SCATTERING THEORY FOR JACOBI OPERATORS WITH GENERAL STEPLIKE QUASI-PERIODIC BACKGROUND

IRYNA EGOROVA, JOHANNA MICHOR, AND GERALD TESCHL

To Vladimir Aleksandrovich Marchenko and Leonid Andreevich Pastur, our teachers and inspiring colleagues.

Abstract. We develop direct and inverse scattering theory for Jacobi operators with steplike coefficients which are asymptotically close to different finite-gap quasi-periodic coefficients on different sides. We give a complete characterization of the scattering data, which allow unique solvability of the inverse scattering problem in the class of perturbations with finite first moment.

1. Introduction

In this paper we consider direct and inverse scattering theory for Jacobi operators with steplike quasi-periodic finite-gap background, using the Marchenko [15] approach.

Scattering theory for Jacobi operators is a classical topic with a long tradition. Originally developed on an informal level by Case in [5], the first rigorous results for the case of a constant background were given by Guseinov [12] with further extensions by Teschl [19], [20]. The case of periodic backgrounds was completely solved in [24] (who in fact handle almost periodic operators with a homogenous Cantor type spectrum) respectively [8] using different approaches. Moreover, the case of a steplike situation, where the coefficients are asymptotically close to two different quasi-periodic finite-gap operators, was solved in [11] (see also [1], [7]) under the restriction that the two background operators are isospectral. It is the purpose of the present paper to remove this restriction.

We should also mention that scattering theory for Jacobi operators is directly applicable to the investigation of the Toda lattice with initial data in the above mentioned classes. See for example [3], [6], [23] for steplike constant backgrounds, and [9], [10], [13], [14], and [16] for periodic backgrounds. For further possible applications and additional references we refer to the discussion in [11].

Finally, let us give a brief overview of the remaining sections. After recalling some necessary facts on algebro-geometric quasi-periodic finite-gap operators in Section 2, we construct the transformation operators and investigate the properties

2000 *Mathematics Subject Classification.* Primary 47B36, 81U40; Secondary 34L25, 39A11.
Key words and phrases. Inverse scattering, Jacobi operators, quasi-periodic, steplike.
Work supported by the Austrian Science Fund (FWF) under Grants No. Y330 and J2655.
Zh. Mat. Fiz. Anal. Geom. **4-1**, 33–62 (2008).

of the scattering data in Section 3. In Section 4 we derive the Gel'fand-Levitan-Marchenko equation and show that it uniquely determines the operator. In addition, we formulate necessary conditions for the scattering data to uniquely determine our Jacobi operator. Our final Section 5 shows that our necessary conditions for the scattering data are also sufficient.

2. Step-like finite-band backgrounds

First we need to recall some facts on quasi-periodic finite-band Jacobi operators which contain all periodic operators as a special case. We refer to [20, Chapter 9] and [8] for details.

Let $H_q^\pm$ be two quasi-periodic finite-band Jacobi operators,[1]

$$H_q^\pm f(n) = a_q^\pm(n) f(n+1) + a_q^\pm(n-1) f(n-1) + b_q^\pm(n) f(n), \quad f \in \ell^2(\mathbb{Z}), \tag{2.1}$$

associated with the Riemann surface of the square root

$$P_\pm(z) = -\prod_{j=0}^{2g_\pm+1} \sqrt{z - E_j^\pm}, \qquad E_0^\pm < E_1^\pm < \cdots < E_{2g_\pm+1}^\pm, \tag{2.2}$$

where $g_\pm \in \mathbb{N}$ and $\sqrt{.}$ is the standard root with branch cut along $(-\infty, 0)$. In fact, $H_q^\pm$ are uniquely determined by fixing a Dirichlet divisor $\sum_{j=1}^{g^\pm} (\mu_j^\pm, \sigma_j^\pm)$, where $\mu_j^\pm \in [E_{2j-1}^\pm, E_{2j}^\pm]$ and $\sigma_j^\pm \in \{-1, 1\}$. The spectra of $H_q^\pm$ consist of $g_\pm + 1$ bands

$$\sigma_\pm := \sigma(H_q^\pm) = \bigcup_{j=0}^{g_\pm} [E_{2j}^\pm, E_{2j+1}^\pm]. \tag{2.3}$$

We will identify the set $\mathbb{C} \setminus \sigma(H_q^\pm)$ with the upper sheet of the Riemann surface. The upper and lower sides of the cuts over the spectrum are denoted by σ^{u} and σ^{l} and the symmetric points on these cuts by λ^{u} and λ^{l}, that is,

$$f(\lambda^u) = \lim_{\epsilon \downarrow 0} f(\lambda + i\epsilon), \quad f(\lambda^l) = \lim_{\epsilon \downarrow 0} f(\lambda - i\epsilon), \quad \lambda \in \sigma_\pm.$$

We will develop the scattering theory for the operator

$$H f(n) = a(n-1) f(n-1) + b(n) f(n) + a(n) f(n+1), \quad n \in \mathbb{Z}, \tag{2.4}$$

whose coefficients are asymptotically close to the coefficients of $H_q^\pm$ on the corresponding half-axes:

$$\sum_{n=0}^{\pm\infty} |n| \Big(|a(n) - a_q^\pm(n)| + |b(n) - b_q^\pm(n)| \Big) < \infty. \tag{2.5}$$

The special case $H_q^- = H_q^+$ has been exhaustively studied in [8] (see also [24]) and the case where H_q^- and H_q^+ are in the same isospectral class $\sigma_- = \sigma_+$ was treated in [11]. Several results are straightforward generalizations, in such situations we will simply refer to [8], [11] and only point out possible differences.

Let $\psi_q^\pm(z, n)$ be the Floquet solutions of the spectral equations

$$H_q^\pm \psi(n) = z\psi(n), \qquad z \in \mathbb{C}, \tag{2.6}$$

---

[1] Everywhere in this paper the sub or super index "+" (resp. "−") refers to the background on the right (resp. left) half-axis.

that decay for $z \in \mathbb{C} \setminus \sigma_\pm$ as $n \to \pm\infty$. They are uniquely defined by the condition $\psi_q^\pm(z,0) = 1$, $\psi_q^\pm(z,\cdot) \in \ell^2(\mathbb{Z}_\pm)$. The solution $\psi_q^+(z,n)$ (resp. $\psi_q^-(z,n)$) coincides with the upper (resp. lower) branch of the Baker–Akhiezer functions of H_q^+ (resp. H_q^-), see [20]. The second solutions $\breve{\psi}_q^\pm(z,n)$ are given by the other branch of the Baker–Akhiezer functions and satisfy $\breve{\psi}_q^\pm(z,\cdot) \in \ell^2(\mathbb{Z}_\mp)$ as $z \in \mathbb{C} \setminus \sigma_\pm$. Their Wronskian is equal to

$$W_q^\pm(\breve{\psi}_q^\pm(z), \psi_q^\pm(z)) = \pm \frac{1}{\rho_\pm(z)}, \tag{2.7}$$

where

$$\rho_\pm(z) = \frac{\prod_{j=1}^{g_\pm}(z - \mu_j^\pm)}{P_\pm(z)} \tag{2.8}$$

satisfy by our choice of the branch for the square root

$$\operatorname{Im}(\rho_\pm(\lambda^{\mathrm{u}})) > 0, \quad \operatorname{Im}(\rho_\pm(\lambda^{\mathrm{l}})) < 0, \quad \lambda \in \sigma_\pm. \tag{2.9}$$

In (2.7) the following notation is used

$$W_{q,n}^\pm(f,g) := a_q^\pm(n)\left(f(n)g(n+1) - f(n+1)g(n)\right). \tag{2.10}$$

Note that $\psi_q^\pm(z,n)$, $\breve{\psi}_q^\pm(z,n)$ have continuous limits as $z \to \lambda^{\mathrm{u,l}} \in \sigma_\pm^{\mathrm{u,l}} \setminus \partial\sigma_\pm$, where

$$\partial\sigma_\pm = \{E_0^\pm, ..., E_{2g_\pm+1}^\pm\},$$

and they satisfy the symmetry property

$$\psi_q^\pm(\lambda^{\mathrm{l}}, n) = \overline{\psi_q^\pm(\lambda^{\mathrm{u}}, n)} = \breve{\psi}_q^\pm(\lambda^{\mathrm{u}}, n), \quad \lambda \in \sigma_\pm. \tag{2.11}$$

The points $(\mu_j^\pm, \sigma_j^\pm)$, $1 \le j \le g_\pm$, form the divisors of poles of the Baker–Akhiezer functions. Correspondingly, the sets of Dirichlet eigenvalues $\{\mu_1^\pm, ..., \mu_{g_\pm}^\pm\}$ can be divided in three disjoint subsets

$$\begin{aligned} M^\pm &= \{\mu_j^\pm \,|\, \mu_j^\pm \in \mathbb{R}\backslash\sigma_\pm \text{ is a pole of } \psi_q^\pm(z,1)\}, \\ \breve{M}^\pm &= \{\mu_j^\pm \,|\, \mu_j^\pm \in \mathbb{R}\backslash\sigma_\pm \text{ is a pole of } \breve{\psi}_q^\pm(z,1)\}, \\ \hat{M}^\pm &= \{\mu_j^\pm \,|\, \mu_j^\pm \in \partial\sigma_\pm\}. \end{aligned} \tag{2.12}$$

In order to remove the singularities of $\psi_q^\pm(z,n)$, $\breve{\psi}_q^\pm(z,n)$ we introduce

$$\begin{aligned} \delta_\pm(z) &:= \prod_{\mu_j^\pm \in M_\pm} (z - \mu_j^\pm), \\ \hat{\delta}_\pm(z) &:= \prod_{\mu_j^\pm \in M_\pm} (z - \mu_j^\pm) \prod_{\mu_j^\pm \in \hat{M}_\pm} \sqrt{z - \mu_j^\pm}, \\ \breve{\delta}_\pm(z) &:= \prod_{\mu_j^\pm \in \breve{M}_\pm} (z - \mu_j^\pm) \prod_{\mu_j^\pm \in \hat{M}_\pm} \sqrt{z - \mu_j^\pm}, \end{aligned} \tag{2.13}$$

where $\prod = 1$ if there are no multipliers, and set

$$\tilde{\psi}_q^\pm(z,n) = \delta_\pm(z)\psi_q^\pm(z,n), \quad \hat{\psi}_q^\pm(z,n) = \hat{\delta}_\pm(z)\psi_q^\pm(z,n). \tag{2.14}$$

Lemma 2.1. *The Floquet solutions $\psi_q^\pm$, $\breve{\psi}_q^\pm$ have the following properties:*

(i) *The functions* $\psi_q^\pm(z,n)$ *(resp.* $\breve\psi_q^\pm(z,n)$*) are holomorphic as functions of* z *in the domain* $\mathbb{C}\setminus(\sigma_\pm\cup M_\pm)$ *(resp.* $\mathbb{C}\setminus(\sigma_\pm\cup\breve M_\pm)$*), take real values on the set* $\mathbb{R}\setminus\sigma_\pm$*, and have simple poles at the points of the set* $M_\pm$ *(resp.* $\breve M_\pm$*). They are continuous up to the boundary* $\sigma_\pm^{\mathrm{u}}\cup\sigma_\pm^{\mathrm{l}}$ *except at the points in* $\hat M_\pm$ *and satisfy the symmetry property* (2.11)*. For* $E\in\hat M_\pm$*, they satisfy*

$$\psi_q^\pm(z,n)=O\left(\frac{1}{\sqrt{z-E}}\right),\quad \breve\psi_q^\pm(z,n)=O\left(\frac{1}{\sqrt{z-E}}\right),\qquad z\to E\in\hat M_\pm.$$

Moreover, the estimate

$$\hat\psi_q^\pm(z,n)-\hat\psi_q^\pm(E,n)=O(\sqrt{z-E}),\qquad E\in\partial\sigma_\pm, \tag{2.15}$$

is valid.

