Habilitationsschrift

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Algebro-geometric solutions and their perturbations

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Preface

Introduction

The unifying theme of the work presented here are algebro-geometric solutions of hierarchies of nonlinear integrable differential-difference equations continuous in time and discrete in space. Algebro-geometric solutions are a natural extension of the class of soliton solutions and can be explicitly constructed using elements of algebraic geometry. The construction of such solutions in terms of specific algebro-geometric data on a compact hyperelliptic Riemann surface is exemplified for one model, the Ablowitz–Ladik hierarchy. Scattering theory with respect to (two different) algebro-geometric background operators and its application to the inverse scattering transform are studied for a second discrete model, the celebrated Toda hierarchy. The key equations defining the corresponding hierarchies are given by[1]

$$\text{Toda:} \qquad \begin{pmatrix} a_t - a(b^+ - b) \\ b_t - 2(a^2 - (a^-)^2) \end{pmatrix} = 0,$$

$$\text{Ablowitz–Ladik:} \qquad \begin{pmatrix} -i\alpha_t - (1-\alpha\beta)(\alpha^- + \alpha^+) + 2\alpha \\ -i\beta_t + (1-\alpha\beta)(\beta^- + \beta^+) - 2\beta \end{pmatrix} = 0.$$

This *Habilitationsschrift* consists of my research papers [**GHMT07**], [**GHMT08b**], [**GHMT10**], [**M10**], [**EMT08**], [**EMT09a**], and [**EMT09b**]. Most of them are joint work either with Fritz Gesztesy, Helge Holden, and Gerald Teschl or with Iryna Egorova and Gerald Teschl. All articles are published in refereed journals. My research papers [**EMT06**], [**EMT07a**], [**EMT07b**], [**GHMT08a**], [**MT04**], [**MT07**], [**MT09**], and [**EMT12**] are not included in the *Habilitationsschrift*, but they are related and at least some are implicitly incorporated, especially in [**EMT08**]. Our research monograph [**GHMT08**] comprises the four papers [**GHMT07**], [**GHMT08a**], [**GHMT08b**], and [**GHMT10**].

The treatise is structured into two parts. Part 1 consists of the papers [**GHMT07**], [**GHMT08b**], [**GHMT10**], and [**M10**] in which we study the **Ablowitz–Ladik hierarchy and its algebro-geometric solutions**. Part 2 entails [**EMT08**], [**EMT09a**], and [**EMT09b**] which contribute to the study of the **inverse scattering transform for the Toda hierarchy** with steplike background.

The following sections in this preface are related to the two parts and provide an overview of the contained articles. The bibliographies are independently attached for each paper and also for the preface. If the cited paper can be found in this collection, the citation is supplemented with a page reference. All other references are confined to the preface.

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[1] Here, and in the following, a, b, α, β, in general called f, are complex-valued functions on $\mathbb{Z} \times \mathbb{R}$, $f_t(n,t) = \frac{d}{dt} f(n,t)$ is the time derivative of the lattice function f, and $f^{\pm}$ denotes the shift on the lattice, that is, $f^{\pm}(n,t) = f(n \pm 1, t)$ for $(n,t) \in \mathbb{Z} \times \mathbb{R}$.

1. The Ablowitz–Ladik hierarchy and its algebro-geometric solutions

The introductions of the first three papers in Part 1 as well as the notes on the Ablowitz–Ladik system in our monograph [**GHMT08**, Sec. 3.9] give detailed and in-depth information on the history and scope of the problems in question. So rather than repeating this information here, I will sketch our approach to the construction of algebro-geometric solutions for the Ablowitz–Ladik hierarchy. This will complement the introductions in Gesztesy and Holden [**GH03**] and [**GHMT08**], where similar sketches were given for the Korteweg–de Vries (KdV) and Toda hierarchies.

Zero-curvature and Lax equations for the Ablowitz–Ladik system. The Ablowitz–Ladik (AL) system,

$$\begin{aligned} &-i\alpha_t - (1-\alpha\beta)(\alpha^- + \alpha^+) = 0, \\ &-i\beta_t + (1-\alpha\beta)(\beta^- + \beta^+) = 0, \end{aligned} \tag{1.1}$$

where $\alpha = \alpha(n,t)$, $\beta = \beta(n,t)$, $(n,t) \in \mathbb{Z}\times\mathbb{R}$, are complex-valued sequences and $f^\pm(n,t) = f(n\pm 1, t)$, $n \in \mathbb{Z}$, can be viewed as an integrable discretisation of the AKNS-ZS system[2] or a complexified version of the discrete nonlinear Schrödinger equation[3]. It was derived by Ablowitz and Ladik [**AL75**]–[**AL76c**] who used inverse scattering techniques to obtain and solve certain classes of nonlinear differential-difference systems. The AL system (1.1) was then found to be equivalent to a zero-curvature equation,

$$U_t + UV - V^+U = 0, \tag{1.2}$$

where $U(z)$ and $V(z)$ are 2×2 matrices with coefficients depending on a spectral parameter $z \in \mathbb{C}\setminus\{0\}$,

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \quad V(z) = i\begin{pmatrix} z - 1 - \alpha\beta^- & \alpha - \alpha^- z^{-1} \\ z\beta^- - \beta & 1 - z^{-1} + \alpha^-\beta \end{pmatrix}. \tag{1.3}$$

The zero-curvature equation arises as the compatibility requirement of the spatial and temporal linear problems[4]

$$\Phi = U\Phi^-, \quad \Phi_t = V^+\Phi.$$

On the other hand, the AL system (1.1) is equivalent to the Lax equation[5]

$$L_t(t) - [P(t), L(t)] = 0, \quad t \in \mathbb{R}, \tag{1.4}$$

where $[\cdot,\cdot]$ denotes the commutator, $[P,L] = PL - LP$. The Lax operator L reads in the standard basis of $\ell^2(\mathbb{Z})$ (abbreviate $\rho = (1-\alpha\beta)^{1/2}$)

$$L = \left(\begin{array}{cccccccc} \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & -\alpha(0)\rho(-1) & -\beta(-1)\alpha(0) & -\alpha(1)\rho(0) & \rho(0)\rho(1) & & \mathbf{0} & \\ & \rho(-1)\rho(0) & \beta(-1)\rho(0) & -\beta(0)\alpha(1) & \beta(0)\rho(1) & 0 & & \\ & & 0 & -\alpha(2)\rho(1) & -\beta(1)\alpha(2) & -\alpha(3)\rho(2) & \rho(2)\rho(3) & \\ & \mathbf{0} & & \rho(1)\rho(2) & \beta(1)\rho(2) & -\beta(2)\alpha(3) & \beta(2)\rho(3) & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right) \tag{1.5}$$

and P is given by

$$P = \tfrac{i}{2}\big(L_+ - L_- + (L^{-1})_- - (L^{-1})_+ + 2Q_d\big).$$

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[2]Ablowitz, Kaup, Newell, and Segur [**AKNS74**], Zakharov and Shabat [**ZS72**]

