SCATTERING THEORY FOR JACOBI OPERATORS WITH QUASI-PERIODIC BACKGROUND

IRYNA EGOROVA, JOHANNA MICHOR, AND GERALD TESCHL

ABSTRACT. We develop direct and inverse scattering theory for Jacobi operators which are short range perturbations of quasi-periodic finite-gap operators. We show existence of transformation operators, investigate their properties, derive the corresponding Gel'fand-Levitan-Marchenko equation, and find minimal scattering data which determine the perturbed operator uniquely.

1. Introduction

Classical scattering theory deals with the reconstruction of a given Jacobi operator

$$Hu(n) = a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n), \tag{1.1}$$

which is a short range perturbation of the *free* one H_0 associated with the coefficients $a(n) = \frac{1}{2}$, $b(n) = 0$. This case has been first developed on an informal level by Case in a series of papers [5] – [10]. The first rigorous results were established by Guseinov [18], who gave necessary and sufficient conditions for the scattering data to determine H uniquely under the assumption

$$\sum_n |n| \left(|a(n) - \frac{1}{2}| + |b(n)| \right) < \infty. \tag{1.2}$$

Further extensions were made by Guseinov [19], [20], and Teschl [27]. Additional details and further references can be found, e.g., in [28].

In addition to being of interest on its own, scattering theory can also be used to solve the initial value problem for the Toda equation via the inverse scattering transform. This has been formally developed by Flaschka [14] (see also [29] and [11] for the case of rapidly decaying sequences) who also worked out the inverse procedure in the reflection-less case. Further results and an extension of the method to the entire Toda hierarchy were given by Teschl in [26] and [27].

The next interesting problem is to replace the free Hamiltonian H_0 by one with a periodic potential. First results in the case of Sturm-Liouville operators have been obtained by Firsova in a series of papers (see [13]). For further results, including potentials with different spatial asymptotics, and additional references see Gesztesy et al. [16]. In the discrete case, the investigation has only recently been started by Boutet de Monvel and Egorova [2] and by Volberg and Yuditskii [31], who treat the case where H has a homogeneous spectrum and is of Szegö class exhaustively

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from an operator point of view. Applications to the Toda lattice can be found in Bazargan and Egorova [1] and Boutet de Monvel and Egorova [3].
Finally, let us give a brief overview of the paper:

Section 2 collects some well-known facts from Riemann surfaces and introduces the necessary notation. Section 3 introduces the Baker-Akhiezer function and investigates the quasi-momentum map. In the periodic case, where the integrals can be explicitly computed, this was first done in [24]. In addition, we characterize the second solution at the band edges. In Section 4 we prove existence of Jost solutions and use them to characterize the spectrum of the perturbed operator. In the periodic case, existence of Jost solutions was first shown by Geronimo and Van Assche [17] and the fact that there are only finitely many eigenvalues in each gap was first proven in Cojuhari [12] and later rediscovered in Teschl [25]. Section 5 introduces the transformation operator and proves the crucial decay estimate on its coefficients. This was first done by Boutet de Monvel and Egorova [2] in the periodic case under the additional assumption that all spectral gaps are open. We fix a problem in the original proof and at the same time simplify and streamline the argument. Section 6 investigates the scattering matrix. Our main result here is the reconstruction of the transmission coefficient from the reflection coefficient, which was not known previously even in the periodic case. Section 7 derives the Gel'fand-Levitan-Marchenko equation and proves positivity of the Gel'fand-Levitan-Marchenko operator. In addition, we formulate necessary conditions for the scattering data to uniquely determine our Jacobi operator. Our final Section 8 shows that our necessary conditions for the scattering data are also sufficient. It should be mentioned that, due to the lack of continuity with respect to the spacial variable n, a significant change in the strategy of the original proof in the continuous case from [22] is needed.

Our approach uses heavily the fact that the Baker-Akhiezer function is a meromorphic function on the Riemann surface associated with the problem. This strategy gives a more streamlined treatment and more elegant proofs even in the special cases which were previously known. In this respect it is important to emphasize that, in contradistinction to the constant background case, the upper sheet of our Riemann surface is not simply connected and in particular not isomorphic to the unit disc.

2. Quasi-periodic finite-gap operators and Riemann surfaces

To set the stage let $\mathbb{M}$ be the Riemann surface associated with the following function

$$R_{2g+2}^{1/2}(z), \qquad R_{2g+2}(z) = \prod_{j=0}^{2g+1} (z - E_j), \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{2.1}$$

$g \in \mathbb{N}$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. We will choose $R_{2g+2}^{1/2}(z)$ as the fixed branch

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j}, \tag{2.2}$$

where $\sqrt{.}$ is the standard root with branch cut along $(-\infty, 0)$.

A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$, or $p = \infty_\pm$, and the projection onto $\mathbb{C} \cup \{\infty\}$ by $\pi(p) = z$. The points $\{(E_j, 0), 0 \le j \le 2g+1\} \subseteq \mathbb{M}$ are called branch points and the sets

$$\Pi_\pm = \{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C} \backslash \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]\} \subset \mathbb{M} \tag{2.3}$$

are called upper, lower sheet, respectively.

Let $\{a_j, b_j\}_{j=1}^g$ be loops on the surface $\mathbb{M}$ representing the canonical generators of the fundamental group $\pi_1(\mathbb{M})$. We require a_j to surround the points E_{2j-1}, E_{2j} (thereby changing sheets twice) and b_j to surround E_0, E_{2j-1} counter-clock wise on the upper sheet, with pairwise intersection indices given by

$$a_i \circ a_j = b_i \circ b_j = 0, \qquad a_i \circ b_j = \delta_{ij}, \qquad 1 \le i, j \le g. \tag{2.4}$$

The corresponding canonical basis $\{\zeta_j\}_{j=1}^g$ for the space of holomorphic differentials can be constructed by

$$\underline{\zeta} = \sum_{j=1}^{g} \underline{c}(j) \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}}, \tag{2.5}$$

where the constants $\underline{c}(.)$ are given by

$$c_j(k) = C_{jk}^{-1}, \qquad C_{jk} = \int_{a_k} \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}} = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1} dz}{R_{2g+2}^{1/2}(z)} \in \mathbb{R}.$$

The differentials fulfill

$$\int_{a_j} \zeta_k = \delta_{j,k}, \qquad \int_{b_j} \zeta_k = \tau_{j,k}, \qquad \tau_{j,k} = \tau_{k,j}, \qquad 1 \le j, k \le g. \tag{2.6}$$

Now pick g numbers (the Dirichlet eigenvalues)

$$(\hat{\mu}_j)_{j=1}^g = (\mu_j, \sigma_j)_{j=1}^g \tag{2.7}$$

whose projections lie in the spectral gaps, that is, $\mu_j \in [E_{2j-1}, E_{2j}]$. Associated with these numbers is the divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ which is one at the points $\hat{\mu}_j$ and zero else. Using this divisor we introduce

$$\underline{z}(p, n) = \underline{\hat{A}}_{p_0}(p) - \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - n \underline{A}_{\infty_-}(\infty_+) - \underline{\hat{\Xi}}_{p_0} \in \mathbb{C}^g, \quad \underline{z}(n) = \underline{z}(\infty_+, n), \tag{2.8}$$

where $\underline{\Xi}_{p_0}$ is the vector of Riemann constants

$$\hat{\Xi}_{p_0,j} = \frac{1 - \sum_{k=1}^{g} \tau_{j,k}}{2}, \qquad p_0 = (E_0, 0), \tag{2.9}$$

and $\underline{A}_{p_0}$ ($\underline{\alpha}_{p_0}$) is Abel's map (for divisors). The hat indicates that we regard it as a (single-valued) map from $\hat{M}$ (the fundamental polygon associated with $\mathbb{M}$) to $\mathbb{C}^g$. We recall that the function $\theta(\underline{z}(p, n))$ has precisely g zeros $\hat{\mu}_j(n)$ (with $\hat{\mu}_j(0) = \hat{\mu}_j$), where $\theta(\underline{z})$ is the Riemann theta function of $\mathbb{M}$.

Then our Jacobi operator H_q is given by

$$\begin{aligned} a_q(n)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1))\theta(\underline{z}(n-1))}{\theta(\underline{z}(n))^2}, \\ b_q(n) &= \tilde{b} + \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln \Big(\frac{\theta(\underline{w} + \underline{z}(n))}{\theta(\underline{w} + \underline{z}(n-1))} \Big) \Big|_{\underline{w}=0}. \end{aligned} \tag{2.10}$$

The constants $\tilde{a}$, $\tilde{b}$ depend only on the Riemann surface and will be defined in the next section.

It is well known that the spectrum of H_q is purely absolutely continuous and consists of $g+1$ bands

$$\sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]. \tag{2.11}$$

For further information and proofs we refer to [28], Section 9.

3. The Baker-Akhiezer function and the quasi-momentum map

The Baker-Akhiezer function $\psi_q(p,n) = \psi_q(p,n,0)$ is given by

$$\psi_q(p,n,n_0) = \sqrt{\frac{\theta(\underline{z}(n_0-1))\theta(\underline{z}(n_0))}{\theta(\underline{z}(n-1))\theta(\underline{z}(n))}} \frac{\theta(\underline{z}(p,n))}{\theta(\underline{z}(p,n_0))} \exp\Big((n-n_0)\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\Big), \tag{3.1}$$

where $\omega_{\infty_+,\infty_-}$ is the normalized Abelian differential of the third kind with simple poles at $\infty_\pm$ and residues ± 1, respectively. They are normalized such that $\psi_q(p,n_0,n_0) = 1$.

The two branches

$$\psi_{q,\pm}(z,n) = \prod_{j=0}^{n-1} \phi_{q,\pm}(z,j) \tag{3.2}$$

of the Baker-Akhiezer function are solutions of $\tau_q u = zu$, $z \in \mathbb{C}$, where τ_q is the difference expression associated with H_q and ([28], (8.87))

$$\phi_{q,\pm}(z,n) = \frac{1}{2a_q(n)}\Big(z - b_q(n) + \sum_{j=1}^{g} \frac{\hat{R}_j(n)}{z-\mu_j(n)} \pm \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^{g}(z-\mu_j(n))}\Big), \tag{3.3}$$

$$R_j(n) = \frac{R_{2g+1}^{1/2}(\mu_j(n))}{\prod_{k\neq j}(\mu_j(n)-\mu_k(n))}, \qquad \hat{R}_j(n) = \sigma_j(n) R_j(n).$$

However, the Wronskian

$$W_q(\psi_{q,-}(z),\psi_{q,+}(z)) = \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^{g}(z-\mu_j)} \tag{3.4}$$

($\mu_j = \mu_j(0)$) shows that they are linearly dependent at the band edge E_j, $0 \le j \le 2g+1$.

The branch $\psi_{q,\sigma_j}(z,n)$ has a first order pole at μ_j if μ_j is away from the band edges

$$\lim_{z\to\mu_j} (z-\mu_j)\psi_{q,\sigma_j}(z,n) = \psi_{q,\sigma_j}(\mu_j,n,1)\frac{\hat{R}_j(0)}{a_q(0)} \tag{3.5}$$

(use (3.3) and $\psi_{q,\pm}(z,n) = \psi_{q,\pm}(z,n,1)\phi_{q,\pm}(z,0)$) and both branches have a square root singularity if μ_j coincides with a band edge E_l

$$\lim_{z\to\mu_j} \sqrt{z-\mu_j}\,\psi_{q,\pm}(z,n) = \pm\frac{\mathrm{i}^l \prod_{k\neq l}\sqrt{|E_l-E_k|}}{2a_q(0)\prod_{k\neq j}\sqrt{E_l-\mu_k}}\psi_{q,+}(E_l,n,1). \tag{3.6}$$

Lemma 3.1. *The solutions of $\tau_q u = zu$ can be characterized as follows.*
(i) If $R_{2g+2}(z) \neq 0$, there exist two solutions satisfying

$$\psi_{q,\pm}(z,n) = \theta_\pm(z,n) w(z)^{\pm n}, \quad w(z) = \exp\Big(\int_{p_0}^{(z,+)} \hat{\omega}_{\infty_+,\infty_-} \Big), \tag{3.7}$$

with $\theta_\pm(z,n)$ quasi-periodic.
(ii) If $R_{2g+2}(z) = 0$, $z = E_l$, there are two solutions satisfying

$$\psi_q(E_l,n) = \psi_{q,+}(E_l,n) = \psi_{q,-}(E_l,n), \qquad \hat{\psi}_q(E_l,n) = \psi_q(E_l,n)(\hat{\theta}_l(n) + n), \tag{3.8}$$

where $\hat{\theta}_l(n)$ is quasi-periodic.

