Proof of the DASASM-conjecture

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Outline

- Alternating sign matrices (ASMs)
- Origin of ASMs: λ-determinant and square ice
- Symmetry classes of ASMs; Last case: DASASMs
- Proof of the DASASM-conjecture
- Proof of Stroganov's refined DASASM-conjecture
- Open problem

ASM=Alternating Sign Matrix

Square matrix with entries in $\{0, 1, -1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

ASM counting is fascinating because it pushes the limits of counting tools!

Origin of ASMs:

λ-determinant and square ice

The origin of ASMs: λ–determinant and square ice

The Desnanot–Jacobi identity:

$$\det(M)\det(M_{1,n}^{1,n}) = \det\begin{pmatrix} \det(M_1^1) & \det(M_1^n) \\ \det(M_n^1) & \det(M_n^n) \end{pmatrix}$$

Notation: For a matrix M, let $M_{i_1,\ldots,i_m}^{j_1,\ldots,j_n}$ denote the matrix that remains when the rows $i_1,\ldots,i_m$ and the columns $j_1,\ldots,j_n$ of M are deleted.

Charles L. Dodgson (Lewis Carroll) used this to devise an algorithm for calculating determinants that required only 2×2 determinants. (Condensation of determinants, 1866)

3×3 determinants

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \frac{1}{a_{2,2}}$$

$$\times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \end{pmatrix}$$

4×4 determinants

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} = \frac{1}{\det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix}}$$
$$\times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \end{pmatrix}$$

3×3 determinants are expressible in terms of 2×2 determinants...and so are 4×4 determinants!

Robbins and Rumsey in the 1980s: What happens if we generalize the definition of a 2×2 determinant to

$$\det_\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21}$$

and, furthermore, use the previous observations to generalize the $n \times n$ determinant?

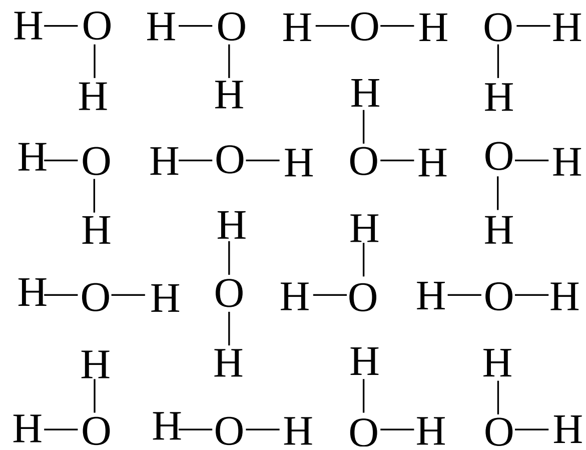
Theorem (Robbins and Rumsey). Let M be an $n \times n$ matrix with entries $a_{i,j}$, $\mathcal{A}_n$ the set of $n \times n$ alternating sign matrices, $\mathcal{I}(B)$ the inversion number of B and $\mathcal{N}(B)$ the number of -1's in B, then

$$\det_\lambda(M) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}.$$

ASMs appear independently in statistical physics

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

ASM



Square ice

Symmetry classes of ASMs

Last case: DASASMs

ASM-Theorem (Zeilberger, 1995)

The number of $n \times n$ ASMs is $\dfrac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!} = \prod_{j=0}^{n-1} \dfrac{\binom{3j+1}{j}}{\binom{2j}{j}}$.

In the 1980s, Richard Stanley suggested the systematical study of symmetry classes of ASMs which led David Robbins to conjecture nice product formulas for many symmetry classes of ASMs.

Symmetry classes of ASMs

- Vertically symmetric ASMs: $a_{i,j} = a_{i,n+1-j}$
  n odd: Kuperberg (2002)

- Half-turn symmetric ASMs: $a_{i,j} = a_{n+1-i,n+1-j}$
  n even: Kuperberg (2002)
  n odd: Razumov/Stroganov (2005)

- Diagonally symmetric ASMs: $a_{i,j} = a_{j,i}$
  no formula ?

- Quarter-turn symmetric ASMs: $a_{i,j} = a_{j,n+1-i}$
  n even: Kuperberg (2002)
  n odd: Razumov/Stroganov (2005)

Symmetry classes of ASMs (Part 2)

- Horizontally and vertically symmetric ASMs: $a_{i,j} = a_{i,n+1-j} = a_{n+1-i,j}$
  n odd: Okada (2004)

- Diagonally and antidiagonally symmetric ASMs: $a_{i,j} = a_{j,i} = a_{n+1-j,n+1-i}$
  n odd: Conjecture by Robbins (1980s)

- All symmetries: $a_{i,j} = a_{j,i} = a_{i,n+1-j}$
  no formula ?