[3]Ablowitz and Ladik showed in [**AL76a**] that the defocusing ($\beta = \overline{\alpha}$) and focusing ($\beta = -\overline{\alpha}$) case of (1.1) correspond to the discrete analogue of the nonlinear Schrödinger equation $iq_t + q_{xx} \pm 2q|q|^2 = 0$.

[4]Here the notation $f^\pm(n) = f(n\pm 1)$, $n \in \mathbb{Z}$, is extended to $\mathbb{C}^2$-valued and (2×2)-matrix-valued sequences with complex-valued entries.

[5]The AL Lax pair in the finite-dimensional defocusing case is due to Nenciu [**N05**], [**N06**].

Here Q_d is the doubly infinite diagonal matrix $Q_d = \big((-1)^k \delta_{k,\ell}\big)_{k,\ell\in\mathbb{Z}}$ and $L_\pm$ denote the upper and lower triangular parts of L,

$$L_\pm = \big(L_\pm(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad L_\pm(m,n) = \begin{cases} L(m,n), & \pm(n-m) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The semi-infinite $\ell^2(\mathbb{N})$-operator realisation of the difference expression L as a five-diagonal matrix was recently rediscovered[6] by Cantero, Moral, and Velazquez [**CMV03**] in the context of orthogonal polynomials on the unit circle in the special defocusing case, where $\beta = \overline{\alpha}$.

The Ablowitz–Ladik hierarchy. To set up the recursive formalism for the AL hierarchy we assume α, β to be C^1-functions in the time variable satisfying $\alpha(n,t)\beta(n,t) \neq \{0,1\}$ for all $(n,t) \in \mathbb{Z}\times\mathbb{R}$.

Both the Lax and zero-curvature approach can be employed to construct a hierarchy of nonlinear evolution equations whose first nonlinear member is (1.1). This means that we construct a hierarchy of matrices $V_{\underline{p}}$, $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2$ with $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, such that the zero-curvature equation

$$U_t + UV_{\underline{p}} - V_{\underline{p}}^+ U = 0 \tag{1.6}$$

defines a hierarchy of differential-difference equations where time evolves continuously and space is considered discrete. The coefficients of the matrices $V_{\underline{p}}$ are defined recursively using Laurent polynomials with respect to the spectral parameter $z \in \mathbb{C}\setminus\{0\}$ as suggested by the appearance of powers of z, z^{-1} in the entries of $V(z)$ in (1.3). This extends a recursive polynomial approach already successfully considered in the continuous and in the discrete context[7].

We take the shortest route[8] to the construction of $V_{\underline{p}}$ by starting from the recursion relations (1.7) and (1.8) below. Define sequences $\{f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ recursively by

$$\begin{aligned} g_{0,+} &= \tfrac{1}{2}c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^+, \quad h_{0,+} = c_{0,+}\beta, \\ g_{\ell+1,+} - g_{\ell+1,+}^- &= \alpha h_{\ell,+}^- + \beta f_{\ell,+}, \quad \ell\in\mathbb{N}_0, \\ f_{\ell+1,+}^- &= f_{\ell,+} - \alpha(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell\in\mathbb{N}_0, \\ h_{\ell+1,+} &= h_{\ell,+}^- + \beta(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell\in\mathbb{N}_0, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} g_{0,-} &= \tfrac{1}{2}c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^+, \\ g_{\ell+1,-} - g_{\ell+1,-}^- &= \alpha h_{\ell,-} + \beta f_{\ell,-}^-, \quad \ell\in\mathbb{N}_0, \\ f_{\ell+1,-} &= f_{\ell,-}^- + \alpha(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell\in\mathbb{N}_0, \\ h_{\ell+1,-}^- &= h_{\ell,-} - \beta(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell\in\mathbb{N}_0. \end{aligned} \tag{1.8}$$

[6] Compare the discussion in Simon [**S07**]: the corresponding unitary half-lattice five-diagonal matrices were first introduced in [**BE91**] and subsequently studied in [**W93**].

[7] For the Korteweg–de Vries and AKNS hierarchies see [**GH03**], the Toda hierarchy is treated by Bulla, Gesztesy, Holden, and Teschl [**BGHT98**] and Teschl [**T00**, Chs. 6, 12]. The Laurent polynomial ansatz was originally developed by Al'ber [**A79**], [**A81**] in the context of the KdV hierarchy.

[8] For the constructive route see [**GHMT08a**], [**GHMT08**, Sec. 3.2]: The zero-curvature equation (1.6) is postulated for $U(z)$, $V_{\underline{p}}(z)$, where a general ansatz is made for the Laurent polynomial entries of $V_{\underline{p}}(z)$. The zero-curvature equation imposes the relationship (1.11) between the Laurent polynomials. A careful comparison of coefficients leads to the refined ansatz (1.9) for the Laurent polynomials. Plugging the refined ansatz into the zero-curvature equation and comparing coefficients yield the recursion relations (1.7) and (1.8) up to $\ell \le p_\pm - 2$ respectively $\ell \le p_\pm - 1$ and the $\underline{p}$th AL equation (1.12), [**GHMT08a**, Lemma 2.3].

