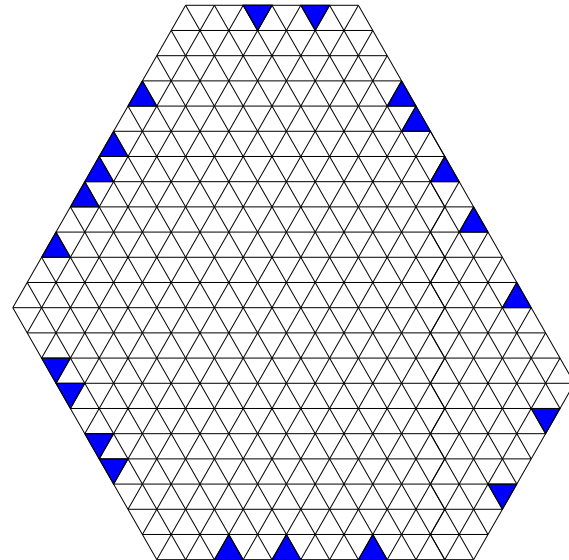


Lozenge tilings of hexagons with boundary dents

Ilse Fischer

Joint work with Mihai Ciucu



Outline

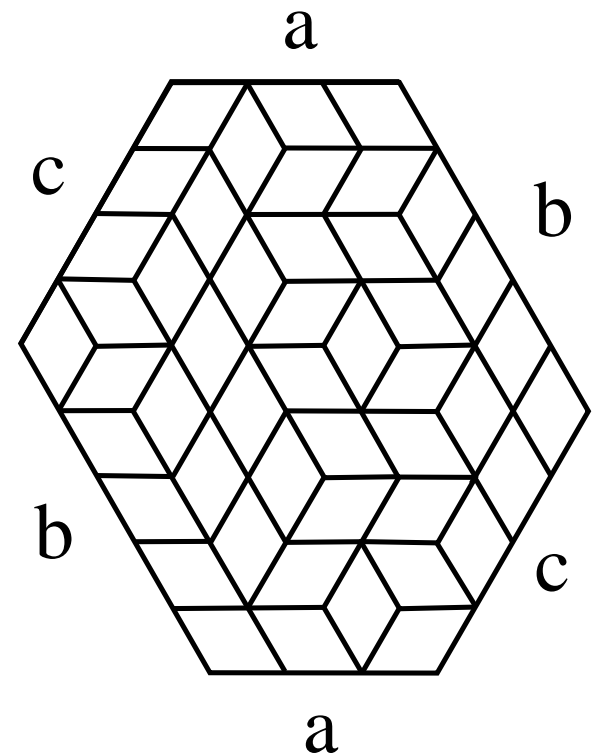
- Introduction
- Tool: Kuo's graphical condensation and Ciucu's generalization
- Balanced hexagons: $\#\triangle = \#\nabla$

Lozenge tilings of a hexagon with side lengths a, b, c, a, b, c

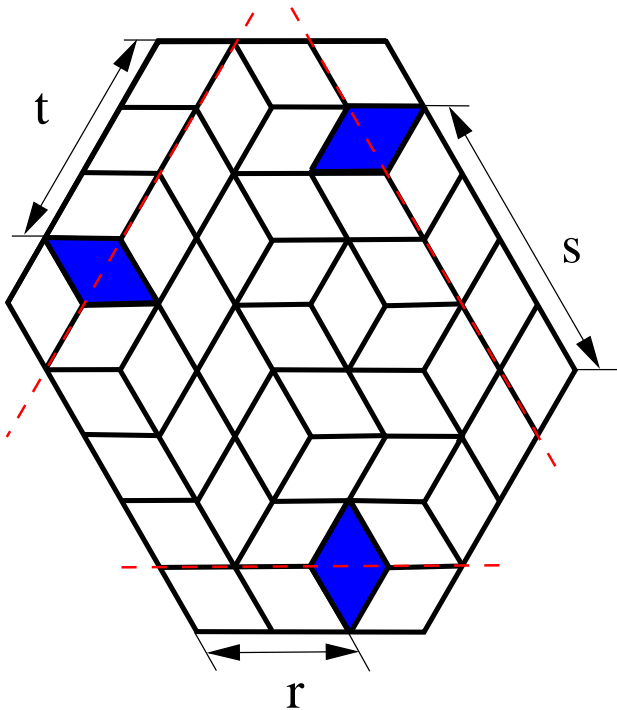
Theorem (MacMahon). The number of lozenge tilings of a hexagon with side lengths a, b, c, a, b, c is equal to

$$\prod_{i=1}^a \frac{(c+i)_b}{(i)_b}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$.



Lozenge tilings of a hexagon with three fixed border tiles



Theresia Eisenkölbl (1999):

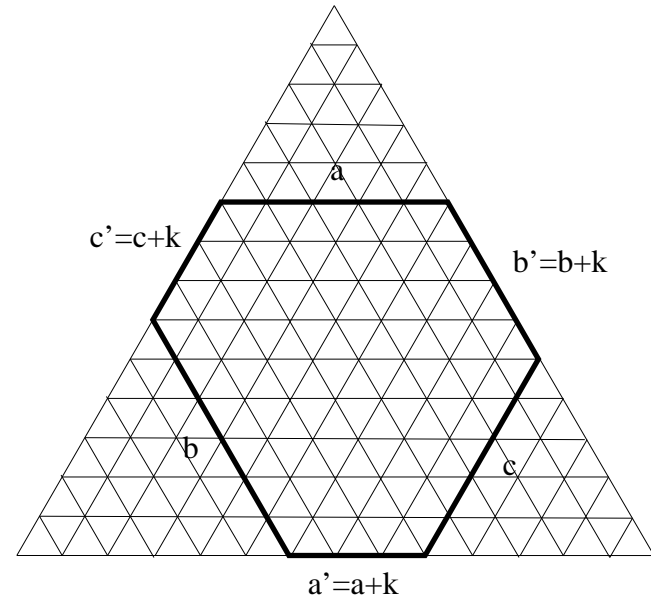
$$\begin{aligned}
 & \frac{\prod_{i=1}^{a-1} \frac{(c+i-1)_{b-1}}{(i)_{b-1}}}{(a+b-2)!(b+c-2)!(a+c-2)!} \\
 & \times (r+1)_{b-2}(s+1)_{c-2}(t+1)_{a-2} \\
 & \times (a+1-r)_{c-2}(b+1-s)_{a-2}(c+1-t)_{b-2} \\
 & \times ((a-1)(b-1)(c-1)(a-r)(b-s)(c-t) \\
 & \quad + (a-1)(b-1)(c-1)rst \\
 & \quad - (a-r)(b-s)(c-t)rst \\
 & \quad + (a-1)(c-1)(b-s)(c-t)rs \\
 & \quad + (a-1)(b-1)(a-r)(c-t)st \\
 & \quad + (b-1)(c-1)(a-r)(b-s)rt)
 \end{aligned}$$

General hexagon on a triangular grid

The side lengths of a hexagon drawn on the triangular lattice are of the form

$$a, b + k, c, a + k, b, c + k$$

for some non-negative integers a, b, c and an integer k . We denote this hexagon by $H_{a,b,c}^k$.

