Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity

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A conjecture

Symmetrizer: $\mathbf{Sym}\, p(x_1, \ldots, x_n) = \sum_{\sigma \in \mathcal{S}_n} p(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$

Conjecture (F., Riegler). For integers $s, t \geq 0$, consider the following rational function in $z_1, \ldots, z_{s+t-1}$

$$P_{s,t} = \prod_{i=1}^{s} z_i^{2s-2i-t+1}(1 - z_i^{-1})^{i-1} \prod_{i=s+1}^{s+t-1} z_i^{2i-2s-t}(1 - z_i^{-1})^{s}$$
$$\times \prod_{1 \leq p < q \leq s+t-1} \frac{1 - z_p + z_p z_q}{z_q - z_p}$$

and let $R_{s,t}(z_1, \ldots, z_{s+t-1}) := \mathbf{Sym}\, P_{s,t}(z_1, \ldots, z_{s+t-1})$. If $s \leq t$ then

$$R_{s,t}(z_1, \ldots, z_i, \ldots, z_{s+t-1}) = R_{s,t}(z_1, \ldots, z_{i-1}, z_i^{-1}, z_{i+1}, \ldots, z_{s+t-1})$$

for all $i \in \{1, 2, \ldots, s+t-1\}$.

Example: $s = 1, t = 3$

$$P_{1,3} = z_1^{-2} z_2^{-2} (z_2 - 1)(z_3 - 1) \\ \times \frac{(1 - z_1 + z_1 z_2)(1 - z_1 + z_1 z_3)(1 - z_2 + z_2 z_3)}{(z_2 - z_1)(z_3 - z_1)(z_3 - z_2)} \\ = \frac{3 + z_1^{-2} - 4 z_1^{-1} + z_2^{-2} + \ldots 32 \text{ terms} \ldots + z_2 z_3^3}{(z_2 - z_1)(z_3 - z_1)(z_3 - z_2)}$$

$$R_{1,3} = -3 + z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_3 + z_3^{-1}$$

Outline

- How did we come up with the conjecture: a refined enumeration of vertically symmetric alternating sign matrices.

- Partial result: it suffices to consider the cases $s = t$ and $s+1 = t$!

- Some remarks on the case $s = 0$.

ASM=Alternating Sign Matrix

Quadratic $0, 1, -1$ matrix such that in each row and each column

- the non–zero entries appear with alternating signs and
- the sum of entries is 1, that is the first and the last non-zero entry is a 1.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

VSASM=Vertically symmetric ASM: $a_{i,j} = a_{i,n+1-j}$

VSASMs

- exist only for odd dimensions and
- the middle column is always $(1, -1, 1, -1, \dots, -1, 1)^T$.

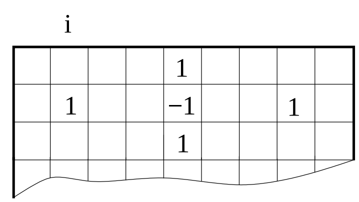
Enumeration of VSASMs

Theorem (Kuperberg, 2002). The number of $(2n+1)\times(2n+1)$ VSASMS is

$$\prod_{i=1}^{n} \frac{(3i-1)(2i-1)!(6i-3)!}{(4i-2)!(4i-1)!}.$$

Conjecture (F., 2009). The number $B_{n,i}$ of $(2n+1)\times(2n+1)$ VSASMs where the first 1 in the second row is in columns i is

$$\frac{\binom{2n+i-2}{2n-1}\binom{4n-i-1}{2n-1}}{\binom{4n-2}{2n-1}} \prod_{j=1}^{n-1} \frac{(3j-1)(2j-1)!(6j-3)!}{(4j-2)!(4j-1)!}.$$



Refined enumeration with respect to the first column

Theorem (Razumov, Stroganov, 2004). The number of $(2n+1)\times(2n+1)$ VSASMs where the first column's unique 1 is located in row i is

$$\prod_{j=1}^{n-1} \frac{(3j-1)(2j-1)!(6j-3)!}{(4j-2)!(4j-1)!} \\ \times \sum_{r=1}^{i-1} (-1)^{i+r-1} \frac{\binom{2n+r-2}{2n-1}\binom{4n-r-1}{2n-1}}{\binom{4n-2}{2n-1}} =: B^*_{n,i},$$

$i = 1, 2, \ldots, 2n+1$.