Here $c_{0,\pm} \in \mathbb{C}$ are given constants. Since $g_{\ell,\pm}$ are only determined up to differences, an arbitrary constant $c_{\ell,\pm} \in \mathbb{C}$ is introduced on each level in the recursion. Using the recursively defined sequences $\{f_{\ell,\pm}, g_{\ell,\pm}, h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ (whose elements are difference polynomials with respect to α, β) one introduces Laurent polynomials $F_{\underline{p}}(z)$, $G_{\underline{p}}(z)$, $H_{\underline{p}}(z)$, and $K_{\underline{p}}(z)$ for $z \in \mathbb{C}\setminus\{0\}$ by setting[9]

$$
\begin{aligned}
F_{\underline{p}}(z) &= \sum_{\ell=1}^{p_-} f_{p_--\ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+-1} f_{p_+-1-\ell,+} z^{\ell}, \\
G_{\underline{p}}(z) &= \sum_{\ell=1}^{p_-} g_{p_--\ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+} g_{p_+-\ell,+} z^{\ell}, \\
H_{\underline{p}}(z) &= \sum_{\ell=0}^{p_--1} h_{p_--1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} h_{p_+-\ell,+} z^{\ell}, \\
K_{\underline{p}}(z) &= G_{\underline{p}}(z) + g_{p_-,-} - g_{p_+,+}.
\end{aligned} \tag{1.9}
$$

Now we define the 2×2 zero-curvature matrices by

$$
U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \quad V_{\underline{p}}(z) = i \begin{pmatrix} G_{\underline{p}}^-(z) & -F_{\underline{p}}^-(z) \\ H_{\underline{p}}^-(z) & -K_{\underline{p}}^-(z) \end{pmatrix}, \tag{1.10}
$$

and postulate the discrete zero-curvature equation (1.6), which results in the equations

$$
\begin{aligned}
\alpha_t &= i\big(zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + K_{\underline{p}}^-) - F_{\underline{p}}\big), \\
\beta_t &= -i\big(\beta(G_{\underline{p}}^- + K_{\underline{p}}) - H_{\underline{p}} + z^{-1}H_{\underline{p}}^-\big), \\
0 &= z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^-, \\
0 &= z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} + K_{\underline{p}}^- - K_{\underline{p}}.
\end{aligned} \tag{1.11}
$$

One verifies that the zero-curvature equation reduces to the basic equations

$$
\begin{aligned}
i\alpha_t &= -\alpha(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^-, \\
i\beta_t &= \beta(g_{p_+,+}^- + g_{p_-,-}) - h_{p_--1,-} + h_{p_+-1,+}^-.
\end{aligned} \tag{1.12}
$$

Varying $\underline{p} \in \mathbb{N}_0^2$, the collection of evolution equations

$$
\mathrm{AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_t - \alpha(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- \\ -i\beta_t + \beta(g_{p_+,+}^- + g_{p_-,-}) - h_{p_--1,-} + h_{p_+-1,+}^- \end{pmatrix} = 0, \quad t \in \mathbb{R}, \tag{1.13}
$$

then defines the *time-dependent Ablowitz–Ladik hierarchy*. The first few equations are explicitly given by (taking $p_- = p_+$ for simplicity),

$$
\begin{aligned}
\mathrm{AL}_{(0,0)}(\alpha,\beta) &= \begin{pmatrix} -i\alpha_t - c_{(0,0)}\alpha \\ -i\beta_t + c_{(0,0)}\beta \end{pmatrix} = 0, \\
\mathrm{AL}_{(1,1)}(\alpha,\beta) &= \begin{pmatrix} -i\alpha_t - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ -i\beta_t + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0, \\
\mathrm{AL}_{(2,2)}(\alpha,\beta) &= \begin{pmatrix} -i\alpha_t - \gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ -i\beta_t + \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix} \\
&\quad + \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \quad \text{etc.},
\end{aligned}
$$

[9] A sum is interpreted as zero whenever the upper limit in the sum is strictly less than its lower limit.

where $c_{\underline{p}} = (c_{p,+} + c_{p,-})/2$. Different ratios $c_{0,+}/c_{0,-}$ of the summation constants lead to different hierarchies. For example, the AL system (1.1) corresponds to the case $\underline{p} = (1,1)$ with $c_{0,\pm} = 1$ and $c_{(1,1)} = -2$. The special choices $\beta = \pm\overline{\alpha}$, $c_{0,\pm} = 1$ lead to the discrete NLS hierarchy, the choices $\beta = \overline{\alpha}$, $c_{0,\pm} = \mp i$ yield the hierarchy of Schur flows.

Alternatively, we can derive the AL hierarchy via Lax pairs $(L, P_{\underline{p}})$. Define

$$P_{\underline{p}} = \frac{i}{2}\sum_{\ell=1}^{p_+} c_{p_+-\ell,+}\big((L^\ell)_+ - (L^\ell)_-\big) - \frac{i}{2}\sum_{\ell=1}^{p_-} c_{p_--\ell,-}\big((L^{-\ell})_+ - (L^{-\ell})_-\big) - \frac{i}{2}c_{\underline{p}}Q_d,$$

where L is the doubly infinite five-diagonal matrix (1.5). The AL hierarchy is defined by imposing the Lax commutator relation

$$\frac{d}{dt}L(t) - [P_{\underline{p}}(t), L(t)] = 0 \tag{1.14}$$

for each $\underline{p} \in \mathbb{N}_0^2$. A fairly tedious computation then shows that the Lax equation (1.14) is equivalent to the $\underline{p}$th time-dependent AL equation $\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$ in (1.13). Moreover, the Lax equation implies existence of a propagator $W_{\underline{p}}(s,t)$ such that the family of operators $L(t)$, $t \in \mathbb{R}$, is similar, $L(s) = W_{\underline{p}}(s,t)L(t)W_{\underline{p}}(s,t)^{-1}$.

Moreover, we introduce the special *stationary Ablowitz–Ladik hierarchy* defined by the stationary zero-curvature equation

$$UV_{\underline{p}} - V_{\underline{p}}^+U = 0, \tag{1.15}$$

respectively, by vanishing of the commutator of $P_{\underline{p}}$ and L, for $\underline{p}$ ranging in $\mathbb{N}_0^2$. To set the stationary AL hierarchy apart from the general time-dependent one, we will denote it by

$$\text{s-AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -\alpha(g_{p_+,+} + g^-_{p_-,-}) + f_{p_+-1,+} - f^-_{p_--1,-} \\ \beta(g^-_{p_+,+} + g_{p_-,-}) - h_{p_--1,-} + h^-_{p_+-1,+} \end{pmatrix} = 0, \quad \underline{p} \in \mathbb{N}_0^2. \tag{1.16}$$

Algebro-geometric solutions. By definition, the class of *algebro-geometric* Ablowitz–Ladik potentials equals the set of solutions α, β of the stationary AL hierarchy (1.16). Algebro-geometric solutions represent a natural extension of soliton solutions. Namely, associated with each equation in the stationary AL hierarchy is a hyperelliptic curve, which we will describe in a moment. Solitons arise as the special case of solutions corresponding to a singular hyperelliptic curve obtained by confluence of pairs of branch points. The theta function associated with this singular curve then degenerates into appropriate determinants with exponential entries. On the other hand, algebro-geometric solutions can be used to approximate more general solutions such as, for instance, almost periodic ones.