(ii) *The following asymptotic expansions hold as* $z\to\pm\infty$

$$\psi_q^\pm(z,n)=z^{\mp n}\Big(\prod_{j=0}^{n-1}{}^{*}\ a_q^\pm(j)\Big)^{\pm1}\Big(1\pm\frac{1}{z}\sum_{j=0}^{n-1}{}^{*}\ b_q^\pm(j+{\scriptstyle{1\atop 0}})+O(\frac{1}{z^2})\Big), \tag{2.16}$$

where

$$\prod_{j=n_0}^{n-1}{}^{*}\ f(j)=\begin{cases}\prod\limits_{j=n_0}^{n-1}f(j), & n>n_0,\\ 1, & n=n_0,\\ \prod\limits_{j=n}^{n_0-1}f(j)^{-1}, & n<n_0,\end{cases}\qquad \sum_{j=n_0}^{n-1}{}^{*}\ f(j)=\begin{cases}\sum\limits_{j=n_0}^{n-1}f(j), & n>n_0,\\ 0, & n=n_0,\\ -\sum\limits_{j=n}^{n_0-1}f(j), & n<n_0.\end{cases}$$

(iii) *The functions* $\psi_q^\pm(\lambda,n)$ *form a complete orthogonal system on the spectrum with respect to the weight*

$$d\omega_\pm(\lambda)=\frac{1}{2\pi\mathrm{i}}\rho_\pm(\lambda)d\lambda, \tag{2.17}$$

namely

$$\oint_{\sigma_\pm}\overline{\psi_q^\pm(\lambda,m)}\psi_q^\pm(\lambda,n)d\omega_\pm(\lambda)=\delta(n,m), \tag{2.18}$$

where

$$\oint_{\sigma_\pm}f(\lambda)d\lambda:=\int_{\sigma_\pm^{\mathrm{u}}}f(\lambda^{\mathrm{u}})d\lambda-\int_{\sigma_\pm^{\mathrm{l}}}f(\lambda^{\mathrm{l}})d\lambda. \tag{2.19}$$

Here $\delta(n,m)=1$ *if* $n=m$ *and* $\delta(n,m)=0$ *else is the Kronecker delta.*

3. Scattering data

Now let H be a steplike operator with coefficients $a(n)$, $b(n)$ satisfying (2.5). The two solutions $\psi_\pm(z,n)$ of the spectral equation

$$H\psi=z\psi,\quad z\in\mathbb{C}, \tag{3.1}$$

which are asymptotically close to the Floquet solutions $\psi_q^\pm(z,n)$ of the background equations (2.6) as $n\to\pm\infty$, are called Jost solutions. They can be represented as (see [8])

$$\psi_\pm(z,n)=\sum_{m=n}^{\pm\infty}K_\pm(n,m)\psi_q^\pm(z,m), \tag{3.2}$$

where the functions $K_\pm(n,.)$ are real valued and satisfy the estimate

$$|K_\pm(n,m)| \le C_\pm(n) \sum_{j=[\frac{m+n}{2}]}^{\pm\infty} \Big(|a(j)-a_q^\pm(j)| + |b(j)-b_q^\pm(j)|\Big), \quad \pm m > \pm n > 0. \tag{3.3}$$

The functions $C_\pm(n) > 0$ decrease monotonically as $n \to \pm\infty$. Moreover, we have

$$\begin{aligned} a(n) &= a_q^+(n)\frac{K_+(n+1,n+1)}{K_+(n,n)},\\ a(n) &= a_q^-(n)\frac{K_-(n,n)}{K_-(n+1,n+1)},\\ b(n) &= b_q^+(n) + a_q^+(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q^+(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)},\\ b(n) &= b_q^-(n) + a_q^-(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q^-(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)}, \end{aligned} \tag{3.4}$$

which implies (cf. [8]) the following asymptotic behavior of the Jost solutions as $z \to \pm\infty$ using (3.2), (2.16),

$$\psi_\pm(z,n) = z^{\mp n} K_\pm(n,n)\Big(\prod_{j=0}^{n-1}{}^{*}\; a_q^\pm(j)\Big)^{\pm 1}\Big(1+\Big(B_\pm(n) \pm \sum_{j=1}^{n}{}^{*}\; b_q^\pm(j-{}^{0}_{1})\Big)\frac{1}{z} + O(\frac{1}{z^2})\Big), \tag{3.5}$$

where

$$B_\pm(n) = \sum_{m=n\pm1}^{\pm\infty}(b_q^\pm(m) - b(m)). \tag{3.6}$$

For $\lambda \in \sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}}$ a second pair of solutions of (3.1) is given by

$$\breve{\psi}_\pm(\lambda,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\breve{\psi}_q^\pm(\lambda,m), \quad \lambda \in \sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}}, \tag{3.7}$$

which cannot be continued to the complex plane. Note that $\breve{\psi}_\pm(\lambda,n) = \overline{\psi_\pm(\lambda,n)}$, $\lambda \in \sigma_\pm$, and from (2.5), (3.2) we conclude

$$W(\overline{\psi_\pm(\lambda)}, \psi_\pm(\lambda)) = W_q^\pm(\breve{\psi}_q^\pm(\lambda), \psi_q^\pm(\lambda)) = \pm\rho_\pm(\lambda)^{-1}. \tag{3.8}$$

The Jost solutions $\psi_\pm$ are holomorphic in the domains $\mathbb{C}\setminus(\sigma_\pm \cup M_\pm)$ and inherit almost all properties of their background counterparts listed in Lemma 2.1. As before, we set

$$\tilde{\psi}_\pm(z,n) = \delta_\pm(z)\psi_\pm(z,n), \quad \hat{\psi}_\pm(z,n) = \hat{\delta}_\pm(z)\psi_\pm(z,n). \tag{3.9}$$

The following Lemma is proven in [8].

Lemma 3.1. *The Jost solutions have the following properties.*

(i) *For all n, the functions $\psi_\pm(z,n)$ are holomorphic in the domain $\mathbb{C}\setminus(\sigma_\pm \cup M_\pm)$ with respect to z and continuous up to the boundary $(\sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}})\setminus\partial\sigma_\pm$, where*

$$\psi_\pm(\lambda^{\mathrm{u}},n) = \overline{\psi_\pm(\lambda^{\mathrm{l}},n)}, \quad \lambda \in (\sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}})\setminus\partial\sigma_\pm. \tag{3.10}$$

The functions $\psi_\pm(z,n)$ are real valued for $z \in \mathbb{R}\setminus\sigma_\pm$ and have simple poles at $\mu_j \in M_\pm$. Moreover, $\hat\psi_\pm$ are continuous up to the boundary $\sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}}$.

(ii) *At the band edges we have for $\lambda \in \sigma_\pm^{\mathrm{u,l}}$*

$$\begin{aligned} \psi_\pm(\lambda,n) - \overline{\psi_\pm(\lambda,n)} &= o(1), && E \in \partial\sigma_\pm \setminus \hat M_\pm, \\ \psi_\pm(\lambda,n) + \overline{\psi_\pm(\lambda,n)} &= o\Big(\tfrac{1}{\sqrt{\lambda-E}}\Big), && E \in \hat M_\pm. \end{aligned} \tag{3.11}$$

Next, we introduce the sets

$$\sigma^{(2)} := \sigma_+ \cap \sigma_-, \quad \sigma_\pm^{(1)} = \mathrm{clos}\,(\sigma_\pm \setminus \sigma^{(2)}), \quad \sigma := \sigma_+ \cup \sigma_-, \tag{3.12}$$

where σ is the (absolutely) continuous spectrum of H and $\sigma_+^{(1)} \cup \sigma_-^{(1)}$ resp. $\sigma^{(2)}$ are the parts which are of multiplicity one resp. two. We will denote the interior of the spectrum by $\mathrm{int}(\sigma)$, that is, $\mathrm{int}(\sigma) := \sigma \setminus \partial\sigma$.

In addition to the continuous part, H has a finite number of eigenvalues situated in the gaps, $\sigma_d = \{\lambda_1, ..., \lambda_p\} \subset \mathbb{R}\setminus\sigma$ (see, e.g., [18]). For every eigenvalue we introduce the corresponding norming constants

$$\gamma_{\pm,k}^{-1} = \sum_{n\in\mathbb{Z}} |\tilde\psi_\pm(\lambda_k,n)|^2, \quad 1 \le k \le p. \tag{3.13}$$

Moreover, $\tilde\psi_\pm(\lambda_k,n) = c_k^\pm \tilde\psi_\mp(\lambda_k,n)$ with $c_k^+ c_k^- = 1$.

Let

$$W(z) := W(\psi_-(z), \psi_+(z)) \tag{3.14}$$

be the Wronskian of two Jost solutions. This function is meromorphic in the domain $\mathbb{C}\setminus\sigma$ with possible poles at the points $M_+ \cup M_- \cup (\hat M_+ \cap \hat M_-)$ and with possible square root singularities at the points $\hat M_+ \cup \hat M_- \setminus (\hat M_+ \cap \hat M_-)$. Set

$$\tilde W(z) = W(\tilde\psi_-(z), \tilde\psi_+(z)), \quad \hat W(z) = W(\hat\psi_-(z), \hat\psi_+(z)), \tag{3.15}$$

then $\hat W(\lambda)$ is holomorphic in the domain $\mathbb{C}\setminus\mathbb{R}$ and continuous up to the boundary. But unlike to $W(z)$ and $\tilde W(z)$, the function $\hat W(\lambda)$ may not take real values on the set $\mathbb{R}\setminus\sigma$ and complex conjugated values on the different sides of the spectrum. That is why it is more convenient to characterize the spectral properties of the operator H by means of the function $\tilde W$, which can have singularities at the points of the sets $\hat M_+ \cup \hat M_-$. We will study the precise character of these singularities in Lemma 3.2 below.

Note that outside the spectrum the function $\tilde W(z)$ vanishes precisely at the eigenvalues. However, it might also vanish inside the spectrum at points in $\partial\sigma_- \cup \partial\sigma_+$. We will call such points virtual levels of the operator H,

$$\sigma_v := \{E \in \sigma : \ \hat W(E) = 0\}, \tag{3.16}$$

and we will show that $\sigma_v \subseteq \partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$ in Lemma 3.2. All other points E of the set $\partial\sigma_+ \cup \partial\sigma_-$ correspond to the generic case $\hat W(E) \neq 0$.

Our next aim is to derive the properties of the scattering matrix. Introduce the scattering relations

$$T_\mp(\lambda)\psi_\pm(\lambda,n) = \overline{\psi_\mp(\lambda,n)} + R_\mp(\lambda)\psi_\mp(\lambda,n), \quad \lambda \in \sigma_\mp^{\mathrm{u,l}}, \tag{3.17}$$

where the transmission and reflection coefficients are defined as usual,

$$T_\pm(\lambda) := \frac{W(\overline{\psi_\pm(\lambda)}, \psi_\pm(\lambda))}{W(\psi_\mp(\lambda), \psi_\pm(\lambda))}, \quad R_\pm(\lambda) := -\frac{W(\psi_\mp(\lambda), \overline{\psi_\pm(\lambda)})}{W(\psi_\mp(\lambda), \psi_\pm(\lambda))}, \quad \lambda \in \sigma_\pm^{\mathrm{u,l}}. \tag{3.18}$$

The equalities in (3.18) imply the identity

$$\frac{1}{T_+(\lambda)\rho_+(\lambda)} = \frac{1}{T_-(\lambda)\rho_-(\lambda)} = W(\lambda), \quad \lambda \in \sigma^{(2)},$$

where $W(\lambda)$ is the Wronskian of two Jost solutions (3.14). This Wronskian plays an important role in the characterization of the properties of the scattering matrix. Namely, the following result is valid.