Relation: $B_{n,i} = B^*_{n,i} + B^*_{n,i+1}, \qquad i = 1, 2, \ldots, n$

Bijective proof ?

$$\begin{array}{ccccccc} & & i & & & & \\ \hline & & & 1 & & & \\ & & 1 & -1 & & & \\ & & & 1 & & & \\ & & & -1 & & & \\ & & & 1 & & & \\ & & & -1 & & & \\ & & & 1 & & & \end{array} \quad = \quad \begin{array}{c|ccccccc} & & & & 1 & & & \\ & & & & -1 & & & \\ i & 1 & & & 1 & & & \\ & & & & -1 & & & \\ & & & & 1 & & & \\ & & & & -1 & & & \\ & & & & 1 & & & \end{array} \quad + \quad \begin{array}{c|ccccccc} & & & & 1 & & & \\ & & & & -1 & & & \\ & & & & 1 & & & \\ i+1 & 1 & & & -1 & & & \\ & & & & 1 & & & \\ & & & & -1 & & & \\ & & & & 1 & & & \end{array}$$

Approach to attack the conjecture on the refined enumeration of VSASMs

Alternative proof of the Refined Alternating sign matrix theorem:
$A_{n,i} = \#$ of $n \times n$ ASMs with $a_{1,i} = 1$

The vector $(A_{n,i})_{1 \leq i \leq n}$ is uniquely determined by the following linear equation system:

$$A_{n,i} = \sum_{j=i}^{n} \binom{2n-i-1}{j-i} (-1)^{j+n} A_{n,j}, \qquad i = 1, \ldots, n$$

$$A_{n,i} = A_{n,n+1-i}, \qquad i = 1, \ldots, n$$

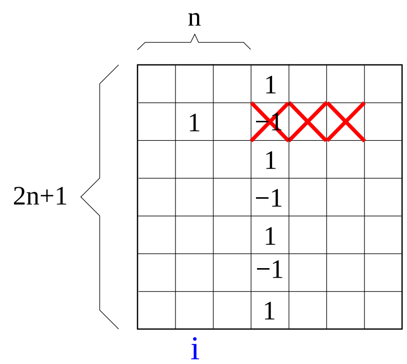
Computer experiments suggest...

...that there is a similar linear equation system for $B_{n,i}$:

$$B_{n,n-i+1} = \sum_{j=i}^{n} \binom{3n-i-1}{j-i} (-1)^{j+n} B_{n,n-j+1}, \quad -n+1 \le i \le n,$$

$$B_{n,n-i+1} = B_{n,n+i}, \quad -n+1 \le i \le n.$$

But: $(B_{n,n-i+1})_{-n+1 \le i \le n} = (B_{n,1}, \ldots, B_{n,n}, B_{n,n+1}, \ldots, B_{n,2n})$



i =position of the first 1 in the second row

- We have extended the combinatorial interpretation of $B_{n,i}$ to $i = n + 1, n + 2, \ldots, 2n$.

- In fact, we have two combinatorial extensions.

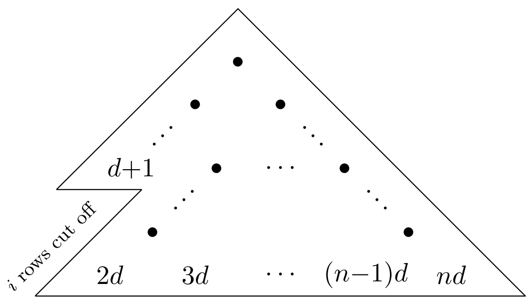
- If the conjecture on the symmetrized rational functions were true then we would know that the number of objects is the same for the two different combinatorial extensions...

