COMPACT ENUMERATION FORMULAS FOR GENERALIZED PARTITIONS

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Counting the number of elements in finite sets S_i (where i typically ranges over some index set I such as the non–negative integers or a cartesian product of the non–negative integers) is surely one of the oldest and most fundamental problems in mathematics. It is in the nature of the subject that only a few enumeration problems have a compact solution in terms of a simple explicit formula in i which (for instance) only makes use of the basic arithmetic operations and factorials. More surprisingly, combinatorialists can still hardly predict when this rather rare event that an enumeration problem has a nice and elegant formula occurs.

A good example to illustrate this (and also to give an idea what an exceptional nice, elegant and compact enumeration formula is) are *alternating sign matrices (ASMs)* [1], which are defined as square matrices with entries in $\{0, 1, -1\}$ such that in each row and column the non–zero entries alternate in sign and add up to one. For instance,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is an ASM. In 1996 Zeilberger [11] was the first who showed that the number of $n \times n$ ASMs is given by the following beautiful product formula

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{1}$$

However, it is still not well understood, why these strangely defined objects have such a compact enumeration formula, whereas for other objects that have a definition of comparable or even less complexity it is simply impossible to write down just any explicit formula of tolerable complexity.

Remarkably, this behaviour extends to many other combinatorial objects such as symmetry classes of ASMs, as well as to *plane partitions (PPs)* and symmetry classes thereof, *rhombus tilings* of various regions, different kinds of *tableaux*, families of *non–intersecting lattice paths*, *fully packed loop configurations* etc. However, the significance of these objects is not only due to the fact their enumeration subject to a variety of different constraints leads to enumeration formulas of compelling simplicity, but also due to their close relations to other areas such as representation theory of classical groups and statistical mechanics.

To be a bit more concrete on what is considered to be a "nice" enumeration formula in this field, let us remark that we are most satisfied if we find formulas that are products of quotients of factorials – as it is the case for ASMs. (Formulas of this type are easy to detect, since in this case the numbers have only small prime factors relative to the size of the parameters.) A bit inferior to these beautiful product formulas are sums of such products – the fewer symbols involved the better. (However, sums are much harder to detect.)

A further indication that not all is already said and done concerning the enumeration of these combinatorial objects is the following: in case that an enumeration problem in this area admits a simple enumeration formula, it is usually possible to guess it by considering small instances of the parameter i and, as a matter of fact, these guesses are almost always correct (in fact I have never seen a counterexample); so it is the standard situation that we know that a certain enumeration formula is true long before someone finds a (in many cases highly nontrivial) proof. (Also (1) was conjectured by Mills, Robbins and Rumsey [9] at the beginning of the 1980s long before it was finally proved.) Is there any explanation for this behaviour? Moreover it can be observed that the proofs often do not give us as much insight that we have a good ability to find new results easily: in many cases, the enumeration problems that allow simple formulas are found by chance. All in all one gets the impression that the proofs in this field rather act as confirmations than as explanations (see also the preface of [1]) and the best approach to treat these problems still has to be discovered.

My current research is centered around the enumeration of PPs, ASMs and related objects [1] – in particular I want to contribute to the understanding of the phenomena mentioned above. One of the reasons for the fact that it is hard to predict when an enumeration problem allows a nice formula is that enumerative combinatorics is nowadays a collection of many different approaches and does not have too many unified methods. Unified methods would make it more easy to compare the various problems. In my habilitation thesis [3] I have already undertaken an attempt to develop a unified approach to the enumeration of PPs and ASMs, where I have emphasized and heavily made use of the fact that the enumeration formulas in this field are often polynomials in certain parameters (if others are fixed).We plan to further develop and apply the ideas and methods presented there. The problems we will consider can roughly be divided into the following three topics.

(1) **Combinatorial analysis of monotone triangles and related objects.** In [5, 6] I have started an analysis of monotone triangles (these are objects that are equivalent to ASMs) to finally give an alternative and elementary proof of the refined ASM theorem [12]. However, recent conjectures of Romik [10] and results of Fischer and Romik [7, 8] indicate that this analysis can be pushed much further and most likely leads to more refined enumerations of ASMs. Moreover, we plan to carry out an analog analysis for related object, thereby attacking some open problems in this field.

(2) **The geometry of simple enumeration formulas.** The second topic concerns the further developement of the polynomial approach to enumeration as presented in [2, 3, 4]. I propose to take the following point of view: the combinatorial objects under consideration are defined or can be translated into planar arrays of integers with certain monotonicity conditions along rows, columns and/or diagonals. This opens the possibilty to furthermore translate these problems into problems of enumerating the integer points in certain rational polytopes. There exists a rather extensive theory on the enumeration of integer points in rational polytopes, which was so far (to the best of my knowledge) not applied to the enumeration of PPs, ASMs and related objects. For instance, there is a multidimensional generalization of Ehrhart's theorem stating that the enumeration of integer points in rational polytopes (with respect to certain parameters) always leads to quasipolynomial enumeration formulas. Remarkably, this explains the quasipolynomiality of the enumeration formulas concerning PPs, ASMs and related objects all at once. There even exists a theorem that characterizes the quasipolynomials (it is usually more than one) that arise for a particular rational polytope. In this part, we will investigate applications of this theory to the enumeration of PPs, ASMs and related objects. Is there any geometric explanation for a nice enumeration formula?

(3) **Bijections.** The combinatorial objects under consideration, i.e. PPs, rhombus tilings, tableaux, non–intersecting lattice paths, ASMs, etc., are highly related objects, however, not all of these relations are so far well explained. The most prominent mystery is that there is the same number of $2n \times 2n \times 2n$ totally symmetric self–complementary PPs as there is of $n \times n$ ASMs, but so far nobody was able to give a bijective proof of this fact and to construct such a bijection is currently one of the most challenging open problems in this field. However, there are many other pairs of objects that enjoy the same property – in this part we systematically study such problems.

[1]The term "generalized partitions" in the title refers to these objects and is motivated by the fact that the combinatorial objects under consideration are usually equivalent to planar arrays of integers with certain monotonicity conditions along rows, columns and/or diagonals.

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