Lemma 3.2. *The entries of the scattering matrix have the following properties:*

I.

$$\begin{array}{lll} (\mathbf{a}) & T_\pm(\lambda^{\mathrm{u}}) = \overline{T_\pm(\lambda^{\mathrm{l}})}, & \lambda \in \sigma_\pm, \\ & R_\pm(\lambda^{\mathrm{u}}) = \overline{R_\pm(\lambda^{\mathrm{l}})}, & \lambda \in \sigma_\pm, \\ (\mathbf{b}) & \dfrac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}} = R_\pm(\lambda), & \lambda \in \sigma_\pm^{(1)}, \\ (\mathbf{c}) & 1 - |R_\pm(\lambda)|^2 = \dfrac{\rho_\pm(\lambda)}{\rho_\mp(\lambda)}\,|T_\pm(\lambda)|^2, & \lambda \in \sigma^{(2)}, \\ (\mathbf{d}) & \overline{R_\pm(\lambda)}T_\pm(\lambda) + R_\mp(\lambda)\overline{T_\pm(\lambda)} = 0, & \lambda \in \sigma^{(2)}. \end{array}$$

II. *The functions $T_\pm(\lambda)$ can be extended analytically to the domain $\mathbb{C}\setminus(\sigma\cup M_\pm\cup \breve{M}_\pm)$ and satisfy*

$$\frac{1}{T_+(z)\rho_+(z)} = \frac{1}{T_-(z)\rho_-(z)} = W(z). \tag{3.19}$$

The function $W(z)$ has the following properties:

(a) *The function $\tilde{W}(z) = \delta_+(z)\delta_-(z)W(z)$ is holomorphic on $\mathbb{C}\setminus\sigma$ with simple zeros at the eigenvalues λ_k, where*

$$\left(\frac{d\tilde{W}}{dz}(\lambda_k)\right)^2 = \frac{1}{\gamma_{+,k}\gamma_{-,k}}. \tag{3.20}$$

Moreover,

$$\overline{\tilde{W}(\lambda^{\mathrm{u}})} = \tilde{W}(\lambda^{\mathrm{l}}), \quad \lambda \in \sigma, \qquad \tilde{W}(z) \in \mathbb{R}, \quad z \in \mathbb{R}\setminus\sigma. \tag{3.21}$$

(b) *The function $\hat{W}(z) = \hat{\delta}_+(z)\hat{\delta}_-(z)W(z)$ is continuous on the set $\mathbb{C}\setminus\sigma$ up to the boundary $\sigma^{\mathrm{u}}\cup\sigma^{\mathrm{l}}$. It can have zeros on the set $\partial\sigma\cup(\partial\sigma_+^{(1)}\cap\partial\sigma_-^{(1)})$ and does not vanish at the other points of the spectrum σ. If $\hat{W}(E) = 0$ as $E \in \partial\sigma\cup(\partial\sigma_+^{(1)}\cap\partial\sigma_-^{(1)})$, then*

$$\frac{1}{\hat{W}(\lambda)} = O\left(\frac{1}{\sqrt{\lambda - E}}\right), \quad \textit{for } \lambda\in\sigma \textit{ close to } E. \tag{3.22}$$

Moreover,

$$\frac{1}{\hat{W}(z)} = O\left((z-E)^{-1/2-\varepsilon}\right), \quad \textit{for } z \textit{ close to } E. \tag{3.23}$$

(c) *In addition,*

$$T_+(\infty) = T_-(\infty) > 0. \tag{3.24}$$

III. **(a)** *The reflection coefficients $R_\pm(\lambda)$ are continuous functions on $\mathrm{int}(\sigma_\pm^{\mathrm{u,l}})$.*

(b) *If $E \in \partial\sigma_+\cap\partial\sigma_-$ and $\hat{W}(E) \neq 0$, then the functions $R_\pm(\lambda)$ are also continuous at E. Moreover,*

$$R_\pm(E) = \begin{cases} -1 & \textit{for} \quad E \notin \hat{M}_\pm, \\ 1 & \textit{for} \quad E \in \hat{M}_\pm. \end{cases} \tag{3.25}$$

Proof. **I**. The symmetry property **(a)** follows from formulas (3.18) and (3.10). For **(b)**, use (3.18) and observe that $\psi_\mp(\lambda)$ are real valued for $\lambda \in \mathrm{int}(\sigma_\pm^{(1)})$. Let $\lambda \in \mathrm{int}(\sigma^{(2)})$. By (3.17),

$$|T_\pm|^2 W(\psi_\mp, \overline{\psi_\mp}) = (|R_\pm|^2 - 1) W(\psi_\pm, \overline{\psi_\pm}),$$

and property **(c)** follows from (3.8). The consistency condition **(d)** can be derived directly from definition (3.18).

II. The identity (3.19) follows from (3.18). **(a)** The Wronskian inherits the properties of $\psi_\pm(z)$, so it remains to show (3.20). If $\hat{W}(z_0) = 0$ for $z_0 \in \mathbb{C} \setminus \sigma$, then

$$\tilde{\psi}_\pm(z_0, n) = c^\pm \tilde{\psi}_\mp(z_0, n) \tag{3.26}$$

for some constants $c^\pm$ (depending on z_0), which satisfy $c^- c^+ = 1$. In particular, each zero of $\tilde{W}$ (or $\hat{W}$) outside the continuous spectrum is a point of the discrete spectrum of H and vice versa.

Let $\gamma_{\pm,j}$ be the norming constants defined in (3.13) for some point of the discrete spectrum λ_j. By virtue of [20], Lemma 2.4,

$$\begin{aligned} \frac{d}{dz} W(\tilde{\psi}_-(z), \tilde{\psi}_+(z))\Big|_{\lambda_j} &= W_n(\tilde{\psi}_-(\lambda_j), \tfrac{d}{dz}\tilde{\psi}_+(\lambda_j)) + W_n(\tfrac{d}{dz}\tilde{\psi}_-(\lambda_j), \tilde{\psi}_+(\lambda_j)) \\ &= -\sum_{k\in\mathbb{Z}} \tilde{\psi}_-(\lambda_j, k)\tilde{\psi}_+(\lambda_j, k) = -\frac{1}{c_j^\pm \gamma_{\pm,j}}. \end{aligned} \tag{3.27}$$

Since $c_j^- c_j^+ = 1$, we obtain (3.20).

(b) Continuity of $\hat{W}$ up to the boundary follows from the corresponding property of $\hat{\psi}_\pm(z, n)$. We begin with the investigation of the possible zeros of this function on the spectrum.

First let $\lambda_0 \in \mathrm{int}(\sigma^{(2)}) := \sigma^{(2)} \setminus \partial\sigma^{(2)}$, that is, $\hat{\delta}_- \neq 0$ and $\hat{\delta}_+ \neq 0$. Suppose $W(\lambda_0) = 0$, then $\psi_+(\lambda_0, n) = c\,\psi_-(\lambda_0, n)$ and $\overline{\psi_+(\lambda_0, n)} = \bar{c}\,\overline{\psi_-(\lambda_0, n)}$, i.e. $W(\psi_+, \overline{\psi_+}) = |c|^2 W(\psi_-, \overline{\psi_-})$. But this implies opposite signs for ρ_+, ρ_- by (3.8), $\mathrm{sign}\,\rho_+(\lambda_0) = -\,\mathrm{sign}\;\rho_-(\lambda_0)$, which contradicts (2.9).

Let $\lambda_0 \in \mathrm{int}(\sigma_\pm^{(1)})$ and $\tilde{W}(\lambda_0) = 0$. The point λ_0 can coincide with a pole $\mu \in M_\mp$. But $\psi_\pm(\lambda_0, n)$ and $\overline{\psi_\pm(\lambda_0, n)}$ are linearly independent and bounded, and $\tilde{\psi}_\mp(\lambda_0, n) \in \mathbb{R}$. If $W(\lambda_0) = 0$, then $\tilde{\psi}_\mp = c_1^\pm \psi_\pm = c_2^\pm \overline{\psi_\pm}$ which implies $W(\psi_\pm, \overline{\psi_\pm})(\lambda_0) = 0$, a contradiction.

In the general mutual location of the background spectra the case $\lambda_0 = E \in (\partial\sigma^{(2)} \cap \mathrm{int}(\sigma_\pm)) \subset \mathrm{int}(\sigma)$ can occur. If $\hat{W}(E) = 0$, then $W(\psi_\pm, \hat{\psi}_\mp)(E) = 0$, where $\hat{\psi}_\mp$ are defined by (3.9). The values of $\hat{\psi}_\mp(E, \cdot)$ are either purely real or purely imaginary, therefore $W(\overline{\psi_\pm}, \hat{\psi}_\mp)(E) = 0$, that is, $\overline{\psi_\pm(E, n)}$ and $\psi_\pm(E, n)$ are linearly dependent, which is impossible at inner points of the set $\sigma_\pm$.

Thus, the virtual level σ_v of H defined in (3.16) can only be located on the set $\partial\sigma_- \cap \partial\sigma_+$, that is,

$$\sigma_v \subseteq \partial\sigma \cup \big(\partial\sigma_-^{(1)} \cap \partial\sigma_+^{(1)}\big). \tag{3.28}$$

To prove (3.22), take $E \in \sigma_v$ and assume for example $E \in \sigma_+$. By (3.17) and (3.19),

$$\frac{\hat{\delta}_+(\lambda)\hat{\psi}_-(\lambda, n)}{\hat{\delta}_-(\lambda)\rho_+(\lambda)W(\lambda)} = \hat{\delta}_+(\lambda)\overline{\psi_+(\lambda, n)} + R_+(\lambda)\hat{\psi}_+(\lambda, n).$$

Choose n_0 such that $\hat{\psi}_-(E,n_0)\neq 0$. By continuity we also have $\hat{\psi}_-(\lambda,n_0)\neq 0$ in a small vicinity of E. Then

$$\frac{\hat{\delta}_+(\lambda)\overline{\psi_+(\lambda,n_0)}+R_+(\lambda)\hat{\psi}_+(\lambda,n_0)}{\hat{\psi}_-(\lambda,n_0)}=O(1),\quad \lambda\to E.$$

Accordingly,

$$\frac{1}{\hat{W}(\lambda)}=O\Big(\frac{\prod_{j=1}^{g_+}(\lambda-\mu_j^+)}{\hat{\delta}_+^2(\lambda)\sqrt{\lambda-E}}\Big)=O\left(\frac{1}{\sqrt{\lambda-E}}\right),\quad \lambda\in\sigma_+,$$

which proves (3.22). To see (3.23) note that

$$g(z,n)=\frac{\psi_+(z,n)\psi_-(z,n)}{W(z)}$$

is a Herglotz function. Moreover, we can assume that $\mu_j\neq E$ and choose n such that $\psi_\pm(E,n_0)\neq 0$. Hence it remains to show the corresponding estimate for $g(z)=g(z,n_0)$. Since the continuous spectrum of H is purely absolutely continuous, we deduce from Stieltjes inversion formula that

$$g(z)=\frac{1}{\pi}\int_{E-\delta}^{E+\delta}\frac{\mathrm{Im}(g(\lambda))}{\lambda-z}d\lambda+\tilde{g}(z),\qquad \delta>0,$$

where $\tilde{g}(z)$ is holomorphic near E. By (3.22) we infer $(\lambda-E)^{1/2+\varepsilon}\mathrm{Im}(g(\lambda))$ is Hölder continuous and the result follows from [17, Eq. (29.8)].

(c) Equation (3.24) follows from (3.5).

III. **(a)** follows from the corresponding properties of $\psi_\pm(z)$ and from **II**, **(b)**. To show **III**, **(b)** we use that by (3.18) the reflection coefficients have the representation

$$R_\pm(\lambda)=-\frac{W(\overline{\psi_\pm(\lambda)},\psi_\mp(\lambda))}{W(\psi_\pm(\lambda),\psi_\mp(\lambda))}=\mp\frac{W(\overline{\psi_\pm(\lambda)},\psi_\mp(\lambda))}{W(\lambda)}\tag{3.29}$$

and are continuous on both sides of the set $\mathrm{int}(\sigma_\pm)\setminus(M_\mp\cup\hat{M}_\mp)$. Moreover,

$$|R_\pm(\lambda)|=\left|\frac{W(\overline{\hat{\psi}_\pm(\lambda)},\hat{\psi}_\mp(\lambda))}{\hat{W}(\lambda)}\right|,$$

where the denominator does not vanish on the set $\sigma_\pm\setminus\sigma_v$. Hence $R_\pm(\lambda)$ are continuous on this set since both numerator and denominator are.