Proof. (ii). We construct a second linearly independent solution at $z = E = E_l$ using (see [28], (1.50))

$$s_q(E,n) = \lim_{z\to E} a_q(0) \frac{\psi_{q,+}(z,n) - \psi_{q,-}(z,n)}{W(\psi_{q,-}(z), \psi_{q,+}(z))}, \tag{3.9}$$

where $s_q(z,n)$ denotes the fundamental solution of $\tau_q u = zu$ with initial conditions $s_q(z,0) = 0$, $s_q(z,1) = 1$. W.l.o.g. we assume that E_l does not coincide with one of the Dirichlet eigenvalues μ_j (otherwise shift the base point). To derive an expression for $\psi_{q,\pm}(z)$ at $z = E + \epsilon^2$ we start with

$$R_{2g+2}^{1/2}(z) = \epsilon(\tilde{R} + O(\epsilon^2)), \qquad \tilde{R} = -\prod_{j\neq l} \sqrt{E - E_j}.$$

Moreover,

$$W_q(\psi_{q,-}(z), \psi_{q,+}(z)) = \frac{\tilde{R}}{\prod_{j=1}^g (E-\mu_j)} \epsilon(1 + O(\epsilon^2))$$

and for $p = (E + \epsilon^2, \pm)$ (see (3.11) below)

$$\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-} = \int_{p_0}^{E} \hat{\omega} \pm \beta\epsilon + O(\epsilon^3), \qquad \beta = \frac{2\prod_{j=1}^g (E-\lambda_j)}{\tilde{R}},$$

$$\underline{z}(p,n) = \underline{z}(E,n) \pm \underline{\gamma}\,\epsilon + O(\epsilon^3), \qquad \underline{\gamma} = \sum_{j=1}^{g} \underline{c}(j) \frac{2E^{j-1}}{\tilde{R}},$$

and

$$\theta(\underline{z}(p,n)) = \theta(\underline{z}(E,n)) \pm \frac{\partial\theta}{\partial \underline{z}}(\underline{z}(E,n))\,\underline{\gamma}\,\epsilon + O(\epsilon^3).$$

Using this to evaluate the limit $\varepsilon \to 0$ shows

$$s_q(E,n) = 2a_q(0) \prod_{j=1}^{g} \frac{E-\mu_j}{E-\lambda_j} \hat{\psi}_q(E,n) = \psi_q(E,n)(\hat{\theta}(n) + n),$$

where

$$\hat{\theta}(n) = \frac{1}{\prod_{j=1}^g (E-\lambda_j)} \sum_{j,k=1}^{g} E^j c_k(j) \frac{\partial}{\partial w_k} \ln\theta(\underline{z}(E,n) + \underline{w}),$$

and finishes the proof. $\square$

Remark 3.2. *(i). Since $\psi_q(z,n)$ has a singularity if $z = \mu_j$ the solutions in Lemma 3.1 are not well-defined for those z. However, you can either remove the singularities of $\psi_q(z,n)$ or choose a different normalization point $n_0 \neq 0$ to see that solutions of the above type exist for every z.*
(ii). In the periodic case Floquet theory tells you that there are two possible cases at a band edge: Either two (linearly independent) periodic solutions or one periodic and one linearly growing solution. The above lemma shows that the first case happens if the corresponding gap is closed and the second if the gap is open.

To understand the properties of $\psi_{q,\pm}(z,n)$ we need to investigate the quasi-momentum map

$$w(z) = \exp\Big(\int_{p_0}^{p} \hat\omega_{\infty_+,\infty_-}\Big), \qquad p = (z,+). \tag{3.10}$$

The differential $\omega_{\infty_+,\infty_-}$ is given by

$$\omega_{\infty_+,\infty_-} = \frac{\prod_{j=1}^{g}(\pi-\lambda_j)}{R_{2g+2}^{1/2}} d\pi, \tag{3.11}$$

where the constants λ_j have to be determined from the normalization

$$\int_{a_j} \omega_{\infty_+,\infty_-} = 2 \int\limits_{E_{2j-1}}^{E_{2j}} \frac{\prod_{j=1}^{g}(z-\lambda_j)}{R_{2g+2}^{1/2}(z)} dz = 0, \tag{3.12}$$

which shows $\lambda_j \in (E_{2j-1}, E_{2j})$.

Since $\lambda_j \in (E_{2j-1}, E_{2j})$ the integrand is a Herglotz function and admits the following representation (c.f. [28], Appendix B)

$$\frac{\prod_{j=1}^{g}(z-\lambda_j)}{R_{2g+2}^{1/2}(z)} = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} d\tilde\mu(\lambda) \tag{3.13}$$

with the probability measure

$$d\tilde\mu(\lambda) = \frac{\prod_{j=1}^{g}(\lambda-\lambda_j)}{\pi\mathrm{i} R_{2g+2}^{1/2}(\lambda)} \chi_{\sigma(H_q)}(\lambda) d\lambda. \tag{3.14}$$

Hence

$$\begin{aligned} g(z,\infty) &= \int_{p_0}^{p} \omega_{\infty_+,\infty_-} = \int_{E_0}^{z} \int_{-\infty}^{\infty} \frac{1}{\lambda-\zeta} d\tilde\mu(\lambda) d\zeta \\ &= \int_{-\infty}^{\infty} \ln\left(\frac{\lambda - E_0}{\lambda - z}\right) d\tilde\mu(\lambda). \end{aligned} \tag{3.15}$$

In particular, note that $-\mathrm{Re}(g(z,\infty))$ is the Green's function of the upper sheet Π_+ with pole at ∞_+ and $\tilde\mu$ is the equilibrium measure of the spectrum (see [30], Thm. III.37). We will abbreviate $g(z) = g(z,\infty)$.

The asymptotic expansion of $\exp(g(z))$ is given by ([28], (9.42))

$$\exp\left(\int_{p_0}^{p} \hat\omega_{\infty_+,\infty_-}\right) = -\frac{\tilde a}{z}\Big(1 + \frac{\tilde b}{z} + O(\frac{1}{z^2})\Big), \qquad z \to \infty, \tag{3.16}$$

where $\tilde{a}$ is the capacity of the spectrum and

$$\tilde{b} = \frac{1}{2}\sum_{j=0}^{2g+1} E_j - \sum_{j=1}^{g} \lambda_j. \tag{3.17}$$

Theorem 3.3. *The map g is a bijection from the upper (resp. lower) half plane $\mathbb{C}^\pm = \{z \in \mathbb{C} \mid \pm\mathrm{Im}(z) > 0\}$ to*

$$S^\pm = \{z \in \mathbb{C} \mid \pm\mathrm{Re}(z) < 0, 0 < \mathrm{Im}(z) < \pi\} \backslash \bigcup_{j=1}^{g} [g(\lambda_j), g(E_{2j+1})] \tag{3.18}$$

such that $\sigma(H_q) = \{z \mid \mathrm{Re}(z) = 0\}$.

Proof. By the Herglotz property of its integrand, the function $g(z,\infty)$ satisfies the conditions of [23], Theorem 1(b) in Chapter VI, which shows that it is one-to-one.

To prove that $g(z,\infty)$ is surjective, it suffices to show that the boundary of $\mathbb{C}^+$ is mapped to the boundary of S^+. Note that $g(\lambda)$ is negative for $\lambda < E_0$ and purely imaginary for $\lambda \in [E_0, E_1]$. At E_1, the real part starts to decrease from zero until it hits its minimum at λ_1 and increases again until it becomes 0 at E_2 (since all a-periods are zero), while the imaginary part remains constant. Proceeding like this we move along the boundary of S^+ as λ moves along the real line. For $\lambda > E_{2g+1}$, $g(\lambda)$ is again negative. □

Remark 3.4. *In the special case where H_q is periodic the quasi-momentum is given by $w(z) = \exp(\mathrm{i}N^{-1}\arccos\Delta(z))$, where $\Delta(z)$ is the Floquet discriminant, and our result is due to* [24].

Therefore the map

$$\begin{aligned} w : \mathbb{C}^\pm &\to W^\pm = \{w \in \mathbb{C} \mid |w| < 1, \pm\mathrm{Im}(w) > 0\} \backslash \bigcup_{j=1}^{g} [w(\lambda_j), w(E_{2j+1})] \\ z &\mapsto \exp(g(z)) \end{aligned} \tag{3.19}$$

is bijective. Denote $W = W^+ \cup W^- \cup (-1,1)$, $W_0 = W\backslash\{0\}$. If we identify corresponding points on the slits $[w(\lambda_j), w(E_{2j+1})]$ we obtain a Riemann surface $\mathbb{W}$ which is isomorphic to the upper sheet Π_+.

Remark 3.5. *In* [24] *the largest band edge E_{2g+1} is chosen for p_0 and w will map $\mathbb{C}^\pm \to W^\mp$ in this case. Moreover, in the periodic case the slits $[w(\lambda_j), w(E_{2j+1})]$ appear at equal angles $\frac{2\pi}{N}$, where N is the period.*

Since $z \mapsto w(z) = \exp(g(z))$ is a bijection, we consider the functions $\psi_{q,\pm}$ as functions of the new parameter w whenever convenient. For notational simplicity we will write $\psi_{q,\pm}(w,n)$ for $\psi_{q,\pm}(\lambda(w),n)$ and similarly for other quantities. The functions $\psi_{q,\pm}(w,n)$ are meromorphic in $\mathbb{W}$ and continuous up to the boundary with the only possible singularities at the images of the Dirichlet eigenvalues $w(\mu_j)$ and at 0. More precisely, denote by $M_\pm$ the sets of poles (and square root singularities if $\mu_j = E_l$) of the Weyl m-functions $\tilde{m}_\pm(\lambda)$, i.e. $M_+ \cup M_- = \{\mu_j\}_{j=1}^g$ (see (3.2) and [28], Section 2.1). Note that $\mu_j \in M_+ \cap M_-$ if and only if $\mu_j = E_l$. Then

(B1) $\psi_{q,\pm}(w,n)$ are holomorphic in $\mathbb{W}\backslash(\{w(\mu_j)\}_{j=1}^g \cup \{0\})$ and continuous on $\partial W\backslash\{w(\mu_j)\}$.

(B2) $\psi_{q,\pm}(w,n)$ has a simple pole at $w(\mu_j)$ if $\mu_j \in M_\pm \backslash \{E_l\}$, no pole if $\mu_j \notin M_\pm$, and if $\mu_j = E_l$,

$$\psi_{q,\pm}(w,n) = \pm \frac{\mathrm{i}^l C(n)}{w - w_l} + O(1),$$

where $C(n)$ is bounded and real.

(B3) $\overline{\psi_{q,\pm}(w,n)} = \psi_{q,\mp}(w,n)$ for $|w| = 1$.

(B4) At $w = 0$ the following asymptotics hold

$$\psi_{q,\pm}(w,n) = (-1)^n \Big(\prod_{m=0}^{n-1}{}^{*} \ a_q(m) \Big)^{\pm 1} (\frac{w}{\tilde{a}})^{\pm n} (1 + O(w)).$$

By Section 2.5 of [28] the vector valued functions

$$\underline{U}(\lambda,n) = \sqrt{\frac{1}{4a_q(0)^2 \pi \mathrm{Im}(\tilde{m}_+(\lambda))}} \begin{pmatrix} \psi_{q,+}(\lambda,n) \\ \psi_{q,-}(\lambda,n) \end{pmatrix} \tag{3.20}$$

form an orthonormal basis for the Hilbert space $L^2(\sigma(H_q), \mathbb{C}^2, d\lambda)$. The Weyl m-functions $\tilde{m}_\pm(z)$ satisfy (see [28], eq. (8.95))

$$\mathrm{Im}(\tilde{m}_\pm(\lambda)) = \frac{\mathrm{i} R_{2g+2}^{1/2}(\lambda)}{2a_q(0)^2 \prod_{j=1}^g (\lambda - \mu_j)}, \qquad \lambda \in \sigma(H_q). \tag{3.21}$$

Using our map $w(z) = \exp(\int_{p_0}^{(z,+)} \hat{\omega}_{\infty_+,\infty_-})$ we can transform this into an orthonormal basis on the unit circle.

