The number of monotone triangles with prescribed bottom row

or

Halfway to a new proof of the refined alternating sign matrix theorem

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Alternating sign matrices

Square matrix of 0s, -1s, 1s for which

- the sum of entries in each row and in each column is 1
- the non-zero entries of each row and of each column alternate in sign.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Theorem (Zeilberger 1996). The number of $n \times n$ alternating sign matrices is

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

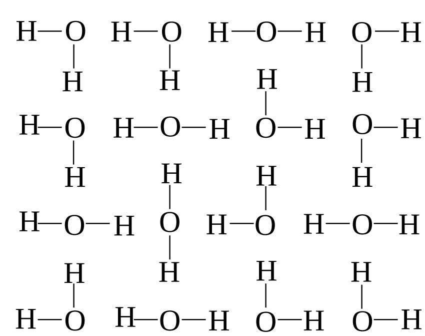
Proofs so far

- Doron Zeilberger 1996: using constant term identities he shows that the number of $n \times n$ alternating sign matrices equals the number of $2n \times 2n \times 2n$ totally symmetric self-complementary plane partitions. 84 pages!

- Greg Kuperberg 1996: using the equivalence of alternating sign matrices with the six-vertex model for "square ice".

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Alternating Sign Matrix



Square Ice

ASMs $\Leftrightarrow$ Monotone triangles

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 2 & & 3 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Monotone triangles

are triangular arrays of integers of the form

$$\begin{array}{ccccccccc} & & & & a_{1,1} & & & & \\ & & & a_{2,1} & & a_{2,2} & & & \\ & & a_{3,1} & & a_{3,2} & & a_{3,3} & & \\ & a_{4,1} & & a_{4,2} & & a_{4,3} & & a_{4,4} & \\ a_{5,1} & & a_{5,2} & & a_{5,3} & & a_{5,4} & & a_{5,5} \end{array}$$

which are

- weakly increasing in northeast direction and southeast direction and
- strictly increasing along rows.

Monotone triangles with bottom row $(1, 2, \ldots, n)$ are in bijection with $n \times n$ alternating sign matrices.

An operator formula for the number of monotone triangles

Theorem (F. 2004). The number of monotone triangles with n rows and bottom row $(k_1, k_2, \ldots, k_n)$ is

$$\left(\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_p} \Delta_{k_q}) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},$$

where $E_x p(x) = p(x+1)$ and $\Delta_x := E_x - \mathrm{id}$.

Questions:

- How to prove this formula?
- How to use this formula?

How to prove the formula

$\alpha(n; k_1, \dots, k_n)$ = Number of monotone triangles with $(k_1, \dots, k_n)$ as bottom row.

Recursion

Deletion of the last row $(k_1, \dots, k_n)$ of a monotone triangle gives a monotone triangle with last row $(l_1, \dots, l_{n-1})$ such that $k_1 \le l_1 \le k_2 \le \dots \le k_{n-1} \le l_{n-1} \le k_n$ and $l_i \ne l_{i+1}$.

$$\alpha(n; k_1, \dots, k_n) = \sum_{\substack{(l_1,\dots,l_{n-1}), l_i \ne l_{i+1} \\ k_1 \le l_1 \le k_2 \le \dots \le k_{n-1} \le l_{n-1} \le k_n}} \alpha(n-1; l_1, \dots, l_{n-1})$$

$$=: \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \alpha(n-1; l_1, \dots, l_{n-1})$$

$$\alpha(1;k_1) = 1$$

$$\alpha(2;k_1,k_2) = 1 - k_1 + k_2$$

$$\begin{aligned}\alpha(3;k_1,k_2,k_3) = \frac{1}{2}(&-3k_1+k_1^2+2k_1k_2-k_1^2k_2-2k_2^2+k_1k_2^2+3k_3-4k_1k_3\\ &+k_1^2k_3+2k_2k_3-k_2^2k_3+k_3^2-k_1k_3^2+k_2k_3^2)\end{aligned}$$