Next, let $E\in\partial\sigma_\pm\setminus\sigma_v$ (in particular $\hat{W}(E)\neq 0$). Then, if $E\notin\hat{M}_\pm$, we use (3.29) in the form

$$R_\pm(\lambda)=-1\mp\frac{\hat{\delta}_\pm(\lambda)W(\psi_\pm(\lambda)-\overline{\psi_\pm(\lambda)},\hat{\psi}_\mp(\lambda))}{\hat{W}(\lambda)},\tag{3.30}$$

which shows $R_\pm(\lambda)\to -1$ since $\psi_\pm(\lambda)-\overline{\psi_\pm(\lambda)}\to 0$ by Lemma 3.1, (2). This settles the first case in (3.25). Similarly, if $E\in\hat{M}_\pm$, we use (3.29) in the form

$$R_\pm(\lambda)=1\pm\frac{\hat{\delta}_\pm(\lambda)W(\psi_\pm(\lambda)+\overline{\psi_\pm(\lambda)},\hat{\psi}_\mp(\lambda))}{\hat{W}(\lambda)},\tag{3.31}$$

which shows $R_\pm(\lambda)\to 1$ since $\hat{\delta}_\pm(\lambda)=O(\sqrt{\lambda-E})$ and $\psi_\pm(\lambda)+\overline{\psi_\pm(\lambda)}=o\big(\frac{1}{\sqrt{\lambda-E}}\big)$ by Lemma 3.1, (2). This settles the second case in (3.25) as well. □

4. The Gel'fand-Levitan-Marchenko equation

The aim of this section is to derive the inverse scattering problem equation (the Gel'fand-Levitan-Marchenko equation) and to discuss some additional properties of the scattering data which are consequences of this equation.

Theorem 4.1. *The inverse scattering problem (the GLM) equation has the form*

$$K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l) F_\pm(l,m) = \frac{\delta(n,m)}{K_\pm(n,n)}, \qquad \pm m \geq \pm n, \tag{4.1}$$

where

$$\begin{aligned} F_\pm(m,n) = &\oint_{\sigma_\pm} R_\pm(\lambda)\psi_q^\pm(\lambda,m)\psi_q^\pm(\lambda,n) d\omega_\pm \\ &+ \int_{\sigma_\mp^{(1),u}} |T_\mp(\lambda)|^2 \psi_q^\pm(\lambda,m)\psi_q^\pm(\lambda,n) d\omega_\mp + \sum_{k=1}^p \gamma_{\pm,k}\tilde\psi_q^\pm(\lambda_k,n)\tilde\psi_q^\pm(\lambda_k,m). \end{aligned} \tag{4.2}$$

Proof. Consider a closed contour Γ_ϵ consisting of a large circular arc and some contours inside this arc, which envelope the spectrum σ at a small distance ε from the spectrum. Let $\pm m \geq \pm n$. The residue theorem, (2.17), (3.5), (3.20), and equality $\tilde\psi_\mp(\lambda_k,n) = c_j^\mp \tilde\psi_\pm(\lambda_k,n)$ imply

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_\epsilon} \frac{\psi_\mp(\lambda,n)\psi_q^\pm(\lambda,m)}{W(\lambda)} d\lambda &= \frac{\delta(n,m)}{K_\pm(n,n)} + \sum_{k=1}^p \mathrm{Res}_{\lambda_k}\left(\frac{\tilde\psi_\mp(\lambda,n)\tilde\psi_q^\pm(\lambda,m)}{\tilde W(\lambda)}\right) \\ &= \frac{\delta(n,m)}{K_\pm(n,n)} - \sum_{k=1}^p \gamma_{\pm,k}\tilde\psi_\pm(\lambda_k,n)\tilde\psi_q^\pm(\lambda_k,m), \end{aligned} \tag{4.3}$$

since the integrand is meromorphic on $\mathbb{C}\backslash\sigma$ with simple poles at the eigenvalues λ_k and at ∞ if $m = n$. It is continuous till the boundary except at the points $E \in \partial\sigma_+ \cup \partial\sigma_-$ where

$$\frac{\psi_\mp(\lambda,n)\psi_q^\pm(\lambda,m)}{W(\lambda)} = O\left(\frac{1}{\sqrt{\lambda - E}}\right), \quad E \in \partial\sigma_+ \cup \partial\sigma_-, \tag{4.4}$$

by (3.22). On the other hand, as $\epsilon \to 0$,

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} &\oint_{\sigma} \frac{\psi_\mp(\lambda,n)\psi_q^\pm(\lambda,m)}{W(\lambda)} d\lambda = \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\sigma_\pm} \frac{\big(\overline{\psi_\pm(\lambda,n)} + R_\pm(\lambda)\psi_\pm(\lambda,n)\big)\psi_q^\pm(\lambda,m)}{T_\pm(\lambda)W(\lambda)} d\lambda \\ &\quad + \frac{1}{2\pi\mathrm{i}} \oint_{\sigma_\mp^{(1)}} \frac{\psi_\mp(\lambda,n)\psi_q^\pm(\lambda,m)}{W(\lambda)} d\lambda \\ &= \oint_{\sigma_\pm} \overline{\psi_\pm(\lambda,n)}\psi_q^\pm(\lambda,m) d\omega^\pm + \oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,n)\psi_q^\pm(\lambda,m) d\omega^\pm \\ &\quad + \frac{1}{2\pi\mathrm{i}} \int_{\sigma_\mp^{(1),u}} \psi_q^\pm(\lambda,m)\left(\frac{\psi_\mp(\lambda,n)}{W(\lambda)} - \frac{\overline{\psi_\mp(\lambda,n)}}{\overline{W(\lambda)}}\right) d\lambda. \end{aligned} \tag{4.5}$$

It remains to treat the last integrand. By (3.17) and Lemma 3.2, **I**,

$$\overline{\psi_{\mp}(\lambda,n)} = T_{\mp}(\lambda)\psi_{\pm}(\lambda,n) - R_{\mp}(\lambda)\psi_{\mp}(\lambda,n) = T_{\mp}(\lambda)\psi_{\pm}(\lambda,n) - \frac{T_{\mp}(\lambda)}{\overline{T_{\mp}(\lambda)}}\psi_{\mp}(\lambda,n),$$

and therefore

$$\frac{\psi_{\mp}(n)}{W} - \frac{\overline{\psi_{\mp}(n)}}{\overline{W}} = \frac{\overline{WT_{\mp}} + WT_{\mp}}{|W|^2\overline{T_{\mp}}}\psi_{\mp}(n) - \frac{T_{\mp}}{\overline{W}}\psi_{\pm}(n) = -\frac{T_{\mp}}{\overline{W}}\psi_{\pm}(n),$$

since $\overline{WT_{\mp}} + WT_{\mp} = 2\mathrm{Re}(WT_{\mp}) = 0$ on $\sigma_{\mp}$. In summary, (4.3) and (4.5) yield

$$\frac{\delta(n,m)}{K_{\pm}(n,n)} = K_{\pm}(n,m) + \oint_{\sigma_{\pm}} R_{\pm}(\lambda)\psi_{\pm}(\lambda,n)\psi_q^{\pm}(\lambda,m)d\omega^{\pm}$$
$$+ \int_{\sigma_{\mp}^{(1),u}} |T_{\mp}(\lambda)|^2\psi_{\pm}(\lambda,n)\psi_q^{\pm}(\lambda,m)d\omega^{\mp} + \sum_{j=1}^{p}\gamma_{\pm,j}\tilde{\psi}_{\pm}(\lambda_j,n)\tilde{\psi}_q^{\pm}(\lambda_j,m),$$

and applying (3.2) finishes the proof. □

As it is shown in [8], the estimate (3.3) for $K_{\pm}(n,m)$ implies the following estimates for $F_{\pm}(n,m)$.

Lemma 4.2. *The kernel of the GLM equation satisfies the following properties.* **IV**. *There exist functions $C_{\pm}(n) > 0$ and $q_{\pm}(n) \geq 0$, $n \in \mathbb{Z}_{\pm}$, such that $C_{\pm}(n)$ decay as $n \to \pm\infty$, $|n|q(n) \in \ell^1(\mathbb{Z}_{\pm})$, and*

$$\begin{aligned} &|F_{\pm}(n,m)| \leq C_{\pm}(n)\sum_{j=n+m}^{\pm\infty} q(j), \\ &\sum_{n=n_0}^{\pm\infty} |n|\big|F_{\pm}(n,n) - F_{\pm}(n\pm 1, n\pm 1)\big| < \infty, \qquad (4.6)\\ &\sum_{n=n_0}^{\pm\infty} |n|\big|a_q^{\pm}(n)F_{\pm}(n,n+1) - a_q^{\pm}(n-1)F_{\pm}(n-1,n)\big| < \infty. \end{aligned}$$

In summary, we have obtained the following necessary conditions for the scattering data:

Theorem 4.3. *The scattering data*

$$\begin{aligned} \mathcal{S} = \Big\{ &R_+(\lambda),\, T_+(\lambda),\, \lambda \in \sigma_+^{\mathrm{u,l}};\, R_-(\lambda),\, T_-(\lambda),\, \lambda \in \sigma_-^{\mathrm{u,l}}; \\ &\lambda_1, \dots, \lambda_p \in \mathbb{R} \setminus (\sigma_+ \cup \sigma_-),\, \gamma_{\pm,1}, \dots, \gamma_{\pm,p} \in \mathbb{R}_+ \Big\} \end{aligned} \qquad (4.7)$$

satisfy the properties **I-III** *listed in Lemma 3.2. The functions $F_{\pm}(n,m)$, defined in (4.2), satisfy property* **IV** *in Lemma 4.2.*

In fact, the conditions on the scattering data given in Theorem 4.3 are both necessary and sufficient for the solution of the scattering problem in the class (2.5). The sufficiency of these conditions as well as the algorithm for the solution of the inverse problem will be discussed in the next section.

5. The inverse scattering problem

Let $H_q^{\pm}$ be two arbitrary quasi-periodic Jacobi operators associated with sequences $a_q^{\pm}(n), b_q^{\pm}(n)$ as introduced in Section 2. Let $\mathcal{S}$ be given scattering data with corresponding kernels $F_{\pm}(n,m)$ satisfying the necessary conditions of Theorem 4.3.

First we show that the GLM equations (4.1) can be solved for $K_{\pm}(n,m)$ if $F_{\pm}(n,m)$ are given.

Lemma 5.1. *Under condition* **IV***, the GLM equations (4.1) have unique real-valued solutions* $K_{\pm}(n,\cdot) \in \ell^1(n,\pm\infty)$ *satisfying the estimates*

$$|K_{\pm}(n,m)| \le C_{\pm}(n) \sum_{j=\left[\frac{n+m}{2}\right]}^{\pm\infty} q(j), \quad \pm m > \pm n. \tag{5.1}$$

Here the functions $q_{\pm}(n)$ *and* $C_{\pm}(n)$ *are of the same type as in (4.6).*

Moreover, the following estimates are valid

$$\begin{aligned} &\sum_{n=n_0}^{\pm\infty} |n|\big|K_{\pm}(n,n) - K_{\pm}(n\pm 1, n\pm 1)\big| < \infty, \\ &\sum_{n=n_0}^{\pm\infty} |n|\big|a_q^{\pm}(n)K_{\pm}(n,n+1) - a_q^{\pm}(n-1)K_{\pm}(n-1,n)\big| < \infty. \end{aligned} \tag{5.2}$$

Proof. The solvability of (4.1) under condition (4.6) and the estimates (5.1), (5.2) follow completely analogous to the corresponding result in [8, Theorem 7.5]. To prove uniqueness, first note that the GLM equations are generated by compact operators. Thus, it is sufficient to prove that the equation

$$f(m) + \sum_{\ell=n}^{\pm\infty} F_{\pm}(\ell,m) f(\ell) = 0 \tag{5.3}$$

has only the trivial solution in the space $\ell^1(n,\pm\infty)$. The proof is similar for the "+" and "−" cases, hence we give it only for the "+" case. Let $f(\ell)$, $\ell > n$, be a nontrivial solution of (5.3) and set $f(\ell) = 0$ for $\ell \le n$. Since $F_+(\ell,n)$ is real-valued, we can assume that $f(\ell)$ is real-valued. Abbreviate by

$$\widehat{f}(\lambda) = \sum_{m\in\mathbb{Z}} \psi_q^+(\lambda,m) f(m) \tag{5.4}$$

the generalized Fourier transform, generated by the spectral decomposition (2.18) (cf. [22]). Recall that $\widehat{f}(\lambda) \in L^1_{loc}(\sigma_+^{\mathrm{u}} \cup \sigma_+^{\mathrm{l}})$.