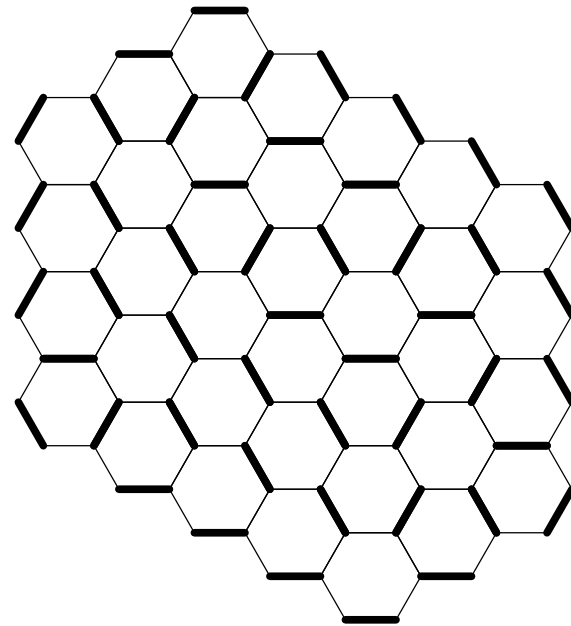
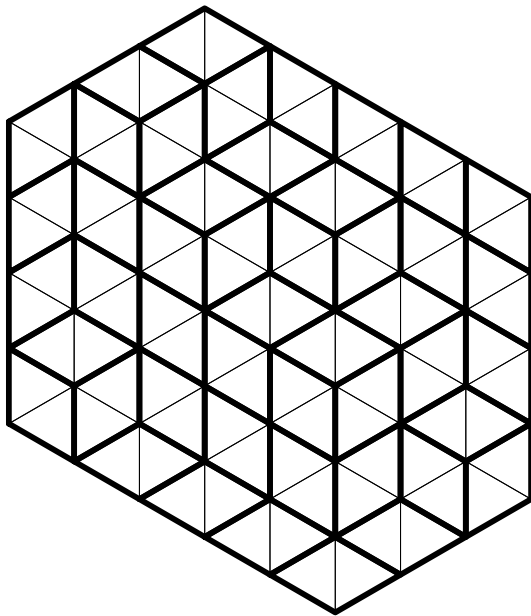


- $\#\triangle = \#\nabla + k$.
- If $k \neq 0$, then $H_{a,b,c}^k$ has no lozenge tiling.
- We need to have k more \triangle -dents than ∇ -dents.

Tool: Ciucu's extension of Kuo's graphical condensation

Kuo's graphical condensation is useful to count **perfect matchings** in planar graphs.

Lozenge tilings are perfect matchings of a hexagonal grid!



Kuo's graphical condensation

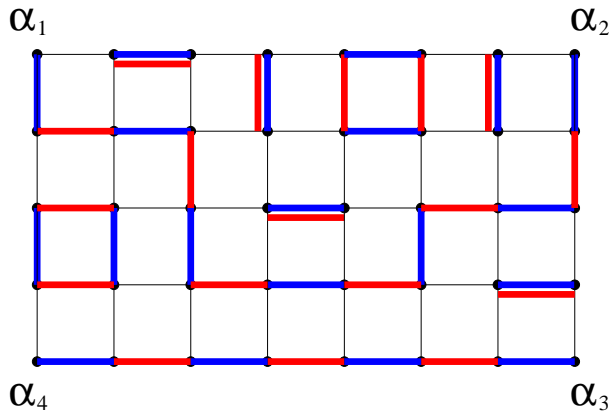
For a graph G , $M(G)$ denotes the number of perfect matchings.

Theorem. Let G be a planar graph with four vertices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that appear in cyclic order on a face of G . Then

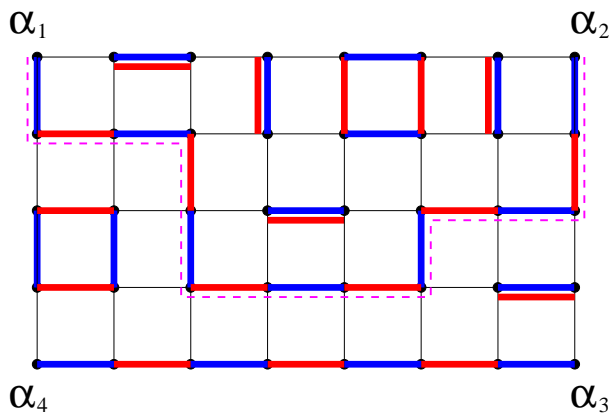
$$\begin{aligned} & M(G) M(G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) + M(G - \{\alpha_1, \alpha_3\}) M(G - \{\alpha_2, \alpha_4\}) \\ &= M(G - \{\alpha_1, \alpha_2\}) M(G - \{\alpha_3, \alpha_4\}) + M(G - \{\alpha_1, \alpha_4\}) M(G - \{\alpha_2, \alpha_3\}). \end{aligned}$$

Idea of the proof

- $M(G) \oplus M(G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$: Superimpose a perfect matching of G (blue) and a perfect matching of $G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (red).

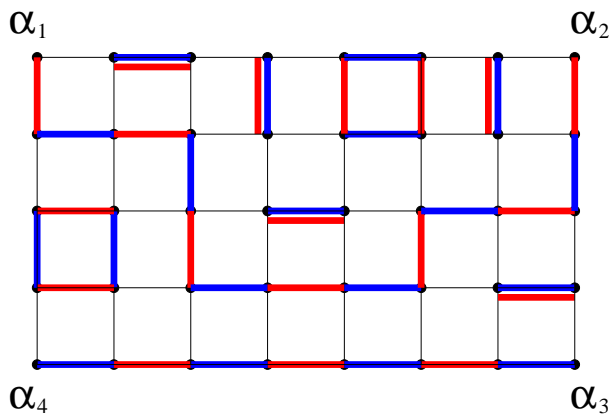


- There is a blue-red-alternating path from α_1 to α_i for an $i \in \{2, 3, 4\}$.



