, x_1-1) = -F_2(x_1, x_2)$. Thus $F_2(x_1, x_2)$ is a linear combination of terms of the form $(x_1)_m (x_2+1)_n - (x_1)_n (x_2+1)_m$ with $m, n \le R$. Now observe that

$$\sum_{x_1=a}^{y} \sum_{x_2=y}^{b} (x_1)_m (x_2+1)_n - (x_1)_n (x_2+1)_m = \\ \frac{1}{m+1} \frac{1}{n+1} \left((a-1)_{n+1} (b+1)_{m+1} - (a-1)_{m+1} (b+1)_{n+1} - (a-1)_{n+1} (y)_{m+1} + \right. \\ \left. (b+1)_{n+1} (y)_{m+1} + (a-1)_{m+1} (y)_{n+1} - (b+1)_{m+1} (y)_{n+1} \right).$$

and thus $\sum_{x_1=a}^{y} \sum_{x_2=y}^{b} F_2(x_1, x_2)$ is a polynomial of degree at most $R+1$ in y. By the assumption in the lemma $\sum_{x_1=a}^{y} \sum_{x_2=y}^{b} F_1(x_1, x_2)$ is of degree at most $R+2$ in y and the assertion follows. ☐

Thus it suffices to show that $D_i F(r, n, c; .)(k_1, \ldots, k_{n-r})$ is of degree at most $2r$ as a polynomial in k_i and k_{i+1}. In Lemma 1.5 we show a much stronger assertion, namely we prove a formula which expresses $D_i F(r, n, c; k_1, \ldots, k_{n-r})$ as a product of $F(r, n-2, c+2; k_1, \ldots, k_{i-1}, k_{i+2} + 2, \ldots, k_{n-r} + 2)$ and an (explicit) polynomial in k_i and k_{i+1} which is obviously of degree $2r$. For the proof of Lemma 1.5 we need another lemma.

Lemma 1.4. *Let $G(l_1, l_2, l_3)$ be a function on $\mathbb{Z}^3$. Then*

$$D_1 \left(\sum_{l_1=k_0}^{k_1} \sum_{l_2=k_1}^{k_2} \sum_{l_3=k_2}^{k_3} G(l_1, l_2, l_3) \right) (k_0, k_1, k_2, k_3) = \\ -\frac{1}{2} \left(\sum_{l_1=k_1+1}^{k_2+1} \sum_{l_2=k_1}^{k_2} \sum_{l_3=k_1-1}^{k_3} D_1 G(l_1, l_2, l_3) + \sum_{l_1=k_0}^{k_1} \sum_{l_2=k_1}^{k_2} \sum_{l_3=k_1-1}^{k_2-1} D_2 G(l_1, l_2, l_3) \right).$$

If $H(l_1,l_2)$ is a function on $\mathbb{Z}^2$ then

$$D_1\left(\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3} H(l_1,l_2)\right)(k_1,k_2,k_3) = -\frac{1}{2}\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_1-1}^{k_2-1} D_1 H(l_1,l_2).$$

Proof. The left-hand side in the first statement of the lemma is equal to

$$\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3} G(l_1,l_2,l_3) + \sum_{l_1=k_0}^{k_2+1}\sum_{l_2=k_2+1}^{k_1-1}\sum_{l_3=k_1-1}^{k_3} G(l_1,l_2,l_3).$$

In this formula, we reverse the middle sum of the second triple sum and split up the first sum in this triple sum to obtain

$$\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3} G(l_1,l_2,l_3) - \sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_3} G(l_1,l_2,l_3) \\ - \sum_{l_1=k_1+1}^{k_2+1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_3} G(l_1,l_2,l_3).$$

Next we cancel some common terms in the first and the second sum and obtain

$$-\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_2-1} G(l_1,l_2,l_3) - \sum_{l_1=k_1+1}^{k_2+1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_1-1}^{k_3} G(l_1,l_2,l_3),$$

which is equal to the right-hand side of the first statement. The proof of the second formula is easy. $\square$

Let $r \geq 1$. We need the following identity.

$$\sum_{x'=x}^{y}\sum_{y'=x-1}^{y-1}(y'-x'-r+3)_{2r-3}(y'-x'+1) = \frac{1}{r(2r-1)}(y-x-r+2)_{2r-1}(y-x+1) \quad (1.4)$$

It follows from

$$(y'-x'-r+3)_{2r-3}(y'-x'+1) = \frac{1}{2}\left((y'-x'-r+2)_{2r-2} - (y'-x'-r+3)_{2r-2}\right),$$

the summation formula

$$\sum_{z=a}^{b}(z+w)_n = \frac{1}{n+1}\left((b+w)_{n+1} - (a-1+w)_{n+1}\right)$$

and the fact that $(-r+1)_{2r} = 0$ and $(-r)_{2r} = 0$.

Lemma 1.5. *Let r,n,i be integers, r non-negative, n positive, $2 \leq n-r$ and $1 \leq i \leq n-r-1$. Then*

$$D_i F(r,n,c;.)(k_1,\ldots,k_{n-r}) = (-1)^r\frac{2}{(2r)!}(k_{i+1}-k_i-r+2)_{2r-1}(k_{i+1}-k_i+1) \\ \times F(r,n-2,c+2;k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_{n-r}+2).$$

Proof. We show the assertion by induction with respect to r. For $r=0$ there is nothing to prove. We assume $r>0$. By (1.2) and Lemma 1.4 the left-hand side in the statement is equal to

$$-\frac{1}{2}\left(\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_i+1}^{k_{i+1}+1}\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+2}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c} D_i F(r-1,n,c;.)(l_1,\ldots,l_{n-r+1})\right.$$
$$\left.+\sum_{l_1=0}^{k_1}\cdots\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+1}-1}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c} D_{i+1} F(r-1,n,c;.)(l_1,\ldots,l_{n-r+1})\right).$$

In this formula, we replace $D_i F(r-1,n,c;.)(l_1,\ldots,l_{n-r+1})$ and $D_{i+1} F(r-1,n,c;.)(l_1,\ldots,l_{n-r+1})$ by the expressions implied by the induction hypothesis. Furthermore we apply (1.4) to obtain

$$(-1)^r\frac{2}{(2r)!}(k_{i+1}-k_i-r+2)_{2r-1}(k_{i+1}-k_i+1)$$
$$\times\left(\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_{i+2}=k_i-1}^{k_{i+2}}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c}\right.$$
$$F(r-1,n-2,c+2;l_1,\ldots,l_{i-1},l_{i+2}+2,l_{i+3}+2,\ldots,l_{n-r+1}+2)$$
$$+\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c}$$
$$\left.F(r-1,n-2,c+2;l_1,\ldots,l_{i-1},l_i,l_{i+3}+2,\ldots,l_{n-r+1}+2)\right).$$

We shift the range of summation of $l_{i+2},l_{i+3},\ldots,l_{n-r+1}$ by two and compensate this shift with the appropriate change in the summand. In the first multiple sum, we rename l_{i+2} to l_i and merge the two multiple sums. We obtain

$$(-1)^r\frac{2}{(2r)!}(k_{i+1}-k_i-r+2)_{2r-1}(k_{i+1}-k_i+1)$$
$$\times\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_{i-1}}^{k_{i+2}+2}\sum_{l_{i+3}=k_{i+2}+2}^{k_{i+3}+2}\cdots\sum_{l_{n-r+1}=k_{n-r}+2}^{c+2} F(r-1,n-2,c+2;l_1,\ldots,l_{i-1},l_i,l_{i+3},\ldots,l_{n-r+1})$$
$$=(-1)^r\frac{2}{(2r)!}(k_{i+1}-k_i-r+2)_{2r-1}(k_{i+1}-k_i+1)$$
$$\times F(r,n-2,c+2;k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_{n-r}+2)$$

and the assertion follows. □

We are finally able to prove the degree lemma.

LEMMA 1.6. *Let r,n,i be integers, r non-negative, n positive and $1\le i\le n-r$. Then $F(r,n,c;k_1,\ldots,k_{n-r})$ is a polynomial in k_i of degree at most $2r$.*

Proof. We prove the assertion by induction with respect to r. For $r=0$ it is trivial. Assume $r>0$ and $1 \le i \le n-r$. The degree of $F(r,n,c;k_1,\ldots,k_{n-r})$ in k_i is the degree of (1.3) in k_i. By Lemma 1.5 the degree of $D_i F(r-1,n,c;l_1,\ldots,l_{n-r+1})$ as a polynomial in l_i and l_{i+1} is $2r-2$. Moreover the degree of $F(r-1,n,c;l_1,\ldots,l_{n-r+1})$ in l_i as well as in l_{i+1} is at most $2r-2$ by the induction hypothesis. The assertion follows from Lemma 1.3. $\square$

4. Exploring the zeros of $F(n-1,n,c;k)$

We finally establish Step 3 of our method for the refinement of the Bender-Knuth (ex-)Conjecture.

LEMMA 1.7. *Let r,n,i be integers, r non-negative, n positive and $1 \le i \le n-r$. Then there exists no (r,n,c)-pattern with first row*

$$(0,k_1,\ldots,k_{n-r},c),$$

if $k_1=-1,-2,\ldots,-r$ or $k_{n-r}=c+1,c+2,\ldots,c+r$.

Proof. Suppose $(a_{i,j})$ is an (r,n,c)-pattern with $a_{r+1,r+1} \in \{-1,-2,\ldots,-r\}$. In particular we have $0 > a_{r+1,r+1}$ and thus the definition of (r,n,c)-patterns implies that $0 > a_{r,r} > a_{r+1,r+1}$. In a similar way we obtain $0 > a_{1,1} > a_{2,2} > \ldots > a_{r,r} > a_{r+1,r+1}$. This is, however, a contradiction, since there exist no r distinct integers strictly between 0 and $a_{r+1,r+1}$ if $a_{r+1,r+1} \in \{-1,-2,\ldots,-r\}$. The case that $a_{r+1,n} \in \{c+1,c+2,\ldots,c+r\}$ is similar. $\square$

We obtain the following corollary.

COROLLARY 1.1. *$F(n-1,n,c;k)/((1+k)_{n-1}(1+c-k)_{n-1})$ is independent of k.*

Proof. By Lemma 1.7, $(1+k)_{n-1}(1+c-k)_{n-1}$ is a factor of $F(n-1,n,c;k)$. By Lemma 1.6, $F(n-1,n,c;k)$ is of degree at most $2n-2$ in k and the assertion follows. $\square$

THEOREM 1.2. *The number of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n is given by*

$$F(n-1,n,c;k) = \frac{(1+k)_{n-1}(1+c-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(c+i+1)_{i-1}}{(i)_i}.$$

Proof. We prove the assertion by induction with respect to n. Observe that the formula is true for $n=1$ since $F(0,1,c;k)=1$. Assume $n>1$. By Corollary 1.1

$$F(n-1,n,c;k) = (1+k)_{n-1}(1+c-k)_{n-1} \frac{F(n-1,n,c;c)}{(1+c)_{n-1}(1)_{n-1}}.$$

Observe that if we have $a_{n,n}=c$ in an $(n-1,n,c)$-pattern $(a_{i,j})_{1\le i\le n, i-1\le j\le n+1}$ then $a_{i,n}=c$ for all i. This implies the recursion

$$F(n-1,n,c;c) = \sum_{k=0}^{c} F(n-2,n-1,c;k). \tag{1.5}$$

We need one other ingredient, namely the following hypergeometric identity

$$\sum_{k=0}^{c}(1+k)_{m-1}(1+c-k)_{m-1} = (1)_{m-1}^2 \sum_{k=0}^{c} \binom{m+k-1}{m-1}\binom{c-k+m-1}{m-1}$$
$$= (1)_{m-1}^2 \binom{c+2m-1}{2m-1} = \frac{(1)_{m-1}^2 (c+1)_{2m-1}}{(1)_{2m-1}}, \quad (1.6)$$

where the second equality is equivalent to the Chu-Vandermonde identity; see [**27**, p. 169, (5.26)]. With the help of the recursion (1.5), the induction hypothesis for $F(n-2,n-1,c;k)$ and the hypergeometric identity we are able to compute $F(n-1,n,c;c)$ and with this $F(n-1,n,c;k)$. □

Remark 1.1. *By the symmetry of Schur functions, the number of strict plane partitions of a fixed shape with x_i parts equal to i is equal to the number of strict plane partitions with $x_{\pi(i)}$ parts equal to i for every permutation π. Thus Theorem 1.2 gives the number of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to i for arbitrary $i \in \{1,2,\ldots,n\}$. However, note that this does not generalize to the generating function of these objects.*

Corollary 1.2 (Andrews [**2**], Gordon [**26**], Macdonald [**38**], Proctor [**46**]). *The number of strict plane partitions with parts in $\{1,2,\ldots,n\}$ and at most c columns is*

$$\prod_{i=1}^{n} \frac{(c+i)_i}{(i)_i}.$$

Proof. By Theorem 1.2 the number of strict plane partitions with parts in $\{1,2,\ldots,n\}$ and at most c columns equals

$$\sum_{k=0}^{c} \frac{(k+1)_{n-1}(1+c-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(c+i+1)_{i-1}}{(i)_i}.$$

The assertion now follows from (1.6).

5. Semistandard tableaux of a fixed shape

In this section we apply our method to the enumeration of semistandard tableaux of a fixed shape. This result is certainly well-known. Nonetheless we think it might be interesting for the reader to see another application of our method which moreover uses more than just one "polynomial parameter" as opposed to the single parameter k in the example above. (At this point the reader may wonder what we mean by a multivariate application of our method, since we have only described the case of a single polynomial parameter in the introduction. However, it is straightforward to generalize this method to a multivariate version, as should become clear in this section.)

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition and r a positive integer. A semistandard tableau of shape λ with entries between 1 and r is a filling of the Ferrers diagram of shape λ with entries in $\{1,2,\ldots,r\}$ such that rows are weakly increasing and columns are strictly increasing. (Semistandard tableaux and strict plane partitions are equivalent objects. Indeed, if we replace every entry e in a semistandard tableau with entries between 1 and r with $r-e$ we clearly obtain a strict plane partition. However, we choose to use the notion of semistandard tableaux in this section for historical reasons.) It is well-known [**58**, p. 375, in (7.105) $q \to 1$] that the number of semistandard tableaux of shape λ with entries between 1 and r is

$$\prod_{1 \le i < j \le k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{i=1}^{k} \frac{(\lambda_i + k + 1 - i)_{r-k}}{(k + 1 - i)_{r-k}}$$

if $k \le r$, otherwise this number is obviously zero by columnstrictness. If $k = r$ the formula simplifies to

$$\prod_{1 \le i < j \le r} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{1.7}$$

It suffices to prove this formula, for the number of semistandard tableaux of shape $(\lambda_1, \dots, \lambda_k)$ is equal to the number of semistandard tableaux of shape $(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$ ($r - k$ zeros).

The expression in (1.7) is a polynomial in the λ_i's which is up to a constant determined by its zeros $\lambda_i = \lambda_j - j + i$, $1 \le i < j \le r$. Clearly the number of semistandard tableaux of shape $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ with entries between 1 and r can be interpreted to be zero if $\lambda_i = \lambda_j - j + i$ for some i, j with $1 \le i < j \le r$, since λ is not a partition in this case. However, we have to find a combinatorial extension to arbitrary $\lambda \in \mathbb{Z}^r$ such that the number of objects is zero if and only if $\lambda_i = \lambda_j - j + i$ for some i, j with $1 \le i < j \le r$, and thus this number is non-zero for many $(\lambda_1, \dots, \lambda_r)$ that are not partitions. Again the proof is divided into three steps.

1. Extension. We extend the combinatorial interpretation of the number of semistandard tableaux of shape λ to arbitrary $\lambda \in \mathbb{Z}^r$. The situation is somehow "reversed" to that in our Bender-Knuth (ex-)Conjecture refinement. In Lemma 1.1 we have showed that semistandard tableaux of shape λ with enries in $\{1, 2, \dots, r\}$ are in bijection with Gelfand-Tsetlin patterns with r rows and prescribed bottom row $(\lambda_r, \lambda_{r-1}, \dots, \lambda_1)$.

A *generalized reversed Gelfand-Tsetlin pattern of size* r (for short: reversed r-pattern) is a triangular array $(a_{i,j})_{1 \le i \le r, i \le j \le r}$ of integers with

(1) if $a_{i,j-1} \le a_{i,j}$ then $a_{i,j-1} \le a_{i+1,j} \le a_{i,j}$
(2) if $a_{i,j-1} > a_{i,j}$ then $a_{i,j-1} > a_{i+1,j} > a_{i,j}$.

For instance,

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 4 & & 4 & & \\ & 5 & & 3 & & 4 & \\ 6 & & 1 & & 5 & & 3 \end{array}$$

is a reversed 4-pattern. Note that a reversed r-pattern $(a_{i,j})_{1 \le i \le r, i \le j \le r}$ with $a_{1,1} \le a_{1,2} \le \ldots \le a_{1,r}$ is a Gelfand-Tsetlin pattern with r rows in the original sense. A pair $(a_{i,j-1}, a_{i,j})$ with $a_{i,j-1} > a_{i,j}$ is called an *inversion* and $(-1)^{\#\text{of inversions}}$ is said to be the sign of the pattern, denoted by $\operatorname{sgn}(a)$. We define

$$A_r(k_1, \dots, k_r) = \sum_a \operatorname{sgn}(a),$$

where the sum is over all reversed r-patterns $a = (a_{i,j})$ with prescribed bottom row $a_{1,i} = k_i$. If $k_1 \le k_2 \le \ldots \le k_r$ then $A_r(k_1, \dots, k_r)$ is the number of semistandard tableaux of shape

$(k_r, \ldots, k_1)$ with entries in $\{1, 2, \ldots, r\}$. This is because a reversed r-pattern with $a_{1,1} \leq a_{1,2} \leq \ldots \leq a_{1,r}$ has no inversion.

2. Polynomial enumeration formula. Observe that the following recursion holds for $A_r(k_1, \ldots, k_r)$.

$$A_r(k_1, \ldots, k_r) = \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \cdots \sum_{l_{r-1}=k_{r-1}}^{k_r} A_{r-1}(l_1, l_2, \ldots, l_{r-1}) \tag{1.8}$$

It is similar to (1.2). Since $A_1(k_1) = 1$ we can conclude that $A_k(k_1, \ldots, k_r)$ is a polynomial in $(k_1, \ldots, k_r)$.

Next we show that $A_r(k_1, \ldots, k_r)$ is of degree at most $r - 1$ in every k_i. This is done by induction with respect to r. Moreover, we show that $D_i A_r(k_1, \ldots, k_r) = 0$ for all i and $r \geq 2$. The assertion on the degree is obviously true if $r = 1, 2$ since $A_2(k_1, k_2) = k_2 - k_1 + 1$. Moreover $D_1 A_2(k_1, k_2) = 0$. Assume that $r > 2$. By induction $A_{r-1}(k_1, \ldots, k_{r-1})$ is of degree at most $r-2$ in every k_i and $D_i A_{r-1}(k_1, \ldots, k_{r-1}) = 0$ for all i. Thus, by (1.8) and Lemma 1.3, $A_r(k_1, \ldots, k_r)$ is of degree at most $r - 1$ in k_i. Moreover, by Lemma 1.4, $D_i A_r(k_1, \ldots, k_r) = 0$ for all i.

3. Linear factors. In this step we find the zeros of $A_r(k_1, \ldots, k_r)$ and use them to deduce the formula. By definition there exists no reversed r-pattern with $a_{1,j} = a_{1,j+1} + 1$ and thus $A_r(k_1, \ldots, k_{i-1}, k_{i+1}+1, k_{i+1}, k_{i+2}, \ldots, k_r) = 0$ for all i. In fact $A_r(k_1, \ldots, k_r) = 0$ if $k_i = k_j + j - i$ and $i < j$:

$$\begin{aligned} &A_r(k_1, \ldots, k_{i-1}, k_j + j - i, k_{i+1}, \ldots, k_j, \ldots, k_r) = \\ &\qquad - A_r(k_1, \ldots, k_{i-1}, k_{i+1} + 1, k_j + j - i - 1, k_{i+2} + 1, \ldots, k_j, \ldots, k_r) = \ldots = \\ &\qquad\qquad (-1)^{j-i-1} A_r(k_1, \ldots, k_{i-1}, k_{i+1} + 1, k_{i+2} + 1, \ldots, k_j + 1, k_j, \ldots, k_r) = 0, \end{aligned}$$

where the l-th equality follows from $D_{i+l-1} A_r(k_1, \ldots, k_r){=}0$. Therefore $A_r(k_1, \ldots, k_r)$ has the factor $\prod_{1 \leq i < j \leq r} (k_j - k_i + j - i)$. This factor is a polynomial of degree $r - 1$ in every k_i and thus

$$A_r(k_1, \ldots, k_r) = C \cdot \prod_{1 \leq i < j \leq r} (k_j - k_i + j - i),$$

where C does not depend on $(k_1, \ldots, k_r)$. Since $A_r(0, \ldots, 0) = 1$ we conclude that $C = \prod_{1 \leq i < j \leq r} \frac{1}{j-i}$ and (1.7) is proved.

Remark 1.2. *The extension of our method introduced in the following section can be used to derive the q-version of* (1.7)*, see* [**58**, p. 375, (7.105)].

6. Extension of the method to q-polynomials

A natural question to ask is whether it is possible to obtain a generating function version of Theorem 1.2. Of course only this would refine the Bender-Knuth (ex-)Conjecture. Clearly our generating function (see Theorem 1.1) is not a polynomial in k, however, we introduce the notion of a *q-polynomial* below and find that the generating function is such a q-polynomial. Thus we adapt our method to q-polynomials in this section.

Let I be an integral domain containing $\mathbb{Q}$. A *q-polynomial* over I in the variables $X_1, X_2, \ldots, X_n$ is an ordinary polynomial over $I(q)$, the field of rational functions in q over I, in $q^{X_1}, q^{X_2}, \ldots, q^{X_n}$.

The ring of these q-polynomials is denoted by $I_q[X_1, \ldots, X_n]$. For expressions of the form

$$q^{a_0}(q^{X_1})^{a_1}(q^{X_2})^{a_2} \ldots (q^{X_n})^{a_n}$$

in a q-polynomial, where the a_i are non-negative integers for $1 \le i \le n$, we also write

$$q^{a_0+a_1X_1+a_2X_2+\ldots+a_nX_n}.$$

We define $[X; q] = (1 - q^X)/(1 - q)$ and $[X; q]_n = \prod_{i=0}^{n-1}[X + i; q]$. Observe that

$$[X_1; q]_{m_1}[X_2; q]_{m_2} \ldots [X_n; q]_{m_n},$$

$(m_1, m_2, \ldots, m_n) \in \mathbb{Z}, m_i \ge 0$, is a basis of $I_q[X_1, \ldots, X_n]$ over $I(q)$. This basis is the most convenient for our purpose.

If we review the proof of Theorem 1.2 we see that the following two basic properties of polynomials were crucial.

- If $p(X)$ is a polynomial over an integral domain containing $\mathbb{Q}$, then there exists a (unique) polynomial $r(X)$ with $\deg r = \deg p + 1$ and
$$\sum_{x=1}^{y} p(x) = r(y)$$
for every integer y.
- If $p(X)$ is a polynomial over an integral domain containing $\mathbb{Q}$ and $a_1, a_2, \ldots, a_r$ are distinct zeros of $p(X)$, then there exists a polynomial $r(X)$ with
$$p(X) = (X - a_1)(X - a_2) \ldots (X - a_r)r(X).$$

The following analogs holds for q-polynomials.

- If $p(X)$ is a q-polynomial, then there exists a (unique) q-polynomial $r(X)$ with $\deg r = \deg p + 1$ and
$$\sum_{x=1}^{y} p(x)\, q^x = r(y)$$
for all integers y. (The degree of a q-polynomial in X is defined as the degree of the corresponding ordinary polynomial in q^X.) In order to see that note
$$[X; q]_{n+1} - [X - 1; q]_{n+1} = q^{X-1}[n + 1; q][X; q]_n,$$
which implies
$$\sum_{x=1}^{y}[x; q]_n q^x = \frac{q}{[n + 1; q]}[y; q]_{n+1} \tag{1.9}$$
for all integers y.
- If $p(X)$ is a q-polynomial and $a_1, a_2, \ldots, a_r$ are distinct integer zeros of $p(X)$, then there exists a q-polynomial $r(X)$ with

$$p(X) = ([X; q] - [a_1; q])([X; q] - [a_2; q]) \cdots ([X; q] - [a_r; q])r(X) =$$
$$q^{a_1+a_2+\ldots+a_r}[X - a_1; q][X - a_2; q] \cdots [X - a_r; q]r(X).$$

The proof is analogous to the proof for ordinary polynomials, namely the fundamental identity is

$$[X;q]^n - [a;q]^n = ([X;q]-[a;q])\sum_{i=0}^{n-1}[X;q]^i[a;q]^{n-1-i} = q^a[X-a;q]\sum_{i=0}^{n-1}[X;q]^i[a;q]^{n-1-i}.$$

REMARK 1.3. *In the following we often use expressions like $[y-x+i;q]_j$ which are no q-polynomials in x and y. However, in our formulas these expressions always come together with a factor $q^{x\cdot j}$ and thus $[y-x+i;q]_j q^{x\cdot j}$ is used as a shorthand for the degree j polynomial $\prod_{l=0}^{j-1}(q^x - q^{y+i+l})/(1-q)$. With this interpretation our formulas are indeed q-polynomials in x and y.*

Using these q-analogs it is quite straightforward to modify the proof of Theorem 1.2 in order to prove Theorem 1.1. In the following we sketch it by stating the q-versions of the definitions and lemmas that are necessary to prove Theorem 1.2.

The norm of an (r,n,c)-pattern is defined as the sum of its parts, where we omit the first and the last part in each row. Our first observation is that the bijection in Lemma 1.1 is norm-preserving. We introduce a q-analog of $F(r,n,c;k_1,\ldots,k_{n-r})$. Let

$$F_q(r,n,c;k_1,\ldots,k_{n-r}) = \left(\sum_a \operatorname{sgn}(a) q^{\operatorname{norm}(a)}\right)/(q^{k_1+k_2+\ldots+k_{n-r}}),$$

where the sum is over all (r,n,c)-patterns $a=(a_{i,j})$ with $a_{r+1,r+i}=k_i$ for $i=1,2,\ldots,n-r$. Observe that $F_q(n-1,n,c;k)\,q^k$ is the generating function of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n. We have $F_q(0,n,c;k_1,\ldots,k_n)=1$ and

$$F_q(r,n,c;k_1,k_2,\ldots,k_{n-r}) = \sum_{l_1=0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3}\cdots\sum_{l_{n-r}=k_{n-r-1}}^{k_{n-r}}\sum_{l_{n-r+1}=k_{n-r}}^{c} F_q(r-1,n,c;l_1,l_2,\ldots,l_{n-r+1})\,q^{l_1+l_2+\ldots+l_{n-r+1}}. \quad (1.10)$$

This shows that $F_q(r,n,c;k_1,k_2,\ldots,k_{n-r})$ is a q-polynomial in $(k_1,\ldots,k_{n-r})$. Next we have to show that $F_q(r,n,c;k_1,\ldots,k_{n-r})$ is of degree at most $2r$ in k_i. For that purpose we need the following q-analog of Lemma 1.3.

LEMMA 1.8. *Let $F(x_1,x_2)$ be a q-polynomial in x_1 and x_2 which is of degree at most R in each of x_1 and x_2. Moreover assume that $D_1F(x_1,x_2)$ is of degree at most R as a q-polynomial in x_1 and x_2, i.e. a linear combination of monomials $(q^{x_1})^m(q^{x_2})^n$ with $m+n\le R$. Then $\sum\limits_{x_1=a}^{y}\sum\limits_{x_2=y}^{b} F(x_1,x_2)q^{x_1+x_2}$ is of degree at most $R+2$ in y.*

Next we state the q-analog of (1.4). Let $r\ge 1$ be an integer. Then

$$\sum_{x'=x}^{y}\sum_{y'=x-1}^{y-1}[y'-x'-r+3;q]_{2r-3}[y'-x'+1;q]\,q^{(2r-2)x'}\,(1+q^{r-1})\,q^{x'+y'} = $$

$$= 2\frac{[y-x-r+2;q]_{2r-1}[y-x+1;q]q^{2rx}}{[2r-1;q][2r;q]}(1+q^r)q^{r-2}. \quad (1.11)$$

The proof is analogous to the proof of (1.4). First one has to use the following transformation.
$[y'-x'-r+3;q]_{2r-3}[y'-x'+1;q]\,(1+q^{r-1}) = [y'-x'-r+2;q]_{2r-2}q^{r-1} + [y'-x'-r+3;q]_{2r-2}$
Then the fundamental identities are

$$\sum_{z=a}^{b}[z+w;q]_n q^z = \frac{q^{-w+1}}{[n+1;q]}\left([b+w;q]_{n+1} - [a-1+w;q]_{n+1}\right)$$

which is an easy consequence of (1.9) and

$$[z;q]_n = (-1)^n q^{n(z+(n-1)/2)}[-z-n+1;q]_n. \quad \square$$

Lemma 1.4 and (1.11) imply the q-analog of Lemma 1.5.