Multiplying (5.3) by $f(m)$, summing over $m \in \mathbb{Z}$, and applying (2.18), (4.2), (5.4), and condition **I**, **(a)**, we have

$$\begin{aligned} &2\int_{\sigma_+^u} |\widehat{f}(\lambda)|^2 d\omega_+(\lambda) + 2\mathrm{Re}\int_{\sigma_+^u} R_+(\lambda)\widehat{f}(\lambda)^2 d\omega_+(\lambda) \\ &\quad + \int_{\sigma_-^{(1),u}} \widehat{f}(\lambda)^2 |T_-(\lambda)|^2 d\omega_-(\lambda) + \sum_{k=1}^{p} \gamma_{+,k}\Big(\sum_{n\in\mathbb{Z}} \tilde{\psi}_q^+(\lambda_k,n) f(n)\Big)^2 = 0. \end{aligned} \tag{5.5}$$

The last two summands in (5.5) are nonnegative since $\widehat{f}(\lambda) \in \mathbb{R}$ for $\lambda \in \sigma_-^{(1)}$ and $\tilde{\psi}_q^+(\lambda_k) \in \mathbb{R}$. We estimate the first two integrands by

$$|\widehat{f}(\lambda)|^2 + \mathrm{Re} R_+(\lambda)\widehat{f}(\lambda)^2 \geq |\widehat{f}(\lambda)|^2 - |R_+(\lambda)\widehat{f}(\lambda)^2| \geq \big(1 - |R_+(\lambda)|\big)|\widehat{f}(\lambda)|^2$$

and drop the last summand in (5.5), thus obtaining

$$2\int_{\sigma^{(2),\mathrm{u}}} (1 - |R_+(\lambda)|)|\widehat{f}(\lambda)|^2 d\omega_+(\lambda) + \int_{\sigma_-^{(1),\mathrm{u}}} \widehat{f}(\lambda)^2 |T_-(\lambda)|^2 d\omega_-(\lambda) \leq 0. \tag{5.6}$$

Here we also used that

$$\int_{\sigma_+^{(1),\mathrm{u}}} (1 - |R_+(\lambda)|)|\widehat{f}(\lambda)|^2 d\omega_+(\lambda) = 0,$$

which follows from condition **I**, **(b)**. Since $|R_+(\lambda)| < 1$ for $\lambda \in \mathrm{int}(\sigma^{(2)})$ and $\omega_-(\lambda) > 0$ for $\lambda \in \mathrm{int}(\sigma_-^{(1)})$ we conclude that

$$\widehat{f}(\lambda) = 0 \quad \text{for} \quad \lambda \in \sigma^{(2)} \cup \sigma_-^{(1)} = \sigma_-.$$

The function $\widehat{f}(z)$ can be defined by formula (5.4) as a meromorphic function on $\mathbb{C} \setminus \sigma_+$. By our analysis it is even meromorphic on $\mathbb{C} \setminus \sigma_+^{(1)}$ and vanishes on σ_-. Thus $\widehat{f}(z)$ and hence also $f(m)$ are equal to zero. □

Next, define the sequences $a_\pm, b_\pm$ by

$$\begin{aligned}
a_+(n) &= a_q^+(n)\frac{K_+(n+1,n+1)}{K_+(n,n)},\\
a_-(n) &= a_q^-(n)\frac{K_-(n,n)}{K_-(n+1,n+1)},\\
b_+(n) &= b_q^+(n) + a_q^+(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q^+(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)},\\
b_-(n) &= b_q^-(n) + a_q^-(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q^-(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)},
\end{aligned} \tag{5.7}$$

and note that estimate (5.2) implies

$$n\left\{|a_\pm(n) - a_q^\pm(n)| + |b_\pm - b_q^\pm(n)|\right\} \in \ell^1(\mathbb{Z}_\pm). \tag{5.8}$$

Lemma 5.2. *The functions $\psi_\pm(z,n)$, defined by*

$$\psi_\pm(z,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_q^\pm(z,m), \tag{5.9}$$

solve the equations

$$a_\pm(n-1)\psi_\pm(z,n-1) + b_\pm(n)\psi_\pm(z,n) + a_\pm(n)\psi_\pm(z,n+1) = z\psi_\pm(z,n), \tag{5.10}$$

where $a_\pm(n), b_\pm(n)$ are defined by (5.7).

Proof. Consider the two operators[2]

$$(H_\pm y)(n) = a_\pm(n-1)y_\pm(n-1) + b_\pm(n)y_\pm(n) + a_\pm(n)y_\pm(n+1), \quad n \in \mathbb{Z}.$$

[2]We don't know that $H_\pm$ is limit point at $\mp\infty$ yet, but this will not be used.

Define two discrete integral operators

$$(\mathcal{K}_\pm f)(n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m) f(m).$$

Then (cf. [8]) the following identity is valid

$$H_\pm \mathcal{K}_\pm = \mathcal{K}_\pm H_q^\pm,$$

which implies (5.10). □

The remaining problem is to show that $a_+(n) \equiv a_-(n)$, $b_+(n) \equiv b_-(n)$ under conditions **II** and **III** on the scattering data $\mathcal{S}$.

Theorem 5.3. *Let the scattering data $\mathcal{S}$, defined as in (4.7), satisfy conditions* **I**, **(a)–(c)**, **II**, **III**, **(a)**, *and* **IV**. *Then each of the GLM equations (4.1) has unique solutions $K_\pm(n,m)$, satisfying the estimate (5.2). The functions $a_\pm(n), b_\pm(n)$, defined by (5.7), satisfy (5.8).*

Under the additional conditions **III**, **(b)** *and* **I**, **(d)**, *these functions coincide, $a_+(n) \equiv a_-(n) = a(n)$, $b_+(n) \equiv b_-(n) = b(n)$, and the data $\mathcal{S}$ are the scattering data for the Jacobi operator associated with the sequences $a(n), b(n)$.*

The proof of Theorem 5.3 takes up the remaining section and is split into several lemmas for the convenience of the reader.

To prove uniqueness of the reconstructed potential we follow the method proposed in [15]. Recall that, by Lemma 2.1 (iii), the functions $\psi_q^\pm(\lambda,n)$ form an orthonormal basis with corresponding generalized Fourier transform. Split the kernel of the GLM equation (4.2) into three summands $F_\pm(m,n) = F_{r,\pm}(m,n) + F_{h,\pm}(m,n) + F_{d,\pm}(m,n)$ and set

$$G_\pm(n,m) := \sum_{l=n}^{\pm\infty} K_\pm(n,l) F_{r,\pm}(l,n). \tag{5.11}$$

Then one obtains as in [8, Theorem 8.2] that the functions $h_\mp(\lambda,n)$, defined by

$$\begin{aligned} h_\mp(\lambda,n) = \frac{1}{T_\pm(\lambda)} \Bigg(& \frac{\breve{\psi}_q^\pm(\lambda,n)}{K_\pm(n,n)} + \sum_{m=n\mp1}^{\mp\infty} G_\pm(n,m)\breve{\psi}_q^\pm(\lambda,m) \\ & \mp \int_{\sigma_\mp^{(1),u}} |T_\mp(\xi)|^2 \psi_\pm(\xi,n) \frac{W_{q,n-1}^\pm(\psi_q^\pm(\xi), \breve{\psi}_q^\pm(\lambda))}{\xi-\lambda} d\omega_\mp(\xi) \\ & \pm \sum_{k=1}^{p} \gamma_{\pm,k} \tilde{\psi}_\pm(\lambda_k,n) \frac{W_{q,n-1}^\pm(\tilde{\psi}_q^\pm(\lambda_k), \breve{\psi}_q^\pm(\lambda))}{\lambda-\lambda_k} \Bigg), \end{aligned} \tag{5.12}$$

satisfy

$$T_\pm(\lambda) h_\mp(\lambda,n) = \overline{\psi_\pm(\lambda,n)} + R_\pm(\lambda)\psi_\pm(\lambda,n), \quad \lambda \in \sigma_\pm^{u,l}. \tag{5.13}$$

Despite the fact that $h_\mp(\lambda,n)$ are defined via the background solutions corresponding to the opposite half-axis $\mathbb{Z}_\pm$, they share a series of properties with $\psi_\mp(\lambda,n)$. Namely, we prove

Lemma 5.4. *Let $h_\mp(z,n)$ be defined by formula (5.12) on the set $\sigma_\pm^{\mathrm{u,l}}$.*

(i) *The functions $\tilde{h}_\mp(z,n) = \delta_\mp(z) h_\mp(z,n)$ admit analytic extensions to the domain $\mathbb{C} \setminus \sigma$.*

(ii) *The functions* $\tilde h_\mp(z,n)$ *are continuous up to the boundary* $\sigma^{\mathrm{u,l}}$ *except possibly at the points* $\partial\sigma_+\cup\partial\sigma_-$. *Furthermore,*

$$\begin{aligned}\tilde h_\mp(\lambda^{\mathrm u},n)&=\tilde h_\mp(\lambda^{\mathrm l},n)\in\mathbb R, &&\lambda\in\mathbb R\setminus\sigma_\mp,\\ \tilde h_\mp(\lambda^{\mathrm u},n)&=\overline{\tilde h_\mp(\lambda^{\mathrm l},n)}, &&\lambda\in\mathrm{int}(\sigma_\mp).\end{aligned}\tag{5.14}$$

(iii) *For large* z *the functions* $h_\mp(z,n)$ *have the following asymptotic behavior*

$$h_\mp(z,n)=\frac{z^{\pm n}}{K_\pm(n,n)T_\pm(\infty)}\Big(\prod_{j=0}^{n-1}{}^{*}\ a_q^\pm(j)\Big)^{\mp1}\Big(1+O(\frac1z)\Big),\quad z\to\infty.\tag{5.15}$$

(iv) *We have*

$$\begin{aligned}W^\pm(h_\mp(z),\psi_\pm(z)):=&a_\pm(n)\big(h_\mp(z,n)\psi_\pm(z,n+1)-h_\mp(z,n+1)\psi_\pm(z,n)\big)\\ \equiv&\pm W(z),\end{aligned}$$

where $W(z)$ *is defined by (3.19).*

Remark 5.5. *Note that we did not establish the connection between the function* $W(z)$ *and the functions* $W^\pm(\psi_+(z,n),\psi_-(z,n))$, *which can depend on* n, *because* ψ_+ *and* ψ_- *are the solutions of Jacobi equations corresponding to possibly different operators* H_+ *and* H_-.

Proof. (i). To show that $\tilde h_\mp(z,n)$ have analytic extensions to $\mathbb C\setminus\sigma$, we study each term in (5.12) separately.