Our strategy to describe the solutions will be the following: First we assume the existence of a solution α, β of the $\underline{p}$th stationary AL equation and derive several of its properties, in particular, the representations of α, β in terms of the Riemann theta function. As a second step we will provide an explicit algorithm to construct the solution given appropriate initial data.

We begin by introducing the hyperelliptic Riemann surface alluded to which is associated with the $\underline{p}$th equation in the stationary AL hierarchy. Let us assume that $c_{0,\pm} \in \mathbb{C}\setminus\{0\}$ and $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$. In the stationary case, the coefficients $g_{p_+,+}$ and $g_{p_-,-}$ are equal[10] up to a lattice constant which can be set equal to zero without loss of generality. It implies that $K_{\underline{p}} = G_{\underline{p}}$ in (1.9) and hence renders $V_{\underline{p}}$ in

[10] [**GHMT08a**, Lemma 2.2].

(1.10) traceless. Taking determinants in (1.15) yields that the expression $R_{\underline{p}}$ defined by

$$R_{\underline{p}} = G_{\underline{p}}^2 - F_{\underline{p}} H_{\underline{p}} \tag{1.17}$$

is a lattice constant and depends only on z. Therefore we may write $R_{\underline{p}}$ as

$$R_{\underline{p}}(z) = (c_{0,+}^2/4) z^{-2p_-} \prod_{m=0}^{2p+1} (z - E_m) \quad \text{for some } \{E_m\}_{m=0,\dots,2p+1} \subset \mathbb{C} \setminus \{0\},$$

where $p = p_- + p_+ - 1$. If we endow the hyperelliptic curve $\mathcal{K}_p$ of (arithmetic) genus p defined by

$$\mathcal{K}_p \colon \mathcal{F}_p(z, y) = y^2 - \prod_{m=0}^{2p+1} (z - E_m) = 0 \tag{1.18}$$

with the usual complex structure[11] and compactify it by joining two points $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, $\mathcal{K}_p$ becomes a two-sheeted hyperelliptic Riemann surface in a standard manner. Points P on $\mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z, y)$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z, y) = 0$. By fixing the curve, i.e., by fixing $E_0, \dots, E_{2p+1}$, the summation constants $\{c_{\ell,\pm}\}_{\ell \in \mathbb{N}}$ are uniquely determined.

A canonical meromorphic function ϕ on $\mathcal{K}_p$ (an analog of the Weyl–Titchmarsh function for the system (1.1)) then serves as the starting point of simultaneously constructing all algebro-geometric solutions for the entire hierarchy. This fundamental function ϕ defined by

$$\begin{aligned} \phi(P, n) &= \frac{(c_{0,+}/2) z^{-p_-} y + G_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n)} \\ &= \frac{-H_{\underline{p}}(z, n)}{(c_{0,+}/2) z^{-p_-} y - G_{\underline{p}}(z, n)}, \quad P = (z, y) \in \mathcal{K}_p,\ n \in \mathbb{Z}, \end{aligned}$$

is linked with the solutions α, β via trace formulas and Dubrovin-type equations for (projections of) the pole divisor of ϕ. It satisfies the Riccati-type equation

$$\alpha \phi(P) \phi^-(P) - \phi^-(P) + z \phi(P) = z \beta. \tag{1.19}$$

We isolate the zeros of $(\cdot)^{p_-} F_{\underline{p}}$ and $(\cdot)^{p_- - 1} H_{\underline{p}}$ by introducing the factorisations

$$F_{\underline{p}}(z) = -c_{0,+} \alpha^+ z^{-p_-} \prod_{j=1}^{p} (z - \mu_j), \quad H_{\underline{p}}(z) = c_{0,+} \beta z^{-p_- + 1} \prod_{j=1}^{p} (z - \nu_j), \tag{1.20}$$

and "lift" the zeros μ_j and ν_j from the complex plane to $\mathcal{K}_p$ using (1.17),

$$\begin{aligned} \hat{\mu}_j(n) &= (\mu_j(n), (2/c_{0,+}) \mu_j(n)^{p_-} G_{\underline{p}}(\mu_j(n), n)), \\ \hat{\nu}_j(n) &= (\nu_j(n), -(2/c_{0,+}) \nu_j(n)^{p_-} G_{\underline{p}}(\nu_j(n), n)), \end{aligned} \quad j = 1, \dots, p,\ n \in \mathbb{Z}. \tag{1.21}$$

We also introduce the points $P_{0,\pm}$ by setting

$$P_{0,\pm} = (0, \pm(c_{0,-}/c_{0,+})) \in \mathcal{K}_p, \quad \frac{c_{0,-}^2}{c_{0,+}^2} = \prod_{m=0}^{2p+1} E_m. \tag{1.22}$$

The divisor $(\phi(\,\cdot\,, n))$ of $\phi(\,\cdot\,, n)$ is given by[12]

$$(\phi(\,\cdot\,, n)) = \mathcal{D}_{P_{0,-} \underline{\hat{\nu}}(n)} - \mathcal{D}_{P_{\infty_-} \underline{\hat{\mu}}(n)}. \tag{1.23}$$

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[11] See for example Appendix B in [**GHMT08**].

[12] We used the following additive notation for divisors: $\mathcal{D}_{Q_0 \underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}$, where $\mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}$ for $\underline{Q} = \{Q_1, \dots, Q_m\} \in \mathrm{Sym}^m(\mathcal{K}_p)$ and $Q_0 \in \mathcal{K}_p$, $m \in \mathbb{N}$.