Half of the cases were dealt with in a famous Annals paper by Kuperberg (2002):

"Symmetry classes of alternating sign matrices under one roof"

Diagonally and antidiagonally symmetric ASMs=DASASMs

Example:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

d(n) = number of $n \times n$ DASASMs

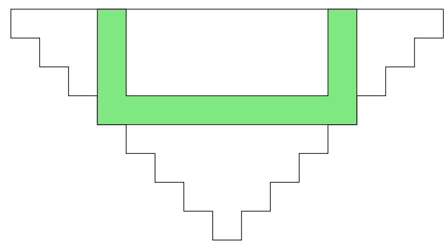
Conjecture (Robbins, 1980s): d$(2n+1) = \prod_{i=1}^{n} \frac{\binom{3i}{i}}{\binom{2i-1}{i}}$

Sequence starts as follows: $1, 3, 15, 126, 1782, 42471, 1706562 \ldots$

(Sketch of) Proof of the DASASM-conjecture

DASASM-triangles

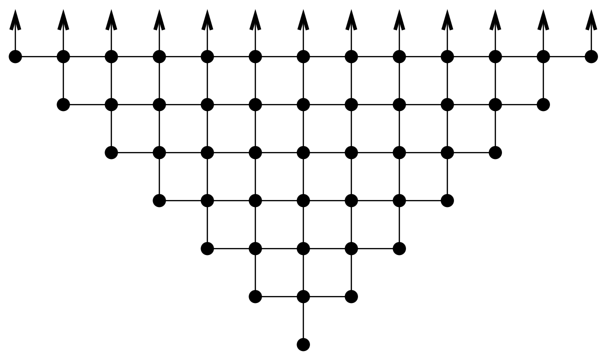
- DASASM $\Rightarrow$ fundamental triangle (DASASM-triangle)



$$
\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & \\
 & & & 0 & 1 & -1 & 0 & 1 & 0 & -1 & & & \\
 & & & & -1 & 0 & 1 & -1 & 0 & & & & \\
 & & & & & 1 & 0 & 0 & & & & & \\
 & & & & & & -1 & & & & & &
\end{array}
$$

Translation into six-vertex model:

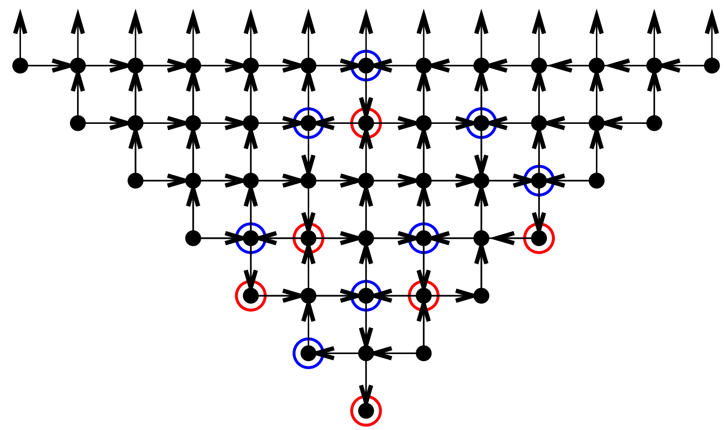
- DASASM-triangle $\Rightarrow$ orientations of triangular graph



Orient edges such that
- all degree 4 vertices are "balanced", and
- all top edges are oriented upward.

1-1 correspondence with fundamental domains of DASASMs

Example



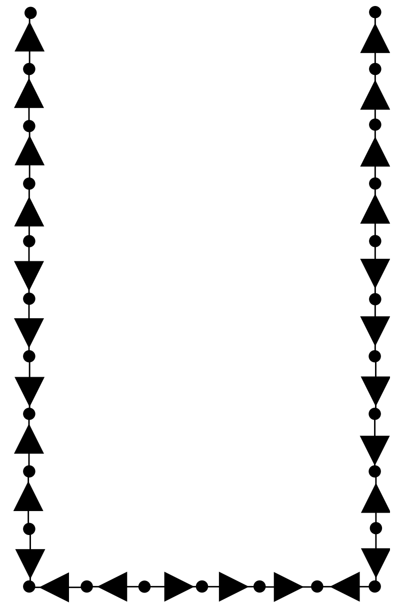
$$\begin{array}{c}
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \\
0 \quad 1 \quad -1 \quad 0 \quad 1 \quad 0 \quad -1 \\
-1 \quad 0 \quad 1 \quad -1 \quad 0 \\
1 \quad 0 \quad 0 \\
-1
\end{array}$$

Dictionary

degree 4 vertices	1	-1	0	0	0	0
left boundary	1	-1		0	0	
right boundary	1	-1	0			0
bottom vertex	1	-1				

Why does this work?

$$\begin{array}{ccccccc}
0 & & & & & & 0 \\
0 & & & & & & 0 \\
0 & & & & & & 0 \\
1 & & & & & & 1 \\
0 & & & & & & 0 \\
0 & & & & & & 0 \\
-1 & & & & & & 0 \\
0 & & & & & & -1 \\
1 & & & & & & 1 \\
0 & 0 & -1 & 0 & 0 & 1 & -1
\end{array}$$



- Along straight lines, change orientation iff you encounter ± 1.
- As for turns, change orientation iff you encounter 0.