$$\begin{aligned}\alpha(4;k_1,k_2,k_3,k_4) = \frac{1}{12}(&20k_2+11k_1k_2-16k_1^2k_2+3k_1^3k_2+4k_1k_2^2+3k_1^2k_2^2\\
&-k_1^3k_2^2+4k_2^3-5k_1k_2^3+k_1^2k_2^3-20k_3+16k_1k_3-4k_1^2k_3-27k_2k_3+9k_1^2k_2k_3\\
&-2k_1^3k_2k_3-3k_1^2k_2^2k_3+k_1^3k_2^2k_3-3k_2^3k_3+4k_1k_2^3k_3-k_1^2k_2^3k_3+16k_1k_3^2-12k_1^2k_3^2\\
&+2k_1^3k_3^2-9k_1k_2k_3^2+6k_1^2k_2k_3^2-k_1^3k_2k_3^2+9k_2^2k_3^2-3k_1k_2^2k_3^2-3k_2^3k_3^2\\
&+k_1k_2^3k_3^2-4k_3^3+8k_1k_3^3-2k_1^2k_3^3-3k_2k_3^3-2k_1k_2k_3^3+k_1^2k_2k_3^3+3k_2^2k_3^3\\
&-k_1k_2^2k_3^3-27k_1k_4+20k_1^2k_4-3k_1^3k_4+16k_2k_4+24k_1k_2k_4-24k_1^2k_2k_4+4k_1^3k_2k_4\\
&-16k_2^2k_4+9k_1k_2^2k_4+3k_1^2k_2^2k_4-k_1^3k_2^2k_4+8k_2^3k_4-6k_1k_2^3k_4+k_1^2k_2^3k_4+11k_3k_4\\
&-24k_1k_3k_4+15k_1^2k_3k_4-2k_1^3k_3k_4-9k_2^2k_3k_4+2k_2^3k_3k_4-4k_3^2k_4+9k_1k_3^2k_4\\
&-6k_1^2k_3^2k_4+k_1^3k_3^2k_4+3k_2^2k_3^2k_4-k_2^3k_3^2k_4-5k_3^3k_4+6k_1k_3^3k_4-k_1^2k_3^3k_4\\
&-4k_2k_3^3k_4+k_2^2k_3^3k_4-20k_1k_4^2+9k_1^2k_4^2-k_1^3k_4^2+4k_2k_4^2+15k_1k_2k_4^2\\
&-9k_1^2k_2k_4^2+k_1^3k_2k_4^2-12k_2^2k_4^2+6k_1k_2^2k_4^2+2k_2^3k_4^2-k_1k_2^3k_4^2+16k_3k_4^2-24k_1k_3k_4^2\\
&+9k_1^2k_3k_4^2-k_1^3k_3k_4^2+9k_2k_3k_4^2-6k_2^2k_3k_4^2+k_2^3k_3k_4^2+3k_3^2k_4^2-3k_1k_3^2k_4^2+3k_2k_3^2k_4^2\\
&-k_3^3k_4^2+k_1k_3^3k_4^2-k_2k_3^3k_4^2-3k_1k_4^3+k_1^2k_4^3+2k_1k_2k_4^3-k_1^2k_2k_4^3-2k_2^2k_4^3+k_1k_2^2k_4^3\\
&+3k_3k_4^3-4k_1k_3k_4^3+k_1^2k_3k_4^3+2k_2k_3k_4^3-k_2^2k_3k_4^3+k_3^2k_4^3-k_1k_3^2k_4^3+k_2k_3^2k_4^3)\end{aligned}$$

A surprising observation.

The degree of $\alpha(n; k_1, \ldots, k_n)$ in k_i is $n - 1$.

Why is the surprising ?

In general, the degree of

$$\sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} a(l_1, \ldots, l_n)$$

in k_i is

$$\deg_{l_{i-1}} a(l_1, \ldots, l_n) + \deg_{l_i} a(l_1, \ldots, l_n) + 2.$$

Thus, one would expect exponential growth of the degree rather than linear growth.

An Operator and ...

$$
\begin{aligned}
T_{x,y} &= (\mathrm{id} + E_y E_x^{-1} S_{x,y}) \\
&\times \frac{E_x^{-1}(\mathrm{id} + E_y \Delta_x)}{E_x^{-1}(\mathrm{id} + E_y \Delta_x) + E_y^{-1}(\mathrm{id} + E_x \Delta_y)}
\end{aligned}
$$

where $S_{x,y} a(x, y) = a(y, x)$.

... two Lemmas about it

Lemma. Let $a(l_1, \ldots, l_n)$ be a polynomial with $T_{l_i, l_{i+1}} a(l_1, \ldots, l_n) = 0$ for all i. Then

$$\deg_{k_i} \sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} a(l_1, \ldots, l_n) \le \max_{j=i-1,i} \deg_{l_j} a(l_1, \ldots, l_n) + 1.$$

Lemma. Let $a(l_1, \ldots, l_n)$ be a polynomial with $T_{l_i, l_{i+1}} a(l_1, \ldots, l_n) = 0$ for all i. Then

$$T_{k_i, k_{i+1}} \left(\sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} a(l_1, \ldots, l_n) \right) = 0$$

for all i.