Here we abbreviated $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}, \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_p\} \in \mathrm{Sym}^p(\mathcal{K}_p)$, where $\mathrm{Sym}^p(\mathcal{K}_p)$ denotes the pth symmetric product of $\mathcal{K}_p$. Comparing coefficients of $F_{\underline{p}}$, $H_{\underline{p}}$ in (1.9) and in the factorisation (1.20) yields trace formulas for the solutions α, β of the $\underline{p}$th stationary AL equation in terms of the zeros μ_j and ν_j,

$$\frac{\alpha}{\alpha^+} = (-1)^{p+1}\frac{c_{0,+}}{c_{0,-}}\prod_{j=1}^{p}\mu_j, \quad \frac{\beta^+}{\beta} = (-1)^{p+1}\frac{c_{0,+}}{c_{0,-}}\prod_{j=1}^{p}\nu_j. \tag{1.24}$$

The basic function ϕ is then used to built the stationary Baker–Akhiezer vector $\Psi = (\psi_1, \psi_2)^T$ on $\mathcal{K}_p$. Its components are defined by[13]

$$\begin{aligned}
\psi_1(P, n, n_0) &= \prod_{j=n_0+1}^{n}{}^{*} \ \big(z + \alpha(j)\phi^-(P, j)\big), \\
\psi_2(P, n, n_0) &= \phi(P, n_0)\prod_{j=n_0+1}^{n}{}^{*} \ \big(z\beta(j)\phi^-(P, j)^{-1} + 1\big).
\end{aligned}$$

For $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, the vector Ψ satisfies $\psi_2(P, n, n_0) = \phi(P, n)\psi_1(P, n, n_0)$ and the zero-curvature equations

$$\begin{aligned}
&U(z)\Psi^-(P) = \Psi(P), \\
&V_{\underline{p}}(z)\Psi^-(P) = -(i/2)c_{0,+}z^{-p_-}y\Psi^-(P).
\end{aligned} \tag{1.25}$$

The ultimate goal of the algebro-geometric approach is a representation of the solutions α, β in terms of the *Riemann theta function*[14] associated with $\mathcal{K}_p$ and an appropriate homology basis of cycles on it. The explicit expressions for ϕ and the Baker-Akhiezer vector Ψ in terms of the theta function are obtained more or less simultaneously with those for the algebro-geometric solutions α, β. First one computes the asymptotic expansions of ϕ and ψ_1 with respect to the spectral parameter near the points $P_{\infty_\pm}$ and $P_{0,\pm}$. The known zeros and poles of ϕ in (1.23) and similarly, the set of zeros $\{P_{0,-}\} \cup \{\hat{\mu}_j(n)\}_{j=1}^p$ and poles $\{P_{\infty_+}\} \cup \{\hat{\mu}_j(n_0)\}_{j=1}^p$ of $\psi_1(\cdot, n, n_0)$, then permit one to find theta function representations for ϕ and ψ_1 by referring to Riemann's vanishing theorem and the Riemann-Roch theorem (the results for $\psi_2(\cdot, n, n_0)$ immediately follow from $\psi_2 = \phi\psi_1$). The corresponding theta function representation of the algebro-geometric solution α, β of the $\underline{p}$th stationary AL equation is obtained from that of ϕ by using the asymptotic expansions near $P_{\infty_\pm}$ and $P_{0,\pm}$. This is the content of [**GHMT07**, Thm. 3.7] on page 21.

The time-dependent Ablowitz–Ladik hierarchy. We now proceed with our discussion to the time-dependent AL hierarchy and describe the steps involved to construct solutions $\alpha = \alpha(n, t), \beta = \beta(n, t)$ of the $\underline{r}$th AL equation with initial values being algebro-geometric solutions of the $\underline{p}$th stationary AL equation. More precisely, we are looking for solutions α, β of the time-dependent algebro-geometric initial value problem

$$\begin{aligned}
&\widetilde{\mathrm{AL}}_{\underline{r}}(\alpha, \beta) = \begin{pmatrix} -i\alpha_t - \alpha(\tilde{g}_{r_+,+} + \tilde{g}_{r_-,-}^-) + \tilde{f}_{r_+-1,+} - \tilde{f}_{r_--1,-}^- \\ -i\beta_t + \beta(\tilde{g}_{r_+,+}^- + \tilde{g}_{r_-,-}) - \tilde{h}_{r_--1,-} + \tilde{h}_{r_+-1,+}^- \end{pmatrix} = 0, \\
&(\alpha, \beta)\big|_{t=t_0} = \big(\alpha^{(0)}, \beta^{(0)}\big), \\
&\text{s-AL}_{\underline{p}}\big(\alpha^{(0)}, \beta^{(0)}\big) = \begin{pmatrix} -\alpha^{(0)}(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- \\ \beta^{(0)}(g_{p_+,+}^- + g_{p_-,-}) - h_{p_--1,-} + h_{p_+-1,+}^- \end{pmatrix} = 0,
\end{aligned} \tag{1.26}$$

[13]If $n = n_0$, the star product is equal to 1; if $n < n_0$, it takes the reciprocal value.

[14]For details on the p-dimensional theta function $\theta(z)$, $z \in \mathbb{C}^p$, we refer to Appendix A in [**GHMT07**] on page 37.

for some $t_0 \in \mathbb{R}$, where a fixed curve $\mathcal{K}_p$ is associated with the stationary solutions $\alpha^{(0)}, \beta^{(0)}$. Since the summation constants in the functionals $f_{\ell,\pm}, g_{\ell,\pm}, h_{\ell,\pm}$ of α, β in the stationary and time-dependent contexts are independent of each other, we indicate this by adding a tilde to all time-dependent quantities. In terms of the zero-curvature formulation, we intend to solve

$$U_t(z,t) + U(z,t)\tilde{V}_{\underline{r}}(z,t) - \tilde{V}_{\underline{r}}^+(z,t)U(z,t) = 0, \tag{1.27}$$

$$U(z,t_0)V_{\underline{p}}(z,t_0) - V_{\underline{p}}^+(z,t_0)U(z,t_0) = 0. \tag{1.28}$$

