Trace Formulas for Jacobi Operators in Connection with Scattering Theory for Quasi-Periodic Background

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Abstract. We investigate trace formulas for Jacobi operators which are trace class perturbations of quasi-periodic finite-gap operators using Krein's spectral shift theory. In particular we establish the conserved quantities for the solutions of the Toda hierarchy in this class.

1. Introduction

Scattering theory for Jacobi operators H with periodic (respectively more general) background has attracted considerable interest recently. In [14] Volberg and Yuditskii have exhaustively treated the case where H has a homogeneous spectrum and is of Szegö class. In [2] Egorova and the authors have established direct and inverse scattering theory for Jacobi operators which are short range perturbations of quasi-periodic finite-gap operators. For further information and references we refer to these articles and [12].

In the case of constant background it is well-known that the transmission coefficient is the perturbation determinant in the sense of Krein [8], see e.g., [11] or [12]. The purpose of the present paper is to establish this result for the case of quasi-periodic finite-gap background, thereby establishing the connection with Krein's spectral shift theory. For related results see also [7], [10].

Moreover, scattering theory for Jacobi operators is not only interesting in its own right, it also constitutes the main ingredient of the inverse scattering transform for the Toda hierarchy (see, e.g., [5], [4], [12], or [13]). Since the transmission coefficient is invariant when our Jacobi operator evolves in time with respect to some equation of the Toda hierarchy, the corresponding trace formulas provide the conserved quantities for the Toda hierarchy in this setting.

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2. Notation

We assume that the reader is familiar with quasi-periodic Jacobi operators. Hence we only briefly recall some notation and refer to [2] and [12] for further information.

Let

$$H_q f(n) = a_q(n) f(n+1) + a_q(n-1) f(n-1) + b_q(n) f(n) \tag{2.1}$$

be a quasi-periodic Jacobi operator in $\ell^2(\mathbb{Z})$ associated with the Riemann surface of the function

$$R_{2g+2}^{1/2}(z), \qquad R_{2g+2}(z) = \prod_{j=0}^{2g+1}(z - E_j), \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{2.2}$$

$g \in \mathbb{N}$. The spectrum of H_q is purely absolutely continuous and consists of $g+1$ bands

$$\sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]. \tag{2.3}$$

For every $z \in \mathbb{C}$ the Baker-Akhiezer functions $\psi_{q,\pm}(z,n)$ are two (weak) solutions of $H_q \psi = z\psi$, which are linearly independent away from the band-edges $\{E_j\}_{j=0}^{2g+1}$, since their Wronskian is given by

$$W_q(\psi_{q,-}(z), \psi_{q,+}(z)) = \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^{g}(z - \mu_j)}. \tag{2.4}$$

Here μ_j are the Dirichlet eigenvalues at base point $n_0 = 0$. We recall that $\psi_{q,\pm}(z,n)$ have the form

$$\psi_{q,\pm}(z,n) = \theta_{q,\pm}(z,n) w(z)^{\pm n},$$

where $\theta_{q,\pm}(z,n)$ is quasi-periodic with respect to n and $w(z)$ is the quasi-momentum. In particular, $|w(z)| < 1$ for $z \in \mathbb{C}\backslash\sigma(H_q)$ and $|w(z)| = 1$ for $z \in \sigma(H_q)$.

3. Asymptotics of Jost solutions

After we have these preparations out of our way, we come to the study of short-range perturbations H of H_q associated with sequences a, b satisfying $a(n) \to a_q(n)$ and $b(n) \to b_q(n)$ as $|n| \to \infty$. More precisely, we will make the following assumption throughout this paper:

Let H be a perturbation of H_q such that

$$\sum_{n \in \mathbb{Z}} \Big(|a(n) - a_q(n)| + |b(n) - b_q(n)| \Big) < \infty, \tag{3.1}$$

that is, $H - H_q$ is trace class.

We first establish existence of Jost solutions, that is, solutions of the perturbed operator which asymptotically look like the Baker-Akhiezer solutions.