- ...and this would conclude our proof of the refined enumeration of VSASMs.

First half of the linear equation system

Theorem (F., Riegler).

$C_{n,i}^{(d)} = \#$ of partial montone triangles of the following shape:



Then

$$C_{n,i+1}^{(d)} = \sum_{j=i}^{n} \binom{(d+1)n-i-1}{j-i} (-1)^{j+n} C_{n,j+1}^{(d)}, \quad i = 1, 2, \ldots, n.$$

Partial result

Recall the conjecture: $P_{s,t}(z_1,\ldots,z_{s+t-1})$ rational function, $R_{s,t} = \mathbf{Sym} P_{s,t}$ then

$$R_{s,t}(z_1,\ldots,z_i,\ldots,z_{s+t-1}) = R_{s,t}(z_1,\ldots,z_i^{-1},\ldots,z_{s+t-1})$$

if $0 \leq s \leq t$.

However, to prove the formula for the refined enumeration of VSASMs, it suffices to show

$$R_{s,t}(z_1,\ldots,z_{s+t-1}) = R_{s,t}(z_1^{-1},\ldots,z_{s+t-1}^{-1}) \quad \text{if } 1 \leq s \leq t.$$

We sketch the proof of the following result:

If the latter identity is true for $t = s$ and $t = s+1$ then it is true for all s,t with $s \leq t$.

Two rational functions:

$$S_{s,t}(z; z_1, \ldots, z_{s+t-2}) := z^{2s-t-1} \prod_{i=1}^{s+t-2} \frac{(1 - z + z_i z)(1 - z_i^{-1})}{(z_i - z)},$$

$$T_{s,t}(z; z_1, \ldots, z_{s+t-2}) := (1 - z^{-1})^s z^{t-2} \prod_{i=1}^{s+t-2} \frac{1 - z_i + z_i z}{(z - z_i) z_i}.$$

Two operators $PS_{s,t}, PT_{s,t}$ on functions f in $s+t-2$ variables:

$$PS_{s,t}[f] := S_{s,t}(z_1; z_2, \ldots, z_{s+t-1}) \cdot f(z_2, \ldots, z_{s+t-1}),$$
$$PT_{s,t}[f] := T_{s,t}(z_{s+t-1}; z_1, \ldots, z_{s+t-2}) \cdot f(z_1, \ldots, z_{s+t-2}).$$

Recursions:

$$P_{s,t} = PS_{s,t}[P_{s-1,t}] \quad \text{and} \quad P_{s,t} = PT_{s,t}[P_{s,t-1}].$$

Two related operators on functions in $s+t-2$ variables:

$$QS_{s,t}[f] := S_{s,t}(z_{s+t-1}^{-1}; z_{s+t-2}^{-1}, z_{s+t-3}^{-1}, \ldots, z_1^{-1}) \cdot f(z_1, \ldots, z_{s+t-2}),$$
$$QT_{s,t}[f] := T_{s,t}(z_1^{-1}; z_{s+t-1}^{-1}, z_{s+t-2}^{-1}, \ldots, z_2^{-1}) \cdot f(z_2, \ldots, z_{s+t-1}).$$

We set $Q_{s,t}(z_1, \ldots, z_{s+t-1}) = P_{s,t}(z_{s+t-1}^{-1}, \ldots, z_1^{-1})$. The recursions from the previous transparency immediately imply