Weighted enumeration

- Principle: sometimes it is easier to prove a generalization!

- Assign to each vertex v a weight $\mathsf{W}(v)$.

- Weight $\mathsf{W}(C)$ of a configuration (=orientation of the triangular graph $\mathcal{T}_n$):

$$\mathsf{W}(C) = \prod_{v \in C} \mathsf{W}(v)$$

- Generating function:

$$Z_n = \sum_{C \text{ admissible orientation of order } n \text{ triangular graph}} \mathsf{W}(C)$$

- Specialization of the parameters in Z_n will give the number of configurations, i.e. the number of $(2n+1) \times (2n+1)$ DASASMs.

Very strange vertex weights

The weight of a vertex depends on the orientations of the surrounding edges and the label of the vertex.

Notation: $x^{-1} = \bar{x}$ and $\sigma(x) = x - \bar{x}$; u is the label and q is a global parameter.

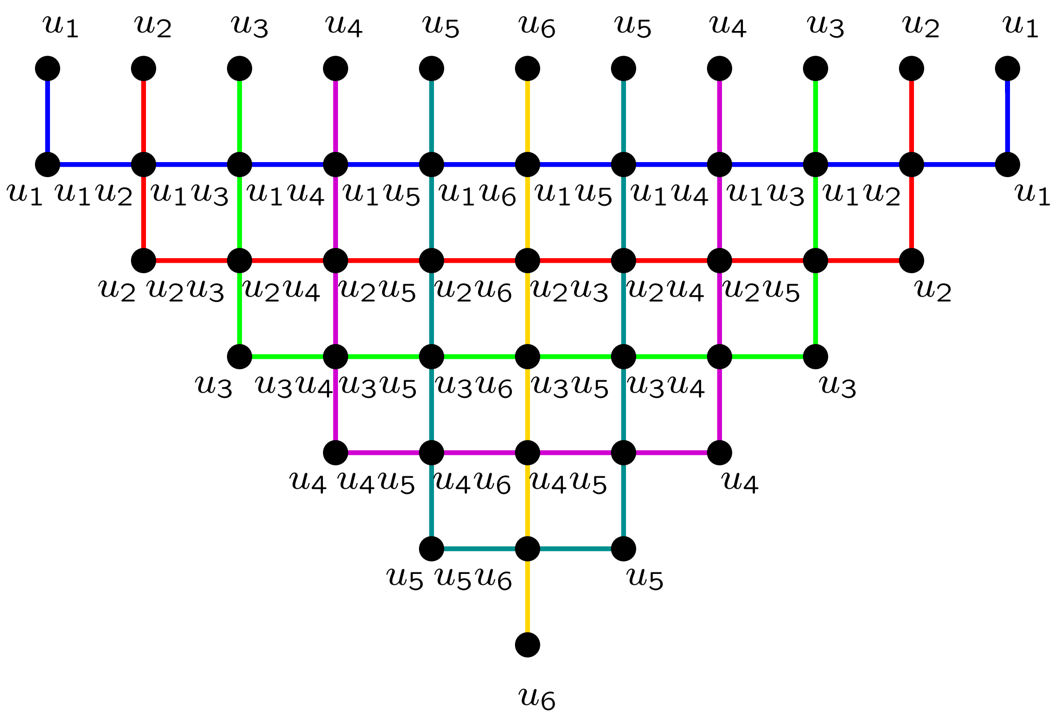
Bulk vertices	Left boundary	Right boundary
$\mathsf{W}(\,, u) = \mathsf{W}(\,, u) = 1$	$\mathsf{W}(\,, u) = \mathsf{W}(\,, u) = 1$	$\mathsf{W}(\,, u) = \mathsf{W}(\,, u) = 1$
$\mathsf{W}(\,, u) = \mathsf{W}(\,, u) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$	$\mathsf{W}(\,, u) = \mathsf{W}(\,, u) = \frac{\sigma(qu)}{\sigma(q)}$	
$\mathsf{W}(\,, u) = \mathsf{W}(\,, u) = \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		$\mathsf{W}(\,, u) = \mathsf{W}(\,, u) = \frac{\sigma(q\bar{u})}{\sigma(q)}$

All degree 1 vertices have weight 1.

