

# Reconstructing Jacobi matrices from three spectra

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**Abstract.** Cut a Jacobi matrix into two pieces by removing the  $n$ -th column and  $n$ -th row. We give necessary and sufficient conditions for the spectra of the original matrix plus the spectra of the two submatrices to uniquely determine the original matrix. Our result contains Hochstadt's theorem as a special case.

## 1. Introduction

The topic of this paper is inverse spectral theory for Jacobi matrices, that is, matrices of the form

$$(1.1) \quad H = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-2} & b_{N-1} & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix}$$

This is an old problem closely related to the moment problem (see [9] and the references therein), which has attracted considerable interest recently (see, e.g., [1] and the references therein, [3], [4], [8]). For analogous results in the case of Sturm-Liouville operators see [2], [6], and [7]. In this note we want to investigate the following question: Remove the  $n$ -th row and the  $n$ -th column from  $H$  and denote the resulting submatrices by  $H_-$  (from  $b_1$  to  $b_{n-1}$ ) respectively  $H_+$  (from  $b_{n+1}$  to  $b_N$ ). When do the spectra of these three matrices determine the original matrix  $H$ ? We will show that this is the case if and only if  $H_-$  and  $H_+$  have no eigenvalues in common.

From a physical point of view such a model describes a chain of  $N$  particles coupled via springs and fixed at both end points (see [11], Section 1.5). Determining the eigenfrequencies of this system and the one obtained by keeping one particle fixed, one can uniquely reconstruct the masses and spring constants. Moreover, these results can be applied to completely integrable systems, in particular the Toda lattice (see e.g., [11]).

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## 2. Main result

To set the stage let us introduce some further notation. We denote the spectra of the matrices introduced in the previous section by

$$(2.1) \quad \sigma(H) = \{\lambda_j\}_{j=1}^N, \quad \sigma(H_-) = \{\mu_k^-\}_{k=1}^{n-1}, \quad \sigma(H_+) = \{\mu_l^+\}_{l=1}^{N-n}.$$

Moreover, we denote by  $(\mu_j)_{j=1}^{N-1}$  the ordered eigenvalues of  $H_-$  and  $H_+$  (listing common eigenvalues twice) and recall the well-known formula (see [1], Theorem 2.4 and Theorem 2.8)

$$(2.2) \quad g(z, n) = -\frac{\prod_{j=1}^{N-1} (z - \mu_j)}{\prod_{j=1}^N (z - \lambda_j)} = \frac{-1}{z - b_n + a_n^2 m_+(z, n) + a_{n-1}^2 m_-(z, n)},$$

where  $g(z, n)$  are the diagonal entries of the resolvent  $(H - z)^{-1}$  and  $m_{\pm}(z, n)$  are the Weyl  $m$ -functions corresponding to  $H_-$  and  $H_+$ . The Weyl functions  $m_{\pm}(z, n)$  are Herglotz and hence have a representation of the following form

$$(2.3) \quad m_-(z, n) = \sum_{k=1}^{n-1} \frac{\alpha_k^-}{\mu_k^- - z}, \quad \alpha_k^- > 0, \quad \sum_{k=1}^{n-1} \alpha_k^- = 1,$$

$$(2.4) \quad m_+(z, n) = \sum_{l=1}^{N-n} \frac{\alpha_l^+}{\mu_l^+ - z}, \quad \alpha_l^+ > 0, \quad \sum_{l=1}^{N-n} \alpha_l^+ = 1.$$

With this notation our main result reads as follows

**Theorem 2.1.** *To each Jacobi matrix  $H$  we can associate spectral data*

$$(2.5) \quad \{\lambda_j\}_{j=1}^N, \quad (\mu_j, \sigma_j)_{j=1}^{N-1},$$

where  $\sigma_j = +1$  if  $\mu_j \in \sigma(H_+) \setminus \sigma(H_-)$ ,  $\sigma_j = -1$  if  $\mu_j \in \sigma(H_-) \setminus \sigma(H_+)$ , and

$$(2.6) \quad \sigma_j = \frac{a_n^2 \alpha_l^+ - a_{n-1}^2 \alpha_k^-}{a_n^2 \alpha_l^+ + a_{n-1}^2 \alpha_k^-}$$

if  $\mu_j = \mu_k^- = \mu_l^+$ .

Then these spectral data satisfy

- (i)  $\lambda_1 < \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots < \lambda_N$ ,
- (ii)  $\sigma_j = \sigma_{j+1} \in (-1, 1)$  if  $\mu_j = \mu_{j+1}$  and  $\sigma_j \in \{\pm 1\}$  if  $\mu_j \neq \mu_i$  for  $i \neq j$

and uniquely determine  $H$ . Conversely, for every given set of spectral data satisfying (i) and (ii), there is a corresponding Jacobi matrix  $H$ .