- Two blue-red-alternating paths cannot cross and thus $i \neq 3$.

- Switch the edges in the path of α_1 and obtain a pair of matchings of $G - \{\alpha_1, \alpha_2\}$ and $G - \{\alpha_3, \alpha_4\}$ or of $G - \{\alpha_1, \alpha_4\}$ and $G - \{\alpha_2, \alpha_3\}$. (This corresponds to the terms on the right-hand side of the identity.)



- When thinking about the reverse mapping, one observes that this is not a bijection and that the term $M(G - \{\alpha_1, \alpha_3\})M(G - \{\alpha_2, \alpha_4\})$ is also necessary.

Pfaffian

Let $A = (a_{i,j})$ be a $2n \times 2n$ skew symmetric matrix and Π_n the set of all perfect matchings of K_{2n} . Then

$$\text{Pf}(A) = \sum_{\pi = \{(i_1, j_1), \dots, (i_n, j_n)\} \in \Pi_n} \text{sgn } \pi \prod_{k=1}^n a_{i_k, j_k}$$

where $\text{sgn } \pi = \text{sgn } i_1 j_1 i_2 j_2 \dots i_n j_n$.

Recall that

$$\text{Pf}(A)^2 = \det(A).$$

$$n = 2: \text{Pf}(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}$$

Compare this to Kuo's condensation:

$$\begin{aligned} M(G - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) &= \frac{1}{M(G)} \\ &\times (M(G - \{\alpha_1, \alpha_2\})M(G - \{\alpha_3, \alpha_4\}) \\ &\quad - M(G - \{\alpha_1, \alpha_3\})M(G - \{\alpha_2, \alpha_4\}) \\ &\quad + M(G - \{\alpha_1, \alpha_4\})M(G - \{\alpha_2, \alpha_3\})) \end{aligned}$$

Ciucu's extension of Kuo's graphical condensation

Theorem (Ciucu, 2014). Let G be a planar graph with the vertices $\alpha_1, \alpha_2, \dots, \alpha_{2n}$ appearing in cyclic order on a face of G . Consider the $2n \times 2n$ skew symmetric matrix $A = (a_{i,j})$ with

$$a_{i,j} = M(G - \{\alpha_i, \alpha_j\}) \quad \text{if } i < j.$$

Then we have that

$$M(G - \{\alpha_1, \dots, \alpha_{2n}\}) = \frac{\text{Pf}(A)}{M(G)^{n-1}}.$$

- **Our application:** $G = H_{a,b,c}^k$, the α_i are the dents. We have computed the matrix entries.
- **Problem:** $M(G) = 0$ if $k \neq 0$! We restrict to the case $k = 0$ in this talk.

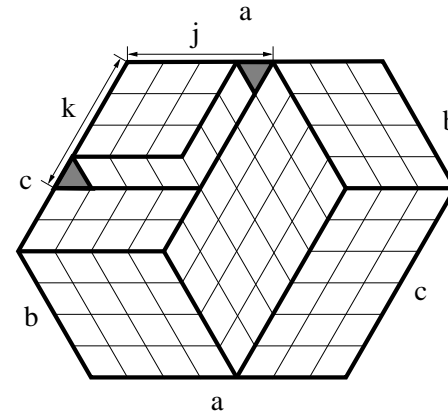
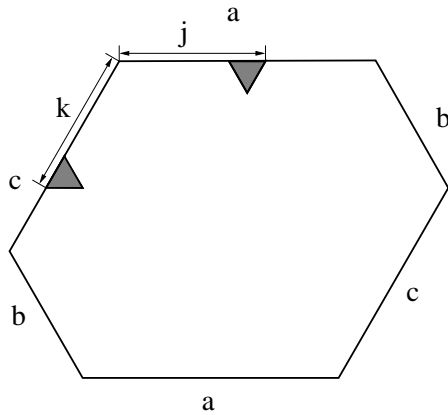
Four cases – two are trivial

Our application: G is the hexagonal grid/triangulated hexagon $H_{a,b,c}^0$,
 α_i are vertices of degree 2/dents along the boundary.

We need to compute $M(G - \{\alpha_i, \alpha_j\})$.

- (1) α_i, α_j are on the same side. The dents are of the same type:
 $M(G - \{\alpha_i, \alpha_j\}) = 0$
- (2) α_i, α_j are on adjacent sides.
- (3) α_i, α_j are on different sides that share an adjacent side:
 $M(G - \{\alpha_i, \alpha_j\}) = 0$
- (4) α_i, α_j are on opposite sides.

Dents on adjacent sides



Theorem 1. Let a, b, c, j, k be non-negative integers with $1 \leq j \leq a$ and $1 \leq k \leq c$. The number of lozenge tilings of the hexagon $H_{a,b,c}^0$ with two dents on adjacent sides of length a and c in positions j and k , respectively, as counted from the common vertex of the two sides is

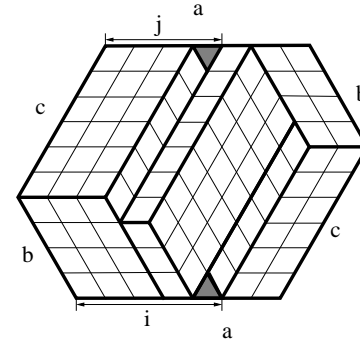
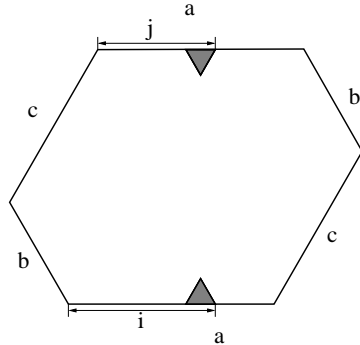
$$\prod_{i=0}^{a-1} \frac{(c+i)_b}{(1+i)_b} {}_3F_2 \left[\begin{matrix} -a+j, b, -c+k \\ 1-a-c, 1+b \end{matrix} ; 1 \right] \frac{(1+b)_{a-j} (j)_{k-1} (1+c-k)_{k-1}}{(1)_{a-j} (1)_{k-1} (1+b+c-k)_{k-1}}.$$