If $u = 1$ and $q = e^{i\pi/6}$, all weights are 1!

Label of a vertex

Each colored path is assigned a parameter u_i as follows.

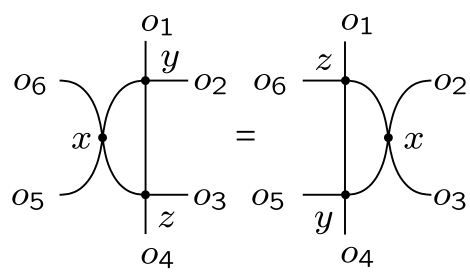


- A degree 4 vertex is contained in two colored paths u_i and u_j $\Rightarrow$ label u_iu_j
- All boundary vertices have a unique path u_i $\Rightarrow$ label u_i

Generating function: $Z_n(u_1, \ldots, u_{n+1})$.

Yang-Baxter equation

Theorem. If $xyz = q^2$ and $o_1, o_2, \ldots, o_6 \in \{\text{in}, \text{out}\}$, then



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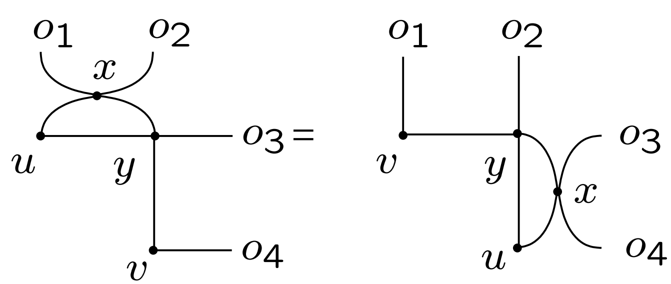
A diagram stands for the generating function of all orientations of the graph such that the external edges have the prescribed orientation $o_1, o_2, \ldots, o_6$, degree 4 vertices are balanced, and the vertex weights are as given in the table, where the letter close to a vertex indicates its label (rotate until the label is in the SW corner).

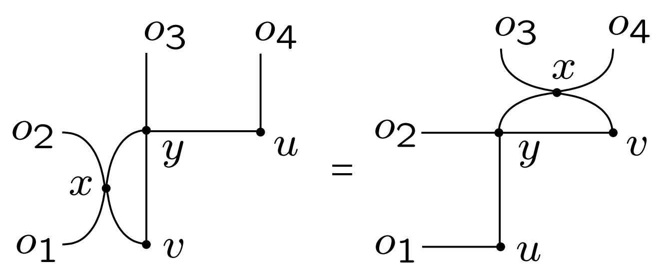
Left and right reflection equation

Theorem (Reflection equations). Suppose $o_1, o_2, o_3, o_4 \in \{\text{in}, \text{out}\}$. If $x = q^2\bar{u}v$ and $y = uv$, then