Theorem 3.1. *Assume (3.1). For every* $z \in \mathbb{C}\backslash\{E_j\}_{j=0}^{2g+1}$ *there exist (weak) solutions* $\psi_\pm(z,.)$ *of* $H\psi = z\psi$ *satisfying*

$$\lim_{n\to\pm\infty} w(z)^{\mp n} \left(\psi_\pm(z,n) - \psi_{q,\pm}(z,n)\right) = 0, \tag{3.2}$$

where $\psi_{q,\pm}(z,.)$ *are the Baker-Akhiezer functions. Moreover,* $\psi_\pm(z,.)$ *are continuous (resp. holomorphic) with respect to* z *whenever* $\psi_{q,\pm}(z,.)$ *are, and have the following asymptotic behavior*

$$\psi_\pm(z,n) = \frac{z^{\mp n}}{A_\pm(n)}\Big(\prod_{j=0}^{n-1}{}^{*}\ a_q(j)\Big)^{\pm 1}\Big(1 + \Big(B_\pm(n) \pm \sum_{j=1}^{n}{}^{*}\ b_q(j - {\scriptstyle{0 \atop 1}})\Big)\frac{1}{z} + O(\frac{1}{z^2})\Big), \tag{3.3}$$

where

$$\begin{aligned} A_+(n) &= \prod_{j=n}^{\infty} \frac{a(j)}{a_q(j)}, & B_+(n) &= \sum_{m=n+1}^{\infty} (b_q(m) - b(m)),\\ A_-(n) &= \prod_{j=-\infty}^{n-1} \frac{a(j)}{a_q(j)}, & B_-(n) &= \sum_{m=-\infty}^{n-1} (b_q(m) - b(m)). \end{aligned} \tag{3.4}$$

Note that since $a_q(n)$ *are bounded away from zero,* $A_\pm(n)$ *are well-defined. Here the star indicates that* $\sum_{j=1}^{n}{}^{*} = -\sum_{j=n+1}^{0}$ *for* $n < 0$ *and similarly for the product.*

Proof. The proof can be done as in the periodic case (see e.g., [2], [6], [9] or [12], Section 7.5). There a stronger decay assumption (i.e., first moments summable) is made, which is however only needed at the band edges $\{E_j\}_{j=0}^{2g+1}$. □

For later use we note the following immediate consequence

Corollary 3.2. *Under the assumptions of the previous theorem we have*

$$\lim_{n\to\pm\infty} w(z)^{\mp n}\left(\psi'_\pm(z,n) \mp n\frac{w'(z)}{w(z)}\psi_\pm(z,n) - \psi'_{q,\pm}(z,n) \pm n\frac{w'(z)}{w(z)}\psi_{q,\pm}(z,n)\right) = 0,$$

where the prime denotes differentiation with respect to z.

Proof. Just differentiate (3.2) with respect to z, which is permissible by uniform convergence on compact subsets of $\mathbb{C}\backslash\{E_j\}_{j=0}^{2g+1}$. □

We remark that if we require our perturbation to satisfy the usual short range assumption as in [2] (i.e., first moments summable), then we even have $w(z)^{\mp n}(\psi'_\pm(z,n) - \psi'_{q,\pm}(z,n)) \to 0$.

From Theorem 3.1 we obtain a complete characterization of the spectrum of H.

Theorem 3.3. *Assume (3.1). Then we have $\sigma_{ess}(H) = \sigma(H_q)$, the point spectrum of H is confined to $\overline{\mathbb{R}\backslash\sigma(H_q)}$. Furthermore, the essential spectrum of H is purely absolutely continuous except for possible eigenvalues at the band edges.*

Proof. This is an immediate consequence of the fact that $H - H_q$ is trace class and boundedness of the Jost solutions inside the essential spectrum. □

Our next result concerns the asymptotics of the Jost solutions at the *other side*.

Lemma 3.4. *Assume (3.1). Then the Jost solutions $\psi_\pm(z,.)$, $z \in \mathbb{C}\backslash\sigma(H)$, satisfy*

$$\lim_{n\to\mp\infty} |w(z)^{\mp n}(\psi_\pm(z,n) - \alpha(z)\psi_{q,\pm}(z,n))| = 0, \tag{3.5}$$

where

$$\alpha(z) = \frac{W(\psi_-(z),\psi_+(z))}{W_q(\psi_{q,-}(z),\psi_{q,+}(z))} = \frac{\prod_{j=1}^g (z-\mu_j)}{R_{2g+2}^{1/2}(z)} W(\psi_-(z),\psi_+(z)). \tag{3.6}$$