*Proof.* We first consider the case where  $H_-$  and  $H_+$  have no eigenvalues in common. The interlacing property (i) is equivalent to the Herglotz property of  $g(z, n)$ .

Furthermore, the residues  $\alpha_i^-$  can be computed from (2.2)

$$(2.7) \quad \frac{\prod_{j=1}^N (z - \lambda_j)}{\prod_{k=1}^{n-1} (z - \mu_k^-) \prod_{l=1}^{N-n} (z - \mu_l^+)} = z - b_n + a_n^2 \sum_{l=1}^{N-n} \frac{\alpha_l^+}{z - \mu_l^+} + a_{n-1}^2 \sum_{k=1}^{n-1} \frac{\alpha_k^-}{z - \mu_k^-}.$$

and are given by  $\alpha_i^- = a_{n-1}^{-2} \beta_i^-$ , where

$$(2.8) \quad \beta_i^- = - \frac{\prod_{j=1}^N (\mu_i^- - \lambda_j)}{\prod_{l \neq i} (\mu_i^- - \mu_l^-) \prod_{l=1}^{N-n} (\mu_i^- - \mu_l^+)}, \quad a_{n-1}^2 = \sum_{i=1}^{n-1} \beta_i^-.$$

Similarly,  $\alpha_l^+ = a_n^{-2} \beta_l^+$ , where

$$(2.9) \quad \beta_l^+ = - \frac{\prod_{j=1}^N (\mu_l^+ - \lambda_j)}{\prod_{k=1}^{n-1} (\mu_l^+ - \mu_k^-) \prod_{p \neq l} (\mu_l^+ - \mu_p^+)}, \quad a_n^2 = \sum_{l=1}^{N-n} \beta_l^+.$$

Hence  $m_{\pm}(z, n)$  are uniquely determined and thus  $H_{\pm}$  by standard results from the moment problem. The only remaining coefficient  $b_n$  follows from the well-known trace formula

$$(2.10) \quad b_n = \operatorname{tr}(H) - \operatorname{tr}(H_-) - \operatorname{tr}(H_+) = \sum_{j=1}^N \lambda_j - \sum_{k=1}^{n-1} \mu_k^- - \sum_{l=1}^{N-n} \mu_l^+.$$

Conversely, suppose we have the spectral data given. Then we can define  $a_n, a_{n-1}, b_n, \alpha_k^-, \alpha_l^+$  as above. By (i),  $\alpha_k^-$  and  $\alpha_l^+$  are positive and hence give rise to  $H_{\pm}$ . Together with  $a_n, a_{n-1}, b_n$  we have thus defined a Jacobi matrix  $H$ . By construction, the eigenvalues  $\mu_k^-, \mu_l^+$  are the right ones and also (2.2) holds for  $H$ . Thus  $\lambda_j$  are the eigenvalues of  $H$ , since they are the poles of  $g(z, n)$ .

Next we come to the general case where  $\mu_{j_0} = \mu_{k_0}^- = \mu_{l_0}^+ (= \lambda_{j_0})$  at least for one  $j_0$ . Now some factors in the left hand side of (2.7) will cancel and we can no longer compute  $\beta_{k_0}^-, \beta_{l_0}^+$ , but only  $\gamma_{j_0} = \beta_{k_0}^- + \beta_{l_0}^+$ . However, by definition of  $\sigma_{j_0}$  we have

$$(2.11) \quad \beta_{k_0}^- = \frac{1 - \sigma_{j_0}}{2} \gamma_{j_0}, \quad \beta_{l_0}^+ = \frac{1 + \sigma_{j_0}}{2} \gamma_{j_0}.$$

Now we can proceed as before to see that  $H$  is uniquely determined by the spectral data.

Conversely, we can also construct a matrix  $H$  from given spectral data, but it is no longer clear that  $\lambda_j$  is an eigenvalue of  $H$  unless it is a pole of  $g(z, n)$ . However, in the case  $\lambda_{j_0} = \mu_{k_0}^- = \mu_{l_0}^+$  we can glue the eigenvectors  $u_-$  of  $H_-$  and  $u_+$  of  $H_+$  to give an eigenvector  $(u_-, 0, u_+)$  corresponding to  $\lambda_{j_0}$  of  $H$ .  $\square$

The special case where we remove the first row and the first column (in which case  $H_-$  is not present) corresponds to Hochstadt's theorem [5]. Similar results for (quasi-)periodic Jacobi operators can be found in [10].

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