,

and if $x = q^2\bar{u}v$ and $y = \bar{u}\bar{v}$, then



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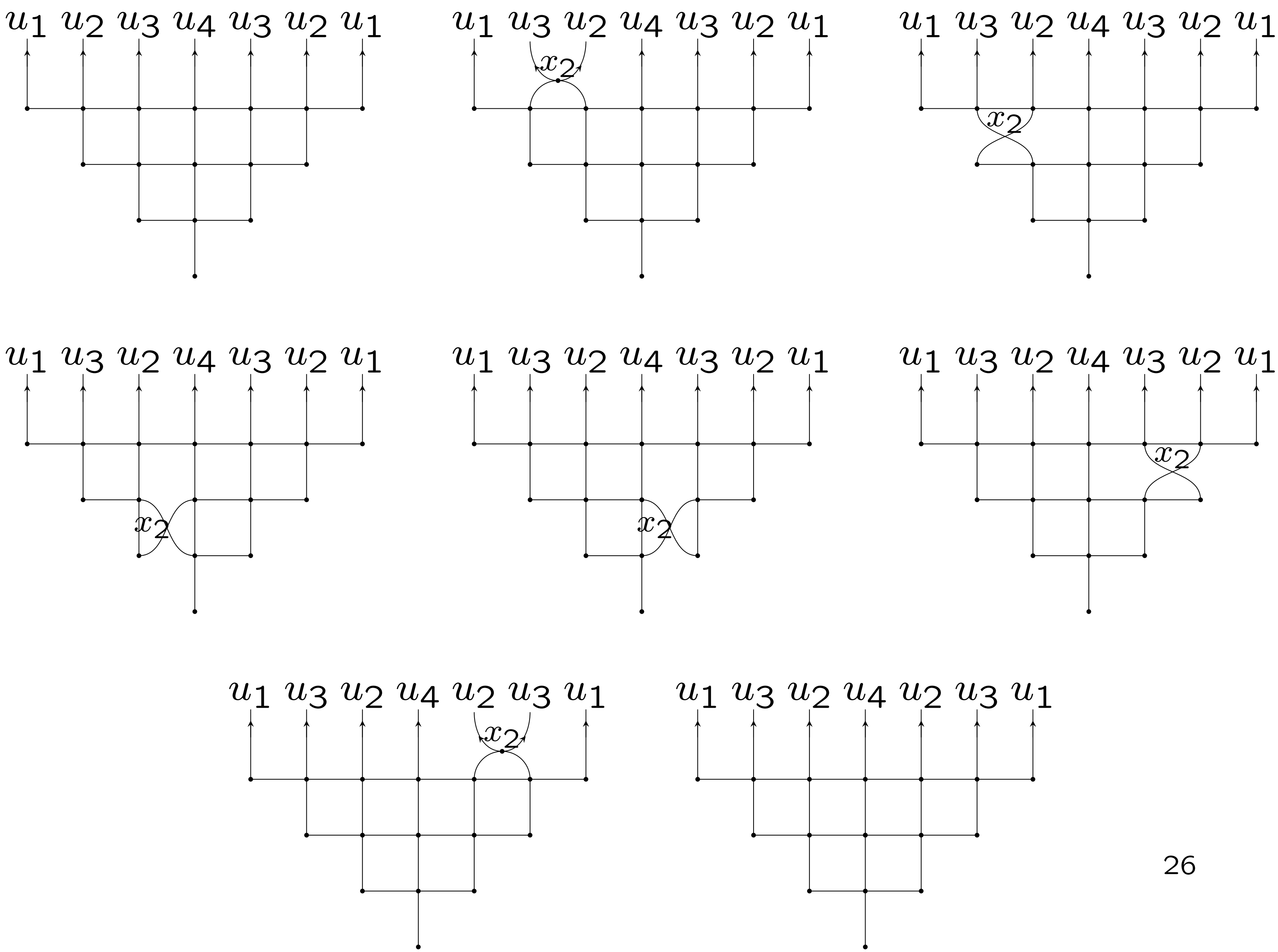
$\Rightarrow$ Symmetry of $Z_n(u_1, \ldots, u_{n+1})$ in $u_1, \ldots, u_n$.


u1 u2 u3 u4 u3 u2 u1
u1 u3 u2 u4 u3 u2 u1
x2
u1 u3 u2 u4 u3 u2 u1
x2
u1 u3 u2 u4 u3 u2 u1
x2
u1 u3 u2 u4 u3 u2 u1
x2
u1 u3 u2 u4 u3 u2 u1
x2
u1 u3 u2 u4 u2 u3 u1
x2
u1 u3 u2 u4 u2 u3 u1

$Z_n(u_1, \ldots, u_n, u_{n+1})$ at $u_{n+1} = 1$

Theorem (BFK 2015).

$$Z_n(u_1, \ldots, u_n, 1)$$
$$= \frac{\sigma(q^2)^n}{\sigma(q)^{2n}\sigma(q^4)^{n^2}} \prod_{i=1}^{n} \sigma(qu_i)\sigma(q\bar{u}_i)\sigma(q^2u_i)\sigma(q^2\bar{u}_i)$$
$$\times \prod_{1\le i<j\le n} \left(\frac{\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(u_i\bar{u}_j)}\right)^2 \det_{1\le i,j\le n}\left(\frac{q^2+\bar{q}^2+u_i^2+\bar{u}_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}\right).$$

Yet another problem: If we set $(u_1, \ldots, u_n) = (1, \ldots, 1)$, then we obtain $\frac{0}{0}$.

Schur function

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a (weakly) decreasing sequence of non-negative integers, then the associated Schur function is defined as

$$\mathrm{s}_\lambda(x_1, \ldots, x_n) = \frac{\det_{1 \le i,j \le n} \left(x_i^{\lambda_j + n - j} \right)}{\prod_{1 \le i < j \le n} (x_j - x_i)}.$$

They are

- an important linear basis for the space of symmetric functions,

- in representation theory the characters of polynomial irreducible representations of the general linear group,

- a generating function of semistandard tableaux.

