Exercises on “Symmetric Functions”

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1. For a poset P, let $\mathcal{J}_f(P)$ be the set of finite order ideals. Argue that this is a lattice and show that $J_f(\mathbb{Z}^2_{\geq 0})$ is isomorphic to the poset of partitions ordered by inclusion of the corresponding Young diagrams (see Section 7.2 of EC2).

2. Prove that the partitions of n equipped with the dominance order are a lattice. Show also that conjugating the partition is an anti-automorphism.

3. Construct an order of the partitions such that $M_{\lambda,\mu}$ is a triangular matrix (that is a missing detail in Theorem 7.4.4).

4. Let $e(n)$ and $o(n)$ be as in the lecture, i.e., $e(n)$ is the number of partitions of n with an even number if even parts and $o(n)$ is the number of partitions of n with an odd number of even parts, and $k(n)$ be the number of self-conjugate partitions of n. Show that
$$(x-1)^{e(n)}(x+1)^{o(n)} = (x^2-1)^{o(n)}(x-1)^{k(n)}.$$

5. Proposition 7.7.4: Argue that
$$\exp \sum_{n\geq 1} \frac{1}{n} p_n(x)p_n(y) = \sum_{\lambda} z_\lambda^{-1} p_\lambda(x)p_\lambda(y),$$
for instance, by providing (and proving) a more general underlying identity.

6. Prove at least three of the entries of the table in Proposition 7.8.3.

7. Express $h_{n+1}(x_1,\ldots,x_n)$ as a polynomial in the $h_0(x_1,\ldots,x_n),\ldots,h_n(x_1,\ldots,x_n)$ for $n = 1,2,3,4,5$. (The use of a computer algebra system is encouraged!) Do you see a pattern? Why isn't $h_{n+1}(x_1,\ldots,x_n) = 0$ although $e_{n+1}(x_1,\ldots,x_n) = 0$ and $\omega(e_{n+1}(x_1,x_2,\ldots)) = h_{n+1}(x_1,x_2,\ldots)$?

8. Example 7.8.5: Let $F(x) = \prod_i (1-x_i)^{-1} \prod_{i<j}(1-x_ix_j)^{-1}$. We have shown that $ex(F(x)) = e^{t+t^2/2}$.

a) Show that $e^{t+t^2/2}$ is the exponential generating function for the number of involutions $e_2(n)$ in $\mathfrak{S}_n$.

b) Show that $[x_1 \cdots x_n]F(x) = e_2(n)$.

9. Find (at least three) generalizations of the combinatorial interpretations of Schur functions in Proposition 7.10.3 in terms of skew Schur functions $f^{\lambda/\mu}$. Illustrate them using a small example.

10. Let q be an indeterminate. Prove the following Schur function expansion:

$$\sum_{\mu \vdash n} q^{\ell(\mu)-1} m_\mu = \sum_{j=0}^{n-1} (q-1)^j s_{n-j,1^j}.$$