A reminder: Hypergeometric notation

The hypergeometric series of parameters a_1, \dots, a_r and b_1, \dots, b_s is defined as

$${}_rF_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}.$$

Dents on opposite sides



Theorem 2. Let a, b, c, i, j be positive integers with $1 \leq i, j \leq a$. The number of lozenge tilings of the hexagon $H_{a,b,c}^0$ with two dents in positions i and j along opposite sides of length a is

$$\prod_{k=0}^{a-2} \frac{(1+c+k)_b}{(1+k)_b} {}_4F_3 \left[\begin{matrix} 1-i, 1-j, 1-c-j, 1+a+b-j \\ 2-c-j, 1+b-j, 2+a-i-j \end{matrix} ; 1 \right] \\ \times \frac{(c)_{j-1} (1+b-j)_{i-1} (2+a-i-j)_{i+j-2}}{(1)_{i-1} (1)_{j-1} (1+a+c-i)_{i-1} (1+a+b-j)_{j-1}}.$$

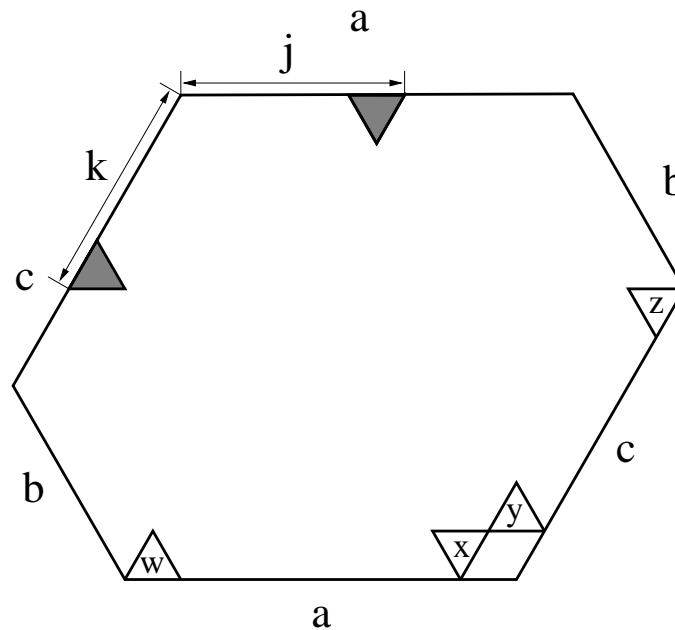
Proof of Theorem 1

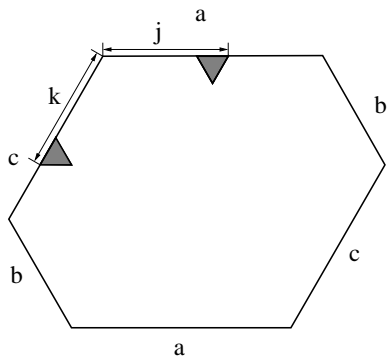
Yet another application of Kuo's condensation.

Theorem. Let $G = (V_1, V_2, E)$ be a bipartite planar graph and w, x, y, z vertices of G that appear in cyclic order on a face of G . If $w, y \in V_1$ and $x, z \in V_2$ then

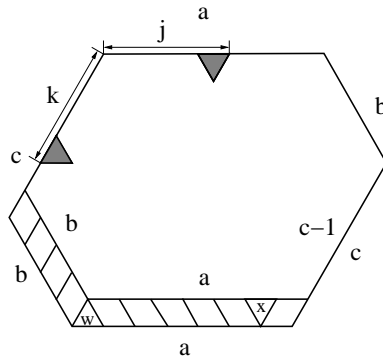
$$M(G) M(G - \{w, x, y, z\}) = M(G - \{w, x\}) M(G - \{y, z\}) + M(G - \{w, z\}) M(G - \{x, y\}).$$

Choice of w, x, y, z in our application:

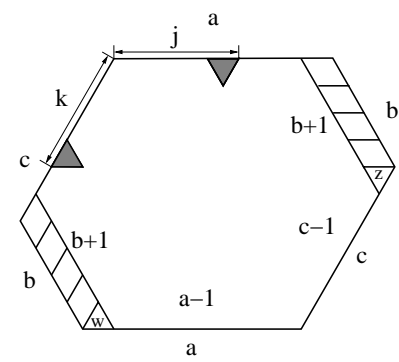




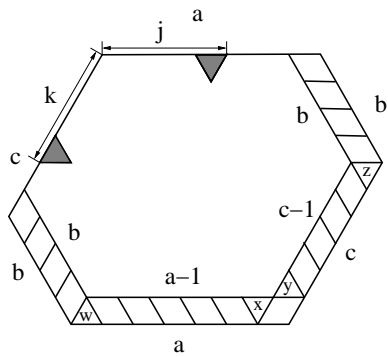
G



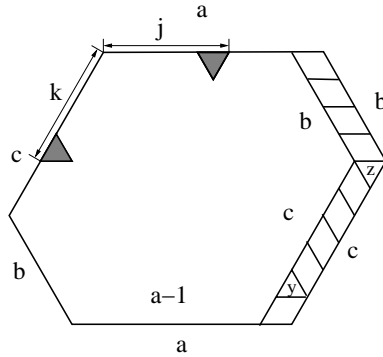
$G - \{w, x\}$



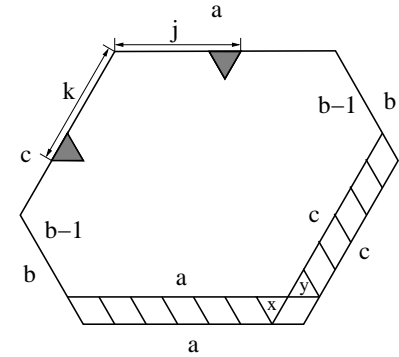
$G - \{w, z\}$



$G - \{w, x, y, z\}$



$G - \{y, z\}$



$G - \{x, y\}$

$$\begin{aligned} & \text{ADJ}(a, b, c)_{j,k} \text{ADJ}(a-1, b, c-1)_{j,k} \\ &= \text{ADJ}(a, b, c-1)_{j,k} \text{ADJ}(a-1, b, c)_{j,k} + \text{ADJ}(a-1, b+1, c-1)_{j,k} \text{ADJ}(a, b-1, c)_{j,k} \end{aligned}$$

Induction w.r.t. $a + b + c$

Base case: Special case of hook-content formula

Induction step:

We need to verify that the formula in the proposition fulfills the recursion.

$$\begin{aligned} & - {}_3F_2 \left[\begin{matrix} -a + j, -1 + b, -c + k \\ 1 - a - c, b \end{matrix} ; 1 \right] {}_3F_2 \left[\begin{matrix} 1 - a + j, 1 + b, 1 - c + k \\ 3 - a - c, 2 + b \end{matrix} ; 1 \right] \times \text{QPP}_1 \\ & - {}_3F_2 \left[\begin{matrix} -a + j, b, 1 - c + k \\ 2 - a - c, 1 + b \end{matrix} ; 1 \right] {}_3F_2 \left[\begin{matrix} 1 - a + j, b, -c + k \\ 2 - a - c, 1 + b \end{matrix} ; 1 \right] \times \text{QPP}_2 \\ & + {}_3F_2 \left[\begin{matrix} -a + j, b, -c + k \\ 1 - a - c, 1 + b \end{matrix} ; 1 \right] {}_3F_2 \left[\begin{matrix} 1 - a + j, b, 1 - c + k \\ 3 - a - c, 1 + b \end{matrix} ; 1 \right] \times \text{QPP}_3 = 0 \end{aligned}$$

$\text{QPP}_i =$ quotient of products of Pochhammer functions

- Arguments differ by small integer values.
- We use various contiguous relations for hypergeometric series!

Open problem

If $k \neq 0$, the result is similar when the dents are confined to five sides of the hexagon. However, in the general case, the formula is a nested Pfaffian.

Can the nested Pfaffian be avoided in the most general case?

Another tool: Contiguous relations for hypergeometric series

$${}_rF_s \left[\begin{matrix} x, (A) \\ y, (B) \end{matrix}; z \right] = {}_rF_s \left[\begin{matrix} x-1, (A) \\ y-1, (B) \end{matrix}; z \right] + \frac{(y-x)z \prod_{i=1}^{r-1} A_i}{(y-1)y \prod_{i=1}^{s-1} B_i} {}_rF_s \left[\begin{matrix} x, (A+1) \\ y+1, (B+1) \end{matrix}; z \right] \quad \text{C40}[x, y]$$

$$\begin{aligned} {}_rF_s \left[\begin{matrix} x, (A) \\ y, (B) \end{matrix}; z \right] &= \frac{(y-2)(y-1) \prod_{i=1}^{s-1} (B_i-1)}{(y-x-1)z \prod_{i=1}^{r-1} (A_i-1)} {}_rF_s \left[\begin{matrix} x, (A-1) \\ y-1, (B-1) \end{matrix}; z \right] \\ &\quad - \frac{(y-2)(y-1) \prod_{i=1}^{s-1} (B_i-1)}{(y-x-1)z \prod_{i=1}^{r-1} (A_i-1)} {}_rF_s \left[\begin{matrix} x-1, (A-1) \\ y-2, (B-1) \end{matrix}; z \right] \end{aligned} \quad \text{C42}[x, y]$$

$${}_rF_s \left[\begin{matrix} w, x, (A) \\ y, (B) \end{matrix}; z \right] = \frac{x(y-w)}{(x-w)y} {}_rF_s \left[\begin{matrix} w, x+1, (A) \\ y+1, (B) \end{matrix}; z \right] + \frac{w(y-x)}{(w-x)y} {}_rF_s \left[\begin{matrix} w+1, x, (A) \\ y+1, (B) \end{matrix}; z \right] \quad \text{C54}[w, x, y]$$

$$\begin{aligned} {}_rF_s \left[\begin{matrix} w, x, (A) \\ y, (B) \end{matrix}; z \right] &= \frac{(1-w+x)(y-1)}{(w-1)(1+x-y)} {}_rF_s \left[\begin{matrix} w-1, x, (A) \\ y-1, (B) \end{matrix}; z \right] \\ &\quad + \frac{x(y-w)}{(w-1)(y-x-1)} {}_rF_s \left[\begin{matrix} w-1, x+1, (A) \\ y, (B) \end{matrix}; z \right] \end{aligned} \quad \text{C55}[w, x, y]$$

(A) and (B) stand for lists A_1, A_2, A_3, \dots and B_1, B_2, B_3, \dots of the appropriate lengths.

C40[x,y]

$$\begin{aligned}
 \frac{(x)_n \prod_{i=1}^{r-1} (A_i)_n z^n}{(y)_n \prod_{i=1}^{s-1} (B_i)_n n!} &= \frac{(x-1)_n \prod_{i=1}^{r-1} (A_i)_n z^n}{(y-1)_n \prod_{i=1}^{s-1} (B_i)_n n!} \\
 &+ \frac{(y-x)z \prod_{i=1}^{r-1} A_i}{(y-1)y \prod_{i=1}^{s-1} B_i} \frac{(x)_{n-1} \prod_{i=1}^{r-1} (A_i+1)_{n-1}}{(y+1)_{n-1} \prod_{i=1}^{s-1} (B_i+1)_{n-1}} \frac{z^{n-1}}{(n-1)!}
 \end{aligned}$$

Simple calculation:

$$\frac{x+n-1}{y \cdot n} = \frac{(x-1)(y+n-1)}{(y-1)y \cdot n} + \frac{y-x}{(y-1)y}$$

Application

C55[1-c+k,b,3-a-c]:

$$\begin{aligned}
 {}_3F_2 \left[\begin{matrix} 1-a+j, b, 1-c+k \\ 3-a-c, 1+b \end{matrix}; 1 \right] &= \frac{(2-a-c)(b+c-k)}{(-2+a+b+c)(-c+k)} {}_3F_2 \left[\begin{matrix} 1-a+j, b, -c+k \\ 2-a-c, 1+b \end{matrix}; 1 \right] \\
 &\quad + \frac{b(2-a-k)}{(2-a-b-c)(-c+k)} {}_3F_2 \left[\begin{matrix} 1-a+j, b+1, -c+k \\ 3-a-c, b+1 \end{matrix}; 1 \right]
 \end{aligned}$$

The second ${}_3F_2$ on the right-hand side can be computed using Chu-Vandermonde summation:

$${}_3F_2 \left[\begin{matrix} 1-a+j, b+1, -c+k \\ 3-a-c, b+1 \end{matrix}; 1 \right] = {}_2F_1 \left[\begin{matrix} 1-a+j, -c+k \\ 3-a-c \end{matrix}; 1 \right] = \frac{(3-a-k)_{a-j+1}}{(3-a-c)_{a-j+1}}$$