Schur function expression for $Z_n(u_1,\ldots,u_n,1)$ at $q = e^{i\pi/6}$

Theorem (BFK 2015).

$$Z_n(u_1,\ldots,u_n,1)\big|_{q=e^{i\pi/6}} = 3^{-\binom{n}{2}} \times \mathsf{s}_{(n,n-1,n-1,n-2,n-2,\ldots,1,1)}(u_1^2,\bar{u}_1^2,\ldots,u_n^2,\bar{u}_n^2,1)$$

Now we may use the formula

$$\mathsf{s}_\lambda(1,\ldots,1) = \prod_{1\le i<j\le k} \frac{\lambda_i - \lambda_j + j - i}{j-i}$$

to conclude the proof of the DASASM (ex-)conjecture.

(Sketch of) Proof of Stroganov's refined DASASM-conjecture

Stroganov's refined DASASM conjecture

Observation: The central entry of an odd order DASASM is ± 1.

$\mathrm{d}_+(2n+1) = \#$ of DASASMs $(a_{i,j})_{1 \le i,j \le 2n+1}$ with $a_{n+1,n+1} = 1$
$\mathrm{d}_-(2n+1) = \#$ of DASASMs $(a_{i,j})_{1 \le i,j \le 2n+1}$ with $a_{n+1,n+1} = -1$

Conjecture (Stroganov, 2008).

$$\frac{\mathrm{d}_-(2n+1)}{\mathrm{d}_+(2n+1)} = \frac{n}{n+1}$$

Combinatorial proof?

Refined generating functions

$Z_n^+(u_1, \ldots, u_{n+1})$ and $Z_n^-(u_1, \ldots, u_{n+1})$ denote the generating function, where we sum over all configurations where the corresponding DASASM has 1 or -1, respectively, as central entry.

Lemma.

$$Z_n^\pm(u_1, \ldots, u_{n+1}) = \frac{1}{2}\Big(Z_n(u_1, \ldots, u_{n+1}) \\ \pm (-1)^n Z_n(u_1, \ldots, u_n, -u_{n+1})\Big)$$

Explicit formula for $Z_n(u_1, \ldots, u_{n+1})$

Theorem (BFK 2015).

$$Z_n(u_1,\ldots,u_{n+1}) = \frac{\sigma(q^2)^n}{\sigma(q)^{2n}\sigma(q^4)^{n^2}} \prod_{i=1}^{n} \frac{\sigma(u_i)\sigma(qu_i)\sigma(q\bar{u}_i)\sigma(q^2 u_i u_{n+1})\sigma(q^2\bar{u}_i\bar{u}_{n+1})}{\sigma(u_i\bar{u}_{n+1})}$$

$$\times \prod_{1\le i<j\le n} \left(\frac{\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(u_i\bar{u}_j)}\right)^2 \left(\mathrm{det}_{1\le i,j\le n+1}\left(\begin{cases} \frac{q^2+\bar{q}^2+u_i^2+\bar{u}_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}, & i\le n \\ \frac{u_{n+1}-1}{u_j^2-1}, & i=n+1 \end{cases}\right)\right.$$

$$\left. + \,\mathrm{det}_{1\le i,j\le n+1}\left(\begin{cases} \frac{q^2+\bar{q}^2+\bar{u}_i^2+u_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}, & i\le n \\ \frac{\bar{u}_{n+1}-1}{\bar{u}_j^2-1}, & i=n+1 \end{cases}\right)\right)$$

Schur function expression for $Z_n(u_1, \ldots, u_{n+1})$ at $q = e^{i\pi/6}$

$$
\begin{aligned}
&Z_n(u_1, \ldots, u_{n+1})\big|_{q=e^{i\pi/6}} \\
&\quad = 3^{-\binom{n}{2}} \Big(\tfrac{u_{n+1}^n}{u_{n+1}+1}\, s_{(n,n-1,n-1,\ldots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \ldots, u_n^2, \bar{u}_n^2, \bar{u}_{n+1}^2) \\
&\qquad\qquad + \tfrac{\bar{u}_{n+1}^n}{\bar{u}_{n+1}+1}\, s_{(n,n-1,n-1,\ldots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \ldots, u_n^2, \bar{u}_n^2, u_{n+1}^2) \Big).
\end{aligned}
$$

This implies Stroganov's conjecture.

Open problem: ASMs and ASTs

Permutation matrices

Binary matrices s.t. each row/column contains precisely one 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are $n!$ permutation matrices of size n.

Permutation triangles

Triangular binary arrays with n rows such that

- each row contains precisely one 1,
- each column contains at most one 1.

$$
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & 1 & 0 & & \\
 & & & 1 & 0 & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

There are $n!$ permutation triangles of size n.

ASM=Alternating Sign Matrix

Square matrix with entries in $\{0, 1, -1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Generalize permutation matrices!