Lemma 1.9. *Let r, n, i be integers, r non-negative, n positive and $1 \le i \le n-r-1$. Then*

$$D_i F_q(r,n,c;.)(k_1,\ldots,k_{n-r}) = (-1)^r \frac{(1+q^r)}{[1;q]_{2r}}[k_{i+1}-k_i-r+2;q]_{2r-1}[k_{i+1}-k_i+1;q]q^{2rk_i} \times q^{r(1+4i-4n+5r)/2}F_q(r,n-2,c+2;k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_{n-r}+2).$$

Lemma 1.9 shows that $D_i F_q(r,n,c;.)(k_1,\ldots,k_{n-r})$ is of degree $2r$ in k_i and in k_{i+1}. In the next lemma we see that this is also true for $F_q(r,n,c;k_1,\ldots,k_{n-r})$ itself.

Lemma 1.10. *Let r, n, i be integers, r non-negative, n positive and $1 \le i \le n-r$. Then $F_q(r,n,c;k_1,\ldots,k_{n-r})$ is a q-polynomial in k_i of degree at most $2r$.*

Proof. Use (1.10), Lemma 1.9 and Lemma 1.8 in the same way as their analogs in Lemma 1.6. $\square$

This lemma, Lemma 1.7 and the second property of q-polynomials imply the following q-analog of Corollary 1.1.

Corollary 1.3. *$F_q(n-1,n,c;k)/([1+k;q]_{n-1}[1+c-k;q]_{n-1}q^{(n-1)k})$ is independent of k.*

We are now able to prove our main theorem.

Proof of Theorem 1.1. We prove the assertion by induction with respect to n. Observe that the formula is true for $n=1$ since $F_q(0,1,c;k)=1$. Applying Corollary 1.3 in the same way as Corollary 1.1 was applied in the proof of Theorem 1.2, it suffices to check the formula for $F_q(n-1,n,c;c)$. For that purpose we need the recursion

$$F_q(n-1,n,c;c) = q^{cn-c}\sum_{k=0}^{c}F_q(n-2,n-1,c;k)q^k$$

and the following identity

$$\sum_{k=0}^{c}[1+k;q]_{m-1}[1+c-k;q]_{m-1}\,q^{mk} = \frac{[1;q]_{m-1}^2[c+1;q]_{2m-1}}{[1;q]_{2m-1}} \tag{1.12}$$

which can be deduced from the q-Chu-Vandermonde identity, see [**5**, (3.3.10)]. $\square$

Finally we are able to prove the Bender-Knuth (ex-)Conjecture.

Corollary 1.4. *The generating function of strict plane partitions with parts in* $\{1, 2, \ldots, n\}$ *and at most* c *columns is*

$$\prod_{i=1}^{n} \frac{[c+i;q]_i}{[i;q]_i}.$$

Proof. By Theorem 1.1 the generating function is equal to

$$\sum_{k=0}^{c} \frac{q^{k\,n}[k+1;q]_{n-1}[1+c-k;q]_{n-1}}{[1;q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1;q]_{i-1}}{[i;q]_i}.$$

The assertion follows from (1.12). □

7. A final observation

A monotone triangle of size n, see [**7**, p. 58], is an $(n-1, n, n+1)$-pattern with strictly increasing rows. Monotone triangles of size n with the central part of the first row equal to k are easily seen to be in bijection with alternating sign matrices of size n, where the unique 1 in the first row is in the k-th column. Let $A(n,k)$ denote the number of these objects. It was conjectured by Mills, Robbins and Rumsey [**42**] (well-known as the Refined Alternating Sign Matrix Theorem) and proved by Zeilberger [**65**] that

$$A(n,k) = \frac{(k)_{n-1}(1+n-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(1)_{3i-2}}{(1)_{n+i-1}}.$$

Surprisingly it turns out that the number of $(n-1, n, n-1)$-patterns $(a_{i,j})$ with $a_{n,n} = k-1$ divided by $A(n,k)$ is independent of k. In fact it is equal to

$$\prod_{1 \le i \le j \le n-1} \frac{i+j+n-2}{i+2j-2},$$

the number of $(n-1) \times (n-1) \times (n-1)$ totally symmetric plane partitions, see [**60**]. As in the case of the enumeration of $(n-1, n, c)$-patterns, it suffices to show that

$$A(n,k)/((k)_{n-1}(1+n-k)_{n-1})$$

is independent of k in order to prove the formula for $A(n,k)$, see [**7**, Sec. 5.2] for an explanation. Therefore we hope to find another proof of the Refined Alternating Sign Matrix Theorem which is along the lines of the proof of Theorem 1.2. The situation is similar to that of strict plane partitions which is under consideration in this paper. First, one would have to find an extension of the combinatorial interpretation of alternating sign matrices of order n such that the unique 1 in the first row is in the k-th column to *arbitrary* integers k. That is to say that one would have to find combinatorial objects indexed by a positive integer n and an arbitrary integer k which are in bijection with alternating sign matrices of order n, where the unique 1 in the first row is in the k-th column for $k \in \{1, 2, \ldots, n\}$. In the view of the fact that generalized $(n-1, n, c)$ Gelfand-Tsetlin patterns were the right extension of the strict plane partitions, one would rather work with monotone triangles than with alternating sign matrices. Next it would have to be shown that for fixed n these objects are enumerated by a polynomial $P_n(k)$ in k of degree $2n-2$. Typically, this could be done by a recursion similar to (1.2). Finally it would have to be shown that there exist none of these extending combinatorial objects if $k = 0, -1, \ldots, -n+2$ or $k = n+1, n+2, \ldots, 2n-1$.

We also wish to remark another way one could bijectively prove the Refined Alternating Sign Matrix Theorem. We have already seen that it would suffice to show that the number of $(n-1, n, n-1)$-patterns $(a_{i,j})$ with $a_{n,n} = k-1$ divided by the number of alternating sign matrices of order n, where the unique 1 in the first row is in the k-th column is independent of k. Thus, a bijection between $(n-1, n, n-1)$-patterns with $0 \le a_{n,n} = k-1 \le n-1$ on one side and pairs consisting of a monotone triangle of size n with the central part in the first row equal to k and $(n-1) \times (n-1) \times (n-1)$ totally symmetric plane partitions would simultaneously prove the formula for $A(n,k)$ and for the number of $(n-1) \times (n-1) \times (n-1)$ totally symmetric plane partitions.

Acknowledgment: The author thanks the referees for their careful reading of the manuscript and many helpful comments.

CHAPTER 2

Another refinement of the Bender-Knuth (ex-)Conjecture

Abstract. We compute the generating function of column-strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns, p rows of odd length and k parts equal to n. This refines both, Krattenthaler's [**30**] and the author's [**15**, resp. Chapter 1] refinement of the Bender-Knuth (ex-)Conjecture. The result is proved by an extension of the method for proving polynomial enumeration formulas which was introduced by the author in [**15**, resp. Chapter 1] to q-quasi-polynomials.

1. Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition, i.e. $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$. A *strict plane partition* of shape λ is an array $\Pi = (\pi_{i,j})_{1 \leq i \leq r, 1 \leq j \leq \lambda_i}$ of non-negative integers such that rows are weakly decreasing and columns are strictly decreasing. For instance,

$$\begin{matrix} 7 & 6 & 5 & 5 & 2 \\ 5 & 4 & 2 & 2 & \\ 4 & 2 & & & \\ 2 & 1 & & & \end{matrix}$$

is a strict plane partition of shape $(5, 4, 2, 2)$. The *norm* $n(\Pi)$ of a strict plane partition is defined as the sum of its parts and Π is said to be a strict plane partition of $n(\Pi)$. Thus, 47 is the norm of our example. Strict plane partitions and closely related objects have been enumerated subject to a variety of different constraints. In [**6**, p.50] Bender and Knuth had conjectured that the generating function of strict plane partitions with at most c columns and parts in $\{1, 2, \ldots, n\}$ is equal to

$$\sum q^{n(\pi)} = \prod_{i=1}^{n} \frac{[c+i;q]_i}{[i;q]_i},$$

where $[n;q] = 1 + q + \cdots + q^{n-1} = (1-q^n)/(1-q)$ and $[a;q]_n = \prod_{i=0}^{n-1} [a+i;q]$. (Here and in the following we consider the generating function with respect to the norm of a strict plane partition.) This conjecture was proved by Andrews [**2**], Gordon [**26**], Macdonald [**38**, Ex. 19, p.53] and Proctor [**46**, Prop. 7.2]. For related papers, which mostly include generalizations of the Bender-Knuth (ex-)Conjecture, see [**12, 13, 15, 29, 30, 48, 59**].

In particular, Krattenthaler [**30**] computed the generating function of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and p rows of odd length. On the other hand the author [**15**, resp. Chapter 1] computed the generating function of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n. In this paper we refine these two results. The main result is the following.

Theorem 2.1. *The generating function of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns, p rows of odd length and k parts equal to n is given by*

$$M_{n,c,p}[k+1;q]_{n-1}[k-c-n+1;q]_{n-1} q^k + L_{n,c,p} \Bigg((-1)^k q^{nk} + (-1)^n q^{(n-1)(2c+n)/2+k}$$
$$\times \sum_{i=1}^{n-1} (-1)^c q^{\binom{i}{2}} \frac{[k+1;q]_{n-1}[k-c-i+1;q]_{i-1}[k-c-n+1;q]_{n-i-1}}{[1;q]_{i-1}[1;q]_{n-1-i}[c+i+1;q]_{n-1}}$$
$$- q^{\binom{i}{2}} \frac{[k+1;q]_{i-1}[k+i+1;q]_{n-i-1}[k-c-n+1;q]_{n-1}}{[1;q]_{i-1}[1;q]_{n-1-i}[c+i+1;q]_{n-1}} \Bigg),$$

where

$$L_{n,c,p}=\begin{cases}\left(\frac{q^{\binom{p+1}{2}}\left[{n-1 \atop p}\right][c;q]}{[c+p;q]_n}-\frac{q^{\binom{p}{2}}\left[{n-1 \atop p-1}\right][c+2n;q]}{[c+p+1;q]_n}\right)\frac{[c+1;q]_{n-1}[1;q]_{n-1}}{2}\prod_{i=1}^{n-1}\frac{[c+2i+1;q]_{n-i}}{[2i;q]_{n-i}[2i;q]} & 2|c\\ \left(q^{\binom{p+1}{2}}\begin{bmatrix}n-1\\p\end{bmatrix}-q^{\binom{p}{2}}\begin{bmatrix}n-1\\p-1\end{bmatrix}\right)\frac{[1;q]_{n-1}}{2}\prod_{i=1}^{n-1}\frac{[c+2i;q]_{n-i}}{[2i;q]_{n-i}[2i;q]} & 2\not|c\end{cases}$$

and

$$M_{n,c,p}=\frac{(-1)^{n-1}q^{(n-1)(2c+n)/2}}{[1;q]_{n-2}}$$
$$\times\begin{cases}\left(\frac{q^{\binom{p+1}{2}}\left[{n-1 \atop p}\right][c;q]}{[c+p;q]_n}\left(\frac{1}{[n-1;q]}-\frac{[c+2n-1;q]}{[c+n;q][2n-2;q]}\right)+\frac{q^{\binom{p}{2}}\left[{n-1 \atop p-1}\right]}{[c+p+1;q]_n}\frac{[c+2n-1;q]_2}{[c+n;q][2n-2;q]}\right)\prod_{i=1}^{n-1}\frac{[c+2i;q]_{n-i}}{[2i;q]_{n-i}} & 2|c\\ \left(q^{\binom{p+1}{2}+n-1}\begin{bmatrix}n-1\\p\end{bmatrix}+q^{\binom{p}{2}}\begin{bmatrix}n-1\\p-1\end{bmatrix}\right)\frac{1}{[2n-2;q]}\prod_{i=1}^{n-1}\frac{[c+2i+1;q]_{n-i-1}}{[2i;q]_{n-i}} & 2\not|c\end{cases}.$$

In these formulas the notion of the q-binomial coefficient is used. It is defined as follows.

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{cases}\frac{[n-k+1;q]_k}{[1;q]_k} & \text{if } 0\le k\le n\\ 0 & \text{otherwise}\end{cases}$$

At the end of Section 6 we show that Theorem 2.1 implies Krattenthaler's and the author's refinement of the Bender-Knuth (ex-)Conjecture and, consequently, the Bender-Knuth (ex-) Conjecture itself.

Our method for proving Theorem 2.1 is an extension of the method for proving polynomial enumeration formulas we have introduced in [**15**, resp. Chapter 1]. It is interesting to note that this elementary method completely avoids the use of determinants, which is unusual for a plane partition enumeration. The method is divided into the following three steps.

(1) **Extension of the combinatorial interpretation.** It only makes sense to ask for the number of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n if $k\in\{0,1,\ldots,c\}$. This is because all n's must be in the first row of the strict plane partition by the columnstrictness. In the first step we find a combinatorial extension of these strict plane partitions to arbitrary integers k, i.e. we find new objects indexed by an arbitrary integer k which are in bijection with strict plane partitions with k parts equal to n if $k\in\{0,1,\ldots,c\}$.
(2) **The extending objects are enumerated by a q-quasi-polynomial in k.** Using a simple recursion we show that the extending objects are enumerated by a q-quasi-polynomial (see Definition 2.1). Moreover the degree of the q-quasi-polynomial is computed.
(3) **Exploring properties of the q-quasi-polynomial that determine it uniquely.** A q-quasi-polynomial of given degree n and period p is determined by a finite number $a(n,p)$ of (independent) properties such as zeros or other evaluations. In the last step we derive enough properties of the q-quasi-polynomial in order to compute it using the degree estimation from the previous step.

Note that this article contains two types of extensions of the method for proving polynomial enumeration formulas presented in [**15**, resp. Chapter 1]. Firstly, the method is extended to q-quasi-polynomials, see Definition 2.1. More remarkable is, however, the following extension:

In [**15**, resp. Chapter 1] we have described a method that is applicable to polynomial enumeration formulas that factorize into distinct linear factors over $\mathbb{Z}$. There the "properties" in the third step are simply the integer zeros together with one (easy to compute) non-zero evaluation. (Thus, the third step was entitled "Exploring natural linear factors".) In this article we demonstrate that the lack of integer zeros can be compensated by other properties of the (q-quasi-)polynomial.

The paper is organized as follows. In Section 2 we give the combinatorial extension of strict plane partitions as proposed in Step 1. In Section 3 we introduce q-quasi-polynomials and establish their properties needed in this paper. In Section 4 we show that the generating function of strict plane partitions which is under consideration in this paper is a q-quasi-polynomial and we compute its degree (Step 2). In Section 5 we deduce enough properties of the q-quasi-polynomial in order to compute it (Step 3). In Section 6 we perform the (complicated) computation and in Section 7 we derive some q-summation formulas which are needed in the computation.

Throughout the whole article we use the extended definition of the summation symbol, namely,

$$\sum_{i=a}^{b} f(i) = \begin{cases} f(a)+f(a+1)+\cdots+f(b) & \text{if } a \le b \\ 0 & \text{if } b = a-1 \\ -f(b+1)-f(b+2)-\cdots-f(a-1) & \text{if } b+1 \le a-1 \end{cases} . \tag{2.1}$$

This assures that for any polynomial $p(X)$ over an arbitrary integral domain I containing $\mathbb{Q}$, there exists a unique polynomial $q(X)$ over I such that $\sum_{x=0}^{y} p(x) = q(y)$ for all integers y. We usually write $\sum_{x=0}^{y} p(x)$ for $q(y)$. We use the analogous extended definition of the product symbol. In particular, this extends the definition of $[a;q]_n$, for example $[a;q]_{-1} = (q-1)/(q^{a-1}-1)$.

2. Extension of the combinatorial interpretation

In this section we establish the combinatorial extension of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n to arbitrary integers k. This extension was already introduced in Section 2 of [**15**, resp. Chapter 1]. We repeat it here in less detail.

Let r,n,c be integers with $0 \le r \le n$. A *generalized* (r,n,c) *Gelfand-Tsetlin pattern* (for short: (r,n,c)-pattern) is an array $(a_{i,j})_{1\le i\le r+1, i-1\le j\le n+1}$ of integers with

(1) $a_{i,i-1}=0$ and $a_{i,n+1}=c$,
(2) if $a_{i,j} \le a_{i,j+1}$ then $a_{i,j} \le a_{i-1,j} \le a_{i,j+1}$
(3) if $a_{i,j} > a_{i,j+1}$ then $a_{i,j} > a_{i-1,j} > a_{i,j+1}$.

The norm of an (r,n,c)-pattern is defined as the sum of its parts, where the first and the last part of each row is omitted. A $(3,6,c)$-pattern for example is of the form

$$\begin{array}{ccccccccccccccc}
 & & & 0 & & a_{4,4} & & a_{4,5} & & a_{4,6} & & c & & & \\
 & & 0 & & a_{3,3} & & a_{3,4} & & a_{3,5} & & a_{3,6} & & c & & \\
 & 0 & & a_{2,2} & & a_{2,3} & & a_{2,4} & & a_{2,5} & & a_{2,6} & & c & \\
0 & & a_{1,1} & & a_{1,2} & & a_{1,3} & & a_{1,4} & & a_{1,5} & & a_{1,6} & & c,
\end{array}$$

such that every entry not in the top row is between its northwest neighbour w and its northeast neighbour e, if $w \le e$ then weakly between, otherwise strictly between. Thus

$$
\begin{array}{ccccccccccccccc}
 & & & 0 & & 3 & & -5 & & 10 & & 4 & & & \\
 & & 0 & & 2 & & -2 & & 3 & & 8 & & 4 & & \\
 & 0 & & 2 & & -1 & & 2 & & 4 & & 7 & & 4 & \\
0 & & 0 & & 0 & & 1 & & 2 & & 5 & & 6 & & 4
\end{array}
$$

is an example of a $(3, 6, 4)$-pattern. Note that a generalized $(n-1, n, c)$ Gelfand-Tsetlin pattern $(a_{i,j})$ with $0 \le a_{n,n} \le c$ is a so-called Gelfand-Tsetlin pattern with n rows and parts in $\{0, 1, \ldots, c\}$, see [**58**, p. 313] or [**20**, (3)] for the original reference. (Observe that $0 \le a_{n,n} \le c$ implies that the third possibility in the definition of a generalized Gelfand-Tsetlin pattern never occurs.) The following correspondence between Gelfand-Tsetlin patterns and strict plane partitions is crucial for our paper.

Lemma 2.1. *There exists a norm-preserving bijection between Gelfand-Tsetlin patterns $(a_{i,j})$ with n rows, parts in $\{0, 1, \ldots, c\}$ and fixed $a_{n,n} = k$, and strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n. In this bijection $(a_{1,n}, a_{1,n-1}, \ldots, a_{1,1})$ is the shape of the strict plane partition.*

Proof. Given such a Gelfand-Tsetlin pattern, the corresponding strict plane partition is such that the shape filled by parts greater than i corresponds to the partition given by the $(n-i)$-th row (counting from the top) of the Gelfand-Tsetlin pattern, where the first and the last part in each row of the pattern are omitted. Thus, the strict plane partition in the introduction corresponds to the following Gelfand-Tsetlin pattern (The first and last part in each row are omitted).

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 2 & & & & & \\
 & & & & 0 & & 1 & & 4 & & & & \\
 & & & 0 & & 1 & & 2 & & 4 & & & \\
 & & 0 & & 0 & & 1 & & 2 & & 4 & & \\
 & 0 & & 0 & & 1 & & 2 & & 4 & & 5 & \\
0 & & 0 & & 0 & & 1 & & 2 & & 4 & & 5
\end{array}
$$

□

Therefore, it suffices to compute the generating function with respect to the norm of $(n-1, n, c)$-patterns with fixed $a_{n,n} = k$, $0 \le k \le c$, and where exactly p values of $a_{1,1}, a_{1,2}, \ldots, a_{1,n}$ are odd. However, $(n-1, n, c)$-patterns are defined for all $a_{n,n} \in \mathbb{Z}$ and thus we have established the combinatorial extension apart from the following technical detail. That is that we actually have to work with a signed enumeration if $a_{n,n} \notin \{0, 1, \ldots, c\}$. Therefore, we define the sign of a pattern.

A pair $(a_{i,j}, a_{i,j+1})$ with $a_{i,j} > a_{i,j+1}$ and $i \ne 1$ is called an *inversion* of the (r, n, c)-pattern and $(-1)^{\#\text{ of inversions}}$ is said to be the *sign* of the pattern, denoted by $\operatorname{sgn}(a)$. The $(3, 6, 4)$-pattern in the example above has altogether 6 inversions and thus its sign is 1. We define the following generating function

$$F_q(r,n,c,p;k_1,k_2,\ldots,k_{n-r}) = \left(\sum_a \operatorname{sgn}(a) q^{\operatorname{norm}(a)}\right) / q^{k_1+k_2+\ldots+k_{n-r}},$$

where the sum is over all (r,n,c)-patterns $(a_{i,j})$ with top row defined by $k_i = a_{r+1,r+i}$ for $i=1,\ldots,n-r$ and such that exactly p of $a_{1,1},a_{1,2},\ldots,a_{1,n}$ are odd. It is crucial that for $0 \le k \le c$ the expression $F_q(n-1,n,c,p;k)\, q^k$ is the generating function of $(n-1,n,c)$-patterns with $a_{n,n}=k$ and where exactly p of $a_{1,1},a_{1,2},\ldots,a_{1,n}$ are odd. This is because an $(n-1,n,c)$-pattern with $0 \le a_{n,n} \le c$ has no inversion. Thus, $F_q(n-1,n,c,p;k)$ is the quantity we want to compute. It has the advantage that it is well defined for all integers k, whereas our original enumeration problem was only defined for $0 \le k \le c$.

3. q-quasi-polynomials and their properties

In the following, let R be a ring containing $\mathbb{C}$. A *quasi-polynomial* (see [**57**, page 210]) in the variables $X_1,X_2,\ldots,X_n$ over R is an expression of the form

$$\sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n, m_i\ge 0} c_{m_1,m_2,\ldots,m_n}(X_1,X_2,\ldots,X_n)\, X_1^{m_1} X_2^{m_2}\cdots X_n^{m_n},$$

where $(X_1,X_2,\ldots,X_n) \to c_{m_1,m_2,\ldots,m_n}(X_1,X_2,\ldots,X_n)$ are periodic functions on $\mathbb{Z}^n$ taking values in R, that is there exists an integer t with

$$c_{m_1,m_2,\ldots,m_n}(k_1,\ldots,k_i,\ldots,k_n) = c_{m_1,m_2,\ldots,m_n}(k_1,\ldots,k_i+t,\ldots,k_n)$$

for all $(k_1,\ldots,k_n)\in\mathbb{Z}^n$ and i, and almost all $c_{m_1,\ldots,m_n}(X_1,\ldots,X_n)$ are zero. Let $(m_1,\ldots,m_n)$ be with $c_{m_1,\ldots,m_n}(X_1,\ldots,X_n) \ne 0$ such that $m_1+\ldots+m_n$ is maximal. Then $m_1+\ldots+m_n$ is said to be the degree of the quasi-polynomial. (The zero-quasi-polynomial is said to be of degree $-\infty$.) The smallest common period of all $c_{m_1,\ldots,m_n}(X_1,\ldots,X_n)$ is the period of the quasi-polynomial. (In this paper we only deal with q-quasi-polynomials of period 1 or 2.) In Section 6 of [**15**, resp. Chapter 1] we have defined q-polynomials. The following definition of q-quasi-polynomials is the merge of these two definitions. In this definition let R_q denote the ring of quotients with elements from $R[q]$ in the numerator and elements from $\mathbb{C}[q]$ in the denominator.

Definition 2.1. *A q-quasi-polynomial in $X_1,X_2,\ldots,X_n$ over R is a quasi-polynomial over R_q in $q^{X_1},q^{X_2},\ldots,q^{X_n}$. Let $R_{qq}[X_1,X_2,\ldots,X_n]$ denote the ring of q-quasi-polynomials in $X_1,\ldots,X_n$ over R.*

Observe that $R_{qq}[X_1,\ldots,X_n]$ is the ring of q-quasi-polynomials in X_i over

$$R_{qq}[X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n].$$

We define $[X;q]=(1-q^X)/(1-q)$ and $[X;q]_n=\prod_{i=0}^{n-1}[X+i;q]$. Observe that

$$[X_1;q]_{m_1}[X_2;q]_{m_2}\cdots[X_n;q]_{m_n},$$

with $(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n$ and $m_i\ge 0$, is a set of independent generators of $R_{qq}[X_1,\ldots,X_n]$ over the periodic functions.

The following two properties of polynomials were crucial for our method for proving polynomial enumeration formulas which we have introduced in [**15**, resp. Chapter 1]. Since we want to extend our method to q-quasi-polynomials, we have to find q-quasi-analogs of these properties.

(1) If $p(X)$ is a polynomial over R, then there exists a (unique) polynomial $r(X)$ with $\deg r = \deg p + 1$ and

$$\sum_{x=0}^{y} p(x) = r(y)$$

for all integers y.

(2) If $p(X)$ is a polynomial over R and a is a zero of $p(X)$, then there exists a polynomial $r(X)$ over R with

$$p(X) = (X - a)r(X).$$

Regarding the first property, we show the following for q-quasi-polynomials.

LEMMA 2.2. *Let $p(X)$ be a q-quasi-polynomial in X over R with degree d and period t. Then $\sum_{x=0}^{y} p(x)q^x$ is a q-quasi-polynomial over R in y with degree at most $d+1$ and period at most t.*

In order to prove this lemma we need a definition and another lemma.

DEFINITION 2.2. *Let $\rho \to f(\rho)$ be a function, e.g. from $\mathbb{C}$ to R_q. Then the q-differential-operator $\frac{d}{d_q\rho}$ is defined as follows*

$$\frac{d}{d_q\rho} f(\rho) = \frac{f(q\,\rho) - f(\rho)}{\rho(q-1)}.$$

The n-fold composition is denoted by $\frac{d}{d_q\rho^n}$.

If we apply the q-differential operator to a laurent polynomial we obtain the following.

$$\frac{d}{d_q\rho} \sum_{i=b}^{c} a_i \rho^i = \sum_{i=b}^{c} [i;q] a_i \rho^{i-1}. \tag{2.2}$$

Note that this is also true if $b > c$.

LEMMA 2.3.

$$\sum_{x=0}^{y} [x;q]_n q^x \sigma^{x-1} = \frac{d}{d_q\sigma^n} \left(\frac{\sigma^{n-1}((\sigma q)^{y+1} - 1)}{(\sigma q - 1)} \right)$$

Proof of Lemma 2.3. By (2.2) we have the following identity.

$$\sum_{x=0}^{y} [x;q]_n q^x \sigma^{x-1} = \frac{d}{d_q\sigma^n} \left(\sum_{x=0}^{y} q^x \sigma^{x+n-1} \right).$$

The assertion follows from

$$\sum_{x=0}^{y} q^x \sigma^{x+n-1} = \frac{\sigma^{n-1}((\sigma q)^{y+1} - 1)}{(\sigma q - 1)}.$$

□

Remark 2.1. *If $\rho = 1$ in the statement of the lemma, it is possible to make a stronger assertion.*

$$\sum_{x=0}^{y} [x;q]_n q^x = \frac{q}{[n+1;q]} [y;q]_{n+1}$$

It follows from $[x;q]_{n+1} - [x-1;q]_{n+1} = [x;q]_n q^{x-1} [n+1;q]$.

Proof of Lemma 2.2. Suppose $p(X)$ is a q-quasi-polynomial with period t. Let $\rho \in \mathbb{C}$ be a primitive t-th root of unity. Then $p(X)$ can be expressed as follows

$$p(X) = p_0(X) + \rho^X p_1(X) + \rho^{2X} p_2(X) + \ldots + \rho^{(t-1)X} p_{t-1}(X),$$

where $p_i(X)$ are q-polynomials, i.e. q-quasi-polynomials with period 1. Suppose d is the degree of $p(X)$. Then, for every i, we have

$$p_i(X) = \sum_{j=0}^{d} a_{i,j} \, [X;q]_j,$$

where $a_{i,j}$ are coefficients in R_q. Thus, by Lemma 2.3,

$$\sum_{x=0}^{y} p(x) q^x \;=\; \sum_{x=0}^{y} \sum_{i=0}^{t-1} \sum_{j=0}^{d} a_{ij} [x;q]_j \rho^{ix} q^x \;=\; \sum_{i=0}^{t-1} \sum_{j=0}^{d} a_{i,j} \rho^i \, \frac{d}{d_q \sigma^j} \left(\frac{\sigma^{j-1}((\sigma q)^{y+1} - 1)}{(\sigma q - 1)} \right) \Bigg|_{\sigma = \rho^i} .$$

The assertion follows after observing that

$$\frac{d}{d_q \sigma^j} \left(\frac{\sigma^{j-1}((\sigma q)^{y+1} - 1)}{(\sigma q - 1)} \right) \Bigg|_{\sigma = \rho^i}$$

is a q-quasi-polynomial in y of degree at most $j+1$. □

Next we consider the second important property of polynomials for our method. It suffices to derive an analog for q-polynomials. Suppose $p(X)$ is a q-polynomial over R and a is an integer zero of $p(X)$. Then there exists a q-polynomial $r(X)$ over R with

$$p(X) = ([X;q] - [a;q]) \, r(X) = q^a [X-a;q] \, r(X).$$

The proof follows from the following identity

$$[X;q]^n - [a;q]^n = ([X;q] - [a;q]) \sum_{i=0}^{n-1} [X;q]^i [a;q]^{n-1-i} = q^a [X-a;q] \sum_{i=0}^{n-1} [X;q]^i [a;q]^{n-1-i}.$$

This property implies that if $a_1, \ldots, a_r$ are distinct integer zeros of a q-polynomial $p(X)$ over an integral domain R, then there exists a q-polynomial $r(X)$ with

$$p(X) = r(X) \prod_{i=1}^{r} [X - a_i; q].$$

This will be fundamental for the "q-Lagrange interpolation", which we use in Lemma 2.14.