Proof. Since $H - H_q$ is trace class, we have for the difference of the Green's functions

$$\lim_{n\to\pm\infty} G(z,n,n) - G_q(z,n,n) = \lim_{n\to\pm\infty} \langle\delta_n, ((H-z)^{-1} - (H_q-z)^{-1})\delta_n\rangle = 0$$

and using

$$G_q(z,n,n) = \frac{\psi_{q,-}(z,n)\psi_{q,+}(z,n)}{W_q(\psi_{q,-}(z),\psi_{q,+}(z))}, \qquad G(z,n,n) = \frac{\psi_-(z,n)\psi_+(z,n)}{W(\psi_{q,-}(z),\psi_{q,+}(z))}$$

we obtain

$$\lim_{n\to-\infty} \psi_{q,-}(z,n)(\psi_+(z,n) - \alpha(z)\psi_{q,+}(z,n)) = 0,$$

which is the claimed result. □

Note that $\alpha(z)$ is just the inverse of the transmission coefficient (see, e.g., [2] or [12], Section 7.5). It is holomorphic in $\mathbb{C}\backslash\sigma(H_q)$ with simple zeros at the discrete eigenvalues of H and has the following asymptotic behavior

$$\alpha(z) = \frac{1}{A}\Big(1 + \frac{B}{z} + O(z^{-2})\Big), \quad A = A_-(0)A_+(0), \quad B = B_-(1) + B_+(0), \tag{3.7}$$

with $A_\pm(n)$, $B_\pm(n)$ from (3.4).

4. Connections with Krein's spectral shift theory and trace formulas

To establish the connection with Krein's spectral shift theory we next show:

Lemma 4.1. *We have*

$$\frac{d}{dz}\alpha(z) = -\alpha(z)\sum_{n\in\mathbb{Z}} \big(G(z,n,n) - G_q(z,n,n)\big), \qquad z \in \mathbb{C}\backslash\sigma(H), \tag{4.1}$$

where $G(z,m,n)$ and $G_q(z,m,n)$ are the Green's functions of H and H_q, respectively.

Proof. Green's formula ([12], eq. (2.29)) implies

$$W_n(\psi_+(z),\psi'_-(z)) - W_{m-1}(\psi_+(z),\psi'_-(z)) = \sum_{j=m}^{n} \psi_+(z,j)\psi_-(z,j), \tag{4.2}$$

hence the derivative of the Wronskian can be written as

$$\begin{aligned}\frac{d}{dz}W(\psi_-(z),\psi_+(z)) &= W_n(\psi'_-(z),\psi_+(z)) + W_n(\psi_-(z),\psi'_+(z))\\ &= W_m(\psi'_-(z),\psi_+(z)) + W_n(\psi_-(z),\psi'_+(z)) - \sum_{j=m+1}^{n}\psi_+(z,j)\psi_-(z,j).\end{aligned}$$

Using Corollary 3.2 and Lemma 3.4 we have

$$\begin{aligned}W_m(\psi'_-(z),\psi_+(z)) &= W_m(\psi'_- + m\frac{w'}{w}\psi_-,\psi_+) - \\ &\quad \frac{w'}{w}\big(m\,W(\psi_-,\psi_+) - a(m)\psi_-(m+1)\psi_+(m)\big)\\ &\to \alpha W_{q,m}(\psi'_{q,-} + m\frac{w'}{w}\psi_{q,-},\psi_{q,+}) - \\ &\quad \alpha\frac{w'}{w}\big(m\,W_q(\psi_{q,-},\psi_{q,+}) - a_q(m)\psi_{q,-}(m+1)\psi_{q,+}(m)\big)\\ &= \alpha(z)W_m(\psi'_{q,-}(z),\psi_{q,+}(z))\end{aligned}$$

as $m\to-\infty$. Similarly we obtain

$$W_n(\psi_-(z),\psi'_+(z)) \to \alpha(z)W_n(\psi_{q,-}(z),\psi'_{q,+}(z))$$

as $n\to\infty$ and again using (4.2) we have

$$W_m(\psi'_{q,-}(z),\psi_{q,+}(z)) = W_n(\psi'_{q,-}(z),\psi_{q,+}(z)) + \sum_{j=m+1}^{n}\psi_{q,+}(z,j)\psi_{q,-}(z,j).$$