AST = Alternating Sign Triangle

Triangular $0, 1, -1$ array such that

- in each row and column the non-zero elements alternate,
- the sum of entries in each row is 1,
- the first non-zero entry of each column is 1 ($\Rightarrow$ c-sums$= 0, 1$).

$$
\begin{array}{ccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & \\
 & & 0 & 0 & 1 & -1 & 1 & & \\
 & & & 1 & -1 & 1 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Generalize permutation triangles!

This is a good generalization in the following sense:

Theorem (Ayyer, Behrend, Fischer, 2016). There is the same number of $n \times n$ ASMs as there is of ASTs with n rows.

Refinement:

Theorem (Ayyer, Behrend, Fischer, 2016). Let n, k be non-negative integers. There is the same number of $n \times n$ ASMs with k occurrences of -1's as there is of ASTs with n rows and k occurrences of -1's.

We have proved the theorem for $k = 0$.

Open problem: Bijective proof !

- Behavior along diagonals and antidiagonals of DASASMs

- Enumeration of extreme configurations

Behavior along diagonals and antidiagonals of DASASMs

$\alpha \in \{-1, 0, 1\}$:
$\mathsf{n}_\alpha(A) =$ Number of α's along the diagonal and the antidiagonal of the fundamental domain

Proposition (AFK, 2014).
Let A be an $(2n+1) \times (2n+1)$ DASASM.

- $n \leq \mathsf{n}_0(A) \leq 2n$
- $0 \leq \mathsf{n}_1(A) \leq n+1$
- $0 \leq \mathsf{n}_{-1}(A) \leq n$

All inequalities are sharp.

Enumerating extreme configurations

Minimal number of zeros:

Theorem 1 (ABF 2015). The number of $(2n+1) \times (2n+1)$ DASASMs A with $\mathsf{n}_0(A) = n$ is equal to the total number of $(n+1) \times (n+1)$ ASMs.

Maximal number of zeros:

Theorem 2 (ABF 2015). The number of $(2n+1) \times (2n+1)$ DASASMs A with $\mathsf{n}_0(A) = 2n$ is equal to the number of $(2n+3) \times (2n+3)$ vertically and horizontally symmetric ASMs (VHASMs).

Cases $\alpha = \pm 1$

Maximal number of 1's:

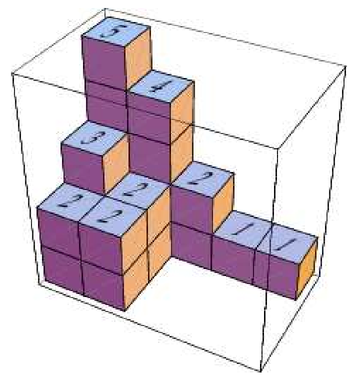
Theorem 3 (ABF 2015). The number of $(2n+1) \times (2n+1)$ DASASMs with $\mathrm{n}_1(A) = n+1$ is equal to the number of cyclically symmetric plane partitions (CSPP) in an $n \times n \times n$ box.

Maximal number of -1's:

Theorem 4 (ABF 2015). The number of $(2n+1) \times (2n+1)$ DASASMs with $\mathrm{n}_{-1}(A) = n$ is equal to the total number of $n \times n$ ASMs.

Theorem 4 is equivalent to the theorem on ASTs!

Plane Partitions



A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$$

with

$$(i, j, k) \in PP \Rightarrow (i', j', k') \in PP \quad \forall (i', j', k') \leq (i, j, k).$$

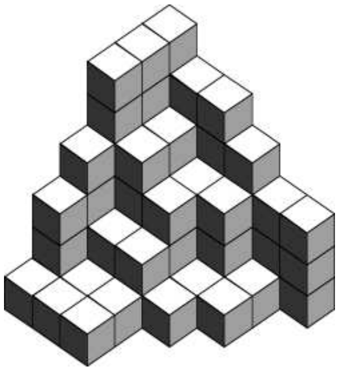
Cyclically symmetric plane partitions

An $n \times n \times n$ PP is cyclically symmetric if

$$(i, j, k) \in PP \Rightarrow (j, k, i) \in PP.$$

In 1979, George Andrews proved that the number of $n \times n \times n$ cyclically symmetric plane partitons is

$$\prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$



Open problem

Refined ASM-Theorem

Observation: There is a unique 1 in the first row of an ASM.

Theorem (Zeilberger, 1996): The number of $n \times n$ ASMs with a 1 in position $(1, r)$ is

$$\binom{n+r-2}{n-1} \frac{(2n-r-1)!}{(n-r)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} = A_{n,r}.$$

Find a statistic on ASTs that has the same distribution as the position of the 1 in the first row of an ASM!

Conjectural statistic

The elements of a column of an AST can add up to 0 or 1. We say that a column is a 1-column if it adds up to 1.