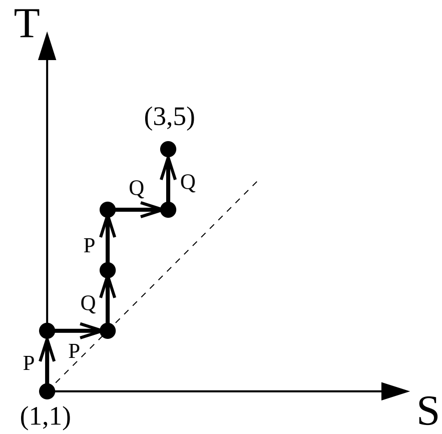
$$Q_{s,t} = QS_{s,t}[Q_{s-1,t}] \quad \text{and} \quad Q_{s,t} = QT_{s,t}[Q_{s,t-1}].$$

We have to show

$$\mathbf{Sym}\, P_{s,t}(z_1, \ldots, z_{s+t-1}) = \mathbf{Sym}\, Q_{s,t}(z_1, \ldots, z_{s+t-1}).$$

Consider words w over the "operator-alphabet" $\mathcal{A} = \{PS, PT, QS, QT\}$ and depict them as labelled lattice paths with starting point in $(1,1)$, step set $\{(1,0),(0,1)\}$ and labels P, Q.

Example: $w = (PT, PS, QT, PT, QS, QT)$



The letters PS, QS correspond to $(1,0)$ steps, while the letters PT, QT correspond to $(0,1)$ steps.

The endpoint of the path is $(|w|_S, |w|_T)$, where

$$|w|_S = \# \text{ of occurrences of } PS, QS + 1,$$
$$|w|_T = \# \text{ of occurrences of } PT, QT + 1.$$

Def. To a word w of length n, we assign a function $F_w(z_1, \ldots, z_{n+1})$ as follows: For instance, if

$$w = (PT, PS, QT, PT, QS, QT)$$

then

$$F_w(z_1, \ldots, z_7) = QT_{3,5} \circ QS_{3,4} \circ PT_{2,4} \circ QT_{2,3} \circ PS_{2,2} \circ PT_{1,2}[1],$$

i.e. apply the operators in reverse order; the indices are the integer points of the lattice path (except for the starting point).

Remark.
- If w is a word over $\{PS, PT\}$ then $F_w = P_{|w|_S,|w|_T}$.
- If w is a word over $\{QS, QT\}$ then $F_w = Q_{|w|_S,|w|_T}$.

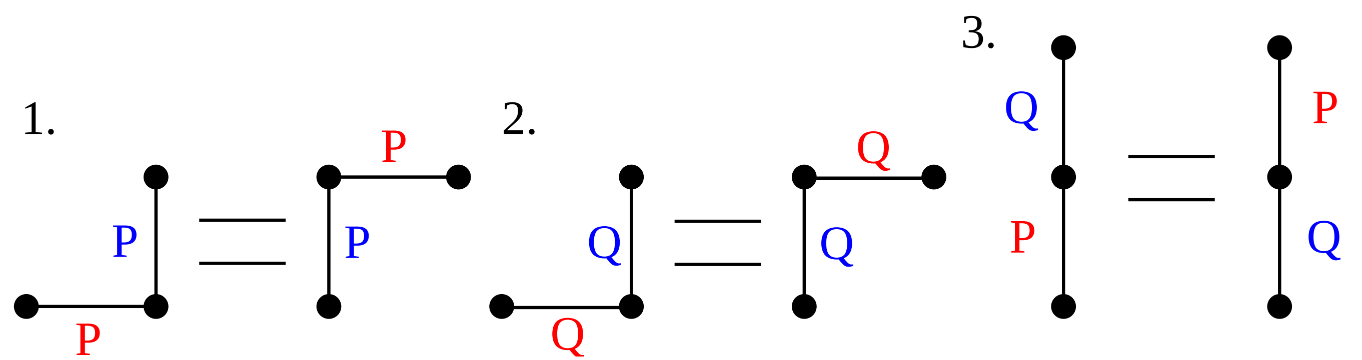
Swapping letters

Key Lemma.