One can show[15] that the stationary Ablowitz–Ladik system (1.28) is actually satisfied for all times $t \in \mathbb{R}$. Thus, we impose

$$\begin{aligned} U_t + U\tilde{V}_{\underline{r}} - \tilde{V}_{\underline{r}}^+ U &= 0, \\ UV_{\underline{p}} - V_{\underline{p}}^+ U &= 0, \end{aligned} \tag{1.29}$$

instead of (1.27) and (1.28). In particular, $R_{\underline{p}} = G_{\underline{p}}^2 - F_{\underline{p}}H_{\underline{p}}$ holds in the present t-dependent setting and we can define the fundamental meromorphic function $\phi(\cdot, n, t)$ on $\mathcal{K}_p$ as before, with the only difference that the Laurent polynomials $F_{\underline{p}}$, $G_{\underline{p}}$, and $H_{\underline{p}}$ now depend on time. In addition to equation (1.19), ϕ satisfies the Riccati-type equation

$$\phi_t(P) = i\tilde{F}_{\underline{r}}\phi^2(P) - i\big(\tilde{G}_{\underline{r}}(z) + \tilde{K}_{\underline{r}}(z)\big)\phi(P) + i\tilde{H}_{\underline{r}}(z), \tag{1.30}$$

which subsequently governs the time evolution of all quantities of interest. The time-dependent Baker–Akhiezer vector Ψ is defined for $(n,t) \in \mathbb{Z} \times \mathbb{R}$ and $P = (z,y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ by

$$\begin{aligned} \Psi(P,n,n_0,t,t_0) &= \exp\left(i \int_{t_0}^t ds \big(\tilde{G}_{\underline{r}}(z,n_0,s) - \tilde{F}_{\underline{r}}(z,n_0,s)\phi(P,n_0,s)\big)\right) \\ &\quad \times \begin{pmatrix} \displaystyle\prod_{j=n_0+1}^{n}{}^{*} \; \big(z + \alpha(j,t)\phi^-(P,j,t)\big) \\ \displaystyle\phi(P,n_0,t) \prod_{j=n_0+1}^{n}{}^{*} \; \big(z\beta(j,t)\phi^-(P,j,t)^{-1} + 1\big) \end{pmatrix}. \end{aligned}$$

The Riccati-type equations for ϕ yield that $V_{\underline{p},t} = \big[\tilde{V}_{\underline{r}}, V_{\underline{p}}\big]$ and

$$\begin{aligned} U(z)\Psi^-(P) &= \Psi(P), \\ V_{\underline{p}}(z)\Psi^-(P) &= -(i/2)c_{0,+}z^{-p_-}y\Psi^-(P), \\ \Psi_t(P) &= \tilde{V}_{\underline{r}}^+(z)\Psi(P). \end{aligned}$$

The corresponding representations of α, β, ϕ, and Ψ in terms of the Riemann theta function associated with $\mathcal{K}_p$ are obtained in close analogy to the stationary case. Since by definition of ϕ, the time parameter t can be viewed as an additional but fixed parameter, it basically remains to analyse the integrand in the exponential term of Ψ. This is done in [**GHMT07**, Thm. 4.6] on page 31.

The algebro-geometric initial value problem. We now turn to the second part of our investigation of solutions and indicate how to construct unique global solutions of the algebro-geometric initial value problem (1.26) for a general class of initial data. This initial data will consist of the coefficient $\alpha(n_0) \in \mathbb{C} \setminus \{0\}$, the constant $c_{0,+} \in \mathbb{C} \setminus \{0\}$, the set $\{E_j\}_{j=0}^{2p+1} \subset \mathbb{C} \setminus \{0\}$ satisfying $E_j \neq E_{j'}$ for $j \neq j'$, and suitably chosen initial Dirichlet data $\{\hat{\mu}_j\}_{j=1,\dots,p}$ at one fixed point n_0. A first approach to the

[15][**GHMT10**, Lemma 6.2], page 80.

actual solution of this initial value problem based for example on the familiar case of real-valued algebro-geometric solutions of the Toda hierarchy[16] might, naively, consist of the following two-step procedure:

(i) An algorithm that constructs admissible (i.e., finite and nonzero) *nonspecial*[17] divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)} \in \mathrm{Sym}^p(\mathcal{K}_p)$ for all $n \in \mathbb{Z}$, starting from a nonspecial initial Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathrm{Sym}^p(\mathcal{K}_p)$. Trace formulas of the type (1.24)[18] should then construct the stationary solutions $\alpha^{(0)}, \beta^{(0)}$ of s-$\mathrm{AL}_{\underline{p}}(\alpha, \beta) = 0$.

(ii) Inserting the factorisation (1.20) of $F_{\underline{p}}$ into $F_{\underline{p},t}(\mu_j) = -2iG_{\underline{p}}(\mu_j)\tilde{F}_{\underline{r}}(\mu_j)$ (which follows from $V_{\underline{p},t} = \big[\tilde{V}_{\underline{r}}, V_{\underline{p}}\big]$ at $z = \mu_j$) yields the first-order Dubrovin-type system of differential equations

$$\mu_{j,t} = -i\tilde{F}_{\underline{r}}(\mu_j)y(\hat{\mu}_j)(\alpha^+)^{-1}\prod_{\substack{k=1\\k\neq j}}^{p}(\mu_j-\mu_k)^{-1}, \quad j = 1,\dots,p, \tag{1.31}$$

for the Dirichlet data $\{\hat{\mu}_j\}_{j=1,\dots,p}$. Augmenting this equations by the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_0)} = \mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ of step (i) together with the analogous trace formulas of (1.24) in the time-dependent context should then yield unique global solutions $\alpha = \alpha(t), \beta = \beta(t)$ of the $\underline{r}$th AL flow $\mathrm{AL}_{\underline{r}}(\alpha, \beta) = 0$ satisfying $\alpha(t_0) = \alpha^{(0)}, \beta(t_0) = \beta^{(0)}$.

However, this approach can be expected to work only if the Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(n,t)} \in \mathrm{Sym}^p(\mathcal{K}_p)$ yield pairwise distinct Dirichlet eigenvalues $\mu_j(n,t)$, $j = 1,\dots,p$, for fixed $(n,t) \in \mathbb{Z}\times\mathbb{R}$, such that formula (1.31) for the time-derivatives $\mu_{j,t}$ is well-defined. Analogous considerations apply to the Neumann divisors $\mathcal{D}_{\underline{\hat{\nu}}} \in \mathrm{Sym}^p(\mathcal{K}_p)$.