Collecting terms we arrive at

$$\begin{aligned}W'(\psi_-(z),\psi_+(z)) = &-\sum_{j\in\mathbb{Z}}\Big(\psi_+(z,j)\psi_-(z,j) - \alpha(z)\psi_{q,+}(z,j)\psi_{q,-}(z,j)\Big)\\ &+ \alpha(z)W'_q(\psi_{q,-}(z)\psi_{q,+}(z)).\end{aligned}$$

Now we compute

$$\begin{aligned}\frac{d}{dz}\alpha(z) = \frac{d}{dz}\Big(\frac{W}{W_q}\Big) &= \Big(\frac{1}{W_q}\Big)'W + \frac{1}{W_q}W'\\ &= -\frac{W'_q}{W_q^2}W + \frac{1}{W_q}\Big(-\sum_{j\in\mathbb{Z}}\Big(\psi_+\psi_- - \alpha\psi_{q,+}\psi_{q,-}\Big) + \alpha W'_q\Big)\\ &= -\frac{1}{W_q}\sum_{j\in\mathbb{Z}}\Big(\psi_+\psi_- - \alpha\psi_{q,+}\psi_{q,-}\Big),\end{aligned}$$

which finishes the proof. □

As an immediate consequence, we can identify $\alpha(z)$ as Krein's perturbation determinant ([8]) of the pair H, H_q.

Theorem 4.2. *The function $A\alpha(z)$ is Krein's perturbation determinant:*

$$\alpha(z) = \frac{1}{A} \det\big(1\!\!1 + (H(t) - H_q(t))(H_q(t) - z)^{-1}\big), \qquad A = \prod_{j\in\mathbb{Z}} \frac{a(j)}{a_q(j)}. \tag{4.3}$$

By [8], Theorem 1, $\alpha(z)$ has the following representation

$$\alpha(z) = \frac{1}{A} \exp\Big(\int_{\mathbb{R}} \frac{\xi_\alpha(\lambda) d\lambda}{\lambda - z}\Big), \tag{4.4}$$

where

$$\xi_\alpha(\lambda) = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \arg \alpha(\lambda + i\epsilon) \tag{4.5}$$

is the spectral shift function.

Hence

$$\tau_j = \mathrm{tr}(H^j - (H_q)^j) = j \int_{\mathbb{R}} \lambda^{j-1} \xi_\alpha(\lambda) d\lambda, \tag{4.6}$$

where τ_j/j are the expansion coefficients of $\ln\alpha(z)$ around $z = \infty$:

$$\ln\alpha(z) = -\ln A - \sum_{j=1}^\infty \frac{\tau_j}{j\, z^j}.$$

They are related to the expansion α_j coefficients of

$$\alpha(z) = \frac{1}{A} \sum_{j=0}^\infty \frac{\alpha_j}{z^j}, \qquad \alpha_0 = 1,$$

via

$$\tau_1 = -\alpha_1, \qquad \tau_j = -j\alpha_j - \sum_{k=1}^{j-1} \alpha_{j-k}\tau_k. \tag{4.7}$$

5. Conserved quantities of the Toda hierarchy

Finally we turn to solutions of the Toda hierarchy TL_r (see, e.g., [1], [4], [12], or [13]). Let $(a_q(t), b_q(t))$ be a quasi-periodic finite-gap solution of some equation in the Toda hierarchy, $\mathrm{TL}_r(a_q(t), b_q(t)) = 0$, and let $(a(t), b(t))$ be another solution, $\mathrm{TL}_r(a(t), b(t)) = 0$, such that (3.1) holds for one (hence any) t.

Since the transmission coefficient $T(z,t) = T(z,0) \equiv T(z)$ is conserved (see [3] – formally this follows from unitary invariance of the determinant), so is $\alpha(z) = T(z)^{-1}$.

Theorem 5.1. *The quantities*

$$A = \prod_{j=-\infty}^{\infty} \frac{a(j,t)}{a_q(j,t)} \tag{5.1}$$

and $\tau_j = \mathrm{tr}(H^j(t) - H_q(t)^j)$*, that is,*

$$\begin{aligned}
\tau_1 &= \sum_{n\in\mathbb{Z}} b(n,t) - b_q(n,t)\\
\tau_2 &= \sum_{n\in\mathbb{Z}} 2(a(n,t)^2 - a_q(n,t)^2) + (b(n,t)^2 - b_q(n,t)^2)\\
&\vdots
\end{aligned}$$

are conserved quantities for the Toda hierarchy.

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