1. $F_{w_1} = F_{w_2}$ if $w_1 = w_L\, PS\, PT w_R$ and $w_2 = w_L\, PT\, PS w_R$.

2. $F_{w_1} = F_{w_2}$ if $w_1 = w_L\, QS\, QT w_R$ and $w_2 = w_L\, QT\, QS w_R$.

3. $F_{w_1} = F_{w_2}$ if $w_1 = w_L\, PT\, QT w_R$ and $w_2 = w_L\, QT\, PT w_R$.



We prove the following more general statement: suppose w_1, w_2 are two words whose labelled paths have the same endpoint and are both prefixes of (rotated) Dyck paths. Then

$$\mathbf{Sym}\, F_{w_1} = \mathbf{Sym}\, F_{w_2}.$$

Induction with respect to the length of the word; nothing to prove for the empty word.

Case 1. The last letters of w_1 and w_2 coincide. W.l.o.g. $w_i = w_i' PS$, $i = 1, 2$. Then

$$\begin{aligned}
\mathbf{Sym}\, F_{w_i} &= \mathbf{Sym}\, PS_{s,t}[F_{w_i'}] = \mathbf{Sym}\, S_{s,t}(z_1; z_2, \dots, z_{s+t-1}) F_{w_i'}(z_2, \dots, z_{s+t-1}) \\
&= \sum_{j=1}^{s+t-1} \sum_{\sigma \in \mathcal{S}_n : \sigma(1) = j} S_{s,t}(z_j; z_1, \dots, \widehat{z_j}, \dots, z_{s+t-1}) F_{w_i'}(z_{\sigma(2)}, \dots, z_{\sigma(s+t-1)}) \\
&= \sum_{j=1}^{s+t-1} S_{s,t}(z_j; z_1, \dots, \widehat{z_j}, \dots, z_{s+t-1}) \mathbf{Sym}\, F_{w_i'}(z_1, \dots, \widehat{z_j}, \dots, z_{s+t-1})
\end{aligned}$$

and, by the induction hypothesis, $\mathbf{Sym} F_{w_1'} = \mathbf{Sym} F_{w_2'}$.

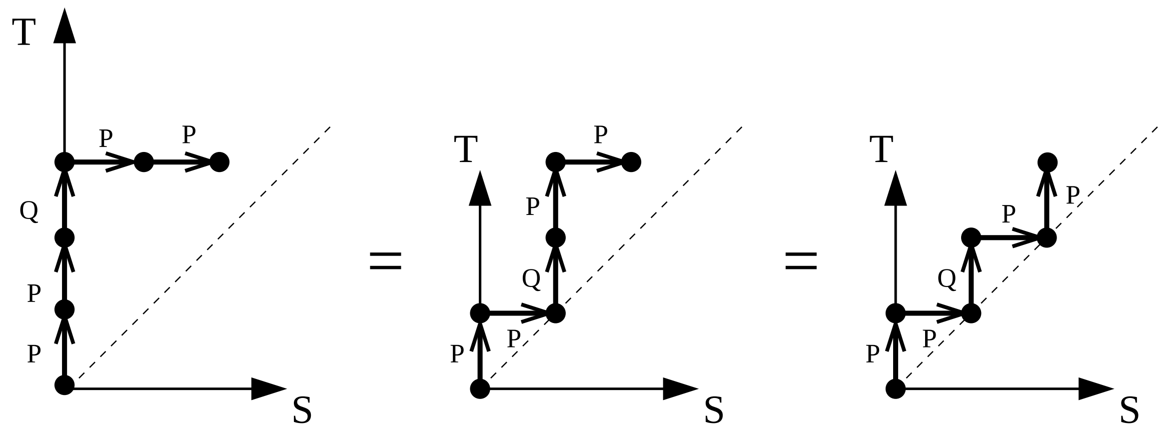
Case 2. The last letters of w_1 and w_2 differ.