Lemma 3.6. *Both functions $\psi_{q,+}(w,n)$ and $\psi_{q,-}(w,n)$ form orthonormal bases in the Hilbert space $L^2(S^1, \frac{1}{2\pi\mathrm{i}} d\omega)$, where*

$$d\omega(w) = \prod_{j=1}^g \frac{\lambda(w) - \mu_j}{\lambda(w) - \lambda_j} \frac{dw}{w}. \tag{3.22}$$

Proof. Just use

$$\frac{dw}{dz} = w \frac{\prod_{j=1}^g (z - \lambda_j)}{R_{2g+2}^{1/2}(z)}. \tag{3.23}$$

□

Observe that $d\omega$ is meromorphic on $\mathbb{W}$ with a simple pole at $w = 0$. In particular, there are no poles at $w(\lambda_j)$.

Remark 3.7. *In [2] a different normalization is used. To establish the connection observe*

$$\sum_{n=1}^N \psi_{q,+}(z,n) \psi_{q,-}(z,n) = N \prod_{j=1}^{N-1} \frac{z - \lambda_j}{z - \mu_j} \tag{3.24}$$

if H_q is periodic with period N.

4. Existence of Jost solutions

After we have these preparations out of our way, we come to the study of short-range perturbations H of H_q associated with sequences a, b satisfying $a(n) \to a_q(n)$ and $b(n) \to b_q(n)$ as $|n| \to \infty$. More precisely, we will make the following assumption throughout this paper.

Hypothesis H. 4.1. *Let H be a perturbation such that*

$$\sum_{n\in\mathbb{Z}} |n| \Big(|a(n) - a_q(n)| + |b(n) - b_q(n)| \Big) < \infty. \tag{4.1}$$

We first establish existence of Jost solutions, that is, solutions of the perturbed operator which asymptotically look like the Baker-Akhiezer solutions.

Theorem 4.2. *Assume (H.4.1). Then there exist solutions $\psi_\pm(z,.)$, $z \in \mathbb{C}$, of $\tau\psi = z\psi$ satisfying*

$$\lim_{n\to\pm\infty} |w(z)^{\mp n}(\psi_\pm(z,n) - \psi_{q,\pm}(z,n))| = 0, \tag{4.2}$$

where $\psi_{q,\pm}(z,.)$ are the Baker-Akhiezer functions. Moreover, $\psi_\pm(z,.)$ are continuous (resp. holomorphic) with respect to z whenever $\psi_{q,\pm}(z,.)$ are and inherit the properties (B1) and (B2), where now $\psi_\pm(z,n) = \frac{\mathrm{i}^l C_\pm(n)}{\sqrt{z-\mu_j}} + O(1)$. (B4) has to be replaced by

$$\psi_\pm(z,n) = \frac{z^{\mp n}}{A_\pm(n)} \Big(\prod_{j=0}^{n-1}{}^{*} \; a_q(j) \Big)^{\pm 1} \Big(1 + \Big(B_\pm(n) \pm \sum_{j=1}^{n}{}^{*} \; b_q(j - {\scriptstyle 0 \atop \scriptstyle 1}) \Big) \frac{1}{z} + O(\frac{1}{z^2}) \Big), \tag{4.3}$$

where

$$\begin{aligned} A_+(n) &= \prod_{j=n}^{\infty} \frac{a(j)}{a_q(j)}, \qquad B_+(n) = \sum_{m=n+1}^{\infty} (b_q(m) - b(m)), \\ A_-(n) &= \prod_{j=-\infty}^{n-1} \frac{a(j)}{a_q(j)}, \qquad B_-(n) = \sum_{m=-\infty}^{n-1} (b_q(m) - b(m)). \end{aligned} \tag{4.4}$$

Proof. The proof can be done as in the periodic case (see e.g., [17], [25] or [28], Section 7.5). The only problem is to show that the second solution at a band edge grows at most linearly. In the periodic case this follows from Floquet theory, here we just use Lemma 3.1. □

From this result we obtain a complete characterization of the spectrum of H.

Theorem 4.3. *Assume (H.4.1). Then we have $\sigma_{ess}(H) = \sigma(H_q)$, the point spectrum of H is finite and confined to the spectral gaps of H_q, that is, $\sigma_p(H) \subset \mathbb{R}\backslash\sigma(H_q)$. Furthermore, the essential spectrum of H is purely absolutely continuous.*

Proof. Again the proof can be done as in the periodic case (see e.g., [25] or [28], Section 7.5). □

5. The transformation operator

We define the kernel of the transformation operator as the Fourier coefficients of the Jost solutions $\psi_\pm(w,n)$ with respect to the orthonormal system given in Lemma 3.6, $\{\psi_{q,\pm}(w,n)\}_{n\in\mathbb{Z}}$,

$$K_\pm(n,m) := \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} \psi_\pm(w,n)\psi_{q,\mp}(w,m) d\omega(w). \tag{5.1}$$

By the Cauchy theorem, this integral equals the residue at $w=0$,

$$K_\pm(n,m) = \mathrm{Res}_0 \frac{1}{w} \psi_\pm(w,n)\psi_{q,\mp}(w,m). \tag{5.2}$$

In particular, since $\psi_\pm(w,n)\psi_{q,\mp}(w,m) = O(w^{\pm(n-m)})$, we conclude

$$K_\pm(n,m) = 0, \qquad \pm(m-n) < 0. \tag{5.3}$$

Lemma 5.1. *Assume H.4.1. The Jost solutions $\psi_\pm(w,n)$ can be represented as*

$$\psi_\pm(w,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_{q,\pm}(w,m), \qquad |w|=1, \tag{5.4}$$

where the kernels $K_\pm(n,.)$ satisfy $K_\pm(n,m)=0$ for $\pm m < \pm n$ and

$$|K_\pm(n,m)| \le C \sum_{j=[\frac{m+n}{2}]\pm 1}^{\pm\infty} \Big(|a(j)-a_q(j)| + |b(j)-b_q(j)| \Big), \quad \pm m > \pm n. \tag{5.5}$$

The constant C depends only on H_q and the value of the sum in (4.1).

Proof. We prove the estimate for $K_+(n,m)$ and omit "+" and "z" whenever possible. Define $\varphi(n) = \psi(n)K(n,n)^{-1}$, then φ fulfills

$$\varphi(n) = \psi_q(n) + \sum_{m=n+1}^{\infty} J(n,m)\varphi(m), \tag{5.6}$$

where

$$J(z,n,m) = \tilde{a}(m-1)\frac{s_q(z,n,m-1)}{a_q(m-1)} + \tilde{b}(m)\frac{s_q(z,n,m)}{a_q(m)} \tag{5.7}$$

with the abbreviation

$$\tilde{a}(m) = \frac{a(m)^2}{a_q(m)} - a_q(m), \qquad \tilde{b}(m) = b(m) - b_q(m). \tag{5.8}$$

On the other hand, $\varphi(n)$ is given by

$$\varphi(n) = \sum_{m=n}^{\infty} \kappa(n,m)\psi_q(m), \qquad \kappa(n,m) = \frac{K(n,m)}{K(n,n)},$$

therefore

$$\sum_{m=n}^{\infty} \kappa(n,m)\psi_q(m) = \sum_{m=n+1}^{\infty} J(n,m)\psi_q(m) + \sum_{m=n+1}^{\infty}\sum_{l=m+1}^{\infty} J(n,m)\kappa(m,l)\psi_q(l). \tag{5.9}$$

Multiplying both sides of (5.9) by $\psi_{q,-}(k)$ and integrating over the unit circle yields

$$\kappa(n,k) = \sum_{m=n+1}^{\infty} \Gamma(n,m,m,k) + \sum_{m=n+1}^{\infty}\sum_{l=n+1}^{\infty} \Gamma(n,m,l,k)\kappa(m,l), \tag{5.10}$$

where

$$\Gamma(n,m,l,k) = \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} J(w,n,m)\psi_{q,+}(w,l)\psi_{q,-}(w,k)d\omega(w). \tag{5.11}$$

Using [28], (1.50),

$$\frac{s_q(n,m)}{a(m)} = \frac{\psi_{q,+}(m)\psi_{q,-}(n) - \psi_{q,+}(n)\psi_{q,-}(m)}{W(\psi_{q,+},\psi_{q,-})}, \tag{5.12}$$

we obtain

$$\Gamma(n,m,l,k) = \tilde{b}(m)\Gamma_q(n,m,l,k) + \tilde{a}(m)\Gamma_q(n,m-1,l,k) \tag{5.13}$$

with

$$\begin{aligned}
\Gamma_q(n,m,l,k) &= \Gamma_0(m,n,l,k) - \Gamma_0(n,m,l,k),\\
\Gamma_0(n,m,l,k) &= \frac{1}{2\pi\mathrm{i}} \int_{w(\gamma)} \frac{\psi_{q,+}(w,n)\psi_{q,-}(w,m)\psi_{q,+}(w,l)\psi_{q,-}(w,k)}{W(\psi_{q,+}(w),\psi_{q,-}(w))} d\omega(w)\\
&= \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \frac{\psi_{q,+}(z,n)\psi_{q,-}(z,m)\psi_{q,+}(z,l)\psi_{q,-}(z,k)}{W(\psi_{q,+}(z),\psi_{q,-}(z))} \frac{\prod(z-\mu_j)}{R_{2g+2}^{1/2}(z)} dz\\
&= \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \frac{\psi_{q,+}(z,n)\psi_{q,-}(z,m)\psi_{q,+}(z,l)\psi_{q,-}(z,k)}{W(\psi_{q,+}(z),\psi_{q,-}(z))^2} dz. \qquad (5.14)
\end{aligned}$$

Here γ is a path on the upper sheet encircling the spectrum. The integrand of Γ_0 is meromorphic on the Riemann surface $\mathbb{M}$ with poles of order one at E_j and poles of order $O(z^{\pm(n-m+l-k)-2})$ near $\infty_\pm$ (there are no poles at the Dirichlet eigenvalues μ_j). We apply the residue theorem twice, first on the side of γ including ∞_+, then on the other side including the spectrum (and thus ∞_-)

$$\begin{aligned}
\Gamma_0(n,m,l,k) &= -\mathrm{Res}_{\infty_+} \frac{\psi_{q,+}(n)\psi_{q,-}(m)\psi_{q,+}(l)\psi_{q,-}(k)}{W(\psi_{q,+},\psi_{q,-})^2}\\
&= \Big(\mathrm{Res}_{\infty_-} + \sum_{j=0}^{2g+1} \mathrm{Res}_{E_j}\Big) \Big(\frac{\psi_{q,+}(n)\psi_{q,-}(m)\psi_{q,+}(l)\psi_{q,-}(k)}{W(\psi_{q,+},\psi_{q,-})^2}\Big). \qquad (5.15)
\end{aligned}$$

The order of the poles at $\infty_\pm$ implies

$$\Gamma_0(n,m,l,k) = \begin{cases} \sum_{j=0}^{2g+1} \mathrm{Res}_{E_j} \frac{\psi_{q,+}(n)\psi_{q,-}(m)\psi_{q,+}(l)\psi_{q,-}(k)}{W(\psi_{q,+},\psi_{q,-})^2} & n-m+l-k<0\\ 0 & n-m+l-k \ge 0, \end{cases}$$

which shows that $\Gamma_0(n,m,l,k)$ is real and bounded since $\psi_{q,+}(E,.) = \psi_{q,-}(E,.)$ are (if $\mu_j = E_l$, use (B2)). Together with (5.14) this yields

$$\Gamma_0(n,m,l,k) = -\overline{\Gamma_0(m,n,k,l)} = -\Gamma_0(m,n,k,l) = -\Gamma_0(n,m,k,l). \tag{5.16}$$

Moreover,

$$\begin{aligned}
\Gamma_q(n,m,l,k) &= 0, \qquad l-k \ge |m-n|,\\
\Gamma_q(n,m,l,k) &= -\Gamma_q(m,n,k,l) \;=\; \Gamma_q(n,m,k,l), \qquad (5.17)
\end{aligned}$$

which then implies

(5.18)

$$\Gamma_q(n,m,l,k) = \begin{cases} \mathrm{sign}(n-m) \sum_{j=0}^{2g+1} \mathrm{Res}_{E_j} \frac{\psi_{q,+}(n)\psi_{q,-}(m)\psi_{q,+}(l)\psi_{q,-}(k)}{W(\psi_{q,+},\psi_{q,-})^2} & |l-k| < |m-n|\\ 0 & |l-k| \ge |m-n| \end{cases}$$

and $\Gamma(n,m,l,k)=0$ for $|l-k|\ge m-n$ if $m>n$. Note that the residue at E_j is given by

$$\frac{2\prod_{\ell=1}^{g}(E_j-\mu_\ell)^2}{\prod_{\ell\neq j}(E_j-E_\ell)}\psi_q(E_j,n)\psi_q(E_j,m)\psi_q(E_j,l)\psi_q(E_j,k). \tag{5.19}$$