4. $F_q(n-1,n,c,p;k)$ is a q-quasi-polynomial in k

In this section we show that $F_q(r,n,c,p;k_1,\ldots,k_{n-r})$ is a q-quasi-polynomial in $k_1,k_2,\ldots,k_{n-r}$ with period at most 2. Moreover, we show that the degree in k_i is at most $2r$.

The following recursion is fundamental.

$$F_q(r,n,c,p;k_1,k_2,\ldots,k_{n-r}) = \sum_{l_1=0}^{k_1}\sum_{l_2=k_1}^{k_2}\sum_{l_3=k_2}^{k_3}\cdots\sum_{l_{n-r}=k_{n-r-1}}^{k_{n-r}}\sum_{l_{n-r+1}=k_{n-r}}^{c} F_q(r-1,n,c,p;l_1,l_2,\ldots,l_{n-r+1})\,q^{l_1+l_2+\ldots+l_{n-r+1}}. \tag{2.3}$$

Furthermore,

$$F_q(0,n,c,p;k_1,\ldots,k_n) = \begin{cases} 1 & \text{if exactly } p \text{ of } k_1,k_2,\ldots,k_n \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$
$$= \sum_{1\le i_1<i_2<\ldots<i_p\le n}\prod_{j=1}^{p}\frac{e_{1,2}(k_{i_j})}{e_{0,2}(k_{i_j})}\prod_{j=1}^{n} e_{0,2}(k_j) =: S(n,p)(k_1,\ldots,k_n).$$

Here, $x\to e_{i,t}(x)$ is the function defined on integers with

$$e_{i,t}(x) = \begin{cases} 1 & x\equiv i \mod t \\ 0 & \text{otherwise} \end{cases} = \prod_{0\le j\le t-1, j\ne i}\frac{\rho^x-\rho^j}{\rho^i-\rho^j},$$

where $\rho\in\mathbb{C}$ is a primitive t-th root of unity. The identity

$$F_q(0,n,c,p;k_1,\ldots,k_i,\ldots,k_n) = F_q(0,n,c,p;k_1,\ldots,k_i+2,\ldots,k_n)$$

for all i, $1\le i\le n$, implies that $F_q(0,n,c,p;k_1,\ldots,k_n)$ is a q-quasi-polynomial with period 2. The recursion (2.3) and Lemma 2.2 implies (inductively with respect to r) that $F_q(r,n,c,p;.)$ is a q-quasi-polynomial in $(k_1,k_2,\ldots,k_{n-r})$ with period at most 2.

For our purpose it is convenient to define the following generalization of $F_q(r,n,c,p;.)$.

Definition 2.3. *Let n,r, $r\le n$, be non-negative integers and $A(k_1,\ldots,k_n)$ a function on $\mathbb{Z}^n$. We define $G_q(r,n,c,A)$ inductively with respect to r: $G_q(0,n,c,A)=A$ and*

$$G_q(r,n,c,A)(k_1,\ldots,k_{n-r}) = \sum_{l_1=0}^{k_1}\sum_{l_2=k_1}^{k_2}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c} G_q(r-1,n,c,A)(l_1,l_2,\ldots,l_{n-r+1})\,q^{l_1+l_2+\ldots+l_{n-r+1}} \tag{2.4}$$

With this definition, we have $F_q(r,n,c,p;k_1,\ldots,k_{n-r}) = G_q(r,n,c,S(n,p))(k_1,\ldots,k_{n-r})$. We define $T(n,i)(k_1,\ldots,k_n) = \sum_{1\le j_1<j_2<\ldots<j_i\le n}(-1)^{k_{j_1}+k_{j_2}+\ldots+k_{j_i}}$. The following lemma shows that $S(n,p)$ is a linear combination of $T(n,1)$,$T(n,2)$,…,$T(n,n)$ and $T(n,0):=1$.

Lemma 2.4.

$$S(n,p) = \frac{1}{2^n}\left(\sum_{i=0}^{n}\sum_{l=\max(0,i-n+p)}^{\min(p,i)}(-1)^l\binom{i}{l}\binom{n-i}{p-l}T(n,i)\right)$$

Proof. Set $[n] := \{1, 2, \dots, n\}$ and fix $P \subseteq [n]$ with $|P| = p$. Then

$$\prod_{j \in P} e_{1,2}(k_j) \prod_{j \in [n] \setminus P} e_{0,2}(k_j) = \prod_{j \in P} \frac{1-(-1)^{k_j}}{2} \prod_{j \in [n]\setminus P} \frac{1+(-1)^{k_j}}{2} =$$

$$= \frac{1}{2^n} \sum_{i=0}^{n} \sum_{l=\max(0,i-n+p)}^{\min(p,i)} (-1)^l \sum_{\substack{1 \le j_1 < \dots < j_l \le n, \\ j_x \in P}} (-1)^{k_{j_1}+\dots+k_{j_l}} \sum_{\substack{1 \le m_1 < \dots < m_{i-l} \le n, \\ m_x \in [n]\setminus P}} (-1)^{k_{m_1}+\dots+k_{m_{i-l}}},$$

where the second equation follows from expanding the product. In the last expression, i counts the number of $\pm(-1)^{k_x}$ we choose from the product of the n factors of the form $1 \pm (-1)^{k_x}$ and l counts the number of $-(-1)^{k_x}$ we choose. Observe that

$$\sum_{\substack{P \subseteq [n], \\ |P|=p}} \sum_{\substack{1 \le j_1 < \dots < j_l \le n, \\ j_x \in P}} (-1)^{k_{j_1}+\dots+k_{j_l}} \sum_{\substack{1 \le m_1 < \dots < m_{i-l} \le n, \\ m_x \in [n]\setminus P}} (-1)^{k_{m_1}+\dots+k_{m_{i-l}}} = \binom{i}{l}\binom{n-i}{p-l} T(n,i),$$

since every $(-1)^{k_{x_1}+\dots+k_{x_i}}$, $1 \le x_1 < \dots < x_i \le n$, appears with multiplicity $\binom{i}{l}\binom{n-i}{p-l}$ on the left-hand-side. This is because there are $\binom{i}{l}$ ways to choose the elements from $\{x_1, \dots, x_i\} =: I$ which lie in P and $\binom{n-i}{p-l}$ ways to choose the elements in $[n] \setminus I$ which lie in P. $\square$

In [**15**, resp. Chapter 1] we have shown that $G_q(n-1, n, c, 1)(k)$ is a q-polynomial of degree at most $2n-2$. Now, we aim to show that, more general, the degree of $G_q(n-1, n, c, T(n,p))(k)$ in k is at most $2n-2$ as well. The linearity of $A \to G_q(r, n, c, A)$ and Lemma 2.4 then implies that the degree of $G_q(n-1, n, c, S(n,p))$ in k is at most $2n-2$.

In fact, we show that the degree of $G_q(r, n, c, T(n,p))$ in k_i is at most $2r$. The computation is rather complicated. Assume by induction with respect to r that the degree of $G_q(r-1, n, c, T(n,p))(k_1, \dots, k_{n-r})$ in each of k_i and k_{i+1} is at most $2r-2$. The degree of $G_q(r, n, c, T(n,p))$ in k_i is at most the degree of

$$\sum_{l_i = k_{i-1}}^{k_i} \sum_{l_{i+1}=k_i}^{k_{i+1}} G_q(r-1, n, c, T(n,p))(l_1, \dots, l_{n-r+1})$$

in k_i (with $k_0 = 0$ and $k_{n-r+1} = c$). Using Lemma 2.2, this allows us to conclude easily that the degree of $G_q(r, n, c, T(n,p))$ in k_i is at most $4r-2$, however, we want to establish that the degree is at most $2r$. The following lemma is fundamental for this purpose. In order to state it, we need to define an operator D_i which is crucial for the analysis of (2.4).

Definition 2.4. *Let $G(k_1, \dots, k_m)$ be a function in m variables and $1 \le i \le m-1$. We set*

$$D_i G(k_1, \dots, k_m) = \\ G(k_1, \dots, k_{i-1}, k_i, k_{i+1}, k_{i+2}, \dots, k_m) + G(k_1, \dots, k_{i-1}, k_{i+1}+1, k_i - 1, k_{i+2}, \dots, k_m).$$

The following lemma shows the significance of the operator in the computation of the degree.

Lemma 2.5. *Let $F(x_1, x_2)$ be a q-quasi-polynomial in x_1 and x_2 which is of degree at most R in each of x_1 and x_2. Moreover assume that $D_1 F(x_1, x_2)$ is of degree at most R as a q-quasi-polynomial in x_1 and x_2, i.e. a linear combination of "monomials" $[x_1; q]_m [x_2; q]_n \rho_1^{x_1} \rho_2^{x_2}$ with*

$m+n \le R$ and where ρ_1 and ρ_2 are roots of unity. Then $\sum_{x_1=a}^{y}\sum_{x_2=y}^{b} F(x_1,x_2)q^{x_1+x_2}$ is of degree at most $R+2$ in y.

Proof. Set $F_1(x_1,x_2) = D_1F(x_1,x_2)/2$ and $F_2(x_1,x_2) = (F(x_1,x_2) - F(x_2+1,x_1-1))/2$. Clearly $F(x_1,x_2) = F_1(x_1,x_2)+F_2(x_1,x_2)$. Observe that $F_2(x_2+1,x_1-1) = -F_2(x_1,x_2)$. Thus $F_2(x_1,x_2)$ is a linear combination of expressions of the form

$$[x_1;q]_m[x_2+1;q]_n\rho_1^{x_1-1}\rho_2^{x_2} - [x_1;q]_n[x_2+1;q]_m\rho_1^{x_2}\rho_2^{x_1-1}$$

with $m,n \le R$ and where ρ_1 and ρ_2 are roots of unity . We set

$$c(y,n,\rho) = \frac{d}{d_q\rho^n}\left(\frac{\rho^{n-1}((\rho q)^{y+1}-1)}{(\rho q-1)}\right).$$

This is a q-quasi-polynomial in y of degree at most $n+1$. Lemma 2.3 implies

$$\begin{aligned}
&\sum_{x_1=a}^{y}\sum_{x_2=y}^{b}([x_1;q]_m[x_2+1;q]_n\rho_1^{x_1-1}\rho_2^{x_2} - [x_1;q]_n[x_2+1;q]_m\rho_1^{x_2}\rho_2^{x_1-1})q^{x_1+x_2+1}\\
&\qquad = (c(y,m,\rho_1)-c(a-1,m,\rho_1))(c(b+1,n,\rho_2)-c(y,n,\rho_2))\\
&\qquad\quad - (c(y,n,\rho_2)-c(a-1,n,\rho_2))(c(b+1,m,\rho_1)-c(y,m,\rho_1))\\
&= c(y,m,\rho_1)c(b+1,n,\rho_2) - c(a-1,m,\rho_1)c(b+1,n,\rho_2) + c(a-1,m,\rho_1)c(y,n,\rho_2) -\\
&\qquad c(y,n,\rho_2)c(b+1,m,\rho_1) + c(a-1,n,\rho_2)c(b+1,m,\rho_1) - c(a-1,n,\rho_2)c(y,m,\rho_1).
\end{aligned}$$

Thus $\sum_{x_1=a}^{y}\sum_{x_2=y}^{b} F_2(x,y)q^{x+y}$ is of degree at most $R+1$ in y. By the assumption in the lemma $\sum_{x_1=a}^{y}\sum_{x_2=y}^{b} F_1(x,y)q^{x+y}$ is of degree at most $R+2$ in y and the assertion follows. ☐

LEMMA 2.6. *Let m, i be a positive integers with $1 \le i \le m-2$, and $G(l_1,\dots,l_m)$ be a function on $\mathbb{Z}^m$. Then*

$$\begin{aligned}
&D_i\left(\sum_{l_1=k_0}^{k_1}\sum_{l_2=k_1}^{k_2}\cdots\sum_{l_m=k_{m-1}}^{k_m} G(l_1,\dots,l_m)\right)(k_0,k_1,\dots,k_m)\\
&\qquad = -\frac{1}{2}\left(\sum_{l_1=k_0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_i+1}^{k_{i+1}+1}\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+2}}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_m=k_{m-1}}^{k_m} D_iG(\mathbf{l})\right.\\
&\qquad\qquad \left. + \sum_{l_1=k_0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+1}-1}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_m=k_{m-1}}^{k_m} D_{i+1}G(\mathbf{l})\right),
\end{aligned}$$

where $\mathbf{l} = (l_1,\dots,l_m)$.

Proof. We set

$$g(l_{i-1},l_i,l_{i+1}) = \sum_{l_1=k_0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_m=k_{m-1}}^{k_m} G(l_1,\dots,l_m).$$

The left-hand side in the statement of the lemma is equal to

$$\sum_{l_i=k_{i-1}}^{k_i} \sum_{l_{i+1}=k_i}^{k_{i+1}} \sum_{l_{i+2}=k_{i+1}}^{k_{i+2}} g(l_i, l_{i+1}, l_{i+2}) + \sum_{l_i=k_{i-1}}^{k_{i+1}+1} \sum_{l_{i+1}=k_{i+1}+1}^{k_i-1} \sum_{l_{i+2}=k_i-1}^{k_{i+2}} g(l_i, l_{i+1}, l_{i+2}).$$

In the expression, we reverse the middle sum in the second triple sum and split up the first sum in this triple sum to obtain

$$\sum_{l_i=k_{i-1}}^{k_i} \sum_{l_{i+1}=k_i}^{k_{i+1}} \sum_{l_{i+2}=k_{i+1}}^{k_{i+2}} g(l_i, l_{i+1}, l_{i+2}) - \sum_{l_i=k_{i-1}}^{k_i} \sum_{l_{i+1}=k_i}^{k_{i+1}} \sum_{l_{i+2}=k_i-1}^{k_{i+2}} g(l_i, l_{i+1}, l_{i+2}) \\ - \sum_{l_i=k_i+1}^{k_{i+1}+1} \sum_{l_{i+1}=k_i}^{k_{i+1}} \sum_{l_{i+2}=k_i-1}^{k_{i+2}} g(l_i, l_{i+1}, l_{i+2}).$$

Next we cancel some common terms in the first and the second sum and obtain

$$- \sum_{l_i=k_{i-1}}^{k_i} \sum_{l_{i+1}=k_i}^{k_{i+1}} \sum_{l_{i+2}=k_i-1}^{k_{i+1}-1} g(l_i, l_{i+1}, l_{i+2}) - \sum_{l_i=k_i+1}^{k_{i+1}+1} \sum_{l_{i+1}=k_i}^{k_{i+1}} \sum_{l_{i+2}=k_i-1}^{k_{i+2}} g(l_i, l_{i+1}, l_{i+2}),$$

and this is equal to the right-hand side in the statement of the lemma. □

We need another definition before we are able to prove the lemma which is crucial for the computation of the degree.

Definition 2.5. *Let r be a non-negative integer and $B(x,y)$ a function in x and y. We define $Z(r,B)(x,y)$ recursively: $Z(0,B)(x,y) = B(x,y)$ and*

$$Z(r,B)(x,y) = \sum_{x'=x+1}^{y+1} \sum_{y'=x}^{y} Z(r-1,B)(x',y') q^{x'+y'}.$$

In the following lemma we establish a recursion which expresses $D_i G_q(r,n,c,A)$ in terms of $G_q(r,n-2,c+2,A'_j)$ and $Z(r,B_j)$ if A fulfills a certain "decomposition condition".

Lemma 2.7. *Let r,n,i be integers, r non-negative, n positive, $2 \le n-r$ and $1 \le i \le n-r-1$. Moreover, let $A(k_1,\ldots,k_n)$ be a function on $\mathbb{Z}^n$. Assume that there exist two families of functions $(B_j(x,y))_{1\le j\le m}$ and $(A'_j(k_1,\ldots,k_{n-2}))_{1\le j\le m}$ with the property that*

$$D_i A(k_1,\ldots,k_n) = \sum_{j=1}^{m} B_j(k_i+i, k_{i+1}+i) A'_j(k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_n+2)$$

for all i, $1 \le i \le n-1$. Then

$$D_i G_q(r,n,c,A)(k_1,\ldots,k_{n-r}) = \sum_{j=1}^{m} \frac{(-1)^r}{2^r} q^{r(-2n+r+1)}\, Z(r,B_j)(k_i+i,k_{i+1}+i) \\ \times G_q(r,n-2,c+2,A'_j)(k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_{n-r}+2).$$

Proof. We show the assertion by induction with respect to r. For $r=0$ there is nothing to prove. Thus, we assume $r>0$. By Lemma 2.6, the left-hand side in the statement is equal to

$$-\frac{1}{2}\left(\sum_{l_1=0}^{k_1}\cdots\sum_{l_i=k_i+1}^{k_{i+1}+1}\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+2}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c} D_i G_q(r-1,n,c,A)(\mathbf{l})q^{l_1+\ldots+l_{n-r+1}}\right.$$
$$\left.+\sum_{l_1=0}^{k_1}\cdots\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+1}-1}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c} D_{i+1} G_q(r-1,n,c,A)(\mathbf{l})q^{l_1+\ldots+l_{n-r+1}}\right).$$

In this expression we replace $D_iG_q(r-1,n,c,A)(\mathbf{l})$ and $D_{i+1}G_q(r-1,n,c,A)(\mathbf{l})$ by the expression implied by the induction hypothesis. We obtain

$$\frac{(-1)^r}{2^r}q^{(r-1)(-2n+r)}\sum_{j=1}^{m}\left(\sum_{l_i=k_i+1}^{k_{i+1}+1}\sum_{l_{i+1}=k_i}^{k_{i+1}} Z(r-1,B_j)(l_i+i,l_{i+1}+i)q^{l_i+l_{i+1}}\right.$$
$$\times\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_{i+2}=k_i-1}^{k_{i+2}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c}$$
$$G_q(r-1,n-2,c+2,A'_j)(l_1,\ldots,l_{i-1},l_{i+2}+2,\ldots,l_{n-r+1}+2)q^{l_1+\ldots+l_{i-1}+l_{i+2}+\ldots+l_{n-r+1}}$$
$$+\sum_{l_{i+1}=k_i}^{k_{i+1}}\sum_{l_{i+2}=k_i-1}^{k_{i+1}-1} Z(r-1,B_j)(l_{i+1}+i+1,l_{i+2}+i+1)q^{l_{i+1}+l_{i+2}}$$
$$\times\sum_{l_1=0}^{k_1}\cdots\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+3}=k_{i+2}}^{k_{i+3}}\cdots\sum_{l_{n-r+1}=k_{n-r}}^{c}$$
$$\left.G_q(r-1,n-2,c+2,A'_j)(l_1,\ldots,l_i,l_{i+3}+2,\ldots,l_{n-r+1}+2)q^{l_1+\ldots+l_i+l_{i+3}+\ldots+l_{n-r+1}}\right).$$

We shift the range of summation of $l_{i+2},l_{i+3},\ldots,l_{n-r+1}$ by two and compensate this shift with the appropriate changes in the summands.

$$\frac{(-1)^r}{2^r}q^{r(-2n+r+1)}\sum_{j=1}^{m} Z(r,B_j)(k_i+i,k_{i+1}+i)$$
$$\times\left(\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_{i+2}=k_i+1}^{k_{i+2}+2}\sum_{l_{i+3}=k_{i+2}+2}^{k_{i+3}+2}\cdots\sum_{l_{n-r+1}=k_{n-r}+2}^{c+2}\right.$$
$$G_q(r-1,n-2,c+2,A'_j)(l_1,\ldots,l_{i-1},l_{i+2},\ldots,l_{n-r+1})q^{l_1+\ldots+l_{i-1}+l_{i+2}+\ldots+l_{n-r+1}}$$
$$+\sum_{l_1=0}^{k_1}\cdots\sum_{l_{i-1}=k_{i-2}}^{k_{i-1}}\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+3}=k_{i+2}+2}^{k_{i+3}+2}\cdots\sum_{l_{n-r+1}=k_{n-r}+2}^{c+2}$$
$$\left.G_q(r-1,n-2,c+2,A'_j)(l_1,\ldots,l_i,l_{i+3},\ldots,l_{n-r+1})q^{l_1+\ldots+l_i+l_{i+3}+\ldots+l_{n-r+1}}\right)$$

Finally, we rename l_{i+2} in the first multiple sum to l_i and merge the two multiple sums. The result is equal to the right-hand side in the statement of the lemma. □

In the next lemma we give a bound for the degree of $Z(r,B)(x,y)$.

LEMMA 2.8. *Suppose $B(x,y)$ is a q-quasi-polynomial in x and y of degree d, i.e. a linear combination of terms of the form $[x;q]_m[y;q]_n\rho_1^x\rho_2^y$ with $m+n \le d$ and where ρ_1,ρ_2 are roots of unity. Then $Z(r,B)(x,y)$ is of degree at most $2r+d$ in x and y.*

Proof. By induction with respect to r it suffices to show that

$$\sum_{x'=x+1}^{y+1}\sum_{y'=x}^{y}[x';q]_m[y';q]_n\rho_1^{x'-1}\rho_2^{y'-1}q^{x'+y'}$$

is of degree at most $m+n+2$ in x and y. Using the notation from Lemma 2.2, we see that this double sum is equal to

$$\left(\sum_{x'=x+1}^{y+1}[x';q]_m\rho_1^{x'-1}q^{x'}\right)\left(\sum_{y'=x}^{y}[y';q]_n\rho_2^{y'-1}q^{y'}\right)$$
$$=(c(y+1,m,\rho_1)-c(x,m,\rho_1))(c(y,n,\rho_2)-c(x-1,n,\rho_2))$$

and the assertion follows. □

In order to be able to apply Lemma 2.7, we show that $T(n,p)(k_1,\ldots,k_n)$ has the "decomposition property". Observe that

$$\begin{aligned}D_iT(n,p) &= 2\,T(n-2,p)(k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_n+2)\\ &\quad+(-1)^{k_i+k_{i+1}}2\,T(n-2,p-2)(k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_n+2),\end{aligned}$$

where $T(m,q)=0$ if $q<0$ or $q>m$. Thus, by Lemma 2.7,

$$\begin{aligned}&D_iG_q(r,n,c,T(n,p)) =\\ &\qquad\frac{(-1)^r}{2^r}q^{r(-2n+r+1)}\big(Z(r,1)(k_i+i,k_{i+1}+i)\\ &\qquad\times G_q(r,n-2,c+2,2\,T(n-2,p))(k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_{n-r}+2)\\ &\qquad\quad+Z(r,(-1)^{k_i+k_{i+1}})(k_i+i,k_{i+1}+i)\\ &\qquad\times G_q(r,n-2,c+2,2\,T(n-2,p-2))(k_1,\ldots,k_{i-1},k_{i+2}+2,\ldots,k_{n-r}+2)\big).\end{aligned}\tag{2.5}$$

By Lemma 2.8, $Z(r,1)(k_i,k_{i+1})$ and $Z(r,(-1)^{k_i+k_{i+1}})(k_i,k_{i+1})$ are q-quasi-polynomials in k_i and k_{i+1} of degree at most $2r$ and thus the same is true for $D_iG_q(r,n,c,T(n,p))$. Finally we show that this implies that $G_q(r,n,c,T(n,p))$ is a q-quasi-polynomial in k_i of degree at most $2r$ for all i.

LEMMA 2.9. *Let n,r be positive integers, $r<n$ and $0\le p\le n$. Then $G_q(r,n,c,T(n,p))$ is a q-quasi-polynomial in k_i of degree at most $2r$ for $i=1,2,\ldots,n-r$.*

Proof. We show the assertion by induction with respect to r. For $r=0$ there is nothing to prove. We assume that $r>0$ and that the assertion is true for $G_q(r-1,n,c,T(n,p))$. The degree of $D_i G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_{n-r+1})$ as a q-quasi-polynomial in l_i and l_{i+1} is at most $2r-2$. Therefore, by Lemma 2.5, the degree of

$$\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}} G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_{n-r+1})q^{l_i+l_{i+1}}$$

in k_i is at most $2r$. By (2.4), the same is true for the degree of $G_q(r,n,c,T(n,p))$ in k_i. $\square$

COROLLARY 2.1. *Let n be a positive integer and $0 \le p \le n$. Then $F_q(n-1,n,c,p;k)$ is a q-quasi-polynomial over $\mathbb{C}$ of degree at most $2n-2$ in k.*

5. Exploring properties of the q-quasi-polynomial $F_q(n-1,n,c,p;k)$

First we observe that $F_q(n-1,n,c,p;k)$ is zero for $k=-1,-2,\ldots,-n+1$ and $k=c+1,c+2,\ldots,c+n-1$.

LEMMA 2.10. *Let r,n,p be integers, $0 \le r < n$ and $0 \le p \le n$. Then $F_q(r,n,c,p;.)$ is zero for $k_1=-1,-2,\ldots,-r$ and $k_{n-r}=c+1,c+2,\ldots,c+r$.*

Proof. It suffices to show that there exists no (r,n,c)-pattern with first row

$$(0,k_1,\ldots,k_{n-r},c),$$

if $k_1=-1,-2,\ldots,-r$ or $k_{n-r}=c+1,c+2,\ldots,c+r$. Indeed, suppose $(a_{i,j})$ is an (r,n,c)-pattern with $a_{r+1,r+1}\in\{-1,-2,\ldots,-r\}$. In particular, we have $0>a_{r+1,r+1}$ and thus the definition of (r,n,c)-patterns implies that $0>a_{r,r}>a_{r+1,r+1}$. In a similar way we obtain $0>a_{1,1}>a_{2,2}>\ldots>a_{r,r}>a_{r+1,r+1}$. This is, however, a contradiction, since there exist no r distinct integers between 0 and $a_{r+1,r+1}$. The proof is analogous if $a_{r+1,n}\in\{c+1,c+2,\ldots,c+r\}$. $\square$

The zeros in Lemma 2.10 do not determine the q-quasi-polynomial $F_q(n-1,n,c,p;k)$ uniquely and thus we need additional properties. To this end we have the following two lemmas.

LEMMA 2.11. *Let r be a non-negative integer. Then we have*

$$Z(r,(-1)^{x+y})=(-1)^{x+y}q^{r(x+y)}C_r+T_r(x,y),$$

where $C_r\in\mathbb{Q}(q)$ and $T_r(x,y)$ is a q-polynomial in x and y over $\mathbb{Q}$.

Proof. The assertion follows from the following identity by induction with respect to r.

$$\sum_{x'=x+1}^{y+1}\sum_{y'=x}^{y}(-1)^{x'+y'}Q^{x'+y'}=\frac{-Q^{2x+1}-Q^{2y+3}-2(-1)^{x+y}Q^{x+y+2}}{(1+Q)^2}\qquad\square$$

Suppose $p(X)$ is a q-quasi-polynomial in X with period 1 or 2. Then there exist unique q-polynomials $p_1(X)$ and $p_2(X)$ with the property that

$$p(X)=(-1)^X p_1(X)+p_2(X).$$

We write $\mathrm{SP}_X(p(X))=p_1(X)$. The following lemma shows that $\mathrm{SP}_{k_i} F_q(r,n,c,p;.)$ has a simple structure.

LEMMA 2.12. *Let r,n,i,p be integers, r non-negative, n positive, $1\le i<n-r$ and $0\le p\le n$. Then*

$$\mathrm{SP}_{k_i} F_q(r,n,c,p;k_1,\ldots,k_{n-r})/q^{rk_i}$$

is independent of k_i.