First of all, note that due to the representation

$$T_\pm(z)=\frac{1}{\rho_\pm(z)W(z)}=\frac{\hat\delta_\mp(z)}{\breve\delta_\pm(z)}\frac{\sqrt{\prod_{j=0}^{2g_\pm+1}(z-E_j^\pm)}}{\hat W(z)},\tag{5.16}$$

the functions $\tilde\zeta_\mp(z,n)=\delta_\mp(z)\zeta_\mp(z,n)$, where

$$\zeta_\mp(z,n):=\frac{\breve\psi_q^\pm(z,n)}{T_\pm(z)},\tag{5.17}$$

can be continued analytically to $\mathbb C\setminus\sigma$. This also holds for the second term since $G_\pm(n,\cdot)\in\ell^1(\mathbb Z)$ are real-valued.

Next we discuss the properties of the Cauchy-type integral in the representation (5.12). We represent the third summand in (5.12) multiplied by $T_\pm^{-1}(z)$ as

$$\Theta_\mp(z,n):=\mp\frac{1}{2\pi\mathrm i}\int_{\sigma_\mp^{(1),\mathrm u}}\theta_\mp(z,\xi,n)\frac{d\xi}{\xi-z},\tag{5.18}$$

where

$$\begin{aligned}\theta_\mp(z,\xi,n)&=-\frac{\delta_\mp(\xi)^2}{\rho_\mp(\xi)|\tilde W(\xi)|^2}\tilde\psi_\pm(\xi,n)W_{q,n-1}^\pm(\tilde\psi_q^\pm(\xi,\cdot),\zeta_\mp(z,\cdot))\\ &=-\frac{|\hat\delta_\mp(\xi)|^2}{\rho_\mp(\xi)|\hat W(\xi)|^2}\frac{|\hat\delta_\pm(\xi)|^2}{\hat\delta_\pm(\xi)^2}\hat\psi_\pm(\xi,n)W_{q,n-1}^\pm(\hat\psi_q^\pm(\xi,\cdot),\zeta_\mp(z,\cdot)).\end{aligned}\tag{5.19}$$

By property **II**, **(a)** the function $\hat W(\xi)$ has no zeros in the interior of $\sigma_\mp^{(1),\mathrm u}$. Thus, for $z\notin\sigma_\mp^{(1)}$, the functions $\theta_\mp(z,.,n)$ are bounded in the interior of $\sigma_\mp^{(1)}$ and the only possible singularities can arise at the boundary. We claim

$$\theta_\mp(z,\xi,n) = \begin{cases} O(\sqrt{\xi - E}) & \text{for } E \notin \sigma_v, \\ O\big(\frac{1}{\sqrt{\xi-E}}\big) & \text{for } E \in \sigma_v, \end{cases} \qquad E \in \partial\sigma_\mp^{(1)},\ z \neq E. \tag{5.20}$$

This follows from $\frac{|\hat\delta_\mp(\xi)|^2}{\rho_\mp(\xi)} = O(\sqrt{\xi - E})$ together with $\hat W(\xi) = O(1)$ if $E \notin \sigma_v$ and $1/\hat W(\xi) = O(1/\sqrt{\xi - E})$ by **II**, **(b)** if $E \in \sigma_v$. Therefore, $\theta_\mp$ are integrable and the third summand of (5.12) also inherits the properties of $\zeta_\mp(z,n)$.

Finally, the last summand in (5.12) again inherits the properties of $\tilde\zeta_\mp(z,n)$ except for possible additional poles at the eigenvalues λ_k. However, these cancel with the zeros of $\tilde W(z)$ at $z = \lambda_k$.

(ii). We consider the boundary values next. The only nontrivial term is of course the Cauchy-type integral (5.18) as $z \to \lambda \in \operatorname{int}(\sigma_\mp^{(1)})$. First of all observe that by (2.7) and (3.19),

$$\frac{W^\pm_{q,n-1}(\tilde\psi_q^\pm(\lambda), \breve\psi_q^\pm(z))}{T_\pm(z)} \to (\delta_\pm W)(\lambda),$$

where the functions $\delta_\pm W$ are bounded and nonzero for $\lambda \in \operatorname{int}(\sigma_\mp^{(1)})$ by **II**, **(a)**. Hence the Plemelj formula applied to (5.18) gives

$$\Theta_\mp(\lambda,n) = \pm\frac{\tilde\psi_\pm(\lambda,n)}{2\delta_\pm(\lambda)\rho_\mp(\lambda)\overline{W(\lambda)}} \mp \fint_{\sigma_\mp^{(1),\mathrm{u}}} \frac{\theta_\mp(\lambda,\xi,n)}{\xi-\lambda} d\xi, \quad \lambda \in \operatorname{int}(\sigma_\mp^{(1),\mathrm{u}}),$$

where both terms are finite. Here $\fint$ denotes the principle value integral. Therefore, the boundary values away from $\partial\sigma_+ \cup \partial\sigma_-$ exist and we have

$$h_\mp(\lambda^{\mathrm{u}}, n) = \overline{h_\mp(\lambda^{\mathrm{l}}, n)}, \quad \lambda \in \sigma_+ \cup \sigma_-. \tag{5.21}$$

By property **I**, **(b)**,

$$h_\mp = T_\pm^{-1}\left(R_\pm\psi_\pm + \overline{\psi_\pm}\right) = \frac{\psi_\pm}{\overline{T_\pm}} + \frac{\overline{\psi_\pm}}{T_\pm} \in \mathbb{R}, \quad \lambda \in \sigma_\pm^{(1)}, \tag{5.22}$$

from which

$$h_\mp(\lambda^{\mathrm{u}}, n) = h_\mp(\lambda^{\mathrm{l}}, n), \quad \lambda \in \sigma_\pm^{(1)}, \tag{5.23}$$

follows. Combining (5.21) and (5.23) yields (5.14).

(iii). Since the last two terms in (5.12) are $O(z^{-1})$, the asymptotic behavior follows from (3.5) and **II**, **(c)**.

(iv). From (5.13), (3.8), and (3.19) we obtain

$$W^\pm(h_\mp(\lambda), \psi_\pm(\lambda)) = \frac{W^\pm(\overline{\psi_\pm(\lambda)}, \psi_\pm(\lambda))}{T_\pm(\lambda)} = \frac{1}{T_\pm(\lambda)\rho_\pm(\lambda)} = \pm W(\lambda), \quad \lambda \in \sigma_\pm.$$

Hence equality holds for all $z \in \mathbb{C}$ by analytic continuation. □

Corollary 5.6. *The functions $\tilde h_\mp(z,n)$ admit analytic extensions to $\mathbb{C} \setminus \sigma_\mp$.*

Proof. Property (i) of Lemma 5.4 holds for $z \in \mathbb{C} \setminus \sigma$. Relation (5.14) implies that $\tilde h_\mp$ have no jumps across $z \in \operatorname{int}(\sigma_\pm^{(1)})$. To finish the proof we need to show that the possible remaining singularities at $E \in \partial\sigma_\pm^{(1)} \cap \partial\sigma$ are removable. This follows from (cf. (5.16))

$$\hat{\zeta}_\mp(z,n) = \frac{\hat{W}(z)}{\sqrt{\prod_{j=0}^{2g_\pm+1}(z-E_j^\pm)}}\breve{\delta}_\pm(z)\breve{\psi}_q^\pm(z,n) \tag{5.24}$$

which shows $\tilde{\zeta}_\mp(z,n) = O((z-E)^{-1/2})$ and hence $\tilde{h}_\mp(z,n) = O((z-E)^{-1/2})$ for $E \in \sigma_\pm^{(1)} \cap \partial\sigma$.

However, let us emphasize at this point that the behavior of $h_\pm(z,n)$ at the remaining edges is a more subtle question to be discussed later. □

Eliminating $\overline{\psi_\pm}$ from

$$\begin{cases} \overline{R_\pm(\lambda)}\,\overline{\psi_\pm(\lambda,n)} + \psi_\pm(\lambda,n) &= \overline{h_\mp(\lambda,n)}\,\overline{T_\pm(\lambda)} \\ R_\pm(\lambda)\,\psi_\pm(\lambda,n) + \overline{\psi_\pm(\lambda,n)} &= h_\mp(\lambda,n)\,T_\pm(\lambda) \end{cases}$$

yields

$$\psi_\pm(\lambda,n)\left(1-|R_\pm(\lambda)|^2\right) = \overline{h_\mp(\lambda,n)}\,\overline{T_\pm(\lambda)} - \overline{R_\pm(\lambda)}\,h_\mp(\lambda,n)\,T_\pm(\lambda).$$

We apply **I**, **(c)**, **II**, and the consistency condition **I**, **(d)** to obtain

$$\begin{aligned} T_\mp(\lambda)\psi_\pm(\lambda,n) &= \overline{h_\mp(\lambda,n)} - \frac{\overline{R_\pm(\lambda)}T_\pm(\lambda)}{\overline{T_\pm(\lambda)}}h_\mp(\lambda,n) \\ &= \overline{h_\mp(\lambda,n)} + R_\mp(\lambda)h_\mp(\lambda,n), \quad \lambda \in \sigma^{(2)}. \end{aligned} \tag{5.25}$$

This equation together with (5.13) gives us a system from which we can eliminate the reflection coefficients $R_\pm$. We obtain

$$T_\pm(\lambda)\big(\psi_\pm(\lambda)\psi_\mp(\lambda) - h_\pm(\lambda)h_\mp(\lambda)\big) = \psi_\pm(\lambda)\overline{h_\pm(\lambda)} - \overline{\psi_\pm(\lambda)}h_\pm(\lambda), \quad \lambda \in \sigma^{(2),\mathrm{u,l}}. \tag{5.26}$$

Now introduce the function

$$G(z) := G(z,n) = \frac{\psi_+(z,n)\psi_-(z,n) - h_+(z,n)h_-(z,n)}{W(z)} \tag{5.27}$$

which is well defined in the domain $z \in \mathbb{C} \setminus (\sigma \cup \sigma_d \cup M_+ \cup M_-)$. By (5.26) and (3.19),

$$G(\lambda) = \Big(\psi_\pm(\lambda)\overline{h_\pm(\lambda)} - \overline{\psi_\pm(\lambda)}h_\pm(\lambda)\Big)\,\rho_\pm(\lambda), \quad \lambda \in \sigma^{(2),\mathrm{u,l}}, \tag{5.28}$$

so we need to study the properties of $G(z,n)$ as a function of z. Our aim is to prove that $G(z,n) = 0$, which will follow from the next lemma.

Lemma 5.7. *The function $G(z,n)$, defined by (5.27), has the following properties.*

(i) $G(\lambda^{\mathrm{u}},n) = G(\lambda^{\mathrm{l}},n) \in \mathbb{R}$ *for* $\lambda \in \mathbb{R} \setminus (\partial\sigma_- \cup \partial\sigma_+ \cup \sigma_d)$.

(ii) *It has removable singularities at the points* $\partial\sigma_- \cup \partial\sigma_+ \cup \sigma_d$, *where* $\sigma_d := \{\lambda_1, \dots, \lambda_p\}$.

Proof. (i). We can rewrite $G(z,n)$ as

$$G(z,n) = \frac{\tilde{\psi}_+(z,n)\tilde{\psi}_-(z,n) - \tilde{h}_+(z,n)\tilde{h}_-(z,n)}{\tilde{W}(z)}, \tag{5.29}$$

where $\tilde h_\pm(z,n)=\delta_\pm(z)h_\pm(z,n)$ as usual. The numerator is bounded near the points under consideration and the denominator does not vanish there. Thus $G(z,n)$ has no singularities at the points $(M_+\cup M_-)\setminus\sigma_d$.