Endpoint: $(|w_i|_S, |w_i|_T) =: (s, t)$

$t = s, s+1$: use $\mathbf{Sym}\, P_{s,s} = \mathbf{Sym}\, Q_{s,s}$ and $\mathbf{Sym}\, P_{s,s+1} = \mathbf{Sym}\, Q_{s,s+1}$.

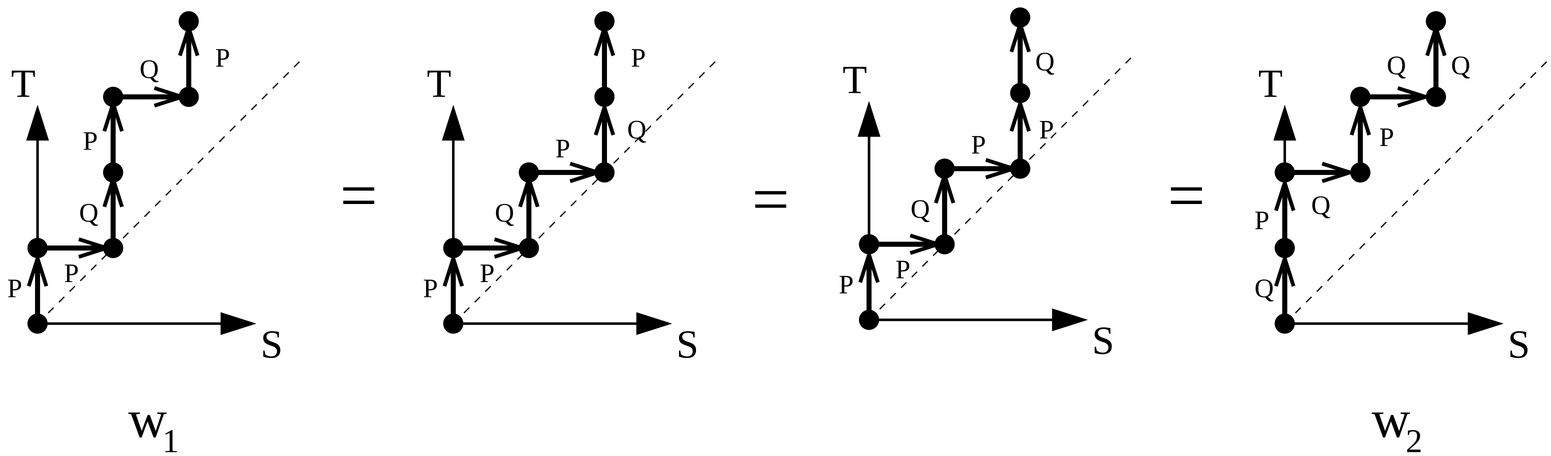
$s+1<t$: The last letter of w_i, $i=1,2$, is w.l.o.g.in $\{PT, QT\}$: Suppose $w_i = w_i' \, PS$ and choose w_i'' such that the path of $w_i'' \, PT$ has the same endpoint as the path of w_i'. By Case 1 and the Key Lemma,

$$\mathbf{Sym}\, F_{w_i} \overset{\text{Def}}{=} \mathbf{Sym}\, F_{w_i' \, PS} \overset{\text{C.1}}{=} \mathbf{Sym}\, F_{w_i'' \, PT \, PS} \overset{\text{K. L}}{=} \mathbf{Sym}\, F_{w_i'' \, PS \, PT}.$$



W.l.o.g. $w_1 = w'_1\, PT$ and $w_2 = w'_2\, QT$. Choose w''_1 such that the path of $w''_1\, QT$ has the same endpoint as the path of w'_1. By Case 1 and the Key Lemma,

$$\mathbf{Sym}\, F_{w_1} \overset{\text{Def}}{=} \mathbf{Sym}\, F_{w'_1\, PT} \overset{\text{C.1}}{=} \mathbf{Sym}\, F_{w''_1\, QT\, PT} \overset{\text{K.L.}}{=} \mathbf{Sym} F_{w''_1\, PT\, QT} \overset{\text{C.1}}{=} \mathbf{Sym}\, F_{w_2}.$$