Now we obtain for $\kappa(n,k)$

$$\begin{aligned}\kappa(n,k) &= \sum_{m=n+1}^{\infty}\Gamma(n,m,m,k)+\sum_{m=n+1}^{\infty}\sum_{l=m+1}^{\infty}\Gamma(n,m,l,k)\kappa(m,l)\\ &= \sum_{m=[\frac{n+k}{2}]+1}^{\infty}\Gamma(n,m,m,k)+\sum_{m=n+1}^{\infty}\sum_{l=n+k-m+1}^{m+k-n-1}\Gamma(n,m,l,k)\kappa(m,l),\end{aligned} \tag{5.20}$$

since $\Gamma(n,m,m,k)\neq 0$ only if $|m-k|<m-n$ implying $m>\frac{n+k}{2}$. In the third sum of (5.20) we need that $|m+\delta-k|<m-n$ for $\delta\ge 1$ which yields $\delta<k-n$ and $\delta>n+k-2m$. Two remarks might be in order: $m+k-n-1\ge n+k-m+1$ since $m-n\ge n-m+2$, and the starting point $l=n+k-m+1$ of the third sum actually has a lower limit, namely $m\le\frac{n+k}{2}$, since we require $l\ge m+1$ for $\kappa(m,l)\neq 0,1$. Note that

$$\begin{aligned}\left|\sum_{m=[\frac{n+k}{2}]+1}^{\infty}\Gamma(n,m,m,k)\right| &\le D\sum_{m=[\frac{n+k}{2}]+1}^{\infty}|\tilde b(m)+\tilde a(m)| =: \hat q(\tfrac{n+k}{2}),\\ \left|\sum_{l=n+k-m+1}^{m+k-n-1}|\Gamma(n,m,l,k)|\right| &\le D\,(m-n-1)|\tilde b(m)+\tilde a(m)| =: \hat c(m)\in\ell^1(\mathbb{Z}),\end{aligned}$$

where D is the estimate provided by (5.18), (5.19). We set up the following iteration procedure

$$\begin{aligned}\kappa_0(n,k) &= \sum_{m=[\frac{n+k}{2}]+1}^{\infty}\Gamma(n,m,m,k),\\ \kappa_j(n,k) &= \sum_{m=n+1}^{\infty}\sum_{l=n+k-m+1}^{m+k-n-1}\Gamma(n,m,l,k)\kappa_{j-1}(m,l).\end{aligned} \tag{5.21}$$

Then using induction one has

$$|\kappa_j(n,k)|\le\hat q(\tfrac{n+k}{2})\frac{\left(\sum_{m=n+1}^{\infty}\hat c(m)\right)^j}{j!} \tag{5.22}$$

and hence the iteration converges and implies the estimate

$$|\kappa(n,k)|=\left|\sum_{j=0}^{\infty}\kappa_j(n,k)\right|\le\hat q(\tfrac{n+k}{2})\exp\left(\sum_{m=n+1}^{\infty}\hat c(m)\right). \tag{5.23}$$

□

Associated with $K_\pm(n,m)$ is the operator

$$(\mathcal{K}_\pm f)(n)=\sum_{m=n}^{\pm\infty}K_\pm(n,m)f(m),\qquad f\in\ell^\infty_\pm(\mathbb{Z},\mathbb{C}), \tag{5.24}$$

which acts as a transformation operator for the pair τ, τ_q.

Theorem 5.2. *Let τ_q and τ be the quasi-periodic and perturbed Jacobi difference expression, respectively. Then*

$$\tau \mathcal{K}_\pm f = \mathcal{K}_\pm \tau_q f, \qquad f \in \ell^\infty_\pm(\mathbb{Z},\mathbb{C}). \tag{5.25}$$

Proof. It suffices to show that $HK_\pm = K_\pm H_q$. Indeed,

$$\begin{aligned} HK_\pm(n,m) &= \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} H\psi_\pm(w,n)\psi_{q,\mp}(w,m) d\omega(w) \\ &= \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} \lambda(w)\psi_\pm(w,n)\psi_{q,\mp}(w,m) d\omega(w) \\ &= \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} \psi_\pm(w,n) H_q \psi_{q,\mp}(w,m) d\omega(w). \end{aligned} \tag{5.26}$$

□

Lemma 5.3. *For $n \in \mathbb{Z}$ we have*

$$\begin{aligned} \frac{a(n)}{a_q(n)} &= \frac{K_+(n+1,n+1)}{K_+(n,n)} = \frac{K_-(n,n)}{K_-(n+1,n+1)}, \\ b(n) - b_q(n) &= a_q(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)} \\ &= a_q(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)}. \end{aligned} \tag{5.27}$$

Proof. Consider the equation of the transformation operator $H\mathcal{K}_\pm = \mathcal{K}_\pm H_q$, which is equivalent to (c.f. (5.26))

$$\begin{aligned} & a(n-1)K_\pm(n-1,m) + b(n)K_\pm(n,m) + a(n)K_\pm(n+1,m) = \\ & = a_q(m-1)K_\pm(n,m-1) + b_q(m)K_\pm(n,m) + a_q(m)K_\pm(n,m+1). \end{aligned}$$

Evaluating at $m = n$ we obtain the first equation and at $m = n \mp 1$ the second. □

In particular, observe

$$K_\pm(n,n) = \frac{1}{A_\pm(n)}, \qquad K_\pm(n,n\pm 1) = \frac{B_\pm(n)}{A_\pm(n) a_q(n - {0 \atop 1})}. \tag{5.28}$$

6. The scattering matrix

Let H_q be a given quasi-periodic Jacobi operator and H a perturbation of H_q satisfying Hypothesis H.4.1. To set up scattering theory for the pair (H, H_q) we proceed as usual.

The Wronskian of our Jost functions can be evaluated as $n \to \pm\infty$ and is given by

(6.1)

$$W(\psi_\pm(\lambda), \overline{\psi_\pm(\lambda)}) = W_q(\psi_{q,\pm}(\lambda), \psi_{q,\mp}(\lambda)) = \mp \frac{R^{1/2}_{2g+2}(\lambda)}{\prod_{j=1}^g (\lambda - \mu_j)}, \qquad \lambda \in \sigma(H_q).$$

Hence $\psi_\pm(\lambda)$, $\overline{\psi_\pm(\lambda)}$ are linearly independent for λ in the interior of $\sigma(H_q)$ and we consider the scattering relations

$$\psi_\pm(\lambda,n) = \alpha(\lambda)\overline{\psi_\mp(\lambda,n)} + \beta_\mp(\lambda)\psi_\mp(\lambda,n), \qquad \lambda \in \sigma(H_q), \tag{6.2}$$

where

$$\begin{aligned}
(6.3)\qquad \alpha(\lambda) &= \frac{W(\psi_\mp(\lambda),\psi_\pm(\lambda))}{W(\psi_\mp(\lambda),\overline{\psi_\mp(\lambda)})} = \frac{\prod_{j=1}^g(\lambda-\mu_j)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_-(\lambda),\psi_+(\lambda)),\\
\beta_\pm(\lambda) &= \frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\pm(\lambda),\overline{\psi_\pm(\lambda)})} = \mp\frac{\prod_{j=1}^g(\lambda-\mu_j)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)}).
\end{aligned}$$

While $\alpha(\lambda)$ is only defined for $\lambda\in\sigma(H_q)$, (6.3) may be used as a definition for $\lambda\in\mathbb{C}\backslash\{E_j\}$. Therefore $\alpha(w)$ can be continued as a holomorphic function on $\mathbb{W}$ and it is continuous up to the boundary except possibly at the band edges.

Remark 6.1. *Note that $\alpha(\lambda)$ does not depend on the normalization of $\psi_\pm(\lambda)$ at the base point $n_0=0$ whereas $\beta_\pm=\beta_{\pm,0}$ does. Using $\psi_\pm(z,n,n_0)=\psi_{q,\pm}(z,n_0)^{-1}\psi_\pm(z,n)$ and*

$$W((\psi_+(\lambda),\psi_-(\lambda)) = \prod_{j=1}^g \frac{\lambda-\mu_j(n_0)}{\lambda-\mu_j} W((\psi_+(\lambda,.,n_0),\psi_-(\lambda,.,n_0))$$

we see

$$\beta_{\pm,0}(\lambda) = \frac{\psi_{q,\mp}(\lambda,n_0)}{\psi_{q,\pm}(\lambda,n_0)}\beta_{\pm,n_0}(\lambda). \tag{6.4}$$

A direct calculation shows

$$\alpha(\overline{w}) = \overline{\alpha(w)}, \qquad \beta_\pm(\overline{w}) = \overline{\beta_\pm(w)} = -\beta_\mp(w) \tag{6.5}$$

and the Plücker identity (c.f. [28], (2.169)) implies

$$|\alpha(w)|^2 = 1 + |\beta_\pm(w)|^2, \qquad |w|=1. \tag{6.6}$$

We will denote the eigenvalues of H by

$$\sigma_p(H) = \{\rho_j\}_{j=1}^q. \tag{6.7}$$

Our next aim is to study the behavior of $\alpha(\lambda)$ at the eigenvalues ρ_j, therefore we modify the Jost solutions $\psi_\pm(\lambda,n)$ according to their poles at μ_j and define the following eigenfunctions $\hat\psi_\pm(\lambda,.)$

$$\hat\psi_+(\lambda,.) = \prod_{\mu_l\in M_+}(\lambda-\mu_l)\,\psi_+(\lambda,.), \quad \hat\psi_-(\lambda,.) = \prod_{\mu_l\in M_-\backslash\{E_j\}}(\lambda-\mu_l)\,\psi_-(\lambda,.). \tag{6.8}$$

Define $\hat\psi_{q,\pm}(\lambda,.)$ accordingly. Moreover, $\hat\psi_\pm(\rho_j,n)=c_j^\pm\hat\psi_\mp(\rho_j,n)$ with $c_j^+c_j^-=1$. The **norming constants** $\gamma_{\pm,j}$ are defined by

$$\frac{1}{\gamma_{\pm,j}} = \sum_{m\in\mathbb{Z}}|\hat\psi_\pm(\rho_j,m)|^2. \tag{6.9}$$

To compute the derivative of $\alpha(\lambda)$ at ρ_j, note that

$$\alpha(\lambda) = \frac{W(\hat\psi_-(\lambda),\hat\psi_+(\lambda))}{R_{2g+2}^{1/2}(\lambda)}. \tag{6.10}$$

By virtue of [28], Lemma 2.4,

$$\frac{d}{d\lambda}W(\hat\psi_-(\lambda),\hat\psi_+(\lambda))\Big|_{\rho_j} = -\sum_{k\in\mathbb{Z}}\hat\psi_-(\rho_j,k)\hat\psi_+(\rho_j,k) = -\frac{1}{c_j^\pm\gamma_{\pm,j}}. \tag{6.11}$$

Therefore

$$\frac{d}{d\lambda}\alpha(\lambda)\Big|_{\rho_j} = \frac{W'(\hat\psi_-(\rho_j),\hat\psi_+(\rho_j))}{R^{1/2}_{2g+2}(\rho_j)} = \frac{-1}{c_j^{\pm}\gamma_{\pm,j}R^{1/2}_{2g+2}(\rho_j)}. \tag{6.12}$$

From (6.12) we obtain a connection between the left and right norming constants

$$\gamma_{+,j}\gamma_{-,j} = \frac{1}{(\alpha'(\rho_j))^2 R_{2g+2}(\rho_j)}. \tag{6.13}$$

As a last preparation, we study the behavior of $\alpha(w)$ as $w\to 0$. By (4.3),

$$W(\psi_-(w),\psi_+(w)) = \frac{1}{A}\tilde a w^{-1} + O(w) \tag{6.14}$$

with $A = A_-(0)A_+(0)$ and

$$\frac{R^{1/2}_{2g+2}(\lambda(w))}{\prod_{j=1}^{g}(\lambda(w)-\lambda_j)} = \tilde a w^{-1} + O(1), \tag{6.15}$$

therefore $\alpha^{-1}(w)$ is bounded at 0 with

$$\alpha(0) = \prod_{j=-\infty}^{\infty}\frac{a_q(j)}{a(j)}. \tag{6.16}$$

We now define the **scattering matrix**

$$S(w) = \begin{pmatrix} T(w) & R_-(w) \\ R_+(w) & T(w)\end{pmatrix}, \qquad |w|=1, \tag{6.17}$$

where $T(w) := \alpha^{-1}(w)$ and $R_\pm(w) := \alpha^{-1}(w)\beta_\pm(w)$ are called **transmission** and **reflection coefficients**. Equations (6.5) and (6.6) imply

Lemma 6.2. *The scattering matrix $S(w)$ is unitary. The coefficients $T(w)$, $R_\pm(w)$ are bounded for $|w|=1$, continuous for $|w|=1$ except at possibly $w_l = w(E_l)$, fulfill*

$$|T(w)|^2 + |R_\pm(w)|^2 = 1, \qquad |w|=1, \tag{6.18}$$

$$T(w)R_+(\overline{w}) + T(\overline{w})R_-(w) = 0, \qquad |w|=1, \tag{6.19}$$

and $\overline{T(w)} = T(\overline{w})$, $\overline{R_\pm(w)} = R_\pm(\overline{w})$ for $|w|=1$.