Let T be an AST with n rows. Define

$$\rho(T) = (\#\text{1-columns in the left half of } T \text{ that have a 1 at the bottom}) + (\#\text{1-columns in the right half of } T \text{ that have a 0 at the bottom}) + 1.$$

Conjecture (B 2015). The number of ASTs T with n rows and $\rho(T) = r$ is equal to $A_{n,r}$.

A constant term identity

Theorem (F 2015). Define

$$P_n(X_1, \ldots, X_{n-1}) = \sum_{0 \le i_1 < i_2 < \ldots < i_{n-1} \le 2n-3} X_1^{-i_1} X_2^{-i_2} \cdots X_{n-1}^{-i_{n-1}}.$$

The constant term of

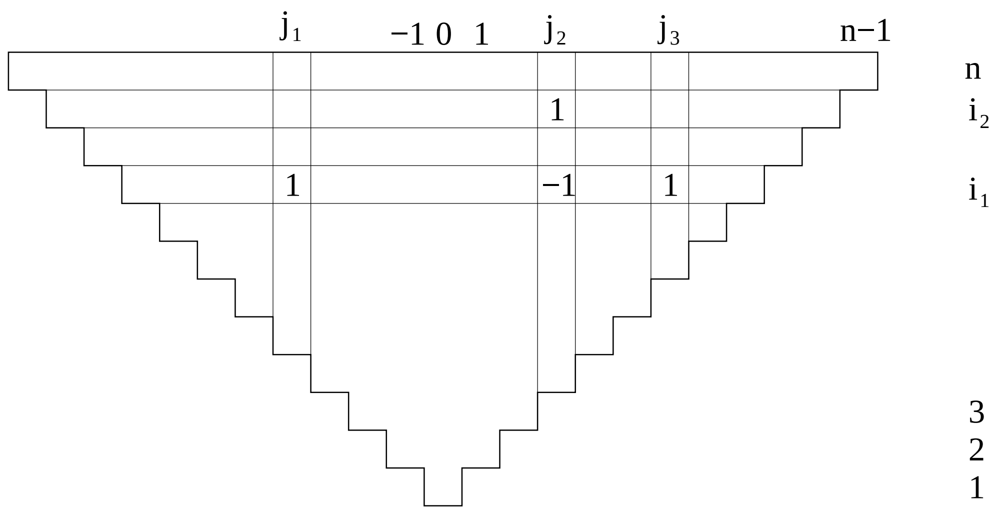
$$P_n(X_1, \ldots, X_{n-1}) \prod_{i=1}^{n-1} (t + X_i) \prod_{1 \le i < j \le n-1} (1 + X_i + X_i X_j)(X_j - X_i)$$

in the variables $X_1, X_2, \ldots, X_{n-1}$ is equal to

$$\sum_{r=1}^{n} \widehat{A}_{n,r} t^{r-1}$$

where $\widehat{A}_{n,r}$ is the number of ASTs T with n rows and $\rho(T) = r$.

$k = 1$: ASTs



Notation:

$$p(a,b) = \begin{cases} a(a+1)\cdots b & \text{if } a \le b, \\ 1 & \text{otherwise.} \end{cases}$$

$$A(j_1,j_3,i_1) = p(1,m)p(m,M-1)p(M-1,i_1-3)$$

$$B(j_1,j_2,j_3,i_1) = p(1,\text{min})p(\text{min},\text{mid}-1) \times p(\text{mid}-1,\text{max}-2)p(\text{max}-2,i_1-4)$$

where $m = \min(|j_1|,|j_3|)$, $M = \max(|j_1|,|j_3|)$, $\text{min} = \min(|j_1|,|j_2|,|j_3|)$, $\text{max} = \max(|j_1|,|j_2|,|j_3|)$, $\text{mid} = |j_1| + |j_2| + |j_3| - \text{min} - \text{max}$.

The number of ASTs with n rows and one -1 is

$$\sum B(j_1,j_2,j_3,i_1)p(i_1-1,i_2-3)p(i_2,n-1) + \left(A(j_1,j_3,i_1) - B(j_1,j_2,j_3,i_1)\right) p(i_1,i_2-2)p(i_2+1,n),$$

where the sum is over all j_1,j_2,j_3 with $-n+1 \le j_1 < j_2 < j_3 < n-1$ and all i_1,i_2 with $\max(|j_1|,|j_2|,|j_3|) + 1 \le i_1 < i_2 \le n$.

Remark. There are two other classes of objects, namely

- totally symmetric self-complementary plane partitions and
- descending plane partitions

that are enumerated by the same numbers.

No bijective proofs are known – all proofs (also ours) are "computational".

Many people consider the problem of finding explicit bijections to be the most important open problem in this field.