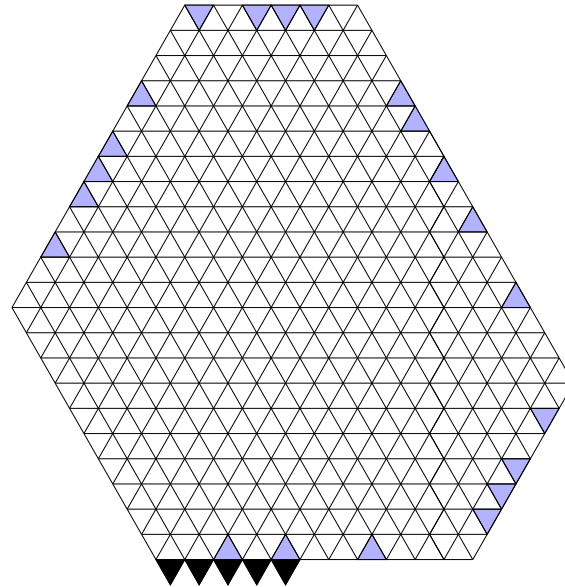
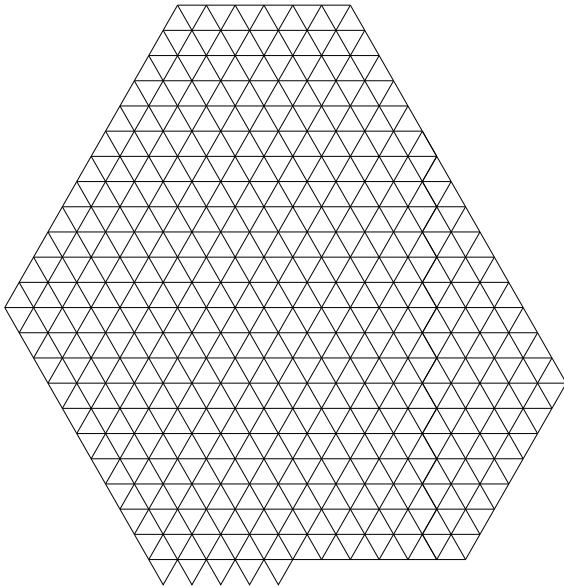
The first ${}_3F_2$ on the right-hand side is another ${}_3F_2$ that appears in our identity. We have reduced the number of ${}_3F_2$ to 5. We apply four more transformations of this type such that in the end we obtain a polynomial of degree at most 2 in

$${}_3F_2 \left[\begin{matrix} 1-a+j, b, -c+k \\ 2-a-c, 1+b \end{matrix}; 1 \right].$$

The coefficients of the polynomial are sums of quotients of products of Pochhammer functions. It is routine to check that the coefficients are in fact zero.

General case $k \geq 0$

$\overline{H}_{a,b,c}^k$ denotes the region obtained from $H_{a,b,c}^k$ by augmenting it by a string of k contiguous ∇ along its bottom row.

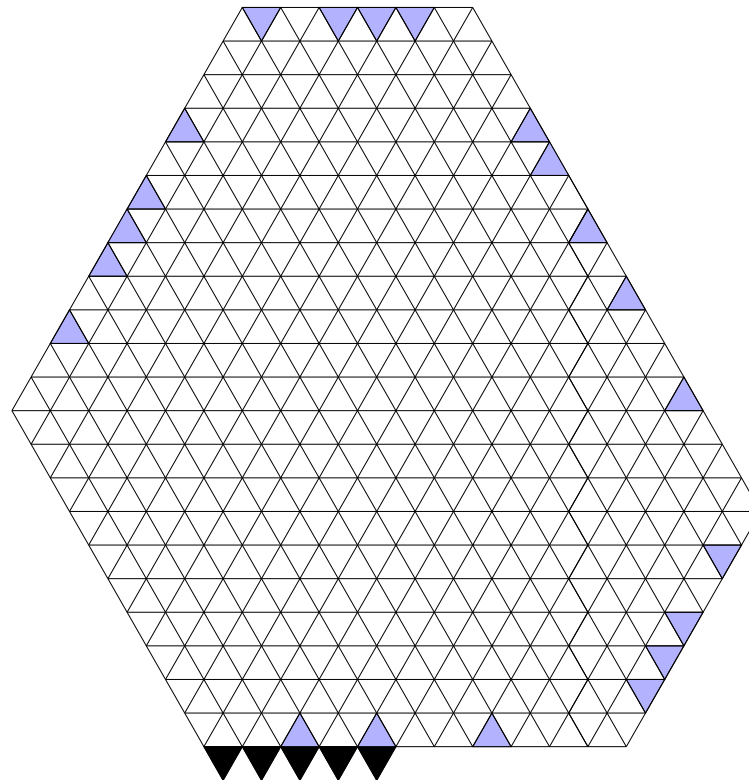


$$M(\overline{H}_{a,b,c}^k) \neq 0$$

Notation:

- $\alpha_1, \alpha_2, \dots, \alpha_{n+k}$: \triangle -dents
- $\beta_1, \beta_2, \dots, \beta_n$: ∇ -dents
- $\gamma_1, \gamma_2, \dots, \gamma_k$: ∇ -triangles below bottom row

Let $\delta_1, \delta_2, \dots, \delta_{2n+2k}$ be a cyclic order of these elements



$\gamma_1, \gamma_2, \alpha_1, \gamma_3, \gamma_4, \alpha_2, \gamma_5, \alpha_3, \beta_1, \beta_2,$
 $\beta_3, \beta_4, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \beta_5, \beta_6, \beta_7,$
 $\beta_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}$

Ciucu's extension of Kuo's graphical condensation implies

$$M(H_{a,b,c}^k - \{\alpha_1, \dots, \alpha_{n+k}, \beta_1, \dots, \beta_n\}) = \frac{\text{Pf}[M(\overline{H}_{a,b,c}^k - \{\delta_i, \delta_j\})_{1 \leq i < j \leq 2n+2k}]}{M(\overline{H}_{a,b,c}^k)^{n+k-1}}.$$

We need to show: Each quantity on the right-hand side can be computed.

Assumption: one of the three sides on which ∇ -dents can occur does not actually have any dents. W.l.o.g. let this be the southwestern side.