Moreover, $R^{1/2}_{2g+2}(w)T(w)^{-1}$ is continuous (in particular $T(w)$ can only vanish at w_l) and

$$\begin{array}{ll} \lim\limits_{w\to w_l} R^{1/2}_{2g+2}(w)\frac{R_\pm(w)+1}{T(w)} = 0, & w_l \neq w(\mu_j) \\ \lim\limits_{w\to w_l} R^{1/2}_{2g+2}(w)\frac{R_\pm(w)-1}{T(w)} = 0, & w_l = w(\mu_j) \end{array}. \tag{6.20}$$

The transmission coefficient $T(w)$ has a meromorphic continuation to $\mathbb{W}$ with simple poles at $w(\rho_j)$,

$$\big(\mathrm{Res}_{\rho_j} T(\lambda)\big)^2 = \gamma_{+,j}\gamma_{-,j}R_{2g+2}(\rho_j). \tag{6.21}$$

In addition, $T(z)\in\mathbb{R}$ as $z\in\mathbb{R}\backslash\sigma(H_q)$ and

$$T(0) = \frac{1}{K_+(n,n)K_-(n,n)} = \prod_{j=-\infty}^{\infty}\frac{a(j)}{a_q(j)}, \tag{6.22}$$

where $K_\pm(n,n)$ are the coefficients of the transformation operators.

Proof. To show (6.20) we use the definition (6.3),

$$R_{2g+2}^{1/2}(\lambda)\frac{R_\pm(\lambda)+1}{T(\lambda)} = \prod_{j=1}^{g}(\lambda-\mu_j)\big(W(\psi_-(\lambda),\psi_+(\lambda)) \mp W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})\big).$$

There are two cases to distinguish: If $\mu_j \neq E_l$ then $\psi_\pm$ are continuous and real at $\lambda = E_l$ and the two Wronskians cancel. Otherwise, if $\mu_j = E_l$ they are purely imaginary (by property (B2) of the Jost functions) and the two terms are equal in the limit and add up. □

The sets

$$S_\pm(H) = \{R_\pm(w), |w|=1;\ (\rho_j,\gamma_{\pm,j}), 1\le j\le q\} \tag{6.23}$$

are called left/right **scattering data** for H.

First we want to show that the transmission coefficient can be reconstructed from either left or right scattering data.

Let $g(w,w_0)$ be the Green function associated with $\mathbb{W}$ and let

$$\mu(w,w_0)dw_0 = \frac{\partial g}{\partial r}(w, r\mathrm{e}^{\mathrm{i}\theta})\Big|_{r=1^-}\mathrm{e}^{\mathrm{i}\theta}d\theta, \qquad w_0=\mathrm{e}^{\mathrm{i}\theta}, \tag{6.24}$$

be the corresponding harmonic measure on the boundary (see, e.g., [30]). Since W_0 is simply connected, we can choose a function $h(w,v)$ such that $\hat{g}(w,w_0) = g(w,w_0) + \mathrm{i}h(w,w_0)$ is analytic in W_0. Clearly $\hat{g}$ is only well-defined up to an imaginary constant and it will not be analytic on $\mathbb{W}\backslash\{0\}$ in general. Similarly we can find a corresponding $\nu(w,w_0)$ and set $\hat{\mu}(w,w_0) = \mu(w,w_0) + \mathrm{i}\nu(w,w_0)$.

Theorem 6.3. *Either one of the sets $S_\pm(H)$ determines the other and $T(w)$ via the Poisson-Jensen type formula*
(6.25)

$$T(w) = \exp\left(\sum_{j=1}^{q}\hat{g}(w,w(\rho_j))\right)\exp\left(\frac{1}{2}\int_{|w|=1}\ln(1-|R_\pm(w_0)|^2)\hat{\mu}(w,w_0)dw_0\right),$$

where the constant of $\hat{g}$ has to be chosen such that $T(0)>0$, and

$$\frac{R_-(w)}{R_+(\overline{w})} = -\frac{T(w)}{T(\overline{w})}, \qquad \gamma_{+,j}\gamma_{-,j} = \frac{\big(\mathrm{Res}_{\rho_j}T(\lambda)\big)^2}{\prod_{l=0}^{2g+1}(\rho_j-E_l)}.$$

Proof. It suffices to prove the formula for $T(w)$, since evaluating the residua provides $\gamma_{\pm,j}$, together with $\{\lambda_l\}$, $\{E_l\}$. The formula for $T(w)$ holds by [32], Theorem 1, at least when taking absolute values. Since both sides are analytic and have equal absolute values, they can only differ by a constant of absolute value one. But both sides are positive at $w=0$ and hence this constant is one. □

Note that neither the Blaschke factors nor the outer function in (6.25) are single valued on $\mathbb{W}$ in general. In particular, the eigenvalues cannot be chosen arbitrarily, which was first observed in [21].

7. The Gel'fand-Levitan-Marchenko equations

In this section we want to derive a procedure which allows the reconstruction of the Jacobi operator H with asymptotically quasi-periodic coefficients from its scattering data $S_\pm(H)$. This will be achieved by deriving an equation for $K_\pm(n,m)$ which is generally known as Gel'fand-Levitan-Marchenko equation.

Since $K_\pm(n,m)$ are essentially the Fourier coefficients of the Jost solutions $\psi_\pm(w,n)$ we compute the Fourier coefficients of the scattering relations (6.2). Therefore we multiply

$$T(w)\psi_\mp(w,n) = R_\pm(w)\psi_\pm(w,n) + \overline{\psi_\pm(w,n)} \tag{7.1}$$

by $(2\pi\mathrm{i})^{-1}\psi_{q,\pm}(w,m)d\omega$, where $\pm m \geq \pm n$, and integrate around the unit circle. First we evaluate the right hand side of (7.1) using (5.1)

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}}\int_{|w|=1} \overline{\psi_+(w,n)}\psi_{q,+}(w,m)d\omega(w) &= K_+(n,m), \\ \frac{1}{2\pi\mathrm{i}}\int_{|w|=1} R_+(w)\psi_+(w,n)\psi_{q,+}(w,m)d\omega(w) &= \sum_{l=n}^{\infty} K_+(n,l)\tilde{F}^+(l,m), \end{aligned} \tag{7.2}$$

where

$$\tilde{F}^+(l,m) = \frac{1}{2\pi\mathrm{i}}\int_{|w|=1} R_+(w)\psi_{q,+}(w,l)\psi_{q,+}(w,m)d\omega(w). \tag{7.3}$$

Note that $\tilde{F}^+(l,m) = \tilde{F}^+(m,l)$ is real.

To evaluate the left hand side of (7.1) we use the residue theorem. The only poles are at the eigenvalues and at 0 if $n = m$, hence

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}}\int_{|w|=1} T(w)\psi_-(w,n)\psi_{q,+}(w,m)d\omega(w) \\ &= \frac{\delta(n,m)}{K_+(n,n)} + \sum_{j=1}^{q} \mathrm{Res}_{\rho_j}\left(\frac{T(\lambda)\hat{\psi}_-(\lambda,n)\hat{\psi}_{q,+}(\lambda,m)}{R_{2g+2}^{1/2}(\lambda)}\right). \end{aligned}$$

Here $\delta(n,m)$ is one for $m = n$ and zero else. By (6.12) the residua at the eigenvalues are given by

$$\mathrm{Res}_{\rho_j}\left(\frac{T(\lambda)\hat{\psi}_-(\lambda,n)\hat{\psi}_{q,+}(\lambda,m)}{R_{2g+2}^{1/2}(\lambda)}\right) = -\gamma_{+,j}\hat{\psi}_+(\rho_j,n)\hat{\psi}_{q,+}(\rho_j,m). \tag{7.4}$$

Collecting all terms yields

$$K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l)\tilde{F}^\pm(l,m) = \frac{\delta(n,m)}{K_\pm(n,n)} - \sum_{j=1}^{q}\gamma_{\pm,j}\hat{\psi}_\pm(\rho_j,n)\hat{\psi}_{q,\pm}(\rho_j,m) \tag{7.5}$$

and we have thus proved the following result.

Theorem 7.1. *The kernel $K_\pm(n,m)$ of the transformation operator satisfies the* **Gel'fand-Levitan-Marchenko equation**

$$K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l)F^\pm(l,m) = \frac{\delta(n,m)}{K_\pm(n,n)}, \qquad \pm m \geq \pm n, \tag{7.6}$$

where

$$F^\pm(l,m) = \tilde{F}^\pm(l,m) + \sum_{j=1}^{q}\gamma_{\pm,j}\hat{\psi}_{q,\pm}(\rho_j,l)\hat{\psi}_{q,\pm}(\rho_j,m). \tag{7.7}$$

Defining the **Gel'fand-Levitan-Marchenko operator**

$$\mathcal{F}_n^{\pm} f(j) = \sum_{l=0}^{\infty} F^{\pm}(n \pm l, n \pm j) f(l), \qquad f \in \ell^2(\mathbb{N}_0, \mathbb{C}), \tag{7.8}$$

yields that the Gel'fand-Levitan-Marchenko equation is equal to

$$(1 + \mathcal{F}_n^{\pm}) K_{\pm}(n, n \pm .) = (K_{\pm}(n,n))^{-1} \delta_0. \tag{7.9}$$

Our next aim is to study the Gel'fand-Levitan-Marchenko operator $\mathcal{F}_n^{\pm}$ in more detail. The structure of the Gel'fand-Levitan-Marchenko equation suggests that the estimate (5.5) for $K_{\pm}(n,m)$ should imply a similar estimate for $F^{\pm}(n,m)$.

Lemma 7.2.

$$|F^{\pm}(n,m)| \le C \sum_{j=[\frac{n+m}{2}]\pm 1}^{\pm\infty} \Big(|a(j) - a_q(j)| + |b(j) - b_q(j)| \Big), \tag{7.10}$$

where the constant C is of the same nature as in (5.5).