Proof. We show the assertion by induction with respect to r. For $r=0$ there is nothing to prove. Let $r>0$. It suffices to prove that

$$\mathrm{SP}_{k_i}\left(\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}} D_i G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_{n-r+1}) q^{l_i+l_{i+1}}\right)/q^{rk_i} \tag{2.6}$$

and

$$\mathrm{SP}_{k_i}\Big(\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}}\Big(G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_i,l_{i+1},\ldots,l_{n-r+1}) \\ - G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_{i+1}+1,l_i-1,\ldots,l_{n-r+1})\Big)q^{l_i+l_{i+1}}\Big)/q^{rk_i} \tag{2.7}$$

are independent of k_i, where $k_0=0$ and $k_{n-r+1}=c$. By (2.5) and Lemma 2.11 $D_i G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_{n-r+1})$ is of the form

$$(-1)^{l_i+l_{i+1}} q^{(r-1)(l_i+l_{i+1})} H(l_1,\ldots,l_{i-1},l_{i+2},\ldots,l_{n-r+1}) + T(l_1,\ldots,l_{n-r+1}),$$

where $H(l_1,\ldots,l_{i-1},l_{i+2},\ldots,l_{n-r+1})$ is a q-quasi-polynomial, $T(l_1,\ldots,l_{n-r+1})$ is a q-quasi-polynomial in $(l_1,\ldots,l_{i-1},l_{i+2},\ldots,l_{n-r+1})$ and a q-polynomial in l_i and l_{i+1}. Therefore T does not contribute to (2.6). Moreover we have

$$\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}} (-1)^{l_i+l_{i+1}} q^{(r-1)(l_i+l_{i+1})} H(l_1,\ldots,l_{i-1},l_{i+2},\ldots,l_{n-r+1})\, q^{l_i+l_{i+1}} \\ = \frac{1}{(1+q^r)^2}((-1)^{k_i} q^{r\,k_i}(-1)^{k_{i-1}} q^{r\,k_{i-1}} + (-1)^{k_i} q^{rk_i}(-1)^{k_{i+1}} q^{r(k_{i+1}+2)} \\ + q^{r(2k_i+1)} + (-1)^{k_{i-1}+k_{i+1}} q^{r\,(k_{i-1}+k_{i+1}+1)})\, H(l_1,\ldots,l_{i-1},l_{i+2},\ldots,l_{n-r+1})$$

and (2.6) follows. For (2.7) observe that by the induction hypothesis

$$G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_{n-r+1})$$

is a linear combination of expressions of the form

$$[l_i;q]_m[l_{i+1}+1;q]_n,\ q^{(r-1)l_i}(-1)^{l_i-1}[l_{i+1}+1;q]_n,\ [l_i;q]_m q^{(r-1)(l_{i+1}+1)}(-1)^{l_{i+1}}$$

and

$$q^{(r-1)l_i}(-1)^{l_i-1} q^{(r-1)(l_{i+1}+1)}(-1)^{l_{i+1}}$$

over $R_{qq}[l_1,\ldots,l_{i-1},l_{i+2},\ldots,l_{n-r}]$. Therefore

$$G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_i,l_{i+1},\ldots,l_{n-r+1}) \\ - G_q(r-1,n,c,T(n,p))(l_1,\ldots,l_{i+1}+1,l_i-1,\ldots,l_{n-r+1})$$

is a linear combination of expressions of the form

$$[l_i;q]_m[l_{i+1}+1;q]_n - [l_i;q]_n[l_{i+1}+1;q]_m, \tag{2.8}$$

$$q^{(r-1)l_i}(-1)^{l_i-1}[l_{i+1}+1;q]_n - [l_i;q]_n q^{(r-1)(l_{i+1}+1)}(-1)^{l_{i+1}} \tag{2.9}$$

and

$$q^{(r-1)l_i}(-1)^{l_i-1} q^{(r-1)(l_{i+1}+1)}(-1)^{l_{i+1}} - q^{(r-1)l_i}(-1)^{l_i-1} q^{(r-1)(l_{i+1}+1)}(-1)^{l_{i+1}} = 0$$

over $R_{qq}[l_1,\ldots,l_{i-1},l_{i+2},\ldots,l_{n-r}]$. Expressions of the form (2.8) do not contribute to (2.7). For expressions of the form (2.9) observe that

$$\sum_{l_i=k_{i-1}}^{k_i}\sum_{l_{i+1}=k_i}^{k_{i+1}}\left(q^{(r-1)l_i}(-1)^{l_i-1}[l_{i+1}+1;q]_n-[l_i;q]_n q^{(r-1)(l_{i+1}+1)}(-1)^{l_{i+1}}\right)q^{l_i+l_{i+1}}=$$
$$-\frac{(-1)^{k_{i-1}}q^{r\,k_{i-1}}+q^r(-1)^{k_i}q^{rk_i}}{1+q^r}\frac{1}{[n+1;q]}([k_{i+1}+1;q]_{n+1}-[k_i;q]_{n+1})$$
$$-\frac{1}{[n+1;q]}([k_i;q]_{n+1}-[k_{i-1}-1;q]_{n+1})\frac{q^r(-1)^{k_i}q^{rk_i}+q^{2r}(-1)^{k_{i+1}}q^{rk_{i+1}}}{1+q^r}=$$
$$\frac{1}{(1+q^r)[n+1;q]}\Big(-(-1)^{k_{i-1}}q^{rk_{i-1}}[k_{i+1}+1;q]_{n+1}-q^r(-1)^{k_i}q^{rk_i}[k_{i+1}+1;q]_{n+1}$$
$$+(-1)^{k_{i-1}}q^{rk_{i-1}}[k_i;q]_{n+1}+[k_{i-1}-1;q]_{n+1}q^r(-1)^{k_i}q^{rk_i}$$
$$-[k_i;q]_{n+1}q^{2r}(-1)^{k_{i+1}}q^{rk_{i+1}}+[k_{i-1}-1;q]_{n+1}q^{2r}(-1)^{k_{i+1}}q^{rk_{i+1}}\Big).$$

The assertion follows. □

Corollary 2.2. *Let n be a positive integer. Then*

$$F(n-1,n,c,p;k)=P_{n,c,p}(k)+(-1)^k q^{(n-1)k}L_{n,c,p},$$

where $P_{n,c,p}(k)$ is a q-polynomial in k and $L_{n,c,p}$ is independent of k.

We define

$$G_{n,c,p}=\sum_{k=0}^{c}F_q(n-1,n,c,p;k)q^k.$$

This is the generating function of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and p rows of odd length. In the following lemma we prove that some special evaluations of $F_q(n-1,n,c,p;k)$ in k can be expressed in terms of the generating function $G_{n-1,c,p}$. This lemma together with Lemma 2.10 and Corollary 2.2 provide enough properties in order to compute $F_q(n-1,n,c,p;k)$ in the following section.

Lemma 2.13. *If $p\neq n$ then*

$$F_q(n-1,n,c,p;0)=G_{n-1,c,p}$$

and if $p\neq 0$ then

$$F_q(n-1,n,c,p;-n)=(-1)^{n-1}q^{-3n(n-1)/2}G_{n-1,c+2,p-1}.$$

Moreover we have

$$F_q(n-1,n,c,n;1)=q^{(n+2)(n-1)/2}G_{n-1,c-1,0}$$

and

$$F_q(n-1,n,c,0;-n-1)=(-1)^{n-1}q^{-(n-1)(2n+1)}G_{n-1,c+3,n-1}.$$

Proof. For the first identity let $(a_{i,j})$ be an $(n-1, n, c)$-pattern with $a_{n,n} = 0$ and exactly p numbers of $a_{1,1}, a_{1,2}, \ldots, a_{1,n}$ are odd. This implies that $a_{i,i} = 0$ for all i and thus $(a_{i,j})_{1 \le i \le n-1, i \le j \le n+1}$ is an $(n-2, n-1, c)$-pattern where p of $a_{1,2}, a_{1,3}, \ldots, a_{1,n}$ are odd. In fact, this induces a norm-preserving and sign-preserving bijection between these $(n-1, n, c)$-patterns and these $(n-2, n-1, c)$-patterns.

For the next identity, let $(a_{i,j})$ be an $(n-1, n, c)$-pattern with $a_{n,n} = -n$ and exactly p of $a_{1,1}, a_{1,2}, \ldots, a_{1,n}$ are odd. This implies that $a_{i,i} = -i$. Therefore $a_{i,i+1} \notin \{-3, -4, \ldots, -n\}$ for $i = 1, \ldots, n-1$. If we set $b_{i,j} := a_{i,j} + 2$ for $i < j$ and $b_{i,i} = 0$ then $(b_{i,j})_{1 \le i \le n-1, i \le j \le n+1}$ is an $(n-2, n-1, c+2)$-pattern, where $p-1$ of $b_{1,2}, b_{1,3}, \ldots, b_{1,n}$ are odd. Again this induces a bijection. However, the bijection is neither norm-preserving nor sign-preserving. The factor $(-1)^{n-1} q^{-3n(n-1)/2}$ takes into account the changes of norm and sign.

For the third identitiy, let $(a_{i,j})$ be an $(n-1, n, c)$-pattern with $a_{n,n} = 1$ and all $a_{1,1}, a_{1,2}, \ldots, a_{1,n}$ be odd. The first assumption implies that $a_{i,i} \in \{0, 1\}$, the second that $a_{1,1} = 1$ and therefore $a_{i,i} = 1$ for all i. If we set $b_{i,j} = a_{i,j} - 1$ then $(b_{i,j})_{1 \le i \le n-1, i \le j \le n+1}$ is an $(n-2, n-1, c-1)$-pattern, where all $b_{1,2}, b_{1,3}, \ldots, b_{1,n}$ are even.

The proof of the fourth identity is similar. $\square$

6. Computation of $F_q(n-1, n, c, p; k)$

In this section we compute $F_q(n-1, n, c, p; k)$ using the properties we have established in the previous section. For these computations we need some q-summation formulas which we derive in Section 7. First we show that Corollary 2.1, Lemma 2.10 and Corollary 2.2 imply a first strong assertion on the form of $F_q(n-1, n, c, p; k)$.

LEMMA 2.14. *Let n be a positive integer and $0 \le p \le n$. Then $F_q(n-1, n, c, p; k)$ is equal to*

$$M_{n,c,p} \cdot [k+1; q]_{n-1} [k-c-n+1; q]_{n-1} + L_{n,c,p} \cdot \Bigg((-1)^k q^{(n-1)k} + (-1)^n q^{(n-1)(2c+n)/2}$$
$$\times \sum_{i=1}^{n-1} \Bigg((-1)^c q^{\binom{i}{2}} \frac{[k+1; q]_{n-1} [k-c-i+1; q]_{i-1} [k-c-n+1; q]_{n-i-1}}{[1; q]_{i-1} [1; q]_{n-1-i} [c+i+1; q]_{n-1}}$$
$$- q^{\binom{i}{2}} \frac{[k+1; q]_{i-1} [k+i+1; q]_{n-i-1} [k-c-n+1; q]_{n-1}}{[1; q]_{i-1} [1; q]_{n-1-i} [c+i+1; q]_{n-1}} \Bigg) \Bigg),$$

where $L_{n,c,p}$ and $M_{n,c,p}$ are independent of k.

Proof. By Lemma 2.10 and Corollary 2.2 we know that for $k \in \{-1, -2, \ldots, -n+1\}$ and $k \in \{c+1, c+2, \ldots, c+n-1\}$ we have

$$P_{n,c,p}(k) = (-1)^{k+1} q^{(n-1)k} L_{n,c,p}.$$

By Corollary 2.1 $P_{n,c,p}(k)$ is a q-polynomial in k of degree at most $2n-2$. By "q-Lagrange interpolation" the following polynomial is the unique q-polynomial $Q_{n,c,p}(k)$ of degree at most $2n-3$ with $Q_{n,c,p}(k) = (-1)^{k+1} q^{(n-1)k} L_{n,c,p}$ for $k \in \{-1, -2, \ldots, -n+1\}$ and $k \in \{c+1, c+

$2,\ldots,c+n-1\}$.

$$\sum_{i=-n+1}^{-1}(-1)^{i+1}\,q^{(n-1)\,i}L_{n,c,p}\prod_{-n+1\le j\le -1,j\ne i}\frac{[k-j;q]}{[i-j;q]}\prod_{j=1}^{n-1}\frac{[k-c-j;q]}{[i-c-j;q]}$$
$$+\sum_{i=1}^{n-1}(-1)^{c+i+1}\,q^{(n-1)(c+i)}L_{n,c,p}\prod_{j=-n+1}^{-1}\frac{[k-j;q]}{[c+i-j;q]}\prod_{1\le j\le n-1,j\ne i}\frac{[k-c-j;q]}{[i-j;q]}$$

This is equal to

$$\sum_{i=1}^{n-1}\Bigg((-1)^{i+1}\,q^{-(n-1)i}L_{n,c,p}\prod_{j=1}^{i-1}\frac{[k+j;q]}{[j-i;q]}\prod_{j=i+1}^{n-1}\frac{[k+j;q]}{[j-i;q]}\prod_{j=1}^{n-1}\frac{[k-c-j;q]}{[-i-c-j;q]}$$
$$+(-1)^{c+i+1}\,q^{(n-1)(c+i)}L_{n,c,p}\prod_{j=1}^{n-1}\frac{[k+j;q]}{[c+i+j;q]}\prod_{j=1}^{i-1}\frac{[k-c-j;q]}{[i-j;q]}\prod_{j=i+1}^{n-1}\frac{[k-c-j;q]}{[i-j;q]}\Bigg)$$
$$=L_{n,c,p}\sum_{i=1}^{n-1}\Bigg((-1)^{i+1}q^{-(n-1)i}\frac{[k+1;q]_{i-1}[k+i+1;q]_{n-1-i}[k-c-n+1;q]_{n-1}}{[1-i;q]_{i-1}[1;q]_{n-1-i}[-n+1-c-i;q]_{n-1}}$$
$$+(-1)^{c+i+1}q^{(n-1)(c+i)}\frac{[k+1;q]_{n-1}[k-c-i+1;q]_{i-1}[k-c-n+1;q]_{n-1-i}}{[c+i+1;q]_{n-1}[1;q]_{i-1}[i-n+1;q]_{n-1-i}}\Bigg).$$

The difference of $P_{n,c,p}(k)$ and the q-polynomial displayed above is a q-polynomial of degree at most $2n-2$ with zeros $-1,\ldots,-n+1$ and $c+1,\ldots,c+n-1$. Thus, this difference is equal to $M_{n,c,p}[k+1;q]_{n-1}[k-c-n+1;q]_{n-1}$, where $M_{n,c,p}$ is a factor independent of k. We use the identity

$$[z;q]_n=[-z-n+1;q]_n(-1)^nq^{n(z+(n-1)/2)}$$

in order to obtain the expression for $P_{n,c,p}(k)$ in the statement of the lemma. □

We set

$$U_{n,c}(k)=$$
$$(-1)^nq^{(n-1)(2c+n)/2}\sum_{i=1}^{n-1}\Bigg((-1)^cq^{\binom{i}{2}}\frac{[k+1;q]_{n-1}[k-c-i+1;q]_{i-1}[k-c-n+1;q]_{n-i-1}}{[1;q]_{i-1}[1;q]_{n-1-i}[c+i+1;q]_{n-1}}$$
$$-q^{\binom{i}{2}}\frac{[k+1;q]_{i-1}[k+i+1;q]_{n-i-1}[k-c-n+1;q]_{n-1}}{[1;q]_{i-1}[1;q]_{n-1-i}[c+i+1;q]_{n-1}}\Bigg)+(-1)^kq^{(n-1)k}$$

and $W_{n,c}(k)=[k+1;q]_{n-1}[k-c-n+1;q]_{n-1}$. Using these definitions, Lemma 2.14 states that

$$F_q(n-1,n,c,p;k)=L_{n,c,p}U_{n,c}(k)+M_{n,c,p}W_{n,c}(k). \tag{2.10}$$

It remains to compute $L_{n,c,p}$ and $M_{n,c,p}$. In the following lemma we give recursive formulas with respect to n for $L_{n,c,p}$ and $M_{n,c,p}$. It is an immediate consequence of Lemma 2.13.

Lemma 2.15. *The initial conditions are $L_{1,c,p}=\frac{(-1)^p}{2}$ and $M_{1,c,p}=\frac{1}{2}$. If $p\notin\{0,n\}$ we have*

$$L_{n,c,p}=\frac{G_{n-1,c,p}W_{n,c}(-n)+(-1)^nq^{-3n(n-1)/2}G_{n-1,c+2,p-1}W_{n,c}(0)}{U_{n,c}(0)W_{n,c}(-n)-U_{n,c}(-n)W_{n,c}(0)}.$$

If $p=0$ we have the following recursion

$$L_{n,c,0}=\frac{(-1)^{n-1}q^{-(n-1)(2n+1)}G_{n-1,c+3,n-1}W_{n,c}(0)-G_{n-1,c,0}W_{n,c}(-n-1)}{U_{n,c}(-n-1)W_{n,c}(0)-U_{n,c}(0)W_{n,c}(-n-1)},$$

and if $p=n$ then

$$L_{n,c,n}=\frac{q^{(n+2)(n-1)/2}G_{n-1,c-1,0}W_{n,c}(-n)+(-1)^n q^{-3n(n-1)/2}G_{n-1,c+2,n-1}W_{n,c}(1)}{U_{n,c}(1)W_{n,c}(-n)-U_{n,c}(-n)W_{n,c}(1)}.$$

Concerning $M_{n,c,p}$ we have

$$M_{n,c,p}=\frac{G_{n-1,c,p}-U_{n,c}(0)L_{n,c,p}}{W_{n,c}(0)},$$

if $p\neq n$ and

$$M_{n,c,p}=\frac{(-1)^{n-1}q^{-3n(n-1)/2}G_{n-1,c+2,p-1}-U_{n,c}(-n)L_{n,c,p}}{W_{n,c}(-n)}$$

if $p\neq 0$.

Proof. Lemma 2.13 and (2.10) implies the following equations. If $p\neq 0$ then

$$L_{n,c,p}U_{n,c}(-n)+M_{n,c,p}W_{n,c}(-n)=(-1)^{n-1}q^{-3n(n-1)/2}G_{n-1,c+2,p-1},$$

and if $p\neq n$ then

$$L_{n,c,p}U_{n,c}(0)+M_{n,c,p}W_{n,c}(0)=G_{n-1,c,p}.$$

If $p=0$ we have

$$L_{n,c,0}U_{n,c}(-n-1)+M_{n,c,0}W_{n,c}(-n-1)=(-1)^{n-1}q^{-(n-1)(2n+1)}G_{n-1,c+3,n-1},$$

and if $p=n$ we have

$$L_{n,c,n}U_{n,c}(1)+M_{n,c,n}W_{n,c}(1)=q^{(n+2)(n-1)/2}G_{n-1,c-1,0}.$$

For every $p\in\{0,1,2,\dots,n\}$ this gives a system of two linearly independent equations in $L_{n,c,p}$ and $M_{n,c,p}$. We use Cramer's rule to obtain the recursions of $L_{n,c,p}$. The recursions for $M_{n,c,p}$ are immediate consequences of the equations. □

In the following lemma we see that the denominators in the recursive formulas for $L_{n,c,p}$ in Lemma 2.15 are products.

Lemma 2.16. *We have*

$$U_{n,c}(0)W_{n,c}(-n)-U_{n,c}(-n)W_{n,c}(0)=\frac{2[1;q^2]_{n-1}(1+q)^{2n-1}}{q^{c(n-1)+2n(n-1)}}\begin{cases}\frac{[(c+2)/2;q^2]_{n-1}}{1+q} & \text{if } c \text{ is even}\\ \frac{[(c+1)/2;q^2]_n}{[c+n;q]} & \text{if } c \text{ is odd}\end{cases},$$

$$U_{n,c}(-n-1)W_{n,c}(0)-U_{n,c}(0)W_{n,c}(-n-1)=-\frac{2[1;q^2]_{n-1}[n-1;q](1+q)^{2n-1}}{q^{(n-1)c+2(n-1)(n+1)}}$$
$$\times\begin{cases}[(c+2)/2;q^2]_n & \text{if } c \text{ is even}\\ \frac{[(c+1)/2;q^2]_{n+1}(1+q)}{[c+n+1;q]} & \text{if } c \text{ is odd}\end{cases}$$

and

$$U_{n,c}(1)W_{n,c}(-n)-U_{n,c}(-n)W_{n,c}(1)=\frac{2[1;q^2]_{n-1}[n-1;q](1+q)^{2n-1}}{q^{(n-1)c+n(2n-3)}[c+n;q]} \times \begin{cases} [c/2;q^2]_n & \textit{if } c \textit{ is even} \\ \frac{[(c-1)/2;q^2]_{n+1}(1+q)}{[c+n-1;q]} & \textit{if } c \textit{ is odd} \end{cases}.$$

Proof. The formulas for $U_{n,c}(0)$, $U_{n,c}(-n)$, $U_{n,c}(1)$ and $U_{n,c}(-n-1)$ in Section 7, (2.12)–(2.15), imply that the denominators from Lemma 2.15 are sums of at most 3 products. A lengthy but straightforward calculation shows that these sums of product simplify to single products. $\square$

We finally give the formulas for $L_{n,c,p}$, $M_{n,c,p}$ and $G_{n,c,p}$.

LEMMA 2.17. *The generating function $G_{n,c,p}$ is equal to*

$$G_{n,c,p}=q^{\binom{p+1}{2}}\begin{bmatrix} n \\ p \end{bmatrix}\begin{cases} \frac{1}{[c+p;q]_{n+1}}\prod_{i=0}^{n}\frac{[c+2i;q]_{n-i+1}}{[2+2i;q]_{n-i}} & 2|c \\ \prod_{i=1}^{n}\frac{[c+2i-1;q]_{n-i+1}}{[2i;q]_{n-i+1}} & 2\not|c \end{cases}.$$

For $L_{n,c,p}$ we have

$$L_{n,c,p}=\begin{cases} \prod_{i=1}^{n-1}\frac{[c+2i+1;q]_{n-i}}{[2i;q]_{n-i}[2i;q]}\frac{[c+1;q]_{n-1}[1;q]_{n-1}}{2}\left(\frac{q^{\binom{p+1}{2}}\begin{bmatrix} n-1 \\ p \end{bmatrix}[c;q]}{[c+p;q]_n}-\frac{q^{\binom{p}{2}}\begin{bmatrix} n-1 \\ p-1 \end{bmatrix}[c+2n;q]}{[c+p+1;q]_n}\right) & 2|c \\ \prod_{i=1}^{n-1}\frac{[c+2i;q]_{n-i}}{[2i;q]_{n-i}[2i;q]}\frac{[1;q]_{n-1}}{2}\left(q^{\binom{p+1}{2}}\begin{bmatrix} n-1 \\ p \end{bmatrix}-q^{\binom{p}{2}}\begin{bmatrix} n-1 \\ p-1 \end{bmatrix}\right) & 2\not|c \end{cases}$$

and for $M_{n,c,p}$ we have

$$M_{n,c,p}=\frac{(-1)^{n-1}q^{(n-1)(2c+n)/2}}{[1;q]_{n-2}} \times\begin{cases} \prod_{i=1}^{n-1}\frac{[c+2i;q]_{n-i}}{[2i;q]_{n-i}}\left(\frac{q^{\binom{p+1}{2}}\begin{bmatrix} n-1 \\ p \end{bmatrix}[c;q]}{[c+p;q]_n}\left(\frac{1}{[n-1;q]}-\frac{[c+2n-1;q]}{[c+n;q][2n-2;q]}\right)+\frac{q^{\binom{p}{2}}\begin{bmatrix} n-1 \\ p-1 \end{bmatrix}}{[c+p+1;q]_n}\frac{[c+2n-1;q]_2}{[c+n;q][2n-2;q]}\right) & 2|c \\ \prod_{i=1}^{n-1}\frac{[c+2i+1;q]_{n-i-1}}{[2i;q]_{n-i}}\frac{1}{[2n-2;q]}\left(q^{\binom{p+1}{2}+n-1}\begin{bmatrix} n-1 \\ p \end{bmatrix}+q^{\binom{p}{2}}\begin{bmatrix} n-1 \\ p-1 \end{bmatrix}\right) & 2\not|c \end{cases}.$$

Proof. We show the assertion by induction with respect to n. For $n=1$ observe that $L_{1,c,0}=1/2$, $L_{1,c,1}=-1/2$ and $M_{1,c,0}=M_{1,c,1}=1/2$. Moreover

$$G_{1,c,0}=\begin{cases} 1+q^2+q^4+\ldots+q^c=\frac{1-q^{2+c}}{1-q^2} & \text{if } c \text{ is even} \\ 1+q^2+q^4+\ldots+q^{c-1}=\frac{1-q^{1+c}}{1-q^2} & \text{if } c \text{ is odd} \end{cases}$$

and

$$G_{1,c,1}=\begin{cases} q+q^3+q^5+\ldots+q^{c-1}=\frac{q(1-q^c)}{1-q^2} & \text{if } c \text{ is even} \\ q+q^3+q^5+\ldots+q^c=\frac{q(1-q^{1+c})}{1-q^2} & \text{if } c \text{ is odd} \end{cases}.$$

Assume that the formulas are already proved for $n-1$. Use the recursions in Lemma 2.15, (2.12)–(2.15) and the formula for $G_{n-1,c,p}$ in order to check the formulas for $L_{n,c,p}$ and $M_{n,c,p}$. By (2.11) and (2.16) we have

$$G_{n,c,p} = \sum_{k=0}^{c} L_{n,c,p} U_{n,c}(k) q^k + M_{n,c,p} W_{n,c}(k) q^k =$$
$$L_{n,c,p} \frac{((1+(-1)^c)[(c+2)/2;q^2]_{n-1}(1+q^{c+n}) + (1-(-1)^c)[(c+1)/2;q^2]_n(1-q^2))}{[1/2;q^2]_{n-1}(1+q^{n-1})(1+q^n)}$$
$$+ M_{n,c,p}(-1)^{n-1} q^{(-n+1)(2c+n)/2} \frac{[1;q]_{n-1}^2 [c+1;q]_{2n-1}}{[1;q]_{2n-1}}.$$

A lengthy but straightforward calculation proves the formula for $G_{n,c,p}$. □

Now we are in the position to explain why Theorem 2.1 implies Krattenthaler's and the author's refinement of the Bender-Knuth (ex-)Conjecture. Krattenthaler's refinement, see [**30**, Theorem 21], is the generating function $G_{n,c,p}$, which we have computed in Lemma 2.17 and thus we have reproved his result with different methods.

The author's refinement, see Theorem 1 (resp. Theorem 1.1) in [**15**, resp. Chapter 1], is the generating function of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n, i.e. the sum over all p's, $0 \le p \le n$, of the generating function in Theorem 2.1. In order to deduce Theorem 1 (resp. Theorem 1.1) in [**15**, resp. Chapter 1] from Theorem 2.1 of the present paper one has to show that

$$\sum_{p=0}^{n} L_{n,c,p} = 0$$

and

$$\sum_{p=0}^{n} (-1)^{n-1} q^{(n-1)(k-c)-\binom{n}{2})+k} M_{n,c,p} = \frac{q^{kn}}{[1;q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1;q]_{i-1}}{[i;q]_i},$$

where

$$[k-c-n+1;q]_{n-1} = (-1)^{n-1} q^{(n-1)(k-c)-\binom{n}{2}} [1+c-k;q]_{n-1}$$

explains the factor accompanying $M_{n,c,p}$. However, Theorem 1 (resp. Theorem 1.1) from [**15**, resp. Chapter 1] was proved with methods similar to the methods we have used to prove Theorem 2.1 in the present paper and thus we omit to show this implication, since it is surely a detour to prove Theorem 1 (resp. Theorem 1.1) from [**15**, resp. Chapter 1] in this way.

7. Some basic hypergeometric identities

In this section we derive some basic hypergeometric identities which were needed above. The notation is adopted from [**19**, page 1–6]. In particular, the basic hypergeometric series is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{s-r+1} z^n,$$

where the rising q-factorial $(a;q)_n$ is given by $(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i)$. (Observe that $[x;q]_n = (q^x;q)_n/(1-q)^n$.) All identities in this section were handled with Krattenthaler's Mathematica package HYPQ [**31**]. A Mathematica-file containing the computations can be downloaded from my webpage (`http://www.mat.univie.ac.at/~ifischer/`).