Easy cases

Denominator: $M(\overline{H}_{a,b,c}^k) = M(H_{a,b+k,c}^0)$

Numerator:

$$M(\overline{H}_{a,b,c}^k - \{\alpha_i, \alpha_j\}) = 0$$

$$M(\overline{H}_{a,b,c}^k - \{\beta_i, \beta_j\}) = 0$$

$$M(\overline{H}_{a,b,c}^k - \{\beta_i, \gamma_j\}) = 0$$

$$M(\overline{H}_{a,b,c}^k - \{\gamma_i, \gamma_j\}) = 0$$

$$M(\overline{H}_{a,b,c}^k - \{\alpha_i, \beta_j\}))$$

The number is **zero** if either

- α_i shares an edge with one of the γ_l , or
- α_i is on the northwestern side, at distance at most $k - 1$ from the western corner.

Otherwise it is given by **Proposition 1** if α_i and β_j are along **adjacent** sides, and by **Proposition 2** if α_i and β_j are along **opposite** sides.

$$M(\overline{H}_{a,b,c}^k - \{\alpha_i, \gamma_j\})$$

The number is **zero** if either

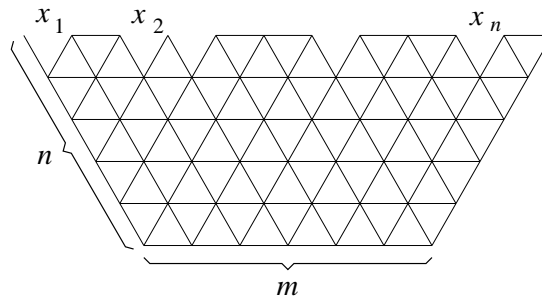
- α_i shares an edge with one of the γ_l with $l \neq j$, or
- α_i is on the northwestern side, at distance at most $j - 2$ from the western corner.

If α_i and γ_j are along the **same** side, the result follows from **Proposition 3** and if α_i and γ_j are along **different** sides, the result follows from **Proposition 4**.

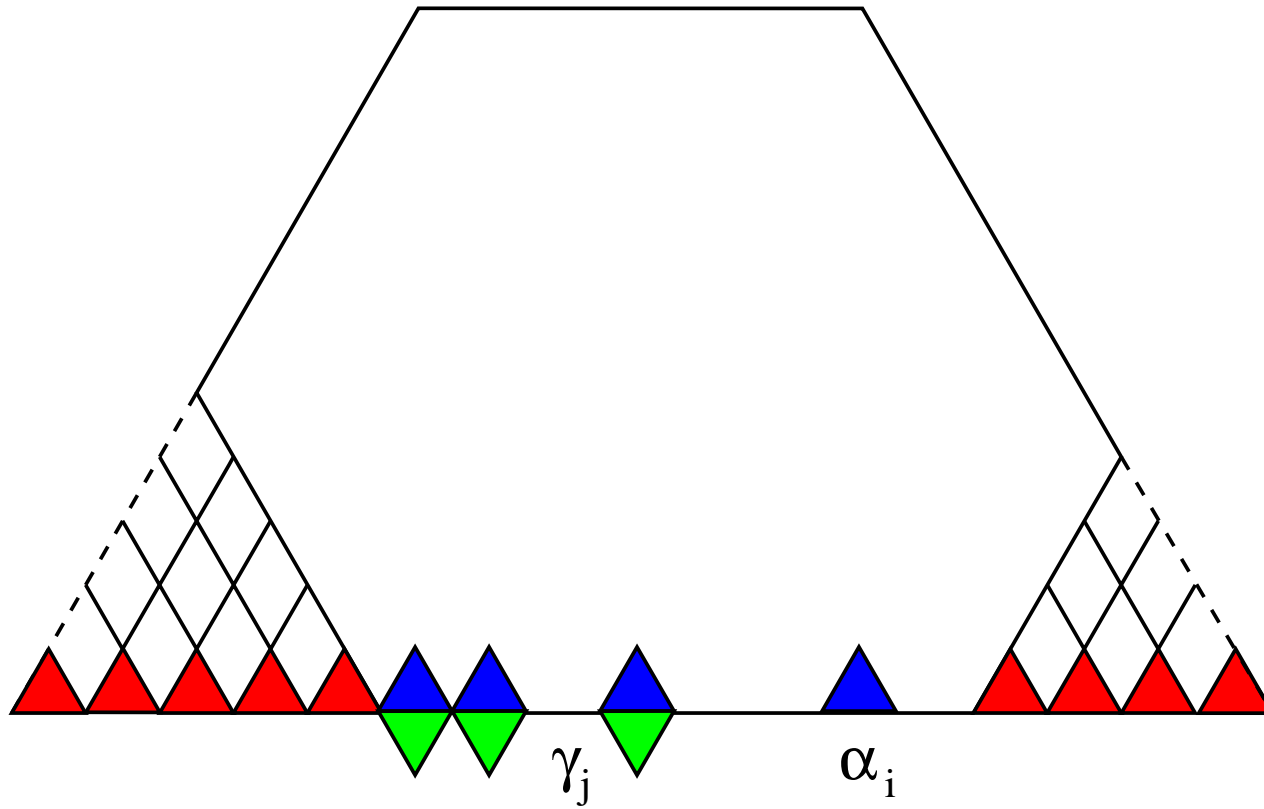
α_i and γ_j are on the same side

Proposition 3. Let $T_{m,n}(x_1, \dots, x_n)$ be the region obtained from the trapezoid of side lengths $m, n, m+n, n$ (clockwise from bottom) by removing the down-pointing unit triangles from along its top that are in positions x_1, x_2, \dots, x_n as counted from left to right. Then