Proof. We abbreviate the estimate (5.5) for $K_+(n,m)$ by

$$|K_+(n,m)| \le C\, C_+(n+m), \tag{7.11}$$

where

$$C_+(n+m) = \sum_{j=[\frac{n+m}{2}]+1}^{\infty} c(j), \qquad c(j) = |a(j) - a_q(j)| + |b(j) - b_q(j)|.$$

Note that $C_+(n+1) \le C_+(n)$. Moreover, $C_+(n) \in \ell^1_+(\mathbb{Z})$ since the summation by parts formula (e.g. [28], (1.18))

$$\sum_{m=n}^{N} g(m)(f(m+1) - f(m)) = g(N) f(N+1) - g(n-1) f(n) + \sum_{m=n}^{N} (g(m-1) - g(m)) f(m) \tag{7.12}$$

implies for $g(m) = m$, $f(m) = C_+(m)$ that

$$\sum_{m=n}^{\infty} m\, c(m) = (n-1) C_+(n) + \sum_{m=n}^{\infty} C_+(m), \tag{7.13}$$

where we used $\lim_{n\to\infty} n\, C_+(n+1) \le \lim_{n\to\infty} \sum_{m=n}^{\infty} m\, c(m) = 0$. Solving the GLM-equation (7.6) for $F^+(n,m)$, $m > n$, we obtain

$$\begin{aligned} |F^+(n,m)| &\le \frac{1}{K_+(n,n)} \left(|K_+(n,m)| + \sum_{l=n+1}^{\infty} \big| K_+(n,l) F^+(l,m) \big| \right) \\ &\le C_1(n) \left(C_+(n+m) + \sum_{l=n+1}^{\infty} C_+(n+l) \big| F^+(l,m) \big| \right), \end{aligned}$$

where $C_1(n) = C\,|K_+(n,n)|^{-1} \to C$ for $n \to \infty$ (see (5.28)). For n large enough, i.e. $C_1(n)C_+(2n) < 1$, we apply the discrete Gronwall-type inequality [28], Lemma 10.8,

$$
\begin{aligned}
|F^+(n,m)| &\leq C_1(n)\left(C_+(n+m) + \sum_{l=n+1}^{\infty} \frac{C_1(l)C_+(l+m)C_+(n+l)}{\prod_{k=n+1}^{l}(1 - C_1(k)C_+(n+k))}\right)\\
&\leq C_1(n)C_+(n+m)\left(1 + \sum_{l=n+1}^{\infty} \frac{C_1(k)C_+(n+l)}{\prod_{k=n+1}^{l}(1 - C_1(n)C_+(n+k))}\right), \qquad (7.14)
\end{aligned}
$$

which finishes the proof. □

Furthermore,

Lemma 7.3. *Let $F^{\pm}(n,m)$ be solutions of the Gel'fand-Levitan-Marchenko equation. Then*

$$
\sum_{n=n_0}^{\pm\infty} |n| \left|F^{\pm}(n,n) - F^{\pm}(n\pm 1, n\pm 1)\right| < \infty, \qquad (7.15)
$$

$$
\sum_{n=n_0}^{\pm\infty} |n| \left|a_q(n)F^{\pm}(n,n+1) - a_q(n-1)F^{\pm}(n-1,n)\right| < \infty. \qquad (7.16)
$$

Proof. We first prove (7.16) for F^+. Lemma 5.3 implies

$$
b(n) - b_q(n) = a_q(n)\kappa_{+,1}(n) - a_q(n-1)\kappa_{+,1}(n-1), \qquad (7.17)
$$

where

$$
\kappa_{+,j}(n) := \kappa_+(n,n+j) := \frac{K_+(n,n+j)}{K_+(n,n)}. \qquad (7.18)
$$

Abbreviate $F_j^+(n) = F^+(n+j,n)$. With this notation, the GLM-equation (7.6) reads

$$
\kappa_{+,l}(n) + F_l^+(n) + \sum_{j=1}^{\infty} \kappa_{+,j}(n)F_{j-l}^+(n+l) = \frac{\delta(l,0)}{K_+(n,n)^2}, \qquad l \geq 0. \qquad (7.19)
$$

Insert the GLM-equation for $F^+(n,n+1)$, $F^+(n-1,n)$ (recall $F^+(n,m) = F^+(m,n)$)

$$
\begin{aligned}
&a_q(n)F_1^+(n) - a_q(n-1)F_1^+(n-1)\\
&\quad = -a_q(n)\kappa_{+,1}(n) + a_q(n-1)\kappa_{+,1}(n-1)\\
&\qquad - \sum_{j=1}^{\infty}\Big(a_q(n)\kappa_{+,j}(n)F_{j-1}^+(n+1) - a_q(n-1)\kappa_{+,j}(n-1)F_{j-1}^+(n)\Big). \qquad (7.20)
\end{aligned}
$$

Since $-a_q(n)\kappa_{+,1}(n)+a_q(n-1)\kappa_{+,1}(n-1)=b_q(n)-b(n)$ the only interesting part is the sum. For N, $J<\infty$,

$$
\begin{aligned}
&\sum_{n=n_0}^{N} n\sum_{j=1}^{J}\Big(a_q(n)\kappa_{+,j}(n)F^+_{j-1}(n+1)-a_q(n-1)\kappa_{+,j}(n-1)F^+_{j-1}(n)\Big)\\
&\quad=\sum_{j=1}^{J}\sum_{n=n_0}^{N} n\Big(a_q(n)\kappa_{+,j}(n)F^+_{j-1}(n+1)-a_q(n-1)\kappa_{+,j}(n-1)F^+_{j-1}(n)\Big)\\
&\quad=\sum_{j=1}^{J}\Big(Na_q(N)\kappa_{+,j}(N)F^+_{j-1}(N+1)-(n_0-1)a_q(n_0-1)\kappa_{+,j}(n_0-1)F^+_{j-1}(n_0)\\
&\qquad+\sum_{n=n_0}^{N}(-1)a_q(n-1)\kappa_{+,j}(n-1)F^+_{j-1}(n)\Big),
\end{aligned}
\tag{7.21}
$$

where we used the summation by parts. Estimates (7.11), (7.14) imply for the first summand

$$
\begin{aligned}
\Big|\sum_{j=1}^{J}Na_q(N)\kappa_{+,j}(N)F^+_{j-1}(N+1)\Big| &\le \sum_{j=1}^{J}|N|a_q(N)\tilde{C}C_+(2N+j)C_+(2N+j+1)\\
&\le |N|a_q(N)\hat{C}C_+(2N+1),
\end{aligned}
$$

which holds uniformly in J, and (compare (7.13))

$$
\lim_{N\to\infty} Na_q(N)\hat{C}C_+(2N+1)=0. \tag{7.22}
$$

Moreover,

$$
\begin{aligned}
&\lim_{N,J\to\infty}\Big|\sum_{j=1}^{J}\sum_{n=n_0}^{N}a_q(n-1)\kappa_{+,j}(n-1)F^+_{j-1}(n)\Big|\\
&\le \lim_{N,J\to\infty}\sum_{j=1}^{J}\sum_{n=n_0}^{N}\Big|a_q(n-1)\kappa_{+,j}(n-1)F^+_{j-1}(n)\Big|\\
&\le \sum_{j=1}^{\infty}\sum_{n=n_0}^{\infty}a_q(n-1)\tilde{C}C_+(2n+j)C_+(2n+j+1)<\infty.
\end{aligned}
$$

Therefore $|n||a_q(n)F^+(n,n+1)-a_q(n-1)F^+(n-1,n)|\in\ell^1_+(\mathbb{Z})$ as desired. To apply Lemma 5.3 for F^- use the symmetry property $F^-(n,m)=F^-(m,n)$. For (7.15), inserting the GLM-equation yields

$$
\begin{aligned}
F^+(n,n)-F^+(n+1,n+1)&=K_+^{-2}(n,n)-K_+^{-2}(n+1,n+1)\\
&+\sum_{j=1}^{\infty}\Big(\kappa_{+,j}(n+1)F^+_j(n+1)-\kappa_{+,j}(n)F^+_j(n)\Big).
\end{aligned}
$$

By (5.28),

$$
\begin{aligned}
\big|K_+^{-2}(n,n)-K_+^{-2}(n+1,n+1)\big| &\le \frac{|a(n)+a_q(n)|}{a(n)^2}\prod_{j=n+1}^{\infty}\frac{a(j)^2}{a_q(j)^2}|a(n)-a_q(n)|\\
&\le C|a(n)-a_q(n)|,
\end{aligned}
\tag{7.23}
$$

and the same considerations as above imply (7.15). $\square$

Remark 7.4. *The Gel'fand-Levitan-Marchenko equation is symmetric in $K_\pm(n,m)$ and $F^\pm(n,m)$, therefore we can invert the analysis done in Lemma 7.3 and obtain estimates for $K_\pm(n,m)$ starting with an analogue of estimate (7.10) for $F^\pm(n,m)$ and the estimates (7.15), (7.16) (c.f. Lemma 8.1).*

Theorem 7.5. *For $n \in \mathbb{Z}$, the Gel'fand-Levitan-Marchenko operator $\mathcal{F}_n^\pm : \ell^2 \to \ell^2$ is Hilbert-Schmidt. Moreover, $1+\mathcal{F}_n^\pm$ is positive and hence invertible.*

In particular, the Gel'fand-Levitan-Marchenko equation (7.9) has a unique solution and $S_+(H)$ or $S_-(H)$ uniquely determine H.

Proof. That $\mathcal{F}_n^\pm$ is Hilbert-Schmidt is a straight-forward consequence of our estimate in Lemma 7.2.

Let $f \in \ell^2(\mathbb{N}_0)$ be real (which is no restriction since $F^+(n,l)$ is real and the real and imaginary part of (7.24) could be treated separately) and abbreviate $f_n(w) = \sum_{j=0}^\infty f(j)\psi_{q,+}(w,n+j)$. Then

$$\begin{aligned}
\sum_{j=0}^\infty f(j)\mathcal{F}_n^+ f(j) &= \sum_{j=0}^\infty f(j)\sum_{l=0}^\infty F^+(n+j,n+l)f(l)\\
&= \frac{1}{2\pi i}\int_{|w|=1} R_+(w)\sum_{j,l=0}^\infty f(j)\psi_{q,+}(w,n+j)\psi_{q,+}(w,n+l)f(l)\,d\omega(w)\\
&\qquad + \sum_{k=1}^q\sum_{j,l=0}^\infty f(j)\gamma_{+,k}\hat\psi_{q,+}(\rho_k,n+j)\hat\psi_{q,+}(\rho_k,n+l)f(l)\\
&= \frac{1}{2\pi i}\int_{|w|=1} R_+(w)f_n(\overline{w})f_n(w)\,d\omega(w) + \sum_{k=1}^q \gamma_{+,k}|\hat f_n(\rho_k)|^2\\
&= \frac{1}{2\pi i}\int_{|w|=1} \tilde R_+(w)|f_n(w)|^2 d\omega(w) + \sum_{k=1}^q \gamma_{+,k}|\hat f_n(\rho_k)|^2, \qquad (7.24)
\end{aligned}$$

where $\tilde R_+(w) = R_+(w)f_n(w)\left(\overline{f_n(w)}\right)^{-1}$ with $|\tilde R_+(w)| = |R_+(w)|$ and $\hat f_n(w) = \sum_{j=0}^\infty f(j)\hat\psi_{q,+}(w,n+j)$. The integral over the imaginary part vanishes since $\overline{\tilde R_+(w)} = \tilde R_+(\overline{w})$ and we replace the real part by

$$\mathrm{Re}(\tilde R_+(w)) = \frac12\big(|1+\tilde R_+(w)|^2 - 1 - |\tilde R_+(w)|^2\big) = \frac12\big(|1+\tilde R_+(w)|^2 + |T(w)|^2\big) - 1,$$

(recall $|\tilde R_+(w)|^2 + |T(w)|^2 = 1$). This yields using $\sum |f(j)|^2 = \frac{1}{2\pi i}\int_{|w|=1}|f_n(w)|^2 d\omega$

$$\begin{aligned}
\sum_{j=0}^\infty \overline{f(j)}(1+\mathcal{F}_n^+)f(j) &= \sum_{k=1}^q \gamma_{+,k}|\hat f_n(\rho_k)|^2\\
&\quad + \frac{1}{4\pi i}\int_{|w|=1}\big(|1+\tilde R_+(w)|^2 + |T(w)|^2\big)|f_n(w)|^2 d\omega(w) \qquad (7.25)
\end{aligned}$$

which establishes $1+\mathcal{F}_n^+ \ge 0$. According to Lemma 6.2, $|T(w)|^2 > 0$ a.e., therefore -1 is not an eigenvalue and $1+\mathcal{F}_n^+ \ge \epsilon_n$ for some $\epsilon_n > 0$. □

To finish the direct scattering step for the Jacobi operator H with asymptotically quasi-periodic coefficients we summarize the properties of the scattering data $S_\pm(H)$.

Hypothesis H. 7.6. *The scattering data*

$$S_\pm(H) = \{R_\pm(w), |w| = 1; (\rho_j, \gamma_{\pm,j}), 1 \le j \le q\} \tag{7.26}$$

satisfy the following conditions.