This implies that...

$$\deg_{k_i} \alpha(n; k_1, \ldots, k_n) = n - 1$$

As a side result we have

$$T_{k_i, k_{i+1}} \alpha(n; k_1, \ldots, k_n) = 0$$

for all i. Equivalently,

$$(\mathrm{id} + S_{k_i, k_{i+1}})(\mathrm{id} + E_{k_{i+1}} \Delta_{k_i}) \, \alpha(n; k_1, \ldots, k_n) = 0.$$

Lemma. Let $a(l_1, \ldots, l_n)$ be a polynomial. Then

$$(\mathrm{id} + E_{k_{i+1}} \Delta_{k_i}) \, a(l_1, \ldots, l_n)$$

is antisymmetric in k_i and k_{i+1} for all i, if and only if

$$\left(\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_q} \Delta_{k_p}) \right) a(k_1, \ldots, k_n)$$

is antisymmetric in $(k_1, \ldots, k_n)$.

Therefore...

$$\left(\prod_{1\le p<q\le n} (\mathrm{id} + E_{k_q}\Delta_{k_p})\right) \alpha(n; k_1, \dots, k_n)$$

is antisymmetric in $(k_1, k_2, \dots, k_n)$ and a polynomial of degree $n-1$ in k_i. Thus, it must be equal to

$$C \cdot \prod_{1\le i<j\le n} (k_j - k_i),$$

where $C = \mathbb{Q}$. (In fact $C = \prod_{1\le i<j\le n} 1/(j-i)$.)

Consequently,

$$\begin{aligned}
&\alpha(n; k_1, \dots, k_n) \\
&= \left(\prod_{1\le p<q\le n} (\mathrm{id} + E_{k_q}\Delta_{k_p})^{-1}\right) \prod_{1\le i<j\le n} \frac{k_j - k_i}{j-i} \\
&\quad = \left(\prod_{1\le p<q\le n} (\mathrm{id} + E_{k_p}\Delta_{k_q})\right) \prod_{1\le i<j\le n} \frac{k_j - k_i}{j-i}.
\end{aligned}$$

How to use the formula

Theorem (Zeilberger 1996). The number of $n \times n$ alternating sign matrices where the unique 1 in the first row is in the k-th column is

$$\frac{(k)_{n-1}(1+n-k)_{n-1}}{(n-1)!} \prod_{j=1}^{n-1} \frac{(3j-2)!}{(n+j-1)!} =: A_{n,k}$$

We need to evaluate

$$\left(\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_p} \Delta_{k_q}) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}$$

at

$$(k_1, k_2, \ldots, k_n) = (1, 2, \ldots, k-1, k+1, \ldots, n+1).$$

Some lemmas

Lemma. Let $P(X_1, \ldots, X_n)$ be a symmetric polynomial in $(X_1, \ldots, X_n)$ over $\mathbb{C}$. Then

$$P(E_{k_1}, \ldots, E_{k_n})\, \alpha(n; k_1, \ldots, k_n) \\ = P(1, \ldots, 1)\, \alpha(n; k_1, \ldots, k_n).$$

Proof. Using the operator formula it suffices to show that

$$P(E_{k_1}, \ldots, E_{k_n}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} \\ = P(1, \ldots, 1) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

A (tricky) application of this lemma to the elementary symmetric functions

$$e_p(X_1, \ldots, X_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_p \le n} X_{i_1} X_{i_2} \ldots X_{i_p},$$

gives the following lemma.

Lemma. Let $0 \le j \le p \le n-1$. Then

$$e_{p-j}(E_{k_1}, \ldots, E_{k_{n-1}})\, \alpha(n; k_1, \ldots, k_n)|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j)}$$
$$= (-1)^j \sum_{i=1}^{n} A_{n,i} \left(\binom{n-i}{p} + \sum_{l=0}^{j-1} \binom{n}{p-l} \binom{i+l-1}{l} (-1)^{l-1} \right).$$

For $p = j$, this simplifies to

$$\alpha(n; 1, \ldots, n-1, n+j) = \sum_{i=1}^{n} A_{n,i} \binom{i+j-1}{i-1}.$$

We set $k = n + j$ and observe that

$$\alpha(n; 1, \ldots, n-1, k) = \sum_{i=1}^{n} A_{n,i} \binom{i+k-n-1}{i-1}$$

for all k.

The following lemma implies that

$$k \to \alpha(n; 1, 2, \ldots, n-1, k)$$

is symmetric of n is odd and antisymmetric if n is even.

Lemma.

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n)$$

Proof. Operator formula

This implies that

$$\sum_{i=1}^{n} A_{n,i} \binom{i+k-n-1}{i-1} = (-1)^{n-1} \sum_{i=1}^{n} A_{n,i} \binom{i-k-n-1}{i-1}$$

for all k. This is equivalent to the fact that $(A_{n,1}, A_{n,2}, \ldots, A_{n,n})$ is an eigenvector of

$$\left((-1)^{j+1} \binom{2n-i-1}{n-i-j+1} \right)_{1 \le i,j \le n}$$

with respect to the eigenvalue 1. The fact that this eigenspace is one-dimensional (Mills, Robbins and Rumsey) completes the proof.