Unfortunately, this scenario of pairwise distinct Dirichlet eigenvalues is not realistic and "collisions" between them can occur at certain values of $(n,t) \in \mathbb{Z}\times\mathbb{R}$. Thus, the stationary algorithm in step (i) as well as the Dubrovin-type system of differential equations (1.31) in step (ii) above breaks down at such values of (n,t). A priori, one has no control over such collisions, especially, it is not possible to identify initial conditions $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_0)}$ at some $(n_0,t_0) \in \mathbb{Z}\times\mathbb{R}$, which avoid collisions for all $(n,t) \in \mathbb{Z}\times\mathbb{R}$. We solve this problem head on by explicitly permitting collisions in the stationary as well as the time-dependent context from the outset. In the stationary context, we introduce an appropriate algorithm alluded to in step (i) by employing a general interpolation formalism for polynomials, going beyond the usual Lagrange interpolation formulas. In the time-dependent context, we replace the Dubrovin-type equations (1.31), augmented with the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_0)}$, by an autonomous first-order system of ordinary differential equations for $f_{\ell,\pm}, g_{\ell,\pm}, h_{\ell,\pm}$ derived from $V_{\underline{p},t} = \big[\tilde{V}_{\underline{r}}, V_{\underline{p}}\big]$ which focuses on symmetric functions of $\mu_1(n,t),\dots,\mu_p(n,t)$ rather than individual Dirichlet eigenvalues $\mu_j(n,t)$, $j = 1,\dots,p$. In this manner collisions of Dirichlet eigenvalues no longer pose a problem.

There arises an additional complication with admissibility: In general, it cannot be guaranteed that $\mu_j(n,t)$ and $\nu_j(n,t)$, $j = 1,\dots,p$, stay away from $P_{\infty_\pm}$ and $P_{0,\pm}$ for all $(n,t) \in \mathbb{Z}\times\mathbb{R}$. We show that the set of initial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_0)}$ for which $\mathcal{D}_{\underline{\hat{\mu}}(n,t)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n,t)}$ are admissible and hence nonspecial for all $(n,t) \in \mathbb{Z}\times\mathbb{R}$ forms a dense set of full measure in $\mathrm{Sym}^p(\mathcal{K}_p)$, therefore most initial divisors are well-behaved. However, in general (unless one is, e.g., in the special periodic case), $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}$ will get arbitrarily

[16]See, e.g., [**BGHT98**], [**GHMT08**, Sec. 1.3], [**T00**, Sect. 8.3] and the extensive literature cited therein.

[17]If $\mathcal{D} = n_1\mathcal{D}_{Q_1} + \dots + n_k\mathcal{D}_{Q_k} \in \mathrm{Sym}^p(\mathcal{K}_p)$ for some $n_\ell \in \mathbb{N}$, $\ell = 1,\dots,k$, with $n_1 + \dots + n_k = p$, then $\mathcal{D}$ is called nonspecial if there is no nonconstant meromorphic function on $\mathcal{K}_p$ which is holomorphic on $\mathcal{K}_p \setminus \{Q_1,\dots,Q_k\}$ with poles at most of order n_ℓ at Q_ℓ, $\ell = 1,\dots,k$.

[18] The second formula in (1.24) requires prior construction of the Neumann divisor $\mathcal{D}_{\underline{\hat{\nu}}}$ from the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ using [**GHMT10**, (4.3)] on page 59.

close to $P_{\infty_\pm}$, $P_{0,\pm}$ since straight motions on the torus are generically dense[19] and hence no uniform bound (and no uniform bound away from zero) on the sequences $\alpha(n,t), \beta(n,t)$ exists as (n,t) varies in $\mathbb{Z} \times \mathbb{R}$. In particular, these complex-valued algebro-geometric solutions of the Ablowitz–Ladik hierarchy initial value problem, in general, will not be *quasi-periodic*[20] with respect to n or t.

In summary, we develop a new algorithm to solve the inverse algebro-geometric spectral problem for general Ablowitz–Ladik Lax operators, starting from a properly chosen dense set of initial divisors of full measure. We combine it with an appropriate first-order system of differential equations with respect to time which allows the construction of global algebro-geometric solutions of the time-dependent AL hierarchy. The approach described here is not limited to the AL hierarchy but applies universally to constructing algebro-geometric solutions of $(1+1)$-dimensional integrable soliton equations. The principal idea of replacing Dubrovin-type equations by a first-order system of the type [**GHMT10**, (6.15), (6.23), (6.24)], page 77, 78, is also relevant in the context of general (non-self-adjoint) Lax operators for the continuous models in $(1+1)$-dimensions.

This concludes our tour into the realm of algebro-geometric solutions.

Conservation laws and the Hamiltonian formalism. In the remainder of this part we discuss general topics on the AL hierarchy such as local conservation laws, the Hamiltonian formalism, and asymptotical properties of solutions. First we employ our Laurent polynomial approach to systematically derive local conservation laws. The AL system (1.1) clearly implies that

$$\frac{d}{dt}\sum_{n\in\mathbb{Z}} \alpha^+(n,t)\beta(n,t) = \frac{d}{dt}\sum_{n\in\mathbb{Z}} \alpha(n,t)\beta^+(n,t) = 0.$$

Indeed, there exists an infinite sequence $\{\rho_{j,\pm}\}_{j\in\mathbb{N}}$ of polynomials of α, β and certain shifts thereof, such that the lattice sum is time-independent,

$$\frac{d}{dt}\sum_{n\in\mathbb{Z}} \rho_{j,\pm}(n,t) = 0, \quad j \in \mathbb{N}.$$

This result is obtained by deriving local conservation laws of the type

$$\frac{d}{dt}\rho_{j,\pm} + (S^+ - I)J_{j,\pm} = 0, \quad j \in \mathbb{N},$$

for certain polynomials $J_{j,\pm}$ of α, β and shifts thereof. The polynomials $J_{j,\pm}$ are constructed via an explicit recursion relation. These results are given in [**GHMT08b**, Thm. 5.7, Rem. 5.8, 5.9] on pages 126–130.

The above analysis extends to the full Ablowitz–Ladik hierarchy as follows. If one sets all summation constants c_ℓ, $\ell \in \mathbb{N}$, equal to zero in the recursive definitions (1.7), (1.8) of $f_{\ell,\pm}, g_{\ell,\pm}, h_{\ell,\pm}$, one obtains homogeneous coefficients $\hat{f}_{\ell,\pm}, \hat{g}_{\ell,\pm}, \hat{h}_{\ell,\pm}$. (A hat $\hat{}$ is added in the notation for all corresponding homogeneous quantities). Then $\hat{f}_{\ell,\pm}, \hat{g}_{\ell,\pm}, \hat{h}_{\ell,\pm}$ can also be expressed explicitly[21] in terms of appropriate matrix elements of powers of the AL Lax finite difference expression L defined in (1.5) and finite difference expressions D and E arising from the factorisation $L = DE$. The conserved

[19] See Arnold [**A89**, Sect. 51] or Katok and Hasselblatt [**KH95**, Sects. 1.4, 1.5].