The list of identities is the following. (For the definitions of $U_{n,c}(k)$ and $W_{n,c}(k)$ see Section 6.)

$$\sum_{k=0}^{c} W_{n,c}(k) q^k = (-1)^{n-1} q^{(-n+1)(2c+n)/2} \frac{[1;q]_{n-1}^2 [c+1;q]_{2n-1}}{[1;q]_{2n-1}} \tag{2.11}$$

$$U_{n,c}(0) = \frac{2\prod_{i=1}^{n-2}(1+q^i)}{(1+q)^{n-1}} \begin{cases} \frac{[c+1;q]_{n-1}}{[(c+1)/2;q]_{n-1}} & \text{if } c \text{ is even} \\ \frac{[c;q]_n}{[c/2;q]_n(1+q)} & \text{if } c \text{ is odd} \end{cases} \tag{2.12}$$

$$U_{n,c}(-n) = \frac{2(1+q)^{n-1}\prod_{i=1}^{n-2}(1+q^i)}{q^{(n-1)(3n-2)/2}} \begin{cases} \frac{[(c+2)/2;q]_{n-1}}{[c+2;q]_{n-1}} & \text{if } c \text{ is even} \\ \frac{[(c+1)/2;q]_n(1+q)}{[c+1;q]_n} & \text{if } c \text{ is odd} \end{cases} \tag{2.13}$$

$$U_{n,c}(1) = \frac{2q^n\prod_{i=1}^{n-2}(1+q^i)}{(1+q)^{n-1}} \times \begin{cases} \frac{[c;q]_{n-1}[n-2;q]}{[(c+1)/2;q^2]_{n-1}} & \text{if } c \text{ is even} \\ \frac{[c-1;q]_{n-1}[n-2;q]}{[c/2,q^2]_{n-1}} - q^{c+n-3}\frac{[c;q]_{n-2}[2\lfloor (n-1)/2\rfloor+1;q][2\lfloor n/2\rfloor;q]}{[(c+2)/2;q^2]_{n-1}} & \text{if } c \text{ is odd} \end{cases} \tag{2.14}$$

$$U_{n,c}(-n-1) = \frac{2(-1)^n(1+q)^{n-1}\prod_{i=1}^{n-2}(1+q^i)}{q^{(n+1)(3n-4)/2}} \times \begin{cases} \frac{[(c+4)/2;q^2]_{n-1}[n-2;q]}{[c+3;q]_{n-1}} & \text{if } c \text{ is even} \\ \frac{[(c+3)/2;q^2]_{n-1}[n-2;q]}{[c+2;q]_{n-1}} + q^{c+n}\frac{[(c+3)/2;q^2]_{n-1}[2\lfloor (n-1)/2\rfloor+1;q][2\lfloor n/2\rfloor;q]}{[c+2;q]_n(1+q)^2} & \text{if } c \text{ is odd} \end{cases} \tag{2.15}$$

$$\sum_{k=0}^{c} U_{n,c}(k) q^k = \frac{2}{[1/2;q^2]_{n-1}(1+q^{n-1})(1+q^n)} \begin{cases} [(c+2)/2;q^2]_{n-1}(1+q^{c+n}) & \text{if } c \text{ is even} \\ [(c+1)/2;q^2]_n(1-q^2) & \text{if } c \text{ is odd} \end{cases} \tag{2.16}$$

First we consider (2.11). Observe that

$$\sum_{k=0}^{c} W_{n,c}(k) q^k = \left(\sum_{k=0}^{c} \frac{(q^n;q)_k (q^{-c};q)_k}{(q;q)_k (q^{1-c-n};q)_k} \right) \frac{(q;q)_{n-1}(q^{1-c-n};q)_{n-1}}{(1-q)^{2n-2}}.$$

Therefore it suffices to show that

$$\sum_{k=0}^{c} \frac{(q^n;q)_k (q^{-c};q)_k}{(q;q)_k (q^{1-c-n};q)_k} = \frac{(q^{2n};q)_c}{(q^n;q)_c}. \tag{2.17}$$

Both sides are q-polynomials in n and therefore it suffices to show the assertion for all n with $n < 1-c$. Observe that in this case

$$\sum_{k=0}^{c} \frac{(q^n;q)_k (q^{-c};q)_k}{(q;q)_k (q^{1-c-n};q)_k} = \sum_{k=0}^{\infty} \frac{(q^n;q)_k (q^{-c};q)_k}{(q;q)_k (q^{1-c-n};q)_k},$$

since $(q^{-c};q)_k = 0$ if $k > c$ and $(q^{1-c-n};q)_k \neq 0$ for all $k > 0$. Using the basic hypergeometric notation introduced above this infinite series can be written as

$$ {}_2\phi_1\left[\begin{matrix} q^n, q^{-c} \\ q^{1-c-n} \end{matrix}; q, q\right]. $$

We use the q-Chu-Vandermonde summation formula [**19**, (1.5.3); Appendix (II.6)]

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q\right] = \frac{a^n (c/a;q)_n}{(c;q)_n}, \tag{2.18} $$

and obtain

$$ \frac{q^{cn}\,(q^{1-c-2n};q)_c}{(q^{1-c-n};q)_c}. \tag{2.19} $$

This is equal to the right-hand side of (2.17).

Next we consider (2.12), (2.13), (2.14) and (2.15). In these summations we first use some contiguous relations before we apply the following summation formula

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ aq/b \end{matrix}; q, -q/b\right] = \frac{(-q;q)_\infty (aq;q^2)_\infty (aq^2/b^2;q^2)_\infty}{(-q/b;q)_\infty (aq/b;q)_\infty}, \tag{2.20} $$

where $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$, see [**19**, (1.8.1); Appendix (II.9)]. We define

$$ X_{n,c}(k) = \sum_{i=1}^{n-1} q^{\binom{i}{2}} \frac{[k+1;q]_{n-1}[k-c-i+1;q]_{i-1}[k-c-n+1;q]_{n-i-1}}{[1;q]_{i-1}[1;q]_{n-1-i}[c+i+1;q]_{n-1}} $$

and

$$ Y_{n,c}(k) = \sum_{i=1}^{n-1} q^{\binom{i}{2}} \frac{[k+1;q]_{i-1}[k+i+1;q]_{n-i-1}[k-c-n+1;q]_{n-1}}{[1;q]_{i-1}[1;q]_{n-1-i}[c+i+1;q]_{n-1}}. $$

Observe that

$$ U_{n,c}(k) = (-1)^n q^{(n-1)(2c+n)/2}((-1)^c X_{n,c}(k) - Y_{n,c}(k)) + (-1)^k q^{(n-1)k}. \tag{2.21} $$

In order to prove (2.12), it suffices to compute $X_{n,c}(0)$ and $Y_{n,c}(0)$. In both sums we can replace the upper limit by ∞, since $(1)_{n-1-i} = \infty$ if $i \geq n$. We write $X_{n,c}(0)$ in basic hypergeometric notation and obtain

$$ {}_2\phi_1\left[\begin{matrix} q^{c+1}, q^{-n+2} \\ q^{c+n+1} \end{matrix}; q, -q^n\right] \frac{(q;q)_{n-1}(q^{1-c-n};q)_{n-2}}{(q;q)_{n-2}(q^{c+2};q)_{n-1}}. $$

If we apply the following contiguous relation

$$ {}_r\phi_s\left[\begin{matrix} a, (A) \\ b, (B) \end{matrix}; q, z\right] = \frac{1-b/q}{a-b/q}\,{}_r\phi_s\left[\begin{matrix} a, (A) \\ b/q, (B) \end{matrix}; q, z/q\right] + \frac{1-a}{b/q-a}\,{}_r\phi_s\left[\begin{matrix} aq, (A) \\ b, (B) \end{matrix}; q, z/q\right] $$

and then (2.20) we obtain the following formula for $X_{n,c}(0)$.

$$ \frac{q^{-1-c}(q^{1-c-n};q)_{n-2}(-q;q)_\infty (q^{c+2};q^2)_\infty (q^{c+2n-1};q^2)_\infty}{(q^{c+2};q)_{n-2}(-q^{n-1};q)_\infty (q^{c+n};q)_\infty} + \frac{q^{-c-n}(q^{c+1};q)_1 (q^{1-c-n};q)_{n-2}(q^{n-1};q)_1(-q;q)_\infty (q^{c+3};q^2)_\infty (q^{c+2n};q^2)_\infty}{(q^{c+2};q)_{n-1}(q^{1-n};q)_1(-q^{n-1};q)_\infty (q^{c+n+1};q)_\infty} $$

Next observe that $Y_{n,c}(0)$ is equal to

$$(-1)^{n-1}{}_3\phi_2\left[\begin{matrix} q^{-n+2}, q^{c+2}, q \\ q^{c+n+1}, q^2 \end{matrix}; q, -q^{n-1}\right] \frac{(q^{n-1};q)_1(q^{c+1};q)_1}{(q;q)_1(q^{c+n};q)_1}.$$

Using the contiguous relation

$${}_r\phi_s\left[\begin{matrix}(A), q \\ (B)\end{matrix}; q, z\right] = \frac{(-1)^{r+s}q^{1-r+s}}{z}\frac{\prod_{i=1}^{s}(1-B_i/q)}{\prod_{i=1}^{r}(1-A_i/q)}\left(1 - {}_r\phi_s\left[\begin{matrix}(A/q), q \\ (B/q)\end{matrix}; q, q^{-1+r-s}z\right]\right), \quad (2.22)$$

we transform the ${}_3\phi_2$ series into a ${}_2\phi_1$ series. Next we apply the following contiguous relation

$${}_r\phi_s\left[\begin{matrix}a, (A) \\ (B)\end{matrix}; q, z\right] = {}_r\phi_s\left[\begin{matrix}aq, (A) \\ (B)\end{matrix}; q, z\right] + (-1)^{r+s}az\frac{\prod_{i=1}^{r-1}(1-A_i)}{\prod_{i=1}^{s}(1-B_i)}{}_r\phi_s\left[\begin{matrix}aq, (qA) \\ (qB)\end{matrix}; q, q^{1-r+s}z\right] \quad (2.23)$$

before we are finally able to apply (2.20) to the remaining two ${}_2\phi_1$ series. We obtain a formula for $Y_{n,c}(0)$ in terms of rising q-factorials. If we combine this with the formula for $X_{n,c}(0)$ we obtain (2.12).

The situation is similar for (2.13), (2.14), (2.15), however, we do not give the proofs in detail. In order to describe a set of contiguous relations which is needed before in each of these cases (2.20) can eventually be applied, we use Krattenthaler's table of contiguous relations, which can be found in the HYPQ documentation [**31**]. For more details see the Mathematica-file with the computation, which can be found on my webpage `http://www.mat.univie.ac.at/~ifischer/`.

$X_{n,c}(-n)$	C34
$Y_{n,c}(-n)$	C15
$X_{n,c}(1)$	C14, C42, C34
$Y_{n,c}(1)$	C16, C15, C02, C41, C11, C37
$X_{n,c}(-n-1)$	C14, C42, C34
$Y_{n,c}(-n-1)$	C14, C15, C36, C11, C37

Finally we consider (2.16), which is a double sum and thus the most complicated identity. By (2.21) it suffices to compute $\sum_{k=0}^{c} X_{n,c}(k)q^k$ and $\sum_{k=0}^{c} Y_{n,c}(k)q^k$. First we compute $\sum_{k=0}^{c} X_{n,c}(k)q^k$. The trick is to consider a more general expression: observe that $X_{n,c}(k)$ is the unique q-polynomial in k of degree at most $2n-3$ with the property that $X_{n,c}(k)=0$ for $k=-1,-2,\dots,-n+1$ and $X_{n,c}(c+i) = (-1)^{n-1}q^{-\binom{n}{2}}(-q^{n-1})^i$ for $i=1,2,\dots,n-1$. (In Lemma 2.14 $X_{n,c}(k)$ is actually constructed such that these conditions are fulfilled.) Consequently $S_{n,c}(d) := \sum_{k=0}^{d} X_{n,c}(k)q^k$ is the unique q-polynomial in d of degree at most $2n-2$ in d with $S_{n,c}(d)=0$ for $d=-1,-2,\dots,-n$ and

$$S_{n,c}(c+i) = S_{n,c}(c) + (-1)^{n-1}q^{-\binom{n}{2}+c+n}\frac{-1+(-q^n)^i}{1+q^n}$$

for $i=0,1,\dots,n-1$. Thus, by q-Lagrange interpolation,

$$S_{n,c}(d) = \sum_{i=1}^{n}\left(S_{n,c}(c) + (-1)^{n-1}q^{-\binom{n}{2}+c+n}\frac{(-q^n)^i-1}{1+q^n}\right)q^{\binom{n-i+1}{2}}(-1)^{n+i} \times \frac{[d+1;q]_n[d-c-i+2;q]_{i-1}[d-c-n+1;q]_{n-i}}{[c+i;q]_n[1;q]_{i-1}[1;q]_{n-i}}.$$

Apriori the degree of this q-polynomial is $2n-1$. Thus, the coefficient of $(q^d)^{2n-1}$

$$\sum_{i=1}^{n}\left(S_{n,c}(c)+(-1)^{n-1}q^{-\binom{n}{2}+c+n}\frac{(-q^n)^i-1}{1+q^n}\right)\frac{q^{\binom{n-i+1}{2}}(-1)^{n+i}q^{-1+c+i+n-cn}}{[c+i;q]_n[1;q]_{i-1}[1;q]_{n-i}}$$

must be zero. We obtain the following expression for $S_{n,c}(c)=\sum_{k=0}^{c}X_{n,c}(k)q^k$.

$$\frac{(-1)^{n-1}q^{-\binom{n}{2}+c+n}}{1+q^n}\left(q^{\binom{n}{2}}\frac{\sum\limits_{i=1}^{n}\frac{q^{\binom{i+1}{2}}}{[c+i;q]_n[1;q]_{i-1}[1;q]_{n-i}}}{\sum\limits_{i=1}^{n}\frac{(-1)^i q^{\binom{n-i+1}{2}+i}}{[c+i;q]_n[1;q]_{i-1}[1;q]_{n-i}}}+1\right)$$

This formula simplifies since

$$\sum_{i=1}^{n}\frac{(-1)^i q^{\binom{n-i+1}{2}+i}}{[c+i;q]_n[1;q]_{i-1}[1;q]_{n-i}}=-q^{(n-1)c+(n+1)n/2}\frac{[1;q]_{2n-2}}{[1;q]_{n-1}^2[c+1;q]_{2n-1}}. \tag{2.24}$$

In order to see that observe that the left-hand-side in this equation is equal to

$$-{}_2\phi_1\left[\begin{matrix}q^{c+1},q^{-n+1}\\ q^{c+n+1}\end{matrix};q,q\right]\frac{q^{n^2/2-n/2+1}(1-q)^{2n-1}}{(q;q)_{n-1}(q^{c+1};q)_n}.$$

Using (2.18) we obtain (2.24). Thus

$$S_{n,c}(c)=\frac{(-1)^{n-1}q^{-\binom{n}{2}+c+n}}{1+q^n}\times\left(-q^{c-n-cn}\sum_{i=0}^{n-1}q^{\binom{i+2}{2}}\frac{[c+1;q]_i[c+i+n+1;q]_{n-i-1}[n-i;q]_i}{[n;q]_{n-1}[1;q]_i}+1\right).$$

Similarly one can show that

$$T_{n,c}(d)=\sum_{i=0}^{n-1}\left(T_{n,c}(0)+\frac{(-1)^n q^{(n-1)(2c+n)/2}(-1+(-q^{-n})^i)}{1+q^n}(-1)^{i+n}q^{(i+i^2+n+2cn+2in+n^2)/2}\times\frac{[d-c-n;q]_n[d;q]_i[d+i+1;q]_{n-i-1}}{[c+i+1;q]_n[1;q]_i[1;q]_{n-1-i}}\right),$$

where $T_{n,c}(d):=\sum_{k=d}^{c}Y_{n,c}(k)q^k$. Again $T_{n,c}(d)$ is a q-polynomial of degree at most $2n-2$, however, the left-hand-side of the equation above is apriori a q-polynomial of degree $2n-1$. Thus we obtain the following formula for $T_{n,c}(0)=\sum_{k=0}^{c}Y_{n,c}(k)q^k$.

$$\frac{(-1)^{n-1}q^{-(n-1)(2c+n)/2}}{(1+q^n)}\left(\frac{q^{\binom{n}{2}}\sum\limits_{i=0}^{n-1}\frac{q^{\binom{i}{2}}}{[c+i+1;q]_n[1;q]_i[1;q]_{n-i-1}}}{\sum\limits_{i=0}^{n-1}\frac{(-1)^i q^{\binom{n+i}{2}}}{[c+i+1;q]_n[1;q]_i[1;q]_{n-i-1}}}-1\right)$$

Again the formula simplifies since

$$\sum_{i=0}^{n-1}\frac{(-1)^i q^{\binom{n+i}{2}}}{[c+i+1;q]_n[1;q]_i[1;q]_{n-i-1}}=\frac{q^{\binom{n}{2}}[1;q]_{2n-2}}{[c+1;q]_{2n-2}[1;q]_{n-1}^2}. \tag{2.25}$$

In order to see that transform the sum into hypergeometric notation

$$ {}_2\phi_1\left[\begin{matrix} q^{c+1}, q^{1-n} \\ q^{c+n+1} \end{matrix}; q, q^{2n-1}\right] \frac{q^{n^2/2-n/2}(1-q)^{2n-1}}{(q;q)_{n-1}(q^{c+1};q)_n} $$

and apply the summation formula, see [**19**, (1.5.2); Appendix (II.7)],

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a}\right] = \frac{(c/a;q)_n}{(c;q)_n} $$

to obtain the result. This implies that

$$ T_{n,c}(0) = \frac{(-1)^{n-1}q^{-(n-1)(2c+n)/2}}{(1+q^n)} \left(\sum_{i=0}^{n-1} q^{\binom{i}{2}} \frac{[c+1;q]_i[c+i+n+1;q]_{n-i-1}[n-i;q]_i}{[n;q]_{n-1}[1;q]_i} - 1 \right). $$

Therefore

$$ \begin{aligned} \sum_{k=0}^{c} U_{n,c}(k)q^k = \frac{1+(-1)^c q^{(1+c)n}}{1+q^n} &+ \frac{(-1)^n q^{-(n-1)(2c+n)/2}}{1+q^n} \\ \times \Bigg(\sum_{i=0}^{n-1} (-1)^c q^{\binom{i+2}{2}+c} &\frac{[c+1;q]_i[c+i+n+1;q]_{n-i-1}[n-i;q]_i}{[n;q]_{n-1}[1;q]_i} \\ &- \sum_{i=0}^{n-1} q^{\binom{i}{2}} \frac{[c+1;q]_i[c+i+n+1;q]_{n-i-1}[n-i;q]_i}{[n;q]_{n-1}[1;q]_i} - (-q^n)^{c+1} + 1 \Bigg). \end{aligned} $$

Consequently, it suffices to compute

$$ \begin{aligned} \sum_{i=0}^{n-1} q^{\binom{i+2}{2}+c} &\frac{[c+1;q]_i[c+i+n+1;q]_{n-i-1}[n-i;q]_i}{[n;q]_{n-1}[1;q]_i} \\ &- \sum_{i=0}^{n-1} q^{\binom{i}{2}} \frac{[c+1;q]_i[c+i+n+1;q]_{n-i-1}[n-i;q]_i}{[n;q]_{n-1}[1;q]_i} \end{aligned} \tag{2.26} $$

and

$$ \begin{aligned} \sum_{i=0}^{n-1} q^{\binom{i+2}{2}+c} &\frac{[c+1;q]_i[c+i+n+1;q]_{n-i-1}[n-i;q]_i}{[n;q]_{n-1}[1;q]_i} \\ &+ \sum_{i=0}^{n-1} q^{\binom{i}{2}} \frac{[c+1;q]_i[c+i+n+1;q]_{n-i-1}[n-i;q]_i}{[n;q]_{n-1}[1;q]_i}. \end{aligned} \tag{2.27} $$

Using basic hypergeometric notation (2.26) is equal to

$$ {}_4\phi_3\left[\begin{matrix} q^{1+c}, q^{3/2+c/2}, -q^{3/2+c/2}, q^{1-n} \\ q^{1/2+c/2}, -q^{1/2+c/2}, q^{1+c+n} \end{matrix}; q, -q^{n-1}\right] \frac{(q^{1+c};q)_1(q^{1+c+n};q)_{n-1}}{(q^n;q)_{n-1}}. $$

We apply the following transformation

$$ {}_4\phi_3\left[\begin{matrix} a, a^{1/2}q, -a^{1/2}q, b \\ a^{1/2}, -a^{1/2}, aq/b \end{matrix}; q, t\right] = {}_2\phi_1\left[\begin{matrix} 1/b, t \\ bqt \end{matrix}; q, aq\right] \frac{(aq;q)_\infty(bt;q)_\infty}{(t;q)_\infty(aq/b;q)_\infty}, \tag{2.28} $$

which can be found in [**19**, Ex. 2.2] and obtain a ${}_2\phi_1$-series. We apply another transformation

$$ {}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,z\right] = {}_2\phi_1\left[\begin{matrix}c/b,z\\ az\end{matrix};q,b\right]\frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}, \tag{2.29}$$

see [**19**, (1.4.1); Appendix (III.1)], before we are able to apply the summation (2.20). We obtain a formula for (2.26) in terms of products of rising q-factorials. In basic hypergeometric notation (2.27) is equal to

$$ {}_4\phi_3\left[\begin{matrix}q^{1+c}, iq^{3/2+c/2}, -iq^{3/2+c/2}, q^{1-n}\\ iq^{1/2+c/2}, -iq^{1/2+c/2}, q^{1+c+n}\end{matrix};q,-q^{n-1}\right]\frac{(q^{2+2c};q)_1(q^{1+c+n};q)_{n-1}}{(q^{c+1};q)_1(q^n;q)_{n-1}}.$$

We apply the following transformation rule

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix}a,b,c,d\\ aq/b,aq/c,aq/d\end{matrix};q,-aq^2/(bcd)\right] = \\ &\quad {}_8\phi_7\left[\begin{matrix}a^2q/(bcd), aq^{3/2}/(bcd)^{1/2}, -aq^{3/2}/(bcd)^{1/2}, a^{1/2}, -a^{1/2}, aq/(cd), aq/(bd), aq/(bc)\\ aq^{1/2}/(bcd)^{1/2}, -aq^{1/2}/(bcd)^{1/2}, a^{3/2}q^2/(bcd), -a^{3/2}q^2/(bcd), aq/b, aq/c, aq/d\end{matrix};q,-q\right] \\ &\qquad\qquad \times\frac{(aq;q)_\infty(-q;q)_\infty(a^{3/2}q^2/(bcd);q)_\infty(-a^{3/2}q^2/(bcd);q)_\infty}{(a^2q^2/(bcd);q)_\infty(-aq^2/(bcd);q)_\infty(a^{1/2}q;q)_\infty(-a^{1/2}q;q)_\infty} \end{aligned}\tag{2.30}$$

see [**19**, Ex. 2.13 (ii)], to obtain a ${}_8\phi_7$-series. Next we apply the transformation rule

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix}a, a^{1/2}q, -a^{1/2}q, b, c, d, e, f\\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f\end{matrix};q,a^2q^2/(bcdef)\right] = \\ &\quad {}_8\phi_7\left[\begin{matrix}a^2q/(bcd), aq^{3/2}/(bcd)^{1/2}, -aq^{3/2}/(bcd)^{1/2}, aq/(cd), aq/(bd), aq/(bc), e, f\\ aq^{1/2}/(bcd)^{1/2}, -aq^{1/2}/(bcd)^{1/2}, aq/b, aq/c, aq/d, a^2q^2/(bcde), a^2q^2/(bcdf)\end{matrix};q,aq/(ef)\right] \\ &\qquad\qquad \times\frac{(aq;q)_\infty(aq/(ef);q)_\infty(a^2q^2/(bcde);q)_\infty(a^2q^2/(bcdf);q)_\infty}{(aq/e;q)_\infty(aq/f;q)_\infty(a^2q^2/(bcd);q)_\infty(a^2q^2/(bcdef);q)_\infty}, \end{aligned}\tag{2.31}$$

see [**19**, (2.10.1); Appendix (III.23)] and finally the summation formula

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix}-(ab/q)^{1/2}c, i(ab)^{1/4}c^{1/2}q^{3/4}, -i(ab)^{1/4}c^{1/2}q^{3/4}, a, b, c, -c, -(abq)^{1/2}/c\\ i(ab/q)^{1/4}c^{1/2}, -i(ab/q)^{1/4}c^{1/2}, -(bq/a)^{1/2}c, -(aq/b)^{1/2}c, -(abq)^{1/2}, (abq)^{1/2}, c^2\end{matrix};q,\frac{cq^{1/2}}{(ab)^{1/2}}\right] = \\ &\quad \frac{(-(abq)^{1/2}c;q)_\infty(-c(q/ab)^{1/2};q)_\infty}{(-(bq/a)^{1/2}c;q)_\infty(-(aq/b)^{1/2}c;q)_\infty}\frac{(aq;q^2)_\infty(bq;q^2)_\infty(c^2q/a;q^2)_\infty(c^2q/b;q^2)_\infty}{(q;q^2)_\infty(abq;q^2)_\infty(c^2q;q^2)_\infty(c^2q/(ab);q^2)_\infty}, \end{aligned}\tag{2.32}$$

see [**19**, Ex. 2.17(i); Appendix (II.16)], in order to obtain a formula for (2.27) in terms of rising q-factorial. We combine the formulas for (2.26) and (2.27) in order to settle (2.16).

CHAPTER 3

The number of monotone triangles with prescribed bottom row

ABSTRACT. We show that the number of monotone triangles with prescribed bottom row $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, $k_1 < k_2 < \ldots < k_n$, is given by a simple product formula which remarkably involves (shift) operators. Monotone triangles with bottom row $(1, 2, \ldots, n)$ are in bijection with $n \times n$ alternating sign matrices.

1. Introduction

An *alternating sign matrix* is a square matrix of 0s, 1s and −1s for which the sum of entries in each row and in each column is 1 and the non-zero entries of each row and of each column alternate in sign. For instance,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

is an alternating sign matrix. In the early 1980s, Robbins and Rumsey [**52**] introduced alternating sign matrices in the course of generalizing a determinant evaluation algorithm. Out of curiosity they posed the question for the number of alternating sign matrices of fixed size and, together with Mills, they came up with the appealing conjecture [**42**] that the number of $n \times n$ alternating sign matrices is

$$\prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!}. \tag{3.1}$$

This turned out to be one of the hardest problems in enumerative combinatorics within the last decades. In 1996 Zeilberger [**64**] finally succeeded in proving their conjecture. Then, some months later, Kuperberg [**34**] realized that alternating sign matrices are equivalent to a model in statistical physics for two-dimensional square ice. Using a determinantal expression for the partition function of this model discovered earlier by physicists, he was able to provide a shorter proof of the formula. For a nice exposition on this topic see [**7**].

Alternating sign matrices can be translated into certain triangular arrays of positive integers, called *monotone triangles*. Monotone triangles are probably the right guise of alternating sign matrices for a recursive treatment [**7**, Section 2.3]. In order to obtain the monotone triangle corresponding to a given alternating sign matrix, replace every entry in the matrix by the sum of elements in the same column above, the entry itself included. In our running example we obtain

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Row by row we record the columns that contain a 1 and obtain the following triangular array.

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

This is the monotone triangle corresponding to the alternating sign matrix above. Observe that it is weakly increasing in northeast direction and in southeast direction. Moreover, it is strictly increasing along rows. In general, a monotone triangle with n rows is a triangular array

$(a_{i,j})_{1\le j\le i\le n}$ of integers such that $a_{i,j} \le a_{i-1,j} \le a_{i,j+1}$ and $a_{i,j} < a_{i,j+1}$ for all i,j. It is not too hard to see that monotone triangles with n rows and bottom row $(1,2,\ldots,n)$, i.e. $a_{n,j}=j$, are in bijection with $n\times n$ alternating sign matrices. Our main theorem provides a formula for the number of monotone triangles with prescribed bottom row $(k_1,k_2,\ldots,k_n)\in\mathbb{Z}^n$.

Theorem 3.1. *The number of monotone triangles with n rows and prescribed bottom row $(k_1,k_2,\ldots,k_n)$ is given by*

$$\left(\prod_{1\le p<q\le n}\left(\mathrm{id}+E_{k_p}\Delta_{k_q}\right)\right)\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i},$$

where E_x denotes the shift operator, defined by $E_x\,p(x)=p(x+1)$, and $\Delta_x := E_x - \mathrm{id}$ denotes the difference operator.

In order to understand this formula, there are a few things to remark. The product of operators is understood as the composition. Moreover note that the shift operators commute, and consequently, it does not matter in which order the operators in the product $\prod_{1\le p<q\le n}\left(\mathrm{id}+E_{k_p}\Delta_{k_q}\right)$ are applied. In order to use this formula to compute the number of monotone triangles with bottom row $(k_1,\ldots,k_n)$, one first has to apply the operator $\prod_{1\le p<q\le n}\left(\mathrm{id}+E_{x_p}\Delta_{x_q}\right)$ to the polynomial $\prod_{1\le i<j\le n}\frac{x_j-x_i}{j-i}$ and then set $x_i=k_i$. Thus, at this time, we do not know how to derive (3.1) from this formula.

Next we discuss the significance of the formula. In the last decades, the enumeration of plane partitions, alternating sign matrices and related objects subject to a variety of different constraints has attracted a lot of interest. This attraction stems from the fact that now and then these enumerations lead to an appealing product formula or hypergeometric series, which is, in spite of their simplicity, pretty hard to prove. Clearly, the search for these simple product formulas gets more and more exhausted. Therefore, a new challenge is the search for possibilities to give enumeration formulas for the vast majority of enumeration problems for which there exists no closed formula in a traditional sense. The formula in Theorem 3.1 contributes to this issue. At this point it is interesting to note that Zeilberger [**64**, Subsublemma 1.2.1] provides a constant term expression for the number of monotone triangles with prescribed rightmost southeast diagonal. (In fact, in his lemma, he considers more general objects, called $n\times k$ – Gog trapezoids, which are monotone triangles with the $n-k$ rightmost northeast diagonals cut off.)