Furthermore, by Lemma 5.4, **II**, **(a)**, and Lemma 3.1 we know that $G(z,n)$ has continuous limiting values on the sets σ_- and σ_+, except possibly at the edges, and satisfies

$$G(\lambda^{\mathrm{u}},n)=\overline{G(\lambda^{\mathrm{l}},n)},\quad \lambda\in\sigma_+\cup\sigma_-.$$

Hence, if we can show that these limits are real, they will be equal and $G(z,n)$ will extend to a meromorphic function on $\mathbb{C}$, that is, (i) holds. To this aim we first observe that (5.14), (5.28), and Lemma 3.1 imply

$$G(\lambda^{\mathrm{u}},n)=G(\lambda^{\mathrm{l}},n)\in\mathbb{R},\quad \lambda\in\operatorname{int}(\sigma^{(2)}). \tag{5.30}$$

Thus, it remains to prove

$$G(\lambda^{\mathrm{u}},n)=G(\lambda^{\mathrm{l}},n)\in\mathbb{R}\quad\text{for}\quad \lambda\in\operatorname{int}(\sigma_-^{(1)})\cup\operatorname{int}(\sigma_+^{(1)}). \tag{5.31}$$

Let us show that $G(\lambda,n)$ has no jump on the set $\operatorname{int}(\sigma_-^{(1)})\cup\operatorname{int}(\sigma_+^{(1)})$. We abbreviate

$$[G]:=G(\lambda)-\overline{G(\lambda)}=\left[\frac{\psi_+\psi_-}{W}\right]-\left[\frac{h_+h_-}{W}\right],\quad \lambda\in\sigma_\pm^{(1),\mathrm{u}}, \tag{5.32}$$

and drop some dependencies until the end of this lemma for notational simplicity.

Let $\lambda\in\operatorname{int}(\sigma_\mp^{(1),\mathrm{u}})$, then $\psi_\pm,h_\pm\in\mathbb{R}$ and $\overline{T}_\mp=-(\overline{W}\,\rho_\mp)^{-1}$. By (3.19), **(I)**, **(b)**, and (5.13) we obtain for $\lambda\in\operatorname{int}(\sigma_\mp^{(1)})$

$$\left[\frac{\psi_+\psi_-}{W}\right]=\psi_\pm\left[\frac{\psi_\mp}{W}\right]=\rho_\mp\psi_\pm\left(\psi_\mp T_\mp+\overline{\psi}_\mp\overline{T}_\mp\right)=\rho_\mp h_\pm\psi_\pm|T_\mp|^2. \tag{5.33}$$

Since $\rho_\pm\in\mathbb{R}$ for $\lambda\in\operatorname{int}(\sigma_\mp^{(1),\mathrm{u}})$, (3.19) implies

$$\left[\frac{h_\mp}{W}\right]=\rho_\pm\left[h_\mp T_\pm\right].$$

The only non-real summand in (5.12) is the Cauchy-type integral. The Plemelj formula applied to this integral gives

$$[h_\mp T_\pm]=-\rho_\mp\psi_\pm|T_\mp|^2W(\psi_q^\pm,\breve\psi_q^\pm)=\rho_\mp\psi_\pm|T_\mp|^2\frac{1}{\rho_\pm},$$

and by (5.33) we get

$$\left[\frac{h_+h_-}{W}\right]=\left[\frac{\psi_+\psi_-}{W}\right]=\rho_\mp\psi_\pm h_\pm|T_\mp|^2,\quad \lambda\in\operatorname{int}(\sigma_\mp^{(1)}). \tag{5.34}$$

Since $\tilde W\neq 0$ for $\lambda\in\operatorname{int}(\sigma_\mp^{(1)})$, the function

$$\rho_\mp\psi_\pm h_\pm|T_\mp|^2=-\frac{\delta_\mp^2}{\rho_\mp}\frac{\tilde\psi_\pm\tilde h_\pm}{|\tilde W|^2}$$

is bounded on the set under consideration. Finally, (5.34) and (5.32) imply (5.31).

(ii). Now we prove that the function $G(z,n)$ has removable singularities at the points $\partial\sigma_-\cup\partial\sigma_+\cup\sigma_d$. We divide this set into four subsets

$$\Omega_1^\pm=\partial\sigma^{(2)}\cap\operatorname{int}(\sigma_\mp),\quad \Omega_2=\partial\sigma^{(2)}\cap\partial\sigma,\quad \Omega_3^\pm=\partial\sigma_\pm^{(1)}\cap\partial\sigma_\pm,\quad \Omega_4=\sigma_d. \tag{5.35}$$

Since all singularities of G are at most isolated poles, it is sufficient to show that

$$G(z)=o\big((z-E)^{-1}\big) \tag{5.36}$$

from some direction in the complex plane.

Ω_1: Consider $E \in \Omega_1^+$ (the case $E \in \Omega_1^-$ being completely analogous). We will study $\lim_{\lambda\to E} G(\lambda,n)$ as $\lambda \in \mathrm{int}(\sigma^{(2)})$ using (5.28) with the "$-$" sign. Note that $\psi_- = O(1)$, $\rho_- = O(1)$, and $\hat W(E) \neq 0$. Moreover, we obtain from Lemma 3.1 respectively **II** that

$$\psi_+(\lambda) = \begin{cases} O(1), & E \notin \hat M_+, \\ O\left(\frac{1}{\sqrt{\lambda - E}}\right), & E \in \hat M_+, \end{cases} \qquad \frac{1}{T_+(\lambda)} = \begin{cases} O\left(\frac{1}{\sqrt{\lambda-E}}\right), & E \notin \hat M_+, \\ O(1), & E \in \hat M_+, \end{cases}$$

which shows

$$h_-(\lambda) = \frac{\overline{\psi_+(\lambda)} + R_+(\lambda)\psi_+(\lambda)}{T_+(\lambda)} = O\left(\frac{1}{\sqrt{\lambda - E}}\right)$$

for $\lambda \in \sigma^{(2)}$. Inserting this into (5.28) shows $G(\lambda,n) = O\big(\frac{1}{\sqrt{\lambda-E}}\big)$ and finishes the case $E \in \Omega_1$.

Ω_2: For $E \in \partial\sigma^{(2)} \cap \partial\sigma$, we use (5.28) and take the limit $\lambda \to E$ from $\sigma^{(2)}$. First of all, observe that

$$\breve\delta_-\left(R_-\psi_- + \overline{\psi}_-\right) = \begin{cases} O(1) & E \in \sigma_v, \\ o(1) & E \notin \sigma_v. \end{cases}$$

The case $E \in \sigma_v$ is evident. If $E \notin \sigma_v$ then (3.11) and (3.25) yield

$$\breve\delta_-\left(R_-\psi_- + \overline{\psi}_-\right) = \begin{cases} \breve\delta_-\big((\psi_- - \overline{\psi}_-) + (R_- + 1)\psi_-\big), & E \notin \hat M_- \\ \big(\breve\delta_-(\psi_- + \overline{\psi}_-) + (R_- - 1)\breve\delta_-\psi_-\big), & E \in \hat M_- \end{cases} = o(1).$$

Therefore, both for virtual and non-virtual levels the estimate

$$\breve\delta_-\left(R_-\psi_- + \overline{\psi}_-\right)\hat W = o(1), \quad E \in \partial\sigma_-, \tag{5.37}$$

is valid. Inserting (5.13) into the summand $\overline{\psi}_+ h_+\rho_+$ of (5.28) (for the second summand we use an analogous approach) we obtain (recall (2.2))

$$\begin{aligned} \overline{\psi}_+ h_+\rho_+ &= \overline{\psi}_+\rho_+\rho_-(\overline{\psi}_- + R_-\psi_-)W = \frac{\overline{\psi}_+\breve\delta_+}{P_+P_-}\hat\delta_+\hat\delta_-\breve\delta_-(\overline{\psi}_- + R_-\psi_-)W \\ &= \frac{\overline{\psi}_+\breve\delta_+}{P_+P_-}\breve\delta_-\left(R_-\psi_- + \overline{\psi}_-\right)\hat W. \end{aligned} \tag{5.38}$$

Combining the estimate

$$\frac{\overline{\psi}_+\breve\delta_+}{P_+P_-} = O\left(\frac{1}{\lambda - E}\right)$$

with (5.37) we have $G(z) = o\big((z-E)^{-1}\big)$ as desired.

Ω_3: Suppose that $E \in \partial\sigma_-^{(1)} \cap \partial\sigma_-$ (the case $E \in \partial\sigma_+^{(1)} \cap \partial\sigma_+$ is again analogous). Now we cannot use (5.28), so we proceed directly from formula (5.27) estimating the summands $\frac{\psi_+\psi_-}{W}$ and $\frac{h_+h_-}{W}$ separately. We investigate the limit as $\lambda \to E$ from the set $\mathrm{int}(\sigma_-^{(1)})$. By Lemma 3.1 and (3.22) we have

$$\frac{\psi_+\psi_-}{W} = \frac{\hat\psi_+\hat\psi_-}{\hat W} = O\left(\frac{1}{\sqrt{\lambda - E}}\right), \tag{5.39}$$

hence the first summand has the desired behavior. To estimate the second summand, we split the function $h_-(\lambda,n)$ according to

$$h_-(\lambda,n) = h_1(\lambda,n) + h_2(\lambda,n),$$

where

$$h_1(\lambda,n)=W^+_{q,n-1}(\zeta_-(\lambda,\cdot),d_-(\lambda,n,\cdot)),\quad h_2(\lambda,n)=h_-(\lambda,n)-h_1(\lambda,n),$$

$$d_-(\lambda,n,.):=\int_{\sigma_-^{(1),u}}\frac{|T_-(\xi)|^2\psi_+(\xi,n)\psi_q^+(\xi,\cdot)}{\xi-\lambda}d\omega_-(\xi). \tag{5.40}$$

It follows from the proof of Lemma 5.4 that $h_2(\lambda)=O(\zeta_-(\lambda))$ for $\lambda\to E$. Recall that at the point under consideration singularities $E\in\{\mu_1^+,\dots,\mu_{g_+}^+\}\cup\hat M_-$ might occur (in the case $\partial\sigma_-^{(1)}\cap\partial\sigma$ one can have $E\in M_+\cup\breve M_+$ and in the case $\partial\sigma_-^{(1)}\cap\partial\sigma_+^{(1)}$ one can have $E\in\hat M_+$). Introduce

$$\phi_q^+(z,n):=\breve\delta_+(z)\breve\psi_q^+(z,n) \tag{5.41}$$

and recall that (2.15) implies

$$\phi_q^+(z,n)-\phi_q^+(E,n)=O(\sqrt{z-E}). \tag{5.42}$$

Then (see (2.2) and (2.13)) we have

$$\frac{h_+\zeta_-}{W}=O\Big(\frac{h_+\breve\psi_q^+}{WT_+}\Big)=O\Big(\frac{h_+\hat\delta_+\breve\delta_+\breve\psi_q^+}{P_+}\Big)=O\Big(\frac{h_+\hat\delta_+}{P_+}\Big)\phi_q^+. \tag{5.43}$$

Now we distinguish two cases: (a) $E\in\partial\sigma_-^{(1)}\cap\partial\sigma_+^{(1)}$ and (b) $E\in\partial\sigma_-^{(1)}\cap\partial\sigma$.

Case (a). By (5.13) and (5.37) we have

$$\hat\delta_+h_+=\frac{(R_-\psi_-+\overline{\psi}_-)\hat\delta_+}{T_-}=\frac{\hat W\breve\delta_-(R_-\psi_-+\overline{\psi}_-)}{P_-}=o\Big(\frac{1}{\sqrt{\lambda-E}}\Big), \tag{5.44}$$

therefore

$$\frac{h_+(\lambda)\zeta_-(\lambda)}{W(\lambda)}=o\Big(\frac{1}{\sqrt{\lambda-E}}\Big)\frac{\phi_q^+(\lambda)}{P_+(\lambda)}. \tag{5.45}$$

As a consequence of $\frac{\phi_q^+}{P_+}=O\big(\frac{1}{\sqrt{\lambda-E}}\big)$ we obtain

$$\frac{h_+h_2}{W}=o\Big(\frac{1}{\lambda-E}\Big),\quad E\in\partial\sigma_-. \tag{5.46}$$

Next, we have to estimate

$$\frac{h_+h_1}{W}=W^+_{q,n-1}\Big(\frac{h_+\zeta_-}{W},d_-\Big). \tag{5.47}$$

By (5.42) we can represent (5.45) as

$$\frac{h_+(\lambda)\zeta_-(\lambda)}{W(\lambda)}=o\Big(\frac{1}{\sqrt{\lambda-E}}\Big)\Big(\frac{\breve\psi_q^+(E)}{\sqrt{\lambda-E}}+O(1)\Big). \tag{5.48}$$

Then (5.47) implies

$$\begin{aligned}\frac{h_+(\lambda,n)h_1(\lambda,n)}{W(\lambda)}=o\Big(\frac{1}{\sqrt{\lambda-E}}\Big)\Big(&O(d_-(\lambda,n))+O(d_-(\lambda,n-1))\\&+\frac{O\left(W^+_{q,n-1}\left(\phi_q^+(E),d_-(\lambda)\right)\right)}{\sqrt{\lambda-E}}\Big).\end{aligned} \tag{5.49}$$

To estimate d_- in the first two summands we distinguish between the resonance case, $E\in\sigma_v$, and non-resonance, $E\notin\sigma_v$. First let $E\notin\sigma_v$, that is, $\hat W(E)\neq 0$.