[20] A sequence $f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is called *quasi-periodic* with fundamental periods $(\omega_1, \dots, \omega_N) \in (0,\infty)^N$ if the frequencies $2\pi/\omega_1, \dots, 2\pi/\omega_N$ are linearly independent over $\mathbb{Q}$ and if there exists a continuous function $F \in C(\mathbb{R}^{\mathbb{N}})$, periodic of period 1 in each of its arguments, such that $f(n) = F(\omega_1^{-1}n, \dots, \omega_N^{-1}n)$, $n \in \mathbb{Z}$. See, e.g., Pastur and Figotin [**PF92**, p. 31].

[21] As described in [**GHMT08b**, Lemma 3.1] on page 104.

densities $\rho_{j,\pm}$ are independent of the equation in the hierarchy while the currents $J_{\underline{p},j,\pm}$ depend on $\underline{p}$; thus one finds

$$\frac{d}{dt}\rho_{j,\pm} + (S^+ - I)J_{\underline{p},j,\pm} = 0, \quad t \in \mathbb{R},\ j \in \mathbb{N},\ \underline{p} \in \mathbb{N}_0^2. \tag{1.32}$$

For $\alpha, \beta \in \ell^1(\mathbb{Z})$ it follows that

$$\frac{d}{dt}\sum_{n\in\mathbb{Z}} \rho_{j,\pm}(n,t) = 0, \quad t \in \mathbb{R},\ j \in \mathbb{N},\ \underline{p} \in \mathbb{N}_0^2.$$

By showing that $\rho_{j,\pm}$ equals $\hat{g}_{j,\pm}$ up to a first-order difference expression and by investigating the time-dependence of $\gamma = 1 - \alpha\beta$, one concludes that

$$\frac{d}{dt}\sum_{n\in\mathbb{Z}} \ln(\gamma(n,t)) = 0, \quad \frac{d}{dt}\sum_{n\in\mathbb{Z}} \hat{g}_{j,\pm}(n,t) = 0, \quad t \in \mathbb{R},\ j \in \mathbb{N},\ \underline{p} \in \mathbb{N}_0^2, \tag{1.33}$$

represent the two infinite sequences of AL conservation laws. Our approach to (1.32) is based on a careful analysis of asymptotic expansions of the Green's function (as the spectral parameter tends to zero and to infinity) for the operator realisation of L in $\ell^2(\mathbb{Z})$.

Next, we briefly touch on our results concerning the Hamiltonian formalism for the AL hierarchy. The $\underline{p}$th equation in the AL hierarchy can be written as

$$\mathrm{AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_t \\ -i\beta_t \end{pmatrix} + \mathcal{D}\nabla\mathcal{H}_{\underline{p}} = 0, \quad \underline{p} \in \mathbb{N}_0^2, \tag{1.34}$$

where the Hamiltonians $\mathcal{H}_{\underline{p}}$ are given by

$$\widehat{\mathcal{H}}_0 = \sum_{n\in\mathbb{Z}} \ln(\gamma(n)), \quad \widehat{\mathcal{H}}_{p_\pm,\pm} = \frac{1}{p_\pm}\sum_{n\in\mathbb{Z}} \hat{g}_{p_\pm,\pm}(n), \quad p_\pm \in \mathbb{N},$$
$$\mathcal{H}_{\underline{p}} = \sum_{\ell=1}^{p_+} c_{p_+-\ell,+}\widehat{\mathcal{H}}_{\ell,+} + \sum_{\ell=1}^{p_-} c_{p_--\ell,-}\widehat{\mathcal{H}}_{\ell,-} + c_{\underline{p}}\widehat{\mathcal{H}}_0, \quad \underline{p} \in \mathbb{N}_0^2.$$

Here $\mathcal{D} = (1-\alpha\beta)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Furthermore, any $\mathcal{H}_{\underline{r}}$ is conserved by the Hamiltonian flows in (1.34), that is,

$$\frac{d\mathcal{H}_{\underline{r}}}{dt} = 0, \quad \underline{r} \in \mathbb{N}_0^2.$$

For general sequences α, β (i.e., not assuming that they satisfy an equation in the AL hierarchy), $\mathcal{H}_{\underline{p}}$ and $\mathcal{H}_{\underline{r}}$ are in involution for all $\underline{p}, \underline{r} \in \mathbb{N}_0^2$, that is,

$$\{\mathcal{H}_{\underline{p}}, \mathcal{H}_{\underline{r}}\} = 0,$$

for a suitably defined Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$. These results are the content of [**GHMT08b**, Thms. 6.5–6.7] on page 138.

Spatial asymptotical properties of solutions. As our last instalment we turn to a topic concerning spatial asymptotical properties of solutions. A question arising for example in the context of the inverse scattering transform is to what extend solutions are preserved by the time evolution. For the inverse scattering method (see Part 2) one intends to prove existence of solutions within the respective class, in particular, short-range perturbations of the background solution should remain short-range during the time evolution. This is the case for solutions of the AL hierarchy; arbitrary bounded solutions which are asymptotically close at the initial time stay close. More precisely, if $\alpha(t), \beta(t)$ and $\tilde{\alpha}(t), \tilde{\beta}(t)$ are bounded solutions of some equation $\mathrm{AL}_{\underline{r}} = 0$ in the AL hierarchy and if

$$\|(\alpha(t) - \tilde{\alpha}(t), \beta(t) - \tilde{\beta}(t))\|_{w,p} < \infty \tag{1.35}$$

holds for one $t = t_0 \in \mathbb{R}$, then it holds for all $t \in (t_0 - T, t_0 + T)$. The weighted norm is defined by

$$\|(\alpha,\beta)\|_{w,p} = \begin{cases} \Big(\sum\limits_{n\in\mathbb{Z}} w(n)\big(|\alpha(n)|^p + |\beta(n)|^p\big)\Big)^{1/p}, & 1 \le p < \infty, \\ \sup\limits_{n\in\mathbb{Z}} w(n)\big(|\alpha(n)| + |\beta(n)|\big), & p = \infty, \end{cases}$$

for a sequence $w(\cdot)$ satisfying $w(n) \ge 1$ and $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$. This result clearly holds for perturbations of steplike background solutions as well.

But even the dominant term of suitably decaying solutions $\alpha(n,t), \beta(n,t)$ of (1