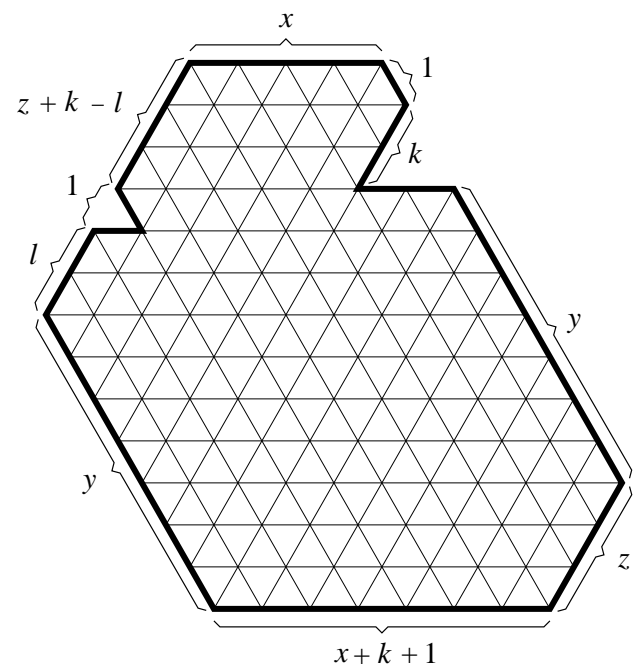
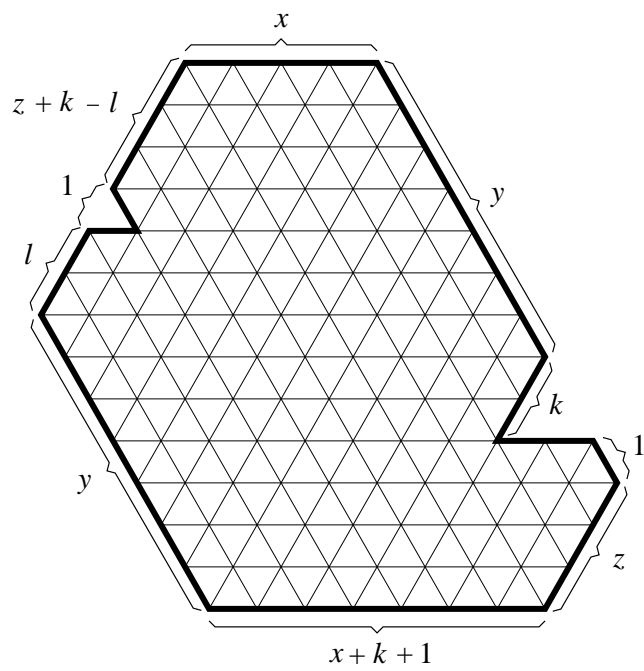
$$M(T_{m,n}(x_1, \dots, x_n)) = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{j - i}.$$



Special case



α_i and γ_j are on different sides



Proposition 4, part 1

Let $H_{a,b,c}(k, l)$ be the region obtained from the hexagon of side lengths $a, b + k + 1, c, a + k + 1, b, c + k + 1$ (clockwise from top) by removing an up-pointing unit triangle from its northwestern side, l units above the western corner, and an up-pointing triangle of side k from its northeastern side, one unit above the eastern corner.

Let $m = \min(a, b)$ and $M = \max(a, b)$. Then we have

$$M(H_{a,b,c}(k, l)) = M(H_{a,b,c}) \frac{p(c, l)}{p(0, 0)},$$

where the polynomial $p(c, l)$ is defined to be

$$\begin{aligned} & (l + 1)_b (c + k - l + 1)_a \\ & \times (c + k + 2)(c + k + 3)^2 \cdots (c + k + m + 1)^m (c + k + m + 2)^m \cdots (c + k + M + 1)^m \\ & \times (c + k + M + 2)^{m-1} (c + k + M + 3)^{m-2} \cdots (c + k + M + m) \\ & \times \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{(i-1)!(k-i+1)!} (l - k + i)_{k-i+1} (l + b + 1)_{i-1} (c + 1)_{i-1} (c + i + 1)_{k-i+1}. \end{aligned}$$

Proposition 4, part 2

Let $H'_{a,b,c}(k,l)$ be the region defined precisely as $H_{a,b,c}(k,l)$, with the one exception that the up-pointing triangle of side k is one unit below the northeastern corner, rather than one unit above the eastern corner.

Let $\nu = \min(b-1, k)$, and define $r(c)$ by

$$r(c) := \begin{cases} (c+2)^1 \cdots (c+\nu+1)^\nu \cdots (c+b+k-\nu)^\nu \cdots (c+b+k-1)^1, & \nu \geq 1 \\ 1, & \nu = 0 \\ \frac{1}{(c+1)_k}, & \nu = -1 \end{cases}$$

(in the first branch the bases are incremented by 1 from each factor to the next; the exponents are incremented by one until they reach ν , stay equal to ν across the middle portion, and then they decrease by one unit from each factor to the next).

Then we have

$$M(H'_{a,b,c}(k, l)) = \binom{a+k}{k} \frac{q(c, l)}{q(0, 0)},$$

where the polynomial $q(c, l)$ is defined to be

$$\begin{aligned} & r(c) (l+1)_b (z+k-l+1)_a \\ & \times (c+k+2)(c+k+3)^2 \cdots (c+k+m+1)^m (c+k+m+2)^m \cdots (c+k+M+1)^m \\ & \times (c+k+M+2)^{m-1} (c+k+M+3)^{m-2} \cdots (c+k+M+m) \\ & \times \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{(i-1)!(k-i+1)!} (l-k+i)_{k-i+1} (l+b+1)_{i-1} (l-k-c)_{i-1} (l-k-c+i)_{k-i+1} \end{aligned}$$

(as in part 1, $m = \min(a, b)$ and $M = \max(a, b)$).

Allowing dents also on the southwestern side: nested Pfaffian

Theorem. Let $\alpha_1, \dots, \alpha_{n+k}$ be arbitrary \triangle -dents and β_1, \dots, β_n arbitrary ∇ -dents along the boundary of $H_{a,b,c}^k$. Then

$$M(H_{a,b,c}^k - \{\alpha_1, \dots, \alpha_{n+k}, \beta_1, \dots, \beta_k\})$$

is equal to a Pfaffian of a $2n \times 2n$ matrix whose entries are Pfaffians of $(2k + 2) \times (2k + 2)$ matrices of the type as in the previous “theorem” (that was never stated).

Proof

$D =$ region obtained from $H_{a,b,c}^k$ by removing the dents $\alpha_{n+1}, \dots, \alpha_{n+k}$.

Apply Ciucu's extension of Kuo's condensation to the "dual" of D , with the removed vertices to be $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n .

We obtain a $2n \times 2n$ Pfaffian with entries of the form $M(D - \{\alpha_i, \beta_j\})$, $i, j \in \{1, 2, \dots, n\}$.

Now $M(D - \{\alpha_i, \beta_j\})$ is a dented hexagon with all dents confined to **four** of its sides (dents of type \triangle can only occur along the north-western, northeastern and southern side, and there is a single dent of type ∇).

By our previous theorem, $M(D - \{\alpha_i, \beta_j\})$ can be expressed as the Pfaffian of a $(2k + 2) \times (2k + 2)$ matrix.