From (5.19) and (5.20) we see that the integrand is bounded as $\lambda \to E \notin \sigma_v$, then $d_-(\lambda) = O(1)$ by [17].

If $E \in \sigma_v$, then (3.22) (see also (5.19)) yields

$$|T_-(\xi)|^2 \rho_-(\xi)\psi_+(\xi,\cdot)\psi_q^+(\xi,\cdot) = O\left(\frac{1}{\sqrt{\xi - E}}\right)$$

and [17, Eq. (29.8)] implies

$$d_-(\lambda) = o\left(\frac{1}{\sqrt{\lambda - E}}\right). \tag{5.50}$$

For the estimate of the last summand in (5.49) we use (5.19) and (5.40) to represent the integrand in $W_{q,n-1}^+\left(\breve{\psi}_q^+(E), d_-(\lambda)\right)$ as

$$\begin{aligned}&|T_-(\xi)|^2 \rho_-(\xi)\psi_+(\xi,n) W_{q,n-1}^+\left(\psi_q^+(\xi), \phi_q^+(E)\right)\\ &\qquad = O\bigg(\frac{\sqrt{\xi - E}}{|\hat{W}(\xi)|^2} W_{q,n-1}^+\big(\hat{\psi}_q^+(\xi), \phi_q^+(E)\big)\bigg).\end{aligned}$$

It follows from (2.15) and (5.41) that

$$W_{q,n-1}^+(\hat{\psi}_q^+(\xi), \phi_q^+(E)) = O\big(\sqrt{\xi - E}\big),$$

which implies together with (3.22) the boundedness of the integrand near E. Thus,

$$W_{q,n-1}^+\left(\phi_q^+(E), d_-(\lambda)\right) = O(1), \tag{5.51}$$

and combining (5.46), (5.49), (5.51), and (5.50) finishes case (a).

Case (b). Now we do not have estimate (5.37) (cf. **III**, **(b)**) at our disposal, but we can proceed as in (5.43), (5.44) since $P_+(E) \neq 0$ and arrive at

$$\frac{h_+\zeta_-}{W} = O(h_+\hat{\delta}_+) = O\bigg(\frac{\hat{W}\breve{\delta}_-(R_-\psi_- + \overline{\psi}_-)}{P_-}\bigg) = O\bigg(\frac{\hat{W}}{\sqrt{\lambda - E}}\bigg). \tag{5.52}$$

This estimate is sufficient to conclude that (5.46) is valid in case (b) as well. For h_1, we use the following estimate (cf. (5.50) and (5.52)) instead of (5.47):

$$\frac{h_+h_1}{W} = O\left(\frac{h_+\zeta_-}{W}\right) O\left(d_-\right) = O\left(\frac{\hat{W}}{\sqrt{\lambda - E}}\right) o\left(\frac{1}{\sqrt{\lambda - E}}\right).$$

Combining this with (5.46) finishes case (b).

Ω_4: Finally we have to show that the singularities of $G(z,n)$ at the points of the discrete spectrum are removable. Since $\tilde{W}(z)$ has simple zeros at $z = \lambda_k$, it suffices by (5.29) to show that

$$\tilde{h}_+(\lambda_k, n)\tilde{h}_-(\lambda_k, n) = \tilde{\psi}_-(\lambda_k, n)\tilde{\psi}_+(\lambda_k, n). \tag{5.53}$$

By Lemma 5.4, the functions $\tilde{h}_\mp = \delta_\mp h_\mp$ given in (5.12) are continuous at the points $\breve{M}_\pm$. Since $(\delta_\mp T_\pm^{-1})(\lambda_k) = 0$ and $(\delta_\mp T_\pm^{-1}\breve{\psi}_q^\pm)(\lambda_k) = 0$, only the last summand in (5.12) is non-zero. We compute the limit of this summand as $\lambda \to \lambda_k$ using (3.19),

$$\tilde{h}_\mp(\lambda_k) = -\gamma_{\pm,k}\tilde{\psi}_\pm(\lambda_k)\frac{d\tilde{W}(\lambda_k)}{d\lambda}, \tag{5.54}$$

and apply (3.20) to obtain (5.53). ∎

The identity $G(z,n) \equiv 0$ implies

$$\psi_+(z,n)\psi_-(z,n) - h_+(z,n)h_-(z,n) \equiv 0, \quad \forall n \in \mathbb{Z}. \tag{5.55}$$

For $z \to \infty$ we obtain by (2.16) and (5.9)

$$\psi_+(z,n)\psi_-(z,n) = K_+(n,n)K_-(n,n) \prod_{j=0}^{n-1}{}^{*} \; \frac{a_q^+(j)}{a_q^-(j)}(1+o(1)).$$

Formulas (5.15) and (3.24) imply

$$h_+(z,n)h_-(z,n) = \frac{1}{T_+(\infty)^2 K_+(n,n)K_-(n,n)} \prod_{j=0}^{n-1}{}^{*} \; \frac{a_q^-(j)}{a_q^+(j)}(1+o(1))$$

and by (5.55),

$$K_+(n,n)K_-(n,n) \prod_{j=0}^{n-1}{}^{*} \; \frac{a_q^+(j)}{a_q^-(j)} = \frac{1}{T_+(\infty)}.$$

The value on the left hand side does not depend on n, so using (5.7) we conclude

$$a_+(n) = a_-(n) \equiv a(n), \quad \forall n \in \mathbb{Z}. \tag{5.56}$$

It remains to prove $b_+(n) = b_-(n)$. If we eliminate the reflection coefficient $R_\pm$ from (5.13) at n and (5.25) at $n+1$ we obtain

$$\begin{aligned} G_1(\lambda,n) &:= \frac{\psi_+(\lambda,n)\psi_-(\lambda,n+1) - h_+(\lambda,n+1)h_-(\lambda,n)}{W(\lambda)} \\ &= \rho_+(\lambda)\big(\overline{h_\pm(\lambda,n+1)}\psi_\pm(\lambda,n) - \overline{\psi_\pm(\lambda,n)}h_\pm(\lambda,n+1)\big), \quad \lambda \in \sigma^{(2),\mathrm{u,l}}. \end{aligned} \tag{5.57}$$

Proceeding as for $G(\lambda,n)$ in Lemma 5.7 we can show that that the function $G_1(z,n)$ is holomorphic in $\mathbb{C}$. From (5.15), (5.9), (3.24), (2.16), (5.56), and the Liouville theorem we conclude that

$$\frac{\psi_+(z,n)\psi_-(z,n+1) - h_+(z,n+1)h_-(z,n)}{W(z)} = -1/a(n).$$

We compute the asymptotics of

$$\bar{W}(z,n) := a(n)\left(\psi_+(z,n)\psi_-(z,n+1) - h_+(z,n+1)h_-(z,n)\right) = -W(z)$$

as $z \to \infty$ and obtain (compare (3.5))

$$0 = \bar{W}(z,n) - \bar{W}(z,n-1) = (b_+(n) - b_-(n))K_+(0,0)K_-(0,0). \tag{5.58}$$

This implies in particular $b_+(n) = b_-(n) \equiv b(n)$, hence the proof of Theorem 5.3 is finished.

Acknowledgments. I.E. and J.M. gratefully acknowledge the extraordinary hospitality of the Faculty of Mathematics of the University of Vienna during their stay in 2007, where parts of this paper were written.

References

[1] J. Bazargan and I. Egorova, *Jacobi operator with step-like asymptotically periodic coefficients*, Mat. Fiz. Anal. Geom. **10**, No.3, 425–442 (2003).

[2] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16**, No.4, 955–977 (2000).

[3] A. Boutet de Monvel, I. Egorova, and E. Khruslov, *Soliton asymptotics of the Cauchy problem solution for the Toda lattice*, Inverse Problems **13**, No.2, 223–237 (1997).

[4] A. Boutet de Monvel, I. Egorova, and G. Teschl, *Inverse Scattering theory for one-dimensional Schrödinger operators with steplike periodic potentials*, Preprint, arXiv:0707.4632.

[5] K. M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **14**, 2166–2175 (1973).

[6] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, No.1, 35–83 (1996).

[7] I. Egorova, *The scattering problem for step-like Jacobi operator*, Mat. Fiz. Anal. Geom. **9**, No.2, 188–205 (2002).

[8] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with quasi-periodic background*, Comm. Math. Phys. **264-3**, 811–842 (2006).

[9] I. Egorova, J. Michor, and G. Teschl, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817–1827 (2007).

[10] I. Egorova, J. Michor, and G. Teschl, *Soliton solutions of the Toda hierarchy on quasi-periodic background revisited*, Math. Nach. (to appear).

[11] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with steplike quasi-periodic background*, Inverse Problems **23**, 905–918 (2007).

[12] G. S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl. **17**, 1684–1688 (1976).

[13] S. Kamvissis and G. Teschl, *Stability of periodic soliton equations under short range perturbations*, Phys. Lett. A, **364-6**, 480–483 (2007).

[14] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice under short range perturbations*, Preprint, arXiv:0705.0346.

[15] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[16] J. Michor and G. Teschl, *Trace formulas for Jacobi operators in connection with scattering theory for quasi-periodic background*, Operator Theory, Analysis, and Mathematical Physics, J. Janas, et al. (eds.), 69–76, Oper. Theory Adv. Appl., 174, Birkhäuser, Basel, 2007.

[17] N. I. Muskhelishvili, *Singular integral equations*, P. Noordhoff Ltd., Groningen, 1953.

[18] G. Teschl, *Oscillation theory and renormalized oscillation theory for Jacobi operators*, J. Diff. Eqs. **129**, 532–558 (1996).

[19] G. Teschl, *On the initial value problem for the Toda and Kac-van Moerbeke hierarchies*, in Differential Equations and Mathematical Physics, R. Weikard and G. Weinstein (eds.), 375–384, AMS/IP Studies in Advanced Mathematics 16, Amer. Math. Soc., Providence, 2000.

[20] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

[21] G. Teschl, *Algebro-geometric constraints on solitons with respect to quasi-periodic backgrounds*, Bull. London Math. Soc. **39**, No.4, 677–684 (2007).

[22] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Vol. 2, Oxford, at the Clarendon Press, 1958.

[23] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. Pure Appl. Math. **44**, No.8-9, 1171–1242 (1991).

[24] A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi Matrices with the Spectrum on an Interval, a finite system of intervals or a Cantor set of positive length*, Comm. Math. Phys. **226**, 567–605 (2002).

Institute for Low Temperature Physics, 47,Lenin ave, 61164 Kharkiv, Ukraine
E-mail address: egorova@ilt.kharkov.ua

Imperial College, 180 Queen's Gate, London SW7 2BZ, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: Johanna.Michor@esi.ac.at
URL: http://www.mat.univie.ac.at/~jmichor/

Faculty of Mathematics, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: Gerald.Teschl@univie.ac.at
URL: http://www.mat.univie.ac.at/~gerald/