Also note that the second product in the formula in Theorem 3.1, i.e. $\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}$, is the number of semistandard tableaux of shape $(k_n-n,k_{n-1}-n+1,\ldots,k_1-1)$ and, equivalently, the number of columnstrict plane partitions of this shape [**58**, p. 375, in (7.105) $q\to 1$]. In fact, these objects are in bijection with monotone triangles with precsribed bottom row $(k_1,k_2,\ldots,k_n)$ that are strictly increasing in southeast direction, see Section 5 of [**15**, resp. Chapter 1]. Thus, our formula once more gives an indication of the relation between plane partitions and alternating sign matrices manifested by a number of enumeration formulas which show up in both fields, a phenomenon which is not yet well (i.e. bijectively) understood.

At this point it is also worth mentioning that we can easily rewrite the formula in Theorem 3.1 such that the second product is the number of semistandard tableaux of shape

$(k_n, \ldots, k_1)$.

$$\alpha(n; k_1, \ldots, k_n) = \left(\prod_{1 \le p < q \le n} \left(E_{k_q}^{-1} + E_{k_q}^{-1} E_{k_p} \Delta_{k_q} \right) \prod_{q=1}^{n} E_{k_q}^{q-1} \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} =$$

$$\left(\prod_{1 \le p < q \le n} \left(\mathrm{id} + E_{k_q}^{-1} \Delta_{k_p} \Delta_{k_q} \right) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}$$

The method for proving our main theorem can roughly be described as follows. In the first step, we introduce a recursion, which relates monotone triangles with n rows to monotone triangles with $n-1$ rows. This recursion immediately implies that the enumeration formula is a polynomial in $k_1, k_2, \ldots, k_n$. In the next step, we compute the degree of the polynomial. Finally, we deduce enough properties of the polynomial in order to compute it, where the polynomial's degree determines how much information is in fact needed. This method is related to the method for proving polynomial enumeration formulas we have introduced in [**15**, resp. Chapter 1] and extended in [**16**, resp. Chapter 2]. In the final section we mention some further projects around Theorem 3.1 we plan to consider next.

2. The recursion

In the following let $\alpha(n; k_1, \ldots, k_n)$, $n \ge 1$, denote the number of monotone triangles with $(k_1, \ldots, k_n)$ as bottom row. If we delete the last row of such a monotone triangle we obtain a monotone triangle with $n-1$ rows and bottom row, say, $(l_1, l_2, \ldots, l_{n-1})$. By the definition of a monotone triangle we have $k_1 \le l_1 \le k_2 \le l_2 \le \ldots \le k_{n-1} \le l_{n-1} \le k_n$ and $l_i \ne l_{i+1}$. Thus

$$\alpha(n; k_1, \ldots, k_n) = \sum_{\substack{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}, \\ k_1 \le l_1 \le k_2 \le \cdots \le k_{n-1} \le l_{n-1} \le k_n, l_i \ne l_{i+1}}} \alpha(n-1; l_1, \ldots, l_{n-1}). \tag{3.2}$$

We introduce the following abbreviation

$$\sum_{\substack{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}, \\ k_1 \le l_1 \le k_2 \le \cdots \le k_{n-1} \le l_{n-1} \le k_n, l_i \ne l_{i+1}}} =: \sum_{(l_1, \ldots, l_{n-1})}^{(k_1, \ldots, k_n)}$$

for $n \ge 2$. This summation operator is well-defined for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 < k_2 < \ldots < k_n$. We extend the definition to arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ by induction with respect to n. If $n = 2$ then

$$\sum_{(l_1)}^{(k_1, k_2)} A(l_1) := \sum_{l_1 = k_1}^{k_2} A(l_1),$$

where here and in the following we use the extended definition of the summation over an interval, namely,

$$\sum_{i=a}^{b} f(i) = \begin{cases} f(a) + f(a+1) + \cdots + f(b) & \text{if } a \le b \\ 0 & \text{if } b = a - 1 \\ -f(b+1) - f(b+2) - \cdots - f(a-1) & \text{if } b + 1 \le a - 1 \end{cases}. \tag{3.3}$$

This assures that for any polynomial $p(X)$ over an arbitrary integral domain I containing $\mathbb{Q}$ there exists a unique polynomial $q(X)$ over I such that $\sum_{x=0}^{y} p(x) = q(y)$ for all integers y. We usually write $\sum_{x=0}^{y} p(x)$ for $q(y)$. (We also use the analog extended definition for the product symbol.) If $n > 2$ then

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) := \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1,\ldots,l_{n-2},l_{n-1}) + \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1}-1)} A(l_1,\ldots,l_{n-2},k_{n-1}).$$

We renew the definition of $\alpha(n;k_1,\ldots,k_n)$ after this extension by setting $\alpha(1;k_1)=1$ and

$$\alpha(n;k_1,\ldots,k_n) = \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1;l_1,\ldots,l_{n-1}).$$

This extends the original function $\alpha(n;k_1,\ldots,k_n)$ to arbitrary $(k_1,\ldots,k_n) \in \mathbb{Z}^n$. The recursion implies that $\alpha(n;k_1,\ldots,k_n)$ is a polynomial in $k_1,\ldots,k_n$. We have used this recursion (and a computer) to compute $\alpha(n;k_1,\ldots,k_n)$ for $n=1,2,3,4$ and obtain the following

$$\begin{aligned}
&1,\, 1-k_1+k_2,\, \frac{1}{2}(-3k_1+k_1^2+2k_1k_2-k_1^2k_2-2k_2^2+k_1k_2^2+3k_3-4k_1k_3+k_1^2k_3+2k_2k_3-k_2^2k_3+\\
&k_3^2-k_1k_3^2+k_2k_3^2),\, \frac{1}{12}(20k_2+11k_1k_2-16k_1^2k_2+3k_1^3k_2+4k_1k_2^2+3k_1^2k_2^2-k_1^3k_2^2+4k_2^3-5k_1k_2^3+k_1^2k_2^3-\\
&20k_3+16k_1k_3-4k_1^2k_3-27k_2k_3+9k_1^2k_2k_3-2k_1^3k_2k_3-3k_1^2k_2^2k_3+k_1^3k_2^2k_3-3k_2^3k_3+4k_1k_2^3k_3-k_1^2k_2^3k_3+\\
&16k_1k_3^2-12k_1^2k_3^2+2k_1^3k_3^2-9k_1k_2k_3^2+6k_1^2k_2k_3^2-k_1^3k_2k_3^2+9k_2^2k_3^2-3k_1k_2^2k_3^2-3k_2^3k_3^2+k_1k_2^3k_3^2-4k_3^3+\\
&8k_1k_3^3-2k_1^2k_3^3-3k_2k_3^3-2k_1k_2k_3^3+k_1^2k_2k_3^3+3k_2^2k_3^3-k_1k_2^2k_3^3-27k_1k_4+20k_1^2k_4-3k_1^3k_4+16k_2k_4+\\
&24k_1k_2k_4-24k_1^2k_2k_4+4k_1^3k_2k_4-16k_2^2k_4+9k_1k_2^2k_4+3k_1^2k_2^2k_4-k_1^3k_2^2k_4+8k_2^3k_4-6k_1k_2^3k_4+k_1^2k_2^3k_4+\\
&11k_3k_4-24k_1k_3k_4+15k_1^2k_3k_4-2k_1^3k_3k_4-9k_2^2k_3k_4+2k_2^3k_3k_4-4k_3^2k_4+9k_1k_3^2k_4-6k_1^2k_3^2k_4+k_1^3k_3^2k_4+\\
&3k_2^2k_3^2k_4-k_2^3k_3^2k_4-5k_3^3k_4+6k_1k_3^3k_4-k_1^2k_3^3k_4-4k_2k_3^3k_4+k_2^2k_3^3k_4-20k_1k_4^2+9k_1^2k_4^2-k_1^3k_4^2+4k_2k_4^2+\\
&15k_1k_2k_4^2-9k_1^2k_2k_4^2+k_1^3k_2k_4^2-12k_2^2k_4^2+6k_1k_2^2k_4^2+2k_2^3k_4^2-k_1k_2^3k_4^2+16k_3k_4^2-24k_1k_3k_4^2+9k_1^2k_3k_4^2-\\
&k_1^3k_3k_4^2+9k_2k_3k_4^2-6k_2^2k_3k_4^2+k_2^3k_3k_4^2+3k_3^2k_4^2-3k_1k_3^2k_4^2+3k_2k_3^2k_4^2-k_3^3k_4^2+k_1k_3^3k_4^2-k_2k_3^3k_4^2-3k_1k_4^3+\\
&k_1^2k_4^3+2k_1k_2k_4^3-k_1^2k_2k_4^3-2k_2^2k_4^3+k_1k_2^2k_4^3+3k_3k_4^3-4k_1k_3k_4^3+k_1^2k_3k_4^3+2k_2k_3k_4^3-k_2^2k_3k_4^3+k_3^2k_4^3-\\
&\qquad k_1k_3^2k_4^3+k_2k_3^2k_4^3).
\end{aligned}$$

From this data it is obviously hard to guess a general formula for $\alpha(n;k_1,\ldots,k_n)$. However, it seems plausible that the degree of $\alpha(n;k_1,\ldots,k_n)$ in k_i is $n-1$. In the following two sections we prove that this is indeed true. Note that at first glance the linear growth of the degree is quite surprising: suppose $A(l_1,\ldots,l_{n-1})$ is a polynomial of degree no greater than R in each of

l_{i-1} and l_i. Then

$$\deg_{k_i}\left(\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1})\right) =$$
$$\deg_{k_i}\left(\sum_{l_{i-1}=k_{i-1}}^{k_i}\sum_{l_i=k_i}^{k_{i+1}} A(l_1,\ldots,l_{n-1}) - A(l_1,\ldots,l_{i-2},k_i,k_i,l_{i+1},\ldots,l_{n-1})\right) \leq 2R+2$$

and there exist polynomials $A(l_1,\ldots,l_{n-1})$ such that the upper bound $2R+2$ is attained. Consequently, $\alpha(n;k_1,\ldots,k_n)$ must be of a very specific shape.

3. Operators related to the recursion

In this section we define some operators that are fundamental for the study of the recursion defined in the previous section. The theory is developed in a bit more generality than is actually needed in order to investigate $\alpha(n;k_1,\ldots,k_n)$. Recall that the *shift operator*, denoted by E_x, is defined as $E_x p(x) = p(x+1)$. Clearly E_x is invertible in the algebra of operators of $\mathbb{C}[x]$ and we denote its inverse by E_x^{-1}. Observe that the shift operators with respect to different variables commute, i.e. $E_x E_y = E_y E_x$. The difference operator Δ_x is defined as $\Delta_x = E_x - \mathrm{id}$. However, the difference operator Δ_x is not invertible since it decreases the degree of a polynomial. If we apply the shift operator or the delta operator to the i-th variable of a function, we sometimes write E_i or Δ_i, respectively, i.e. $\Delta_{k_i} f(k_1,\ldots,k_n) = \Delta_i f(k_1,\ldots,k_n)$. Moreover, $\Delta_2 f(k_3,k_3,k_3)$, for instance, is shorthand for

$$(\Delta_{l_2} f(l_1,l_2,l_3))|_{l_1=k_3,l_2=k_3,l_3=k_3}\,.$$

The *swapping operator* $S_{x,y}$ is applicable to functions in (at least) two variables and defined as $S_{x,y} f(x,y) = f(y,x)$. If we apply $S_{x,y}$ to the i-th and j-th variable of a function we sometimes write $S_{i,j}$.

In the following we consider rational functions in shift operators. In order to guarantee that the inverse of the denominator always exists, we need the following lemma.

LEMMA 3.1. *Let $p(x_1,\ldots,x_n)$ be a polynomial in $x_1,x_1^{-1},x_2,x_2^{-1},\ldots,x_n,x_n^{-1}$ over $\mathbb{C}$, and fix an integer i, $1\leq i\leq n$. Consider the operator*

$$\mathrm{id} + \Delta_{k_i} p(E_{k_1},E_{k_2},\ldots,E_{k_n}) =: \mathrm{Op}$$

on $\mathbb{C}[k_1,\ldots,k_n]$. Then Op is invertible and the inverse is

$$\mathrm{Op}^{-1} = \sum_{l=0}^{\infty}(-1)^l \Delta_{k_i}^l p(E_{k_1},E_{k_2},\ldots,E_{k_n})^l,$$

where $\Delta_{k_i}^0 p(E_{k_1},E_{k_2},\ldots,E_{k_n})^0 = \mathrm{id}$. Moreover

$$\deg_{k_i} G(k_1,\ldots,k_n) = \deg_{k_i} \mathrm{Op}\, G(k_1,\ldots,k_n) = \deg_{k_i} \mathrm{Op}^{-1} G(k_1,\ldots,k_n).$$

Proof. Let $G(k_1,\ldots,k_n)\in\mathbb{C}[k_1,\ldots,k_n]$. First observe that

$$\deg_{k_i} G(k_1,\ldots,k_n) = \deg_{k_i} \mathrm{Op}\, G(k_1,\ldots,k_n), \tag{3.4}$$

since Δ_{k_i} decreases the degree in k_i and $p(E_{k_1}, E_{k_2}, \ldots, E_{k_n})$ does not increase the degree. It is easy to see that

$$F(k_1, \ldots, k_n) = \sum_{l=0}^{\infty} (-1)^l \Delta_{k_i}^l p(E_{k_1}, E_{k_2}, \ldots, E_{k_n})^l G(k_1, \ldots, k_n)$$

is a polynomial with the property that $\operatorname{Op} F = G$. (Observe that the sum is finite since $\Delta_{k_i}^l G(k_1, \ldots, k_n) = 0$ if $l > \deg_{k_i} G$.) Assume there is another polynomial $F' \in \mathbb{C}[k_1, \ldots, k_n]$ with the property that $\operatorname{Op} F' = G$. Then $\operatorname{Op} H = 0$ with $H = F - F'$. Thus, by (3.4), $\deg_{k_i} H = \deg_{k_i} 0 = -\infty$, a contradiction. We obtain $\deg_{k_i} \operatorname{Op}^{-1} G = \deg_{k_i} G$ if we apply (3.4) to $\operatorname{Op}^{-1} G$. □

Next we define two operators applicable to polynomials $G(k_1, \ldots, k_n) \in \mathbb{C}[k_1, \ldots, k_n]$. We set

$$V_{k_i,k_j} = \operatorname{id} + E_{k_i}^{-1} \Delta_{k_i} \Delta_{k_j} = E_{k_i}^{-1}(\operatorname{id} + E_{k_j} \Delta_{k_i})$$

and

$$T_{k_i,k_{i+1}} = (\operatorname{id} + E_{k_{i+1}} E_{k_i}^{-1} S_{k_i,k_{i+1}}) \frac{V_{k_i,k_{i+1}}}{V_{k_i,k_{i+1}} + V_{k_{i+1},k_i}}.$$

By Lemma 3.1, the inverse $(V_{k_i,k_{i+1}} + V_{k_{i+1},k_i})^{-1}$ is well-defined. The following lemma explains the significance of $T_{k_i,k_{i+1}}$ for the recursion (3.2).

LEMMA 3.2. *Let $A(l_1, l_2)$ be a polynomial in l_1 and l_2 which is of degree at most R in each of l_1 and l_2. Moreover assume that $T_{l_1,l_2} A(l_1, l_2)$ is of degree at most R as a polynomial in l_1 and l_2, i.e. a linear combination of monomials $l_1^m l_2^n$ with $m + n \le R$. Then*

$$\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1, l_2) = \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} A(l_1, l_2) - A(k_2, k_2)$$

is of degree at most $R+2$ in k_2. Moreover, if $T_{l_1,l_2} A(l_1, l_2) = 0$ then $\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1, l_2)$ is of degree at most $R+1$ in k_2.

Proof. We decompose $A(l_1, l_2) = T_{l_1,l_2} A(l_1, l_2) + (\operatorname{id} - T_{l_1,l_2}) A(l_1, l_2)$. If we define $A^*(l_1, l_2) = (\operatorname{id} - T_{l_1,l_2})((l_1)_p (l_2)_q / (p! q!))$, it suffices to show that the degree of

$$\sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} A^*(l_1, l_2) - A^*(k_2, k_2) \tag{3.5}$$

in k_2 is no greater than $\max(p, q) + 1$, where $(x)_p = \prod_{i=0}^{p-1} (x + i)$. Observe that

$$\operatorname{id} - T_{l_1,l_2} = \frac{V_{l_2,l_1}}{V_{l_1,l_2} + V_{l_2,l_1}} (\operatorname{id} - E_{l_2} E_{l_1}^{-1} S_{l_1,l_2}),$$

where we use the fact that $S_{l_1,l_2} R(E_{l_1}, E_{l_2}) = R(E_{l_2}, E_{l_1}) S_{l_1,l_2}$ if $R(x,y)$ is a rational function in x, x^{-1}, y, y^{-1}. Lemma 3.1 implies that

$$\begin{aligned} A^*(l_1,l_2) = \frac{1}{2} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \left(-\frac{1}{2}\right)^i \binom{i}{j} \\ \times \Bigg(\frac{(l_1+i-j)_{p-i}(l_2+j)_{q-i}}{(p-i)!(q-i)!} - \frac{(l_2+j+1)_{p-i}(l_1+i-j-1)_{q-i}}{(p-i)!(q-i)!} \\ + \frac{(l_1+i-j+1)_{p-i-1}(l_2+j)_{q-i-1}}{(p-i-1)!(q-i-1)!} - \frac{(l_2+j+1)_{p-i-1}(l_1+i-j)_{q-i-1}}{(p-i-1)!(q-i-1)!} \Bigg). \end{aligned}$$

Using the summation formula

$$\sum_{z=a}^{b} (z+w)_n = \frac{1}{n+1}((b+w)_{n+1} - (a-1+w)_{n+1}) \tag{3.6}$$

we observe that (3.5) is equal to

$$\begin{aligned} &\frac{1}{2} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \left(-\frac{1}{2}\right)^i \binom{i}{j} \\ &\times \Bigg(-\binom{k_2+p-j}{p-i+1} \binom{k_2+q+j-i-1}{q-i+1} + \binom{k_2+q-j-1}{q-i+1} \binom{k_2+p+j-i}{p-i+1} \\ &- \binom{k_2+p-j}{p-i} \binom{k_2+q+j-i-2}{q-i} + \binom{k_2+q-j-1}{q-i} \binom{k_2+p+j-i-1}{p-i} \\ &- \binom{k_2+p-j-1}{p-i} \binom{k_2+q+j-i-1}{q-i} + \binom{k_2+q-j-2}{q-i} \binom{k_2+p+j-i}{p-i} \\ &- \binom{k_2+p-j-1}{p-i-1} \binom{k_2+q+j-i-2}{q-i-1} + \binom{k_2+q-j-2}{q-i-1} \binom{k_2+p+j-i-1}{p-i-1} \Bigg) \\ &+ R(k_1,k_2,k_3), \end{aligned}$$

where $R(k_1,k_2,k_3)$ is a polynomial in k_1,k_2,k_3 of degree no greater than $\max(p,q)+1$ in k_2. If we replace j by $i-j$ in every other product of two binomial coefficients we see that this expression simplifies to $R(k_1,k_2,k_3)$ and the lemma is proved. $\square$

In order to use Lemma 3.2 to compute the degree of $\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1,l_2)$ in k_2, one has to compute the degree of $T_{l_1,l_2} A(l_1,l_2)$ in l_1 and l_2. However, the operator T_{l_1,l_2} is complicated, and thus it is convenient to consider a simplified version of T_{l_1,l_2} for this purpose, which is obtained by multiplication with an operator that preserves the degree.

$$\begin{aligned} T'_{k_i,k_{i+1}} := E_{k_i}(V_{k_i,k_{i+1}} + V_{k_{i+1},k_i}) T_{k_i,k_{i+1}} = \\ (\mathrm{id} + S_{k_i,k_{i+1}}) E_{k_i} V_{k_i,k_{i+1}} = (\mathrm{id} + S_{k_i,k_{i+1}})(\mathrm{id} + E_{k_{i+1}} \Delta_{k_i}) \end{aligned}$$

Observe that $\deg_{k_i,k_{i+1}} T_{k_i,k_{i+1}} G(k_1,\dots,k_n) = \deg_{k_i,k_{i+1}} T'_{k_i,k_{i+1}} G(k_1,\dots,k_n)$, since

$$V_{k_i,k_{i+1}} + V_{k_{i+1},k_i} = 2\,\mathrm{id} + (E_{k_i}^{-1} + E_{k_{i+1}}^{-1}) \Delta_{k_i} \Delta_{k_{i+1}}$$

and $\Delta_{k_i}\Delta_{k_{i+1}}$ decreases the degree of a polynomial in k_i and k_{i+1}. In particular,

$$T_{k_i,k_{i+1}}G(k_1,\dots,k_n)=0 \quad\Leftrightarrow\quad T'_{k_i,k_{i+1}}G(k_1,\dots,k_n)=0.$$

4. The fundamental lemma

Suppose $A(l_1,\dots,l_n)$ is a function on $\mathbb{Z}^n$. In this section we prove a lemma that expresses

$$T'_{k_i,k_{i+1}}\left(\sum_{(l_1,\dots,l_n)}^{(k_1,\dots,k_{n+1})} A(l_1,\dots,l_n)\right)(k_1,\dots,k_{n+1})$$

in terms of $T'_{l_{i-1},l_i}A(l_1,\dots,l_n)$ and $T'_{l_i,l_{i+1}}A(l_1,\dots,l_n)$. In particular, this shows that if

$$T'_{l_i,l_{i+1}}A(l_1,\dots,l_n)=0$$

for all $i=1,\dots,n-1$ then

$$T'_{k_i,k_{i+1}}\left(\sum_{(l_1,\dots,l_n)}^{(k_1,\dots,k_{n+1})} A(l_1,\dots,l_n)\right)(k_1,\dots,k_{n+1})=0$$

for all $i=1,\dots,n$.

Lemma 3.3. *Let $f(k_1,k_2,k_3)$ be a function from $\mathbb{Z}^3$ to $\mathbb{C}$ and define*

$$g(k_1,k_2,k_3,k_4):=\sum_{(l_1,l_2,l_3)}^{(k_1,k_2,k_3,k_4)} f(l_1,l_2,l_3).$$

Then

$$\begin{aligned}
&T'_{2,3}\,g(k_1,k_2,k_3,k_4)=\\
&\qquad -\frac{1}{2}\left(\sum_{l_1=k_2+1}^{k_3}\sum_{l_2=k_2+1}^{k_3}\sum_{l_3=k_2}^{k_4} T'_{1,2}\,f(l_1,l_2,l_3)+\sum_{l_1=k_1}^{k_2+1}\sum_{l_2=k_2}^{k_3-1}\sum_{l_3=k_2}^{k_3-1} T'_{2,3}\,f(l_1,l_2,l_3)\right)\\
&\quad+\frac{1}{2}\left(\sum_{l_1=k_2}^{k_3-1}\sum_{l_2=k_2}^{k_3-1}\Delta_2(\mathrm{id}+E_1)T'_{1,2}\,f(l_1,l_2,k_2)-\sum_{l_2=k_2}^{k_3-1}\sum_{l_3=k_2}^{k_3-1}\Delta_2(\mathrm{id}+E_3)T'_{2,3}\,f(k_2+1,l_2,l_3)\right)\\
&\quad+\frac{1}{2}\Big(T'_{1,2}\,f(k_2,k_2,k_2+1)-T'_{1,2}\,f(k_2,k_2,k_3+1)+T'_{2,3}\,f(k_2,k_2,k_2)-T'_{2,3}\,f(k_3,k_2,k_2)\Big)\\
&\qquad\qquad -T'_{1,2}\,f(k_2,k_3,k_2+1)-T'_{2,3}\,f(k_2,k_2,k_3).
\end{aligned}$$

Moreover, for a function $h(l_1,l_2)$ on $\mathbb{Z}^2$,

$$T'_{1,2}\left(\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} h(l_1,l_2)\right)(k_1,k_2,k_3)=-\frac{1}{2}\sum_{l_1=k_1}^{k_2-1}\sum_{l_2=k_1}^{k_2-1}T'_{1,2}\,h(l_1,l_2).$$

Proof. We only sketch the proof of the first formula since the proof of the second formula is easy. Observe that in this formula $T'_{1,2}f(k_2,k_2,k_2+1)$, for instance, is shorthand for

$$T'_{l_1,l_2}f(l_1,l_2,k_2+1)\big|_{l_1=k_2,l_2=k_2}.$$

By definition

$$g(k_1,k_2,k_3,k_4)=\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}\sum_{l_3=k_3}^{k_4}f(l_1,l_2,l_3)-\sum_{l_3=k_3}^{k_4}f(k_2,k_2,l_3)-\sum_{l_1=k_1}^{k_2}f(l_1,k_3,k_3).$$

It is easy to see that

$$\Delta_2 g(k_1,k_2,k_3,k_4)=-\sum_{l_1=k_1}^{k_2-1}\sum_{l_3=k_3}^{k_4}f(l_1,k_2,l_3)+\sum_{l_2=k_2+2}^{k_3}\sum_{l_3=k_3}^{k_4}f(k_2+1,l_2,l_3)-f(k_2+1,k_3,k_3).$$

This implies that

$$\begin{aligned}(\mathrm{id}+\Delta_2 E_3)g(k_1,k_2,k_3,k_4)&=\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}\sum_{l_3=k_3}^{k_4}f(l_1,l_2,l_3)\\&-\sum_{l_3=k_3}^{k_4}f(k_2,k_2,l_3)-\sum_{l_1=k_1}^{k_2}f(l_1,k_3,k_3)-f(k_2+1,k_3+1,k_3+1)\\&-\sum_{l_1=k_1}^{k_2-1}\sum_{l_3=k_3+1}^{k_4}f(l_1,k_2,l_3)+\sum_{l_2=k_2+2}^{k_3+1}\sum_{l_3=k_3+1}^{k_4}f(k_2+1,l_2,l_3).\end{aligned}$$

Next we want to apply the operator $(\mathrm{id}+S_{2,3})$. Observe that

$$\begin{aligned}(\mathrm{id}+S_{2,3})&\left(\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}\sum_{l_3=k_3}^{k_4}f(l_1,l_2,l_3)\right)(k_1,k_2,k_3,k_4)\\&=\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}\sum_{l_3=k_3}^{k_4}f(l_1,l_2,l_3)-\sum_{l_1=k_1}^{k_3}\sum_{l_2=k_2}^{k_3}\sum_{l_3=k_2}^{k_4}f(l_1,l_2,l_3)\\&=-\sum_{l_1=k_2+1}^{k_3}\sum_{l_2=k_2+1}^{k_3}\sum_{l_3=k_2}^{k_4}f(l_1,l_2,l_3)-\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3-1}\sum_{l_3=k_2}^{k_3-1}f(l_1,l_2,l_3)+\sum_{l_1=k_1}^{k_2}\sum_{l_3=k_2}^{k_4}f(l_1,k_2,l_3)\\&+\sum_{l_1=k_1}^{k_3}\sum_{l_3=k_3}^{k_4}f(l_1,k_3,l_3)+\sum_{l_1=k_2+1}^{k_3}\sum_{l_3=k_2}^{k_3-1}f(l_1,k_3,l_3)\\&=-\frac{1}{2}\left(\sum_{l_1=k_2+1}^{k_3}\sum_{l_2=k_2+1}^{k_3}\sum_{l_3=k_2}^{k_4}(\mathrm{id}+S_{1,2})f(l_1,l_2,l_3)+\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3-1}\sum_{l_3=k_2}^{k_3-1}(\mathrm{id}+S_{2,3})f(l_1,l_2,l_3)\right)\\&+\sum_{l_1=k_1}^{k_2}\sum_{l_3=k_2}^{k_4}f(l_1,k_2,l_3)+\sum_{l_1=k_1}^{k_3}\sum_{l_3=k_3}^{k_4}f(l_1,k_3,l_3)+\sum_{l_1=k_2+1}^{k_3}\sum_{l_3=k_2}^{k_3-1}f(l_1,k_3,l_3).\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&(\mathrm{id}+S_{2,3})(\mathrm{id}+\Delta_2 E_3)g(k_1,k_2,k_3,k_4)\\
&\qquad = -\frac{1}{2}\left(\sum_{l_1=k_2+1}^{k_3}\sum_{l_2=k_2+1}^{k_3}\sum_{l_3=k_2}^{k_4} T'_{1,2}\, f(l_1,l_2,l_3) + \sum_{l_1=k_1}^{k_2+1}\sum_{l_2=k_2}^{k_3-1}\sum_{l_3=k_2}^{k_3-1} T'_{2,3}\, f(l_1,l_2,l_3)\right)\\
&\qquad + \sum_{l_3=k_2}^{k_3} f(k_2+1,k_2+1,l_3) - \sum_{l_3=k_2}^{k_3+1} f(k_2+1,k_3+1,l_3) - \sum_{l_2=k_2}^{k_3-1} f(k_2+1,l_2,k_3)\\
&\qquad\quad + \sum_{l_2=k_2+2}^{k_3+1} f(k_3+1,l_2,k_2) + \sum_{l_1=k_2+2}^{k_3-1} f(l_1,k_3,k_2) - \sum_{l_1=k_2+2}^{k_3} f(l_1,k_2,k_2)\\
&\qquad\qquad - f(k_2,k_2,k_3) - f(k_3+1,k_2+1,k_2+1).
\end{aligned}
$$