(i). The reflection coefficients $R_\pm(w)$ *are continuous except possibly at* $w_l = w(E_l)$ *and fulfill*

$$\overline{R_\pm(w)} = R_\pm(\overline{w}). \tag{7.27}$$

Moreover, $|R_\pm(w)| < 1$ *for* $w \ne w_l$ *and*

$$1 - |R_\pm(w)|^2 \ge C \prod_{l=0}^{2g+1} |w - w_l|^2. \tag{7.28}$$

The Fourier coefficients

$$\tilde{F}^\pm(l,m) = \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} R_\pm(w) \psi_{q,\pm}(w,l) \psi_{q,\pm}(w,m) d\omega(w) \tag{7.29}$$

satisfy

$$\begin{aligned}
&|\tilde{F}^\pm(n,m)| \le \sum_{j=n+m}^{\pm\infty} q(j), \qquad q(j) \ge 0, \qquad |j| q(j) \in \ell^1(\mathbb{Z}),\\
&\sum_{n=n_0}^{\pm\infty} |n| \big| \tilde{F}^\pm(n,n) - \tilde{F}^\pm(n \pm 1, n \pm 1) \big| < \infty,\\
&\sum_{n=n_0}^{\pm\infty} |n| \big| a_q(n) \tilde{F}^\pm(n,n+1) - a_q(n-1) \tilde{F}^\pm(n-1,n) \big| < \infty.
\end{aligned}$$

(ii). The values $\rho_j \in \mathbb{R} \backslash \sigma(H_q)$, $1 \le j \le q$, *are distinct and the norming constants* $\gamma_{\pm,j}$, $1 \le j \le q$, *are positive.*

(iii). $T(w)$ *defined via equation (6.25) extends to a single valued function on* $\mathbb{W}$ *(i.e., it has equal values on the corresponding slits).*

(iv). Transmission and reflection coefficients satisfy

$$\begin{aligned}
&\lim_{w \to w_l} (w - w_l) \tfrac{R_\pm(w)+1}{T(w)} = 0, \qquad w_l \ne w(\mu_j),\\
&\lim_{w \to w_l} (w - w_l) \tfrac{R_\pm(w)-1}{T(w)} = 0, \qquad w_l = w(\mu_j),
\end{aligned} \tag{7.30}$$

and the consistency conditions

$$\frac{R_-(w)}{R_+(\overline{w})} = -\frac{T(w)}{T(\overline{w})}, \qquad \gamma_{+,j}\, \gamma_{-,j} = \frac{\big(\mathrm{Res}_{\rho_j} T(\lambda)\big)^2}{\prod_{l=0}^{2g+1} (\rho_j - E_l)}.$$

Remark 7.7. *Note that (7.28) implies that* $\ln(1 - |R_\pm(w)|^2)$ *is integrable and ensures that (6.25) is well-defined, at least as a multi valued function. Condition (iii), which is void in the constant background case, shows that the reflection coefficient and the eigenvalues cannot be chosen independent of each other.*

8. Inverse scattering theory

In this section we want to invert the process of scattering theory, that is, we want to reconstruct the operator H from a given set $S_\pm$ and a given quasi-periodic Jacobi operator H_q.

If $S_\pm$ (satisfying H.7.6 (i)–(ii)) and H_q are known, we can construct $F^\pm(l,m)$ via formula (7.7) and thus derive the Gel'fand-Levitan-Marchenko equation, which has a unique solution by Theorem 7.5. This solution

$$\begin{aligned} K_\pm(n,n) &= \langle \delta_0, (1+\mathcal{F}_n^\pm)^{-1}\delta_0\rangle^{1/2} \\ K_\pm(n,n\pm j) &= \frac{1}{K_\pm(n,n)}\langle \delta_j, (1+\mathcal{F}_n^\pm)^{-1}\delta_0\rangle \end{aligned} \tag{8.1}$$

is the kernel of the transformation operator. Since $1+\mathcal{F}_n^\pm$ is positive, $K_\pm(n,n)$ is positive and we can set in accordance with Lemma 5.3

$$\begin{aligned} a_+(n) &= a_q(n)\frac{K_+(n+1,n+1)}{K_+(n,n)}, \\ a_-(n) &= a_q(n)\frac{K_-(n,n)}{K_-(n+1,n+1)}, \\ b_+(n) &= b_q(n) + a_q(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)}, \\ b_-(n) &= b_q(n) + a_q(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)}. \end{aligned} \tag{8.2}$$

Let H_+, H_- be the associated Jacobi operators.

Lemma 8.1. *Suppose a given set $S_\pm$ satisfies H.7.6 (i)–(ii). Then the sequences defined in (8.2) satisfy $n|a_\pm(n) - a_q(n)|$, $n|b_\pm(n) - b_q(n)| \in \ell^1_\pm(\mathbb{N})$.*

Moreover, $\psi_\pm(\lambda,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_{q,\pm}(\lambda,m)$, where $K_\pm(n,m)$ is the solution of the Gel'fand-Levitan-Marchenko equation, satisfies $\tau_\pm\psi_\pm = \lambda\psi_\pm$.

Proof. We only prove the statements for the "+" case. Define $F^+(n,m)$ by (c.f. (7.7))

$$F^+(l,m) = \tilde{F}^+(l,m) + \sum_{j=1}^{q} \gamma_{+,j}\hat{\psi}_{q,+}(\rho_j,l)\hat{\psi}_{q,+}(\rho_j,m).$$

Hypothesis H.7.6 (i) implies

$$|F^+(n,m)| \le C \sum_{j=n+m}^{\infty} q(j) =: C_+(n+m), \tag{8.3}$$

$$\sum_{n=n_0}^{\infty} |n|\big|F^+(n,n) - F^+(n+1,n+1)\big| < \infty, \tag{8.4}$$

$$\sum_{n=n_0}^{\infty} |n|\big|a_q(n)F^+(n,n+1) - a_q(n-1)F^+(n-1,n)\big| < \infty, \tag{8.5}$$

since $\hat{\psi}_{q,+}(\rho_j,n)$ decay exponentially as $n\to\infty$ and $\sum_j \gamma_{+,j}\hat{\psi}_{q,+}(\rho_j,.)\hat{\psi}_{q,+}(\rho_j,.)$ form a telescopic sum. Note that $C_+(n+1) < C_+(n)$.

Set $\kappa_+(n,m) := K_+(n,m)K_+(n,n)^{-1}$. Then as in the proof of Lemma 7.2 we obtain

$$|\kappa_+(n,m)| \le C_+(n+m)(1+O(1)). \tag{8.6}$$

Now we have all estimates at our disposal to prove $n|b_+(n)-b_q(n)| \in \ell^1(\mathbb{N})$. By definition (c.f. (8.2)),

$$b_+(n) - b_q(n) = a_q(n)\kappa_+(n,n+1) - a_q(n-1)\kappa_+(n-1,n). \tag{8.7}$$

We insert the GLM-equation for $\kappa_+(n,n+1)$, $\kappa_+(n-1,n)$ and use estimate (8.5), the summation by parts formula, and estimates (8.3), (8.6) in the same way as in Lemma 7.3. Similarly using (8.4) we see

$$\sum_{n=n_0}^{\infty} |n| \left| \frac{1}{K_+^2(n,n)} - \frac{1}{K_+^2(n+1,n+1)} \right| < \infty. \tag{8.8}$$

Equation (8.2) yields

$$\left| \frac{1}{K_+^2(n,n)} - \frac{1}{K_+^2(n+1,n+1)} \right| = \frac{1}{a_q(n)^2} \Big(\prod_{j=n+1}^{\infty} \frac{a_+(j)^2}{a_q(j)^2} \Big) |a_+(n)^2 - a_q(n)^2|.$$

The product converges and therefore $|n||a_+(n)^2 - a_q(n)^2| \in \ell^1(\mathbb{N})$.

Next we consider $\psi_+(\lambda,n)$. Abbreviate

$$\begin{aligned}(\Delta K_+)(n,m) &= a_q(n-1)\kappa_+(n-1,m) + a_+^2(n)a_q^{-1}(n)\kappa_+(n+1,m) \\ &\quad - a_q(m-1)\kappa_+(n,m-1) - a_q(m)\kappa_+(n,m+1) + (b_+(n)-b_q(m))\kappa_+(n,m).\end{aligned} \tag{8.9}$$

$\Delta K_+ = 0$ is equivalent to the operator equality $H_+K_+ = K_+H_q$, which in turn implies that $\psi_+(\lambda,n)$ satisfies $H_+\psi_+ = \lambda\psi_+$

$$H_+\psi_+ = H_+K_+\psi_{q,+} = K_+H_q\psi_{q,+} = K_+\lambda\psi_{q,+} = \lambda K_+\psi_{q,+} = \lambda\psi_+. \tag{8.10}$$

To show that $\Delta K_+ = 0$ we insert the GLM-equation into (8.9) and obtain

$$(\Delta K_+)(n,m) + \sum_{l=n+1}^{\infty} (\Delta K_+)(n,l)F^+(l,m) = 0, \qquad m > n+1. \tag{8.11}$$

In the calculations we used

$$\begin{aligned}&a_q(n-1)F^+(n-1,m) + b_q(n)F^+(n,m) + a_q(n)F^+(n+1,m) = \\ &a_q(m-1)F^+(n,m-1) + b_q(m)F^+(n,m) + a_q(m)F^+(n,m+1)\end{aligned}$$

which follows from (7.7). By Theorem 7.5 equation (8.11) has only the trivial solution $\Delta K_+ = 0$ and hence the proof is complete. □

Now we can prove the main result of this section.

Theorem 8.2. *Hypothesis H.7.6 is necessary and sufficient for a set $S_\pm$ to be the left/right scattering data of a unique Jacobi operator H associated with sequences a, b satisfying H.4.1.*

Proof. Necessity has been established in the previous section. By Lemma 8.1, we know existence of sequences $a_\pm$, $b_\pm$ and corresponding solutions $\psi_\pm(w,n)$ associated with S_+ (or S_-). Hence it remains to establish $a_+(n) = a_-(n)$ and $b_+(n) = b_-(n)$.

Consider the following part of the GLM-equation

$$\Phi_+(n,.) := \sum_{l=n}^{\infty} K_+(n,l)\tilde{F}^+(l,.) \in \ell^1_+(\mathbb{Z}). \tag{8.12}$$

Then by use of (7.2) and Lemma 3.6,

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \Phi_+(n,m)\psi_{q,-}(w,m) &= \sum_{m\in\mathbb{Z}} \Big(\sum_{l=n}^{\infty} K_+(n,l)\tilde{F}^+(l,m)\Big)\psi_{q,-}(w,m)\\
&= \sum_{m\in\mathbb{Z}} \Big(\frac{1}{2\pi\mathrm{i}}\int_{|w|=1} R_+(w)\psi_+(w,n)\psi_{q,+}(w,m)d\omega(w)\Big)\psi_{q,-}(w,m)\\
&= \sum_{m\in\mathbb{Z}} \langle\psi_{q,-}(w,m), R_+(w)\psi_+(w,n)\rangle\psi_{q,-}(w,m)\\
&= R_+(w)\psi_+(w,n). \qquad (8.13)
\end{aligned}$$

On the other hand, inserting the GLM-equation yields for $|w|=1$

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \Phi_+(n,m)\psi_{q,-}(w,m) &=\\
&= \sum_{m=-\infty}^{n-1} \Phi_+(n,m)\psi_{q,-}(w,m) + \sum_{m=n}^{\infty}\Big[\delta(n,m)K_+^{-1}(n,n) - K_+(n,m)\\
&\quad -\sum_{l=n}^{\infty} K_+(n,l)\sum_{j=1}^{q}\gamma_{+,j}\hat{\psi}_{q,+}(\rho_j,l)\hat{\psi}_{q,+}(\rho_j,m)\Big]\psi_{q,-}(w,m)\\
&= \sum_{m=-\infty}^{n-1} \Phi_+(n,m)\psi_{q,-}(w,m) + \psi_{q,-}(w,n)K_+^{-1}(n,n) - \overline{\psi_+(w,n)}\\
&\quad -\sum_{j=1}^{q}\gamma_{+,j}\hat{\psi}_+(\rho_j,n)\sum_{m=n}^{\infty}\hat{\psi}_{q,+}(\rho_j,m)\psi_{q,-}(w,m), \qquad (8.14)
\end{aligned}$$

(recall the definition of $\hat{\psi}_{q,\pm}$ from (6.8)) and therefore

$$T(w)h_-(w,n) = \overline{\psi_+(w,n)} + R_+(w)\psi_+(w,n), \qquad |w|=1, \tag{8.15}$$

where

$$\begin{aligned}
h_-(w,n) &= \frac{\psi_{q,-}(w,n)}{T(w)}\Big(\frac{1}{K_+(n,n)} + \sum_{m=-\infty}^{n-1}\Phi_+(n,m)\frac{\psi_{q,-}(w,m)}{\psi_{q,-}(w,n)}\\
&\quad + \sum_{j=1}^{q}\gamma_{+,j}\hat{\psi}_+(\rho_j,n)\frac{W_{n-1}(\hat{\psi}_{q,+}(\rho_j),\psi_{q,-}(w))}{\psi_{q,-}(w,n)(\lambda(w)-\rho_j)}\Big), \qquad (8.16)
\end{aligned}$$

since Green's formula ([28], eq. (1.20)) implies for $\lambda\in\sigma(H_q)$

$$(\lambda-\rho_j)\sum_{m=n}^{\infty}\hat{\psi}_{q,+}(\rho_j,m)\psi_{q,-}(\lambda,m) = -W_{n-1}(\hat{\psi}_{q,+}(\rho_j),\psi_{q,-}(\lambda)).$$