Some remarks on the case $s = 0$

$$P_{0,n+1} = \prod_{1 \le i < j \le n} \frac{z_i^{-1} + z_j - 1}{1 - z_i z_j^{-1}}$$

Question: Are there also other rational functions $T(x, y)$ such that symmetrizing $\prod_{1 \le i < j \le n} T(z_i, z_j)$ leads to a Laurent polynomial that is invariant under replacing z_i by z_i^{-1}?

Computer experiments:

$$T(x, y) = \frac{[a(x^{-1} + y) + c][b(x + y^{-1}) + c]}{1 - xy^{-1}} + abx^{-1}y + d, \;\; a, b, c, d \in \mathbb{C}.$$

Some special cases are easy...for instance:

$$\mathbf{Sym} \prod_{1\le i<j\le n} \frac{z_i^{-1}+z_j}{1-z_iz_j^{-1}} = \ldots$$

$$= \prod_{1\le i<j\le n}(1+z_iz_j)\prod_{i=1}^{n} z_i^{-n+1}\mathbf{Sym}\frac{\prod_{i=1}^{n} z_i^{2i-2}}{\prod_{1\le i<j\le n}(z_j-z_i)}$$

$$= \prod_{1\le i<j\le n}(1+z_iz_j)\prod_{i=1}^{n} z_i^{-n+1}\frac{\det_{1\le i,j\le n}((z_i^2)^{j-1})}{\prod_{1\le i<j\le n}(z_j-z_i)} = \ldots$$

$$= \prod_{1\le i<j\le n}(1+z_iz_j)(z_i+z_j)\prod_{i=1}^{n} z_i^{-n+1}$$

Two final theorems

$$R_{0,n+1} = \mathbf{Sym} \prod_{1 \le i < j \le n} \frac{z_i^{-1} + z_j - 1}{1 - z_i z_j^{-1}} =: \text{VSASM}(1; z_1, \ldots, z_n) \prod_{i=1}^{n} z_i^{-n+1}$$

Computer experiments: $R_{0,n+1}(1, 1, \ldots, 1)$ is the number of $(2n+1) \times (2n+1)$ VSASMs.

Theorem. Let $\text{VSASM}(X; z_1) = 1$ and, for $n > 1$,

$$\text{VSASM}(X; z_1, \ldots, z_n) = \sum_{j=1}^{n} z_j^{2n-2} \prod_{1 \le i \le n, i \ne j} \frac{1 + z_i(X-2) + z_i z_j}{z_j - z_i} \times \text{VSASM}(X; z_1, \ldots, \widehat{z_j}, \ldots, z_n).$$

Then the coefficient of $z^i X^j$ in $\text{VSASM}(X; z, 1, 1, \ldots, 1)$ is the number of $(2n+1) \times (2n+1)$ VSASMs with $a_{i,1} = 1$ and j occurrences of -1 in the first n columns.

Theorem. Let $\mathrm{ASM}(X; z_1) = 1$ and, for $n > 1$,

$$\mathrm{ASM}(X; z_1, \ldots, z_n) = \sum_{j=1}^{n} z_j^{n-1} \prod_{1 \le i \le n, i \ne j} \frac{1 + z_i(X-2) + z_i z_j}{z_j - z_i} \times \mathrm{ASM}(X; z_1, \ldots, \widehat{z_j}, \ldots, z_n).$$

Then the coefficient of $z^i X^j$ in $\mathrm{ASM}(X; z, 1, 1, \ldots, 1)$ is the number of $n \times n$ ASMs with $a_{1,i} = 1$ and j occurrences of -1.

To reprove the alternating sign matrix theorem, it would suffice to show that

$$\mathrm{ASM}(1; 1, 1, \ldots, 1) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Thank you!