Finally check that the right-hand-side of this equation is equal to the right-hand-side in the statement of the lemma. □

This proves the statement preceding the lemma for $n = 2, 3$. It can easily be extended to general n by deriving a merging rule for the recursion (3.2). For this purpose we need another operator. Let $f(x,z)$ be a function on $\mathbb{Z}^2$. Then the operator $I^y_{x,z}$ transforms $f(x,z)$ into a function on $\mathbb{Z}$ by

$$I^y_{x,z} f(x,z) := f(y-1,y) + f(y,y+1) - f(y-1,y+1) = V_{x,z} f(x,z)\big|_{x=y,z=y}.$$

With this definition we have

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = I^{k_i}_{k'_i,k''_i} \sum_{(l_1,\ldots,l_{i-1})}^{(k_1,\ldots,k_{i-1},k'_i)} \sum_{(l_i,\ldots,l_{n-1})}^{(k''_i,k_{i+1},\ldots,k_n)} A(l_1,\ldots,l_n). \tag{3.7}$$

Fix a function $A(l_1,\ldots,l_n)$ on $\mathbb{Z}^n$ and an i with $2 \le i \le n-1$. Let

$$A'_{x,y}(l_1,\ldots,l_{i-2},k_i,k_{i+1},l_{i+2},\ldots,l_n) = \sum_{(l_{i-1},l_i,l_{i+1})}^{(x,k_i,k_{i+1},y)} A(l_1,\ldots,l_n)$$

and

$$A''_{w,x,y,z}(k_1,\ldots,k_{i-2},k_i,k_{i+1},k_{i+3},\ldots,k_{n+1}) \quad = \quad \sum_{(l_1,\ldots,l_{i-2})}^{(k_1,\ldots,k_{i-2},w)} \sum_{(l_{i+2},\ldots,l_n)}^{(z,k_{i+3},\ldots,k_{n+1})} A'_{x,y}(l_1,\ldots,l_n).$$

Then, by (3.7),

$$\sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} A(l_1,\ldots,l_n) = I^{k_{i-1}}_{w,x} I^{k_{i+2}}_{y,z} A''_{w,x,y,z}(k_1,\ldots,k_{i-2},k_i,k_{i+1},k_{i+3},\ldots,k_{n+1}).$$

Define

$$
\begin{aligned}
&A^*_{x,y}(l_1,\dots,l_{i-2},k_i,k_{i+1},l_{i+2},\dots,l_n) = \\
&\quad -\frac{1}{2}\left(\sum_{l_{i-1}=k_i+1}^{k_{i+1}}\sum_{l_i=k_i+1}^{k_{i+1}}\sum_{l_{i+1}=k_i}^{y} T'_{i-1,i}A(l_1,\dots,l_n) + \sum_{l_{i-1}=x}^{k_i+1}\sum_{l_i=k_i}^{k_{i+1}-1}\sum_{l_{i+1}=k_i}^{k_{i+1}-1} T'_{i,i+1}A(l_1,\dots,l_n)\right) \\
&\qquad\qquad + \frac{1}{2}\left(\sum_{l_{i-1}=k_i}^{k_{i+1}-1}\sum_{l_i=k_i}^{k_{i+1}-1} \Delta_i(\mathrm{id}+E_{i-1})T'_{i-1,i}A(l_1,\dots,l_i,k_i,l_{i+2},\dots,l_n)\right. \\
&\qquad\qquad \left. - \sum_{l_i=k_i}^{k_{i+1}-1}\sum_{l_{i+1}=k_i}^{k_{i+1}-1} \Delta_i(\mathrm{id}+E_{i+1})T'_{i,i+1}A(l_1,\dots,l_{i-2},k_i+1,l_i,\dots,l_n)\right) \\
&\quad \frac{1}{2}\Big(T'_{i-1,i}A(\dots,l_{i-2},k_i,k_i,k_i+1,l_{i+2},\dots) - T'_{i-1,i}A(\dots,l_{i-2},k_i,k_i,k_{i+1}+1,l_{i+2},\dots) \\
&\qquad + T'_{i,i+1}A(\dots,l_{i-2},k_i,k_i,k_i,l_{i+2},\dots) - T'_{i,i+1}A(\dots,l_{i-2},k_{i+1},k_i,k_i,l_{i+2},\dots)\Big) \\
&\qquad\quad - T'_{i-1,i}A(\dots,l_{i-2},k_i,k_{i+1},k_i+1,l_{i+2},\dots) - T'_{i,i+1}A(\dots,l_{i-2},k_i,k_i,k_{i+1},l_{i+2},\dots)
\end{aligned}
$$

and

$$
A^{**}_{w,x,y,z}(k_1,\dots,k_{i-2},k_i,k_{i+1},k_{i+3},\dots,k_{n+1}) \quad = \quad \sum_{(l_1,\dots,l_{i-2})}^{(k_1,\dots,k_{i-2},w)} \sum_{(l_{i+2},\dots,l_n)}^{(z,k_{i+3},\dots,k_{n+1})} A^*_{x,y}(l_1,\dots,l_n).
$$

Then, by the first formula in Lemma 3.3, we have

$$
\begin{aligned}
&T'_{k_i,k_{i+1}}\left(\sum_{(l_1,\dots,l_n)}^{(k_1,\dots,k_{n+1})} A(l_1,\dots,l_n)\right)(k_1,\dots,k_{n+1}) \\
&\qquad\qquad = I^{k_{i-1}}_{w,x} I^{k_{i+2}}_{y,z} A^{**}_{w,x,y,z}(k_1,\dots,k_{i-2},k_i,k_{i+1},k_{i+3},\dots,k_{n+1}). \qquad (3.8)
\end{aligned}
$$

If we use the second formula in Lemma 3.3, we obtain a similar formula for the case $i = 1$. By symmetry an analog formula follows for $i = n$. These formulas imply the following corollary.

Corollary 3.1. *Suppose $A(l_1,\dots,l_n)$ is a function on $\mathbb{Z}^n$ with $T'_{l_i,l_{i+1}}A(l_1,\dots,l_n) = 0$ for all i, $1 \le i < n$. Then*

$$
T'_{k_i,k_{i+1}}\left(\sum_{(l_1,\dots,l_n)}^{(k_1,\dots,k_{n+1})} A(l_1,\dots,l_n)\right)(k_1,\dots,k_{n+1}) = 0
$$

for all i, $1 \le i \le n$.

We come back to $\alpha(n;k_1,\dots,k_n)$. By induction with respect to n we conclude that

$$
T'_{k_i,k_{i+1}}\alpha(n;k_1,\dots,k_n) = 0
$$

for all i, $1 \le i < n$, if $n \ge 2$. (Note that $\alpha(2;k_1,k_2) = k_2 - k_1 + 1$.) Thus $T_{k_i,k_{i+1}}\alpha(n;k_1,\dots,k_n) = 0$ for all i. Therefore, by Lemma 3.2 and by induction with respect to n, the polynomial $\alpha(n;k_1,\dots,k_n)$ is of degree no greater than $n-1$ in every k_i.

5. Proof of the theorem

In the previous two sections we have seen that the fact that $T'_{k_i,k_{i+1}}\alpha(n;k_1,\ldots,k_n)=0$ for all i is fundamental for the computation of the polynomial's degree. In this section, however, we demonstrate that this property already determines $\alpha(n;k_1,\ldots,k_n)$ up to a multiplicative constant. Observe that $T'_{k_i,k_{i+1}}A(k_1,\ldots,k_n)=0$ is equivalent with the fact that $(\mathrm{id}+E_{k_{i+1}}\Delta_{k_i})A(k_1,\ldots,k_n)$ is antisymmetric in k_i and k_{i+1}. In the following lemma we characterize functions $A(k_1,\ldots,k_n)$ with the property that

$$(\mathrm{id}+E_{k_{i+1}}\Delta_{k_i})A(k_1,\ldots,k_n)$$

is antisymmetric in k_i and k_{i+1} for all i.

Lemma 3.4. *Let $A(k_1,\ldots,k_n)$ be a polynomial in $(k_1,\ldots,k_n)$. Then*

$$(\mathrm{id}+E_{k_{i+1}}\Delta_{k_i})A(k_1,\ldots,k_n)$$

is antisymmetric in k_i and k_{i+1} for all i, $1\le i\le n-1$, if and only if

$$\left(\prod_{1\le p<q\le n}(\mathrm{id}+E_{k_q}\Delta_{k_p})\right)A(k_1,\ldots,k_n)$$

is antisymmetric in $k_1,\ldots,k_n$.

Proof. First assume that $(\mathrm{id}+E_{k_{i+1}}\Delta_{k_i})A(k_1,\ldots,k_n)$ is antisymmetric in k_i and k_{i+1} for all i. We have to show that

$$(\mathrm{id}+S_{k_i,k_{i+1}})\left(\prod_{1\le p<q\le n}(\mathrm{id}+E_{k_q}\Delta_{k_p})\right)A(k_1,\ldots,k_n)=0$$

for all i. For this purpose observe that

$$\begin{aligned}(\mathrm{id}+S_{k_i,k_{i+1}})&\left(\prod_{1\le p<q\le n,(p,q)\neq(i,i+1)}(\mathrm{id}+E_{k_q}\Delta_{k_p})\right)(\mathrm{id}+E_{k_{i+1}}\Delta_{k_i})A(k_1,\ldots,k_n)\\ &=\left(\prod_{1\le p<q\le n,(p,q)\neq(i,i+1)}(\mathrm{id}+E_{k_q}\Delta_{k_p})\right)(\mathrm{id}+S_{k_i,k_{i+1}})(\mathrm{id}+E_{k_{i+1}}\Delta_{k_i})A(k_1,\ldots,k_n)=0,\end{aligned}$$

because

$$\begin{aligned}\prod_{1\le p<q\le n,(p,q)\neq(i,i+1)}(\mathrm{id}+E_{k_q}\Delta_{k_p})&=\left(\prod_{\substack{1\le p<q\le n,\\ p,q\notin\{i,i+1\}}}(\mathrm{id}+E_{k_q}\Delta_{k_p})\right)\left(\prod_{i+1<q\le n}(\mathrm{id}+E_{k_q}\Delta_{k_i})\right)\\ &\times\left(\prod_{i+1<q\le n}(\mathrm{id}+E_{k_q}\Delta_{k_{i+1}})\right)\left(\prod_{1\le p<i}(\mathrm{id}+E_{k_i}\Delta_{k_p})\right)\left(\prod_{1\le p<i}(\mathrm{id}+E_{k_{i+1}}\Delta_{k_p})\right)\end{aligned}$$

is symmetric in k_i and k_{i+1}. Conversely, assume that

$$\left(\prod_{1\le p<q\le n}(\mathrm{id}+E_{k_q}\Delta_{k_p})\right)A(k_1,\ldots,k_n)$$

is antisymmetric in $k_1, \ldots, k_n$. Consequently,

$$\left(\prod_{1 \le p < q \le n, (p,q) \neq (i,i+1)} (\mathrm{id} + E_{k_q} \Delta_{k_p}) \right) (\mathrm{id} + S_{k_i,k_{i+1}})(\mathrm{id} + E_{k_{i+1}} \Delta_{k_i}) A(k_1, \ldots, k_n) = 0,$$

for all i, $1 \le i \le n-1$. By Lemma 3.1 the operator $\prod_{\substack{1 \le p < q \le n,\\ (p,q) \neq (i,i+1)}} (\mathrm{id} + E_{k_q} \Delta_{k_p})$ is invertible, and therefore $(\mathrm{id} + S_{k_i,k_{i+1}})(\mathrm{id} + E_{k_{i+1}} \Delta_{k_i}) A(k_1, \ldots, k_n) = 0$. $\square$

Using this lemma we see that

$$\left(\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_q} \Delta_{k_p}) \right) \alpha(k_1, \ldots, k_n) \tag{3.9}$$

is an antisymmetric polynomial in $k_1, \ldots, k_n$. A product of shift operators does not increase a polynomial's degree, and thus the degree of (3.9) in every k_i is no greater than $n-1$. Every antisymmetric function in $k_1, \ldots, k_n$ is a multiple of $\prod_{1 \le i < j \le n} (k_j - k_i)$, and since this product is of degree $n-1$ in every k_i, the expression in (3.9) is equal to $C \prod_{1 \le i < j \le n} (k_j - k_i)$, where C is a rational constant. By Lemma 3.1 $\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_q} \Delta_{k_p})$ is invertible, and therefore

$$\alpha(n; k_1, \ldots, k_n) = \left(\prod_{1 \le p < q \le n} \frac{1}{\mathrm{id} + E_{k_q} \Delta_{k_p}} \right) C \prod_{1 \le i < j \le n} (k_j - k_i).$$

We compute the constant C. We expand $\alpha(n; k_1, \ldots, k_n)$ with respect to the basis $\prod_{i=1}^{n} (k_i)_{m_i}$ and consider the (non-zero) coefficient of the basis element with maximal $(m_n, m_{n-1}, \ldots, m_1)$ in lexicographic order. We show by induction with respect to n that $(m_n, m_{n-1}, \ldots, m_1) = (n-1, n-2, \ldots, 1, 0)$ and that the coefficient is $\prod_{i=1}^{n} \frac{1}{(i-1)!}$. Assume that the assertion is true for $n-1$. A careful analysis of the definition of $\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)}$ shows that the "maximal" basis element of $\alpha(n; k_1, \ldots, k_n)$ with respect to the lexicographic order is the "maximal" basis element of

$$\sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2+1}^{k_3} \cdots \sum_{l_{n-1}=k_{n-1}+1}^{k_n} \prod_{i=1}^{n-1} \frac{(l_i)_{i-1}}{(i-1)!}.$$

The assertion follows and thus $C = \prod_{i=1}^{n} \frac{1}{(i-1)!} = \prod_{1 \le i < j \le n} \frac{1}{j-i}$. We obtain the following Theorem.

THEOREM 3.2. *The number of monotone triangles with n rows and prescribed bottom row $(k_1, k_2, \ldots, k_n)$ is equal to*

$$\left(\prod_{1 \le p < q \le n} \frac{1}{\mathrm{id} + E_{k_q} \Delta_{k_p}} \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

By the formula for the geometric series, the inverse of the operator $\mathrm{id} + E_{k_q}\Delta_{k_p}$ appearing in this formula is equal to

$$\sum_{l=0}^{\infty} (-1)^l E_{k_q}^l \Delta_{k_p}^l.$$

This follows from the proof of Lemma 3.1. However, it is also possible to give a similar formula for $\alpha(n; k_1, \ldots, k_n)$ which does not involve inverses of operators. In order to derive it, we need the following lemma.

Lemma 3.5. *Let $P(X_1, \ldots, X_n)$ be a polynomial in $(X_1, \ldots, X_n)$ over $\mathbb{C}$ which is symmetric in $(X_1, \ldots, X_n)$. Then*

$$P(E_{k_1}, \ldots, E_{k_n}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} = P(1, \ldots, 1) \cdot \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

Proof. Let $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ be with $m_i \ge 0$ for all i and $m_i \ne 0$ for at least one i. It suffices to show that

$$\sum_{\pi \in \mathcal{S}_n} \Delta_{k_1}^{m_{\pi(1)}} \Delta_{k_2}^{m_{\pi(2)}} \ldots \Delta_{k_n}^{m_{\pi(n)}} \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} = 0.$$

By the Vandermonde determinant evaluation, we have

$$\prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} = \det_{1 \le i,j \le n} \left(\binom{k_i}{j-1} \right).$$

Therefore, it suffices to show that

$$\sum_{\pi, \sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \binom{k_1}{\sigma(1) - m_{\pi(1)} - 1} \binom{k_2}{\sigma(2) - m_{\pi(2)} - 1} \cdots \binom{k_n}{\sigma(n) - m_{\pi(n)} - 1} = 0.$$

If, for fixed $\pi, \sigma \in \mathcal{S}_n$, there exists an i with $\sigma(i) - m_{\pi(i)} - 1 < 0$ then the corresponding summand vanishes. We define a sign reversing involution on the set of non-zero summands. Fix $\pi, \sigma \in \mathcal{S}_n$ such that the summand corresponding to π and σ does not vanish. Consequently, $\{\sigma(1) - m_{\pi(1)} - 1, \sigma(2) - m_{\pi(2)} - 1, \ldots, \sigma(n) - m_{\pi(n)} - 1\} \subseteq \{0, 1, \ldots, n-1\}$ and since $(m_1, \ldots, m_n) \ne (0, \ldots, 0)$ there are i, j, $1 \le i < j \le n$, with $\sigma(i) - m_{\pi(i)} - 1 = \sigma(j) - m_{\pi(j)} - 1$. Among all pairs (i, j) with this property, let (i', j') be the pair which is minimal with respect to the lexicographic order. Then the summand corresponding to $\pi \circ (i', j')$ and $\sigma \circ (i', j')$ is the negativ of the summand corresponding to π and σ. $\square$

Observe that $\prod_{1 \le p,q \le n} (1 + X_q(X_p - 1))$ is symmetric in $(X_1, \ldots, X_n)$. Thus, by Lemma 3.5,

$$\prod_{1 \le p,q \le n} \left(\mathrm{id} + E_{k_q}\Delta_{k_p}\right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} = \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

Therefore, by Theorem 3.2,

$$\begin{aligned}\alpha(n;k_1,\ldots,k_n) &= \left(\prod_{1\le p<q\le n} \frac{1}{\mathrm{id}+E_{k_q}\Delta_{k_p}}\right) \prod_{1\le i<j\le n} \frac{k_j-k_i}{j-i} \\ &= \left(\prod_{1\le p<q\le n} \frac{1}{\mathrm{id}+E_{k_q}\Delta_{k_p}}\right)\left(\prod_{1\le p,q\le n} \left(\mathrm{id}+E_{k_q}\Delta_{k_p}\right)\right) \prod_{1\le i<j\le n} \frac{k_j-k_i}{j-i} \\ &= \left(\prod_{1\le p<q\le n} \left(\mathrm{id}+E_{k_p}\Delta_{k_q}\right)\right) \prod_{1\le i<j\le n} \frac{k_j-k_i}{j-i}\end{aligned}$$

and this completes the proof of Theorem 3.1.

6. Some further projects

In this section we list some further projects around the formula given in Theorem 3.1 we plan to pursue.

(1) A natural question to ask is whether it is possible to derive the formula for the number of $n\times n$ alternating sign matrices (3.1) from Theorem 3.1, i.e. to show that

$$\left[\left(\prod_{1\le p<q\le n} \left(\mathrm{id}+E_{k_p}\Delta_{k_q}\right)\right) \prod_{1\le i<j\le n} \frac{k_j-k_i}{j-i}\right]\Bigg|_{(k_1,k_2,\ldots,k_n)=(1,2,\ldots,n)} = \prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!}$$

More generally, one could try to reprove the refined alternating sign matrix theorem [**65**], which states that the number of $n\times n$ alternating sign matrices in which the unique 1 in the top row is in the k-th column is given by

$$\frac{(k)_{n-1}(1+n-k)_{n-1}}{(n-1)!}\prod_{j=1}^{n-1}\frac{(3j-2)!}{(n+j-1)!}. \tag{3.10}$$

An analysis of the correspondence between alternating sign matrices and monotone triangles shows that $\alpha(n-1;1,2,\ldots,k-1,k+1,\ldots,n)$ is the number of $n\times n$ alternating sign matrices in which the unique 1 in the bottom row is in the k-th column and this is by symmetry equal to (3.10). This could be a consequence of an even more general theorem: computer experiments suggest that there are other $(k_1,k_2,\ldots,k_n)\in\mathbb{Z}^n$ "near" $(1,2,\ldots,n)$ for which $\alpha(n;k_1,\ldots,k_n)$ has only small prime factors. Small prime factors are an indication for a simple product formula. A similar phenomenon can be observed for some $(k_1,k_2,\ldots,k_n)\in\mathbb{Z}^n$ "near" $(1,3,\ldots,2n-1)$. It is not too hard to see that $\alpha(n;1,3,\ldots,2n-1)$ is the number of $(2n+1)\times(2n+1)$ alternating sign matrices, which are symmetric with respect to the reflection along the vertical axis. Kuperberg [**35**] showed that the number of these objects is

$$\frac{n!}{(2n)!2^n}\prod_{j=1}^{n}\frac{(6j-2)!}{(2n+2j-1)!}.$$

(2) Let $\beta(n;k_1,\dots,k_n)$ denote the number of monotone triangles with prescribed bottom row $(k_1,\dots,k_n)$ that are strictly increasing in southeast direction. With this notation, Theorem 3.1 states that

$$\alpha(n;k_1,\dots,k_n) = \left(\prod_{1\le p<q\le n} (\mathrm{id} + E_{k_p}\Delta_{k_q})\right) \beta(n;k_1,\dots,k_n). \tag{3.11}$$

It would be interesting to find a bijective proof of this formula in the following sense: if we expand the product of operators on the left hand side we obtain a sum of expressions of the form

$$E_{k_1}^{a_1} E_{k_2}^{a_2} \dots E_{k_n}^{a_n} \Delta_{k_1}^{b_1} \Delta_{k_2}^{b_2} \dots \Delta_{k_n}^{b_n} \beta(n;k_1,\dots,k_n)$$

with $a_i, b_i \in \{0,1,2,\dots\}$. We can interpret these expressions as sums and differences of cardinalities of certain subsets of monotone triangles with n rows. For instance,

$$\Delta_{k_q}\beta(n;k_1,\dots,k_n)$$

is the number of monotone triangles that are strictly increasing in southeast direction and with bottom row $(k_1,\dots,k_q+1,\dots,k_n)$ such that the $(q-1)$-st part of the $(n-1)$-st row is equal to k_q minus the number of monotone triangles that are strictly increasing in southeast direction and with bottom row $(k_1,\dots,k_n)$ such that the q-th part of the $(n-1)$-st row is equal to k_q. In order to prove (3.11), one has to show that these cardinalities add up to the number of monotone triangles. Equivalently, one could follow a similar strategy for the identity

$$\left(\prod_{1\le p<q\le n} (\mathrm{id} + E_{k_q}\Delta_{k_p})\right) \alpha(n;k_1,\dots,k_n) = \beta(n;k_1,\dots,k_n)$$

which is equivalent to Theorem 3.2.

(3) This is more a remark than another project: to prove Theorem 3.1 I have more or less carried out an analysis of the recursion (3.2). I originally started this analysis when considering a somehow reversed situation: let an (r,n) monotone trapezoid be a monotone triangle with the top $n-r$ rows cut off and bottom row $(1,2,\dots,n)$. Let $\gamma(r,n;k_1,\dots,k_{n-r+1})$ denote the number of (r,n) monotone trapzoids with prescribed top row $(k_1,\dots,k_{n-r+1})$. In particular, $\gamma(n,n;k)$ is the number of monotone triangles with n rows, bottom row $(1,2,\dots,n)$ and k as entry in the top row. In the bijection between alternating sign matrices and monotone triangles, the entry in the top row of the monotone triangle corresponds to the column of the unique 1 in the first row of the alternating sign matrix. Thus, $\gamma(n,n;k)$ must be equal to (3.10). On the other hand, we can also use (3.2) to compute $\gamma(r,n;k_1,\dots,k_{n-r+1})$: $\gamma(1,n;k_1,\dots,k_n)=1$ and

$$\gamma(r,n;k_1,\dots,k_{n-r+1}) = \sum_{(l_1,\dots,l_{n-r+2})}^{(1,k_1,\dots,k_{n-r+1},n)} \gamma(r-1,n;l_1,\dots,l_{n-r+2}).$$

With this extended definition, $\gamma(n,n;k)$ is a polynomial in k. In the following we list it for $n=1,2,\dots,6$.

$$
\begin{aligned}
\gamma(1,1;k) &= 1\\
\gamma(2,2;k) &= -1+3\,k-k^2\\
\gamma(3,3;k) &= \frac{1}{12}(48-92\,k+103\,k^2-40\,k^3+5\,k^4)\\
\gamma(4,4;k) &= \frac{1}{72}(-2160+5910\,k-5407\,k^2+2940\,k^3\\
&\quad -919\,k^4+150\,k^5-10\,k^6)\\
\gamma(5,5;k) &= \frac{1}{1440}(584640-1644072\,k+1970008\,k^2\\
&\quad -1211172\,k^3+456863\,k^4-111708\,k^5\\
&\quad +17462\,k^6-1608\,k^7+67\,k^8)\\
\gamma(6,6;k) &= \frac{1}{7560}(-73316880+225502200\,k\\
&\quad -284097336\,k^2+204504097\,k^3\\
&\quad -91897169\,k^4+27466950\,k^5\\
&\quad -5651016\,k^6+805518\,k^7\\
&\quad -77646\,k^8+4655\,k^9-133\,k^{10})
\end{aligned}
$$

Unfortunately, these polynomials are not equal to (3.10). (For instance, they do not factor over $\mathbb{Z}$.) They only coincide on the combinatorial range $\{1,2,\dots,n\}$ of k. However, it might still be possible to compute $\gamma(n,n;k)$ for general n.

Strikingly the degree of $\gamma(n,n;k)$ in k is $2n-2$ as the degree of (3.10). This linear growth is again unexpected because the application of (3.2) can more than double a polynomial's degree, see Section 2. However, one can use Lemma 3.2 and an extension of Lemma 3.3 to show that, more generally, the degree of $\gamma(r,n;k_1,\dots,k_{n-r+1})$ is $2r-2$ in every k_i.

(4) Finally we have started to investigate a q-version of the formula in Theorem 3.1, i.e. a weighted enumeration of monotone triangles with prescribed bottom row $(k_1,\dots,k_n)$ which reduces to our formula as q tends to 1.

CHAPTER 4

A new proof of the refined alternating sign matrix theorem

Abstract. In the early 1980s, Mills, Robbins and Rumsey conjectured, and in 1996 Zeilberger proved a simple product formula for the number of $n \times n$ alternating sign matrices with a 1 at the top of the i-th column. We give an alternative proof of this formula using our operator formula for the number of monotone triangles with prescribed bottom row. In addition, we provide the enumeration of certain 0-1-(-1) matrices generalizing alternating sign matrices.

1. Introduction

An alternating sign matrix is a square matrix of 0s, 1s and -1s for which the sum of entries in each row and in each column is 1 and the non-zero entries of each row and of each column alternate in sign. For instance,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is an alternating sign matrix. In [**41, 42**] Mills, Robbins and Rumsey conjectured that there are

$$\prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!} \tag{4.1}$$

$n \times n$ alternating sign matrices. This was first proved by Zeilberger [**64**]. Another, shorter, proof was given by Kuperberg [**34**] using the equivalence of alternating sign matrices with the six-vertex model for "square ice", which was earlier introduced in statistical mechanics. Zeilberger [**65**] then used Kuperberg's observations to prove the following refinement generalizing (4.1).

Theorem 4.1. *The number of $n \times n$ alternating sign matrices where the unique 1 in the first row is at the top of the i-th column is*

$$\frac{(i)_{n-1}(1+n-i)_{n-1}}{(n-1)!} \prod_{j=1}^{n-1} \frac{(3j-2)!}{(n+j-1)!}. \tag{4.2}$$

In this formula, $(a)_n = \prod_{i=0}^{n}(a+i)$.

The task of this paper is to give an alternative proof of Theorem 4.1. As a byproduct, we obtain the enumeration of certain objects generalizing alternating sign matrices.

Theorem 4.2. *The number of $n \times k$ matrices of 0s, 1s and -1s for which the non-zero entries of each row and each column alternate in sign and the sum in each row and each column is 1, except for the columns $n, n+1, \dots, k-1$, where we have sum 0 and the first non-zero element is a 1, is*

$$\prod_{j=1}^{n-1} \frac{(3j-2)!}{(n+j-1)!} \sum_{i=1}^{n} \frac{(i)_{n-1}(1+n-i)_{n-1}}{(n-1)!} \binom{i+k-n-1}{i-1}.$$

Our proofs are based on the formula in Theorem 1 (resp. Theorem 3.1) of [**17**, resp. Chapter 3] for the number of monotone triangles with given bottom row $(k_1, k_2, \dots, k_n)$, which we have recently derived. Monotone triangles with bottom row $(1, 2, \dots, n)$ are in bijection with $n \times n$ alternating sign matrices. The monotone triangle corresponding to a given alternating sign matrix can be obtained as follows: Replace every entry in the matrix by the sum of elements in the same column above, the entry itself included. In our running example we have

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Row by row we record the columns that contain a 1 and obtain the following triangular array.