Similarly, we obtain

$$\begin{aligned}
h_+(w,n) &= \frac{\psi_{q,+}(w,n)}{T(w)}\Big(\frac{1}{K_-(n,n)} + \sum_{m=n+1}^{\infty}\Phi_-(n,m)\frac{\psi_{q,+}(w,m)}{\psi_{q,+}(w,n)}\\
&\quad - \sum_{j=1}^{q}\gamma_{-,j}\hat{\psi}_-(\rho_j,n)\frac{W_n(\hat{\psi}_{q,-}(\rho_j),\psi_{q,+}(w))}{\psi_{q,+}(w,n)(\lambda(w)-\rho_j)}\Big) \qquad (8.17)
\end{aligned}$$

with

$$\Phi_-(n,m) = \sum_{l=-\infty}^{n} K_-(n,l)\tilde F^-(l,m).$$

For $n \in \mathbb{Z}$, $|w| = 1$, we see that $h_\mp(w^{-1},n) = \overline{h_\mp(w,n)}$, since $K_\pm(n,m)$ and $\Phi_\pm(n,m)$ are real. The functions $h_\mp(w,n)$ are continuous for $|w| = 1$, $w \neq w(E_j)$, since $T^{-1}(w)$ is continuous on this set by the Poisson-Jensen formula (6.25) ($|R_\pm(w)| < 1$ for $w \neq w(E_j)$ by H.7.6 (i)) and $\psi_{q,\mp}(w,m)$ are continuous on $\partial W\backslash\{w(\mu_k)\}$. The functions $h_\mp(w,n)$ have a meromorphic continuation to $\mathbb{W}\backslash\{0\}$ with the only possible poles at $w(\rho_j)$ and $w(\mu_j)$. At $w(\rho_j)$ there are no poles, due to the zeros of $T^{-1}(w)$ at $w(\rho_j)$. For $w = w(\mu_j)$ we have the same type of singularity as $\psi_{q,\pm}$. In summary, $h_\pm(w,n)$ have simple poles at $w(\mu_j)$ and are continuous at the boundary except possibly at $w(E_j)$.

To study the behavior of $h_\pm(w,n)$ as $w \to 0$, we recall $z^{-1} = -w/\tilde a\,(1+O(w))$. Then

$$\begin{aligned}
&\big(\frac{w}{\tilde a} + O(w^2)\big)W_{n-1}(\hat\psi_{q,+}(\rho_j),\psi_{q,-}(w))\\
&\quad= \frac{(-1)^n \tilde a^{n-1}}{\prod_{j=0}^{n-2} a_q(j)} w^{-n+1}(\hat\psi_{q,+}(\rho_j,n-1) + O(w)),\\
&\big(\frac{-w}{\tilde a} + O(w^2)\big)W_n(\hat\psi_{q,-}(\rho_j),\psi_{q,+}(w))\\
&\quad= \frac{(-1)^n \prod_{j=0}^{n} a_q(j)}{\tilde a^{n+1}} w^{n+1}(\hat\psi_{q,-}(\rho_j,n+1) + O(w)),
\end{aligned}$$

and property (B4) implies

$$\sum_{m=n\mp1}^{\mp\infty} \Phi_\pm(n,m)\psi_{q,\mp}(w,m)\psi_{q,\mp}^{-1}(w,n) = O(w), \qquad w \to 0. \tag{8.18}$$

We conclude that

$$\lim_{w\to0} h_\mp(w,n)\psi_{q,\pm}(w,n) = \frac{1}{T(0)K_\pm(n,n)}. \tag{8.19}$$

H.7.6 (iv) and (6.1) imply the following behavior of $\hat h_\mp(\lambda,n)$ as $\lambda \to \rho_j$

$$\begin{aligned}
\lim_{\lambda\to\rho_j} \hat h_\mp(\lambda,n) &= \pm\gamma_{\pm,j}\hat\psi_\pm(\rho_j,n)\lim_{\lambda\to\rho_j}\frac{W_{n-1}(\hat\psi_{q,\pm}(\rho_j),\hat\psi_{q,\mp}(\lambda))}{(\lambda-\rho_j)T(\lambda)}\\
&= \gamma_{\pm,j}\hat\psi_\pm(\rho_j,n)\big(\mathrm{Res}_{\rho_j}T(\lambda)\big)^{-1}\prod_{l=0}^{2g+1}\sqrt{\rho_j - E_l}, \qquad (8.20)
\end{aligned}$$

where $\hat h_\pm$ are defined as in (6.8).

By virtue of the consistency condition $T(w)\overline{R_+(w)} = -\overline{T(w)}R_-(w)$ we obtain

$$\begin{aligned}
&\overline{h_\pm(w,n)} + R_\pm(w)h_\pm(w,n) =\\
&= \frac{1}{\overline{T(w)}}\Big(\psi_\mp(w,n) + \overline{R_\mp(w)\psi_\mp(w,n)}\Big) + \frac{R_\pm(w)}{T(w)}\Big(\overline{\psi_\mp(w,n)} + R_\mp(w)\psi_\mp(w,n)\Big)\\
&= \psi_\mp(w,n)\Big(\frac{1}{\overline{T(w)}} + \frac{R_\pm(w)R_\mp(w)}{T(w)}\Big) + \overline{\psi_\mp(w,n)}\Big(\frac{\overline{R_\mp(w)}}{\overline{T(w)}} + \frac{R_\pm(w)}{T(w)}\Big)\\
&= \psi_\mp(w,n)T(w), \qquad |w| = 1.
\end{aligned}$$

If we eliminate $R_\mp(w)$ from the last equation and (8.15) we see

$$
\begin{aligned}
&T(w)R_{2g+2}^{-1/2}(w)\big(\hat\psi_+(w,n)\hat\psi_-(w,n)-\hat h_+(w,n)\hat h_-(w,n)\big) \quad = \\
(8.21)\quad &\frac{\prod_j(\lambda(w)-\mu_j)}{R_{2g+2}^{1/2}(w)}\big(\overline{h_\pm(w,n)}\psi_\pm(w,n)-\overline{\psi_\pm(w,n)}h_\pm(w,n)\big) \quad =: \quad G(w,n)
\end{aligned}
$$

for $|w|=1$. Observe that $G(\overline{w},n)=\overline{G(w,n)}=G(w,n)$, $|w|=1$, since $\overline{h_\pm}\psi_\pm-\overline{\psi_\pm}h_\pm$ and $R_{2g+2}^{-1/2}(w)$ are odd functions for $|w|=1$. The function $G(w,n)$ can be continued analytically on $\mathbb{W}$ since the difference $\hat\psi_+\hat\psi_- - \hat h_+\hat h_-$ vanishes at the poles $w(\rho_j)$ of $T(w)$ by (8.20). Note that the product $\hat\psi_+\hat\psi_-$ and hence also $\hat h_+\hat h_-$ do not have poles at $w(\mu_j)$. Moreover, since $\mathbb{W}$ is just the image of the upper sheet, we can extend it to a compact Riemann surface $\tilde{\mathbb{W}}$ by adding the image of the lower sheet. Now by $G(\overline{w},n)=G(w,n)$ we can extend G to $\tilde{\mathbb{W}}$ by setting $G(w,n)=G(w^{-1},n)$ for $|w|>1$.

Now let us investigate the behavior at the band edges: If $w_l\neq w(\mu_j)$, we obtain by (7.30), (8.15), and real-valuedness of $\hat\psi_\pm$ at the band edges that

$$
\begin{aligned}
&\lim_{w\to w_l} R_{2g+2}^{1/2}(w)\prod_j(\lambda(w)-\mu_j)h_\mp(w,n)\overline{\psi_\mp(w,n)} \\
&= \lim_{w\to w_l}\frac{R_{2g+2}^{1/2}\prod_j(\lambda-\mu_j)}{T}\Big(\overline{\psi_\pm}+R_\pm\psi_\pm\Big)\overline{\psi_\mp} \\
&= \lim_{w\to w_l}\frac{R_{2g+2}^{1/2}\prod_j(\lambda-\mu_j)}{T}\Big((R_\pm+1)\psi_\pm+\overline{\psi_\pm}-\psi_\pm\Big)\overline{\psi_\mp}=0.
\end{aligned}
$$

If $w_l=w(\mu_j)$, the same calculation shows that

$$
\begin{aligned}
&\lim_{w\to w_l} R_{2g+2}^{1/2}(w)\prod_j(\lambda(w)-\mu_j)h_\pm(w,n)\overline{\psi_\pm(w,n)} \\
&= (-1)^{l+1}C_+(n)C_-(n)\lim_{w\to w_l}R_{2g+2}^{1/2}(w)\frac{R_\pm(w)-1}{T(w)}=0
\end{aligned}
$$

by (7.30), where we used $\psi_\pm(w,n)=\mathrm{i}^l C_\pm(n)(\lambda(w)-\mu_j)^{-1/2}+O(1)$.

Consequently, $R_{2g+2}(w)G(w,n)$ is continuous at $w=w_l$ and vanishes at the band edges. Thus the singularities of $R_{2g+2}^{1/2}(w)G(w,n)$ at w_l are removable. Furthermore, $R_{2g+2}^{1/2}(w)G(w,n)$ is purely imaginary for $|w|=1$ and real on the slits and hence must vanish at w_l by continuity. So the singularities of $G(w,n)$ at w_l are removable as well. Thus G is holomorphic on all of $\tilde{\mathbb{W}}$ and vanishes at $w=0$, that is, $G(w,n)\equiv 0$ which implies (compare (B4))

$$
\begin{aligned}
&\lim_{w\to 0}\Big(\psi_+(w,n)\psi_-(w,n)-h_+(w,n)h_-(w,n)\Big) \\
&\quad = \quad K_+(n,n)K_-(n,n)-(T(0)^2K_+(n,n)K_-(n,n))^{-1}=0.
\end{aligned}
$$

Using (8.2) we finally obtain from $T(0)^2=\big(K_+(n,n)K_-(n,n)\big)^{-2}$ that

$$
(8.22)\qquad a_+(n)=a_-(n)\equiv a(n),\qquad \forall n\in\mathbb{Z}.
$$

It remains to prove $b_+(n) = b_-(n)$. Proceeding as for $G(w,n)$ we can show that

$$T(w)R_{2g+2}^{-1/2}(w)\big(\hat{\psi}_+(w,n)\hat{\psi}_-(w,n+1) - \hat{h}_+(w,n+1)\hat{h}_-(w,n)\big) = \tag{8.23}$$
$$\frac{\prod_j(\lambda(w)-\mu_j)}{R_{2g+2}^{1/2}(w)}\big(\overline{h_+(w,n+1)}\psi_+(w,n) - \overline{\psi_+(w,n)}h_+(w,n+1)\big)$$

is a constant equal to $-1/a(n)$. Thus

$$\begin{aligned} W(w,n) &:= a(n)\left(\psi_+(w,n)\psi_-(w,n+1) - h_+(w,n+1)h_-(w,n)\right) \\ &= -\frac{R_{2g+2}^{1/2}(w)}{T(w)\prod_j(\lambda(w)-\mu_j)}. \end{aligned} \tag{8.24}$$

Computing the asymptotics at $w=0$ (compare (4.3)) we see

$$0 = W(w,n) - W(w,n-1) = \frac{1}{A}(b_+(n) - b_-(n)) \tag{8.25}$$

and in particular $b_+(n) = b_-(n) \equiv b(n)$.

Our operator H has the correct norming constants since as in (6.12) it follows

$$\sum_{n\in\mathbb{Z}} \hat{\psi}_+(\rho_j,n)\hat{\psi}_-(\rho_j,n) = \big(\mathrm{Res}_{\rho_j} T(\lambda)\big)^{-1} \prod_{l=0}^{2g+1} \sqrt{\rho_j - E_l} \tag{8.26}$$

and by (8.20),

$$\sum_{n\in\mathbb{Z}} \hat{\psi}_\pm(\rho_j,n)\hat{\psi}_\pm(\rho_j,n) = \gamma_{\pm,j}^{-1}.$$

□

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Kharkiv National University, 47,Lenin ave, 61164 Kharkiv, Ukraine
E-mail address: `egorova@ilt.kharkov.ua`

Faculty of Mathematics, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: `Johanna.Michor@esi.ac.at`

Faculty of Mathematics, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: `Gerald.Teschl@univie.ac.at`
URL: `http://www.mat.univie.ac.at/~gerald/`