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

This is the monotone triangle corresponding to the alternating sign matrix above. Observe that it is weakly increasing in northeast direction and in southeast direction. Moreover, it is strictly increasing along rows. In general, a monotone triangle with n rows is a triangular array $(a_{i,j})_{1 \le j \le i \le n}$ of integers such that $a_{i,j} \le a_{i-1,j} \le a_{i,j+1}$ and $a_{i,j} < a_{i,j+1}$ for all i, j. It is not too hard to see that monotone triangles with n rows and $a_{n,j} = j$ are in bijection with $n \times n$ alternating sign matrices. Moreover, monotone triangles with $n-1$ rows and bottom row $(1, 2, \ldots, i-1, i+1, \ldots, n)$ are in bijection with $n \times n$ alternating sign matrices with a 1 at the bottom (or equivalently the top) of the i-th column. In [**17**, resp. Chapter 3] we gave the following operator formula for the number of monotone triangles with prescribed bottom row $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$.

Theorem 4.3 ([**17**], Theorem 1). *The number of monotone triangles with n rows and prescribed bottom row $(k_1, k_2, \ldots, k_n)$ is given by*

$$\left(\prod_{1 \le p < q \le n} \left(\mathrm{id} + E_{k_p} \Delta_{k_q} \right) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},$$

where E_x denotes the shift operator, defined by $E_x \, p(x) = p(x+1)$, and $\Delta_x := E_x - \mathrm{id}$ denotes the difference operator. (In this formula, the product of operators is understood as the composition.)

Thus, for instance, the number of monotone triangles with bottom row (k_1, k_2, k_3) is

$$\begin{aligned} &(\mathrm{id} + E_{k_1} \Delta_{k_2})(\mathrm{id} + E_{k_1} \Delta_{k_3})(\mathrm{id} + E_{k_2} \Delta_{k_3}) \frac{1}{2} (k_2 - k_1)(k_3 - k_1)(k_3 - k_2) \\ &\qquad = \frac{1}{2} (-3k_1 + k_1^2 + 2k_1 k_2 - k_1^2 k_2 - 2k_2^2 + k_1 k_2^2 + 3k_3 - 4k_1 k_3 \\ &\qquad\qquad + k_1^2 k_3 + 2k_2 k_3 - k_2^2 k_3 + k_3^2 - k_1 k_3^2 + k_2 k_3^2). \end{aligned}$$

We outline our proof of Theorem 4.1. To prove the theorem using the formula in Theorem 4.3 clearly means that we have to evaluate the formula at $(k_1, k_2, \ldots, k_n) = (1, 2, \ldots, i-1, i+1, \ldots, n+1)$. Let $A_{n,i}$ denote the number of $n \times n$ alternating sign matrices with a 1 at the top of the i-th column and let $\alpha(n; k_1, \ldots, k_n)$ denote the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$. Using the formula in Theorem 4.3, we extend the interpretation of $\alpha(n; k_1, \ldots, k_n)$ to arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. In our proof, we first give a formula for

$\alpha(n;1,2,\ldots,n-1,k)$ in terms of $A_{n,i}$. Next we show that $\alpha(n;1,2,\ldots,n-1,k)$ is an even polynomial in k if n is odd and an odd polynomial if n is even. These two facts will then imply that $(A_{n,i})_{1\le i\le n}$ is an eigenvector with respect to the eigenvalue 1 of a certain matrix with binomial coefficients as entries. Finally we see that this determines $A_{n,i}$ up to a constant, which can easily be computed by induction with respect to n.

We believe that this second approach to prove the refined alternating sign matrix theorem not only provides us with a better understanding of known theorems, but will also enable us to obtain new results in the field of plane partition and alternating sign matrix enumeration in the future. A general strategy might be to derive analogous multivariate operators formulas for other (triangular) arrays of integers, which correspond to certain classes of plane partitions and alternating sign matrices and which simplify to nice product formulas if we specialize the parameters in the right way. Hopefully these operator formulas can then also be used to derive these product formulas.

2. A formula for $\alpha(n;1,2,\ldots,n-1,k)$

We start by stating a fundamental recursion for $\alpha(n;k_1,\ldots,k_n)$. If we delete the last row of a monotone triangle with bottom row $(k_1,k_2,\ldots,k_n)$, we obtain a monotone triangle with $n-1$ rows and bottom row, say, $(l_1,l_2,\ldots,l_{n-1})$. By the definition of a monotone triangle, we have $k_1\le l_1\le k_2\le l_2\le\ldots\le k_{n-1}\le l_{n-1}\le k_n$ and $l_i\ne l_{i+1}$. Thus

$$\alpha(n;k_1,\ldots,k_n)=\sum_{\substack{(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1},\\ k_1\le l_1\le k_2\le\cdots\le k_{n-1}\le l_{n-1}\le k_n, l_i\ne l_{i+1}}}\alpha(n-1;l_1,\ldots,l_{n-1}). \tag{4.3}$$

In the following lemma, we explain the action of operators, which are symmetric polynomials in $E_{k_1},E_{k_2},\ldots,E_{k_n}$, on $\alpha(n;k_1,\ldots,k_n)$. It will be used twice in our proof of Theorem 4.1.

LEMMA 4.1. *Let $P(X_1,\ldots,X_n)$ be a symmetric polynomial in $(X_1,\ldots,X_n)$ over $\mathbb{C}$. Then*

$$P(E_{k_1},\ldots,E_{k_n})\alpha(n;k_1,\ldots,k_n)=P(1,1,\ldots,1)\cdot\alpha(n;k_1,\ldots,k_n).$$

Proof. By Theorem 4.3 and the fact that shift operators with respect to different variables commute, it suffices to show that

$$P(E_{k_1},\ldots,E_{k_n})\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=P(1,1,\ldots,1)\cdot\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}.$$

Let $(m_1,\ldots,m_n)\in\mathbb{Z}^n$ be with $m_i\ge 0$ for all i and $m_i\ne 0$ for at least one i. It suffices to show that

$$\sum_{\pi\in\mathcal{S}_n}\Delta_{k_1}^{m_{\pi(1)}}\Delta_{k_2}^{m_{\pi(2)}}\ldots\Delta_{k_n}^{m_{\pi(n)}}\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=0.$$

By the Vandermonde determinant evaluation, we have

$$\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=\det_{1\le i,j\le n}\left(\binom{k_i}{j-1}\right).$$

Therefore, it suffices to show that

$$\sum_{\pi,\sigma\in\mathcal{S}_n}\operatorname{sgn}\sigma\binom{k_1}{\sigma(1)-m_{\pi(1)}-1}\binom{k_2}{\sigma(2)-m_{\pi(2)}-1}\cdots\binom{k_n}{\sigma(n)-m_{\pi(n)}-1}=0.$$

If, for fixed $\pi,\sigma\in\mathcal{S}_n$, there exists an i with $\sigma(i)-m_{\pi(i)}-1<0$ then the corresponding summand vanishes. We define a sign reversing involution on the set of non-zero summands. Fix $\pi,\sigma\in\mathcal{S}_n$ such that the summand corresponding to π and σ does not vanish. Consequently, $\{\sigma(1)-m_{\pi(1)}-1,\sigma(2)-m_{\pi(2)}-1,\dots,\sigma(n)-m_{\pi(n)}-1\}\subseteq\{0,1,\dots,n-1\}$ and since $(m_1,\dots,m_n)\neq(0,\dots,0)$, there are i,j, $1\le i<j\le n$, with $\sigma(i)-m_{\pi(i)}-1=\sigma(j)-m_{\pi(j)}-1$. Among all pairs (i,j) with this property, let (i',j') be the pair, which is minimal with respect to the lexicographic order. Then the summand corresponding to $\pi\circ(i',j')$ and $\sigma\circ(i',j')$ is the negativ of the summand corresponding to π and σ. $\square$

Let

$$e_p(X_1,\dots,X_n)=\sum_{1\le i_1<i_2<\dots i_p\le n}X_{i_1}X_{i_2}\dots X_{i_p}$$

denote the p-th elementary symmetric function. Lemma 4.1 will be used to deduce a formula for

$$e_{p-j}(E_{k_1},E_{k_2},\dots,E_{k_{n-1}})\alpha(n;k_1,\dots,k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n+j)} \tag{4.4}$$

in terms of $A_{n,i}$ if $0\le j\le p\le n-1$ (Lemma 4.3). If we specialize $p=j$ in this identity we obtain the desired formula for $\alpha(n;1,2,\dots,n-1,k)$, which we have mentioned in the outline of our proof. The formula for (4.4) will be shown by induction with respect to j. In the following lemma, we deal with the initial case of the induction.

Lemma 4.2. *Let $0\le p\le n-1$. Then we have*

$$e_p(E_{k_1},E_{k_2},\dots,E_{k_{n-1}})\,\alpha(n;k_1,\dots,k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n)}=\sum_{i=1}^{n}\binom{n-i}{p}A_{n,i}.$$

Proof. First observe that for $1\le i_1<i_2<\dots i_p\le n-1$ we have

$$\begin{aligned}&E_{k_{i_1}}E_{k_{i_2}}\dots E_{k_{i_p}}\alpha(n;k_1,\dots,k_n)\Big|_{(k_1,\dots,k_n)=(1,2,\dots,n)}\\&=\alpha(n;1,2,\dots,i_1-1,i_1+1,i_1+1,\dots,i_2-1,i_2+1,i_2+1,\dots,i_p-1,i_p+1,i_p+1,\dots,n)\\&\qquad=\sum_{\substack{1\le j_1\le 2\le j_2\le\dots\le i_1-1\le j_{i_1-1}\le i_1+1\\ j_1<j_2<\dots<j_{i_1-1}}}\alpha(n-1;j_1,j_2,\dots,j_{i_1-1},i_1+1,i_2+1,\dots,n)\\&\qquad\qquad=\alpha(n;1,2,\dots,i_1-1,i_1+1,i_1+1,i_1+2,i_1+3,\dots,n),\end{aligned}$$

where the second equality follow from (4.3). In particular, we see that

$$E_{k_{i_1}}E_{k_{i_2}}\dots E_{k_{i_p}}\alpha(n;k_1,\dots,k_n)\Big|_{(k_1,\dots,k_n)=(1,2,\dots,n)}$$

does not depend on $i_2,\dots,i_p$. Consequently, the left-hand-side in the statement of the lemma is equal to

$$\sum_{j=1}^{n-1}\binom{n-1-j}{p-1}\alpha(n;1,2,\dots,j-1,j+1,j+1,j+2,\dots,n). \tag{4.5}$$

Next observe that by (4.3)

$$\alpha(n;1,2,\dots,j-1,j+1,j+1,j+2,\dots,n) = \sum_{i=1}^{j} \alpha(n-1;1,2,\dots,i-1,i+1,\dots,n) = \sum_{i=1}^{j} A_{n,i}.$$

Thus, (4.5) is equal to

$$\sum_{j=1}^{n-1} \binom{n-1-j}{p-1} \sum_{i=1}^{j} A_{n,i} = \sum_{i=1}^{n}\sum_{j=i}^{n} \binom{n-1-j}{p-1} A_{n,i} - \binom{-1}{p-1} \sum_{i=1}^{n} A_{n,i}.$$

We complete the proof by using the following summation formula

$$\sum_{j=a}^{b} \binom{x+j}{n} = \sum_{j=a}^{b} \left(\binom{x+j+1}{n+1} - \binom{x+j}{n+1} \right) = \binom{x+b+1}{n+1} - \binom{x+a}{n+1}.$$

□

LEMMA 4.3. *Let $0 \le j \le p \le n-1$. Then*

$$e_{p-j}(E_{k_1}, E_{k_2}, \dots, E_{k_{n-1}}) \alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n+j)} = (-1)^j \sum_{i=1}^{n} A_{n,i} \left(\binom{n-i}{p} + \sum_{l=0}^{j-1} \binom{n}{p-l} \binom{i+l-1}{l} (-1)^{l-1} \right)$$

For $p = j$ this simplifies to

$$\alpha(n;1,\dots,n-1,n+j) = \sum_{i=1}^{n} A_{n,i} \binom{i+j-1}{i-1}.$$

Proof. First we show how the first formula implies the second. For this purpose, we have to consider

$$(-1)^j \left(\binom{n-i}{j} + \sum_{l=0}^{j-1} \binom{n}{j-l} \binom{i+l-1}{l} (-1)^{l-1} \right).$$

Using $\binom{i+l-1}{l} = \binom{-i}{l}(-1)^l$, we see that this is equal to

$$(-1)^j \left(\binom{n-i}{j} + (-1)^j \binom{i+j-1}{j} - \sum_{l=0}^{j} \binom{n}{j-l} \binom{-i}{l} \right).$$

We apply the Chu-Vandermonde summation [**27**, p. 169, (5.22)], in order to see that this is simplifies to $\binom{i+j-1}{j} = \binom{i+j-1}{i-1}$.

Now we consider the first formula. By Lemma 4.1, we have

$$e_{p-j}(E_{k_1}, \dots, E_{k_n}) \alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n+j)} = \binom{n}{p-j} \alpha(n;1,2,\dots,n-1,n+j).$$

On the other hand, we have

$$\begin{aligned}
&e_{p-j}(E_{k_1}, \dots, E_{k_n})\alpha(n; k_1, \dots, k_n)|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n+j)} \\
&\qquad = e_{p-j-1}(E_{k_1}, \dots, E_{k_{n-1}})\alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n+j+1)} \\
&\qquad\qquad + e_{p-j}(E_{k_1}, \dots, E_{k_{n-1}})\alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n+j)}.
\end{aligned}$$

This implies the recursion

$$\begin{aligned}
&e_{p-j-1}(E_{k_1}, \dots, E_{k_{n-1}})\alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n+j+1)} \\
&\qquad = \binom{n}{p-j}\alpha(n; 1, \dots, n-1, n+j) \\
&\qquad\qquad - e_{p-j}(E_{k_1}, \dots, E_{k_{n-1}})\alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n+j)}.
\end{aligned}$$

Now we can prove the first formula in the lemma by induction with respect to j. The case $j = 0$ was dealt with in Lemma 4.2. $\square$

By Theorem 4.3, $\alpha(n; 1, 2, \dots, n-1, n+j)$ is a polynomial in j of degree no greater than $n-1$. By Lemma 4.3, it coincides with $\sum_{i=1}^{n} A_{n,i}\binom{i+j-1}{i-1}$ for $j = 0, 1, \dots, n-1$ and, since $\sum_{i=1}^{n} A_{n,i}\binom{i+j-1}{i-1}$ is a polynomial in j of degree no greater than $n-1$ as well, the two polynomials must be equal. This constitutes the following.

Lemma 4.4. *Let $n \geq 1$ and $k \in \mathbb{Z}$. Then we have*

$$\alpha(n; 1, 2, \dots, n-1, k) = \sum_{i=1}^{n} A_{n,i}\binom{i+k-n-1}{i-1}.$$

3. The symmetry of $k \to \alpha(n; 1, 2, \dots, n-1, k)$

In the following lemma we prove a transformation formula for $\alpha(n; k_1, \dots, k_n)$, which implies as a special case that $k \to \alpha(n; 1, 2, \dots, n-1, k)$ is even if n is odd and odd otherwise.

Lemma 4.5. *Let $n \geq 1$. Then we have*

$$\alpha(n; k_1, \dots, k_n) = (-1)^{n-1}\alpha(n; k_2, \dots, k_n, k_1 - n).$$

Proof. By Theorem 4.3

$$\begin{aligned}
&(-1)^{n-1}\alpha(n; k_2, \dots, k_n, k_1 - n) \\
&\quad = (-1)^{n-1} \prod_{2 \leq p < q \leq n} (\mathrm{id} + E_{k_p}\Delta_{k_q}) \prod_{p=2}^{n} (\mathrm{id} + E_{k_p}\Delta_{k_1}) \prod_{2 \leq i < j \leq n} \frac{k_j - k_i}{j - i} \prod_{i=2}^{n} \frac{k_1 - k_i - n}{i - 1} \\
&\quad = \prod_{2 \leq p < q \leq n} (\mathrm{id} + E_{k_p}\Delta_{k_q}) \prod_{p=2}^{n} (\mathrm{id} + E_{k_p}\Delta_{k_1}) \prod_{2 \leq i < j \leq n} \frac{k_j - k_i}{j - i} \prod_{i=2}^{n} \frac{k_i + n - k_1}{i - 1} \\
&\quad = \prod_{2 \leq p < q \leq n} (\mathrm{id} + E_{k_p}\Delta_{k_q}) \prod_{p=2}^{n} (\mathrm{id} + E_{k_p}\Delta_{k_1}) E_{k_1}^{-n} \prod_{1 \leq i < j \leq n} \frac{k_j - k_i}{j - i}.
\end{aligned}$$

Thus, we have to show that

$$\prod_{2\le p<q\le n}(\mathrm{id}+E_{k_p}\Delta_{k_q})\left(\prod_{q=2}^{n}\left(\mathrm{id}+E_{k_1}\Delta_{k_q}\right)-E_{k_1}^{-n}\prod_{q=2}^{n}(\mathrm{id}+E_{k_q}\Delta_{k_1})\right)\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i} \quad = \quad 0.$$

Clearly, it suffices to prove that

$$\left(E_{k_1}^{n}\prod_{q=2}^{n}\left(\mathrm{id}+E_{k_1}\Delta_{k_q}\right)-\prod_{q=2}^{n}(\mathrm{id}+E_{k_q}\Delta_{k_1})\right)\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=0.$$

We replace Δ_{k_i} by X_i and, accordingly, E_{k_i} by (X_i+1) in the operator in this expression and obtain

$$(X_1+1)^n\prod_{q=2}^{n}(1+(X_1+1)X_q)-\prod_{q=2}^{n}(1+(X_q+1)X_1). \tag{4.6}$$

By the proof of Lemma 4.1, the assertion follows if we show that this polynomial is in the ideal, which is generated by the symmetric polynomials in $X_1, X_2, \dots, X_n$ without constant term. Observe that (4.6) is equal to

$$\begin{aligned}
&\sum_{j=0}^{n-1}(X_1+1)^{n+j}e_j(X_2,\dots,X_n)-\sum_{j=0}^{n-1}X_1^j\,e_j(X_2+1,X_3+1,\dots,X_n+1)\\
&\qquad=\sum_{j=0}^{n-1}(X_1+1)^{n+j}e_j(X_2,\dots,X_n)-\sum_{j=0}^{n-1}X_1^j\sum_{i=0}^{j}\binom{n-i-1}{j-i}e_i(X_2,\dots,X_n)\\
&\qquad\qquad=\sum_{j=0}^{n-1}\left((X_1+1)^{n+j}-X_1^j(X_1+1)^{n-j-1}\right)e_j(X_2,\dots,X_n).
\end{aligned} \tag{4.7}$$

We recursively define a sequence $(q_j(X))_{j\ge 0}$ of Laurent polynomials. Let $q_0(X)=0$ and

$$q_{j+1}(X)=(X+1)^{2j+1}-X^j-q_j(X)-q_j(X)\,X^{-1}. \tag{4.8}$$

We want to show that this is in fact a sequence of polynomials having a zero at $X=0$. For this purpose, we consider

$$Q(X,Y):=\sum_{j\ge 0}q_j(X)Y^j.$$

Using (4.8) and the initial condition, we obtain the following

$$Q(X,Y)=\frac{X\,Y}{(1-X\,Y)(1-(X+1)^2Y)},$$

which immediately implies the assertion. We set

$$p_j(X)=q_j(X)(X+1)^{n-j}X^{-1}$$

and observe that, for all j with $j\le n$, $p_j(X)$ is a polynomial in X. The recursion (4.8) clearly implies

$$p_{j+1}(X)X=(X+1)^{n+j}-X^j(X+1)^{n-j-1}-p_j(X).$$

Thus, (4.7) is equal to

$$\sum_{j=0}^{n-1} \left(p_j(X_1) + p_{j+1}(X_1)X_1\right) e_j(X_2, \ldots, X_n) = \sum_{j=0}^{n} p_j(X_1) e_j(X_1, \ldots, X_n).$$

Since $p_0(X) = 0$, this expression is in the ideal generated by the symmetric polynomials in $(X_1, \ldots, X_n)$ without constant term and the assertion of the lemma is proved. □

If $(a_{i,j})_{1 \le j \le i \le n}$ is a monotone triangle with bottom row $(k_1, \ldots, k_n)$ then $(-a_{i,n+1-j})_{1 \le j \le i \le n}$ is a monotone triangle with bottom row $(-k_n, \ldots, -k_1)$. This implies the following identity.

$$\alpha(n; k_1, k_2, \ldots, k_n) = \alpha(n; -k_n, -k_{n-1}, \ldots, -k_1) \tag{4.9}$$

Similarly, it is easy to see that

$$\alpha(n; k_1, k_2, \ldots, k_n) = \alpha(n; k_1 + c, k_2 + c, \ldots, k_n + c) \tag{4.10}$$

for every integer constant c. Therefore

$$\begin{aligned} \alpha(n; 1, 2, \ldots, n-1, k) &= \alpha(n; -k, -n+1, -n+2, \ldots, -1) \\ &= (-1)^{n-1} \alpha(n; -n+1, -n+2, \ldots, -1, -k-n) = (-1)^{n-1} \alpha(n; 1, 2, \ldots, n-1, -k), \end{aligned}$$

where the first equality follows from (4.9), the second from Lemma 4.5 and the third from (4.10) with $c = n$. This, together with Lemma 4.4, implies the following identity

$$\sum_{i=1}^{n} A_{n,i} \binom{i+k-n-1}{i-1} = (-1)^{n-1} \sum_{i=1}^{n} A_{n,i} \binom{i-k-n-1}{i-1}$$

for all integers k. In this identity, we replace $\binom{i-k-n-1}{i-1}$ by $(-1)^{i-1}\binom{k+n-1}{i-1}$ and $\binom{i+k-n-1}{i-1}$ by

$$\sum_{j=1}^{n} \binom{i-2n}{i-j} \binom{k+n-1}{j-1},$$

which is possible by the Chu-Vandermonde summation [**27**, p. 169, (5.22)] if $i \le n$. We interchange the role of i and j on the left-hand-side and obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{n,j} \binom{j-2n}{j-i} \binom{k+n-1}{i-1} = \sum_{i=1}^{n} A_{n,i} (-1)^{n+i} \binom{k+n-1}{i-1}$$

for all integers k. Since $(\binom{k+n-1}{i-1})_{i \ge 1}$ is a basis of $\mathbb{C}[k]$ as a vectorspace over $\mathbb{C}$, this implies

$$\sum_{j=1}^{n} A_{n,j} \binom{j-2n}{j-i} = A_{n,i} (-1)^{n+i}$$

for $i = 1, 2, \ldots, n$. We replace $\binom{j-2n}{j-i}$ by $(-1)^{j-i}\binom{2n-i-1}{j-i}$. Moreover we replace j by $n+1-j$ and use the fact that $A_{n,j} = A_{n,n+1-j}$ in order to obtain

$$\sum_{j=1}^{n} A_{n,j} (-1)^{j+1} \binom{2n-i-1}{n-i-j+1} = A_{n,i}. \tag{4.11}$$

Phrased differently, $(A_{n,1}, A_{n,2}, \ldots, A_{n,n})$ is an eigenvector of $((-1)^{j+1}\binom{2n-i-1}{n-i-j+1})_{1\le i,j\le n}$ with respect to the eigenvalue 1. In the following section we see that this determines $(A_{n,1}, A_{n,2}, \ldots, A_{n,n})$ up to a multiplicative constant, which we are able to compute.

4. The eigenspace of $\left((-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\right)_{1\le i,j\le n}$ with respect to 1

Since $(A_{n,1}, \ldots, A_{n,n})$ is an eigenvector of $\left((-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\right)_{1\le i,j\le n}$, it suffices to show that the dimension of the eigenspace with respect to 1 is no greater than 1. Thus we have to show that the rank of

$$\left((-1)^{j}\binom{2n-i-1}{n-i-j+1} + \delta_{i,j}\right)_{1\le i,j\le n}$$

is at least $n-1$. It suffices to show that

$$\det_{2\le i,j\le n}\left((-1)^{j}\binom{2n-i-1}{n-i-j+1} + \delta_{i,j}\right) \neq 0.$$

We shift i and j by one in this determinant and obtain

$$\det_{1\le i,j\le n-1}\left((-1)^{j+1}\binom{2n-i-2}{n-i-j-1} + \delta_{i,j}\right). \tag{4.12}$$

Let B_n denote the matrix underlying the determinant. We define $R_n = \left(\binom{n+j-i-1}{j-i}\right)_{1\le i,j\le n-1}$. Observe that $R_n^{-1} = \left((-1)^{i+j}\binom{n}{j-i}\right)_{1\le i,j\le n-1}$. Moreover, we have $R_n^{-1}B_nR_n = B_n^* + I_{n-1}$, where I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix and B_n^* is the $(n-1)\times(n-1)$ matrix with $\binom{i+j}{j-1}$ as entry in the i-th row and j-th column except for the last row, where we have all zeros. (This transformation is due to Mills, Robbins and Rumsey [**41**].) Thus the determinant in (4.12) is equal to

$$\det(B_n^* + I_{n-1}) = \det_{1\le i,j\le n-2}\left(\binom{i+j}{j-1} + I_{n-2}\right),$$

where we have expanded $B_n^* + I_{n-1}$ with respect to the last row. Andrews [**3**] has shown that this determinant gives the number of desecending plane partitions with no part greater than $n-1$ and, therefore, the determinant does not vanish. Recently, Krattenthaler [**33**] showed that descending plane partitions can be geometrically realized as cyclically symmetric rhombus tilings of a certain hexagon of which a centrally located equilateral triangle of side length 2 has been removed.

In order to complete our proof, we have to show that $(B_{n,i})_{1\le i\le n}$ with

$$B_{n,i} = \frac{(i)_{n-1}(1+n-i)_{n-1}}{(n-1)!}\prod_{k=1}^{n-1}\frac{(3k-2)!}{(n+k-1)!}$$

is an eigenvector of $\left((-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\right)_{1\le i,j\le n}$ with respect to the eigenwert 1, i.e. we have to show that

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\frac{(j)_{n-1}(1+n-j)_{n-1}}{(n-1)!}\prod_{k=1}^{n-1}\frac{(3k-2)!}{(n+k-1)!}= \frac{(i)_{n-1}(1+n-i)_{n-1}}{(n-1)!}\prod_{k=1}^{n-1}\frac{(3k-2)!}{(n+k-1)!}.$$

This is equivalent to showing that

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{2n-i-1}{n-j-i+1}\binom{n+j-2}{n-1}\binom{2n-j-1}{n-1}=\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}. \quad (4.13)$$

Observe that the left hand side of this identity is equal to

$$\binom{2n-i-1}{n-1}\binom{n+i-2}{n-1}\binom{n-1}{i-1}^{-1}\sum_{j=1}^{n}(-1)^{j+1}\binom{2n-j-1}{n-j-i+1}\binom{n-1}{j-1}=$$
$$\binom{2n-i-1}{n-1}\binom{n+i-2}{n-1}\binom{n-1}{i-1}^{-1}\sum_{j=1}^{n}(-1)^{n+i}\binom{-n-i+1}{n-j-i+1}\binom{n-1}{j-1}=$$
$$\binom{2n-i-1}{n-1}\binom{n+i-2}{n-1}\binom{n-1}{i-1}^{-1}(-1)^{n+i}\binom{-i}{n-i}=\binom{2n-i-1}{n-1}\binom{n+i-2}{n-1},$$

where the first and third equality follows from $\binom{a}{b}=(-1)^b\binom{b-a-1}{b}$ and the second equality follows from the Chu-Vandermonde identity; see [**27**, p. 169, (5.26)]. Consequently, $A_{n,i}=C_n\cdot B_{n,i}$ for $C_n\in\mathbb{Q}$. It is easy to check that $C_1=1$. Observe that $\sum_{i=1}^{n-1}A_{n-1,i}=A_{n,1}$, since $(n-1)\times(n-1)$ alternating sign matrices are bijectively related to $n\times n$ alternating sign matrices with a 1 at the top of the first column. Moreover, observe that we also have $\sum_{i=1}^{n-1}B_{n-1,i}=B_{n,1}$, since

$$\sum_{i=1}^{n-1}B_{n-1,i}=\prod_{k=1}^{n-2}\frac{(3k-2)!}{(n+k-2)!}\sum_{i=1}^{n-1}\frac{(i)_{n-2}(n-i)_{n-2}}{(n-2)!}=$$
$$\prod_{k=1}^{n-2}\frac{(3k-2)!}{(n+k-2)!}(n-2)!\sum_{i=1}^{n-1}\binom{i+n-3}{n-2}\binom{2n-i-3}{n-2}=$$
$$\prod_{k=1}^{n-2}\frac{(3k-2)!}{(n+k-2)!}(n-2)!\binom{3n-5}{2n-3}=B_{n,1},$$

where the second equality follows from an identity, which is equivalent to the Chu-Vandermonde identity; see [**27**, p. 169, (5.26)]. Therefore, by induction with respect to n, we have $C_n=1$ for all n. This completes our proof of the refined alternating sign matrix theorem. Theorem 4.2 follows if we combine Theorem 4.1 and Lemma 4.4, since a careful analysis of the bijection between alternating sign matrices and monotone triangles shows that $\alpha(n; 1, 2, \ldots, n-1, k)$ is the number of objects described in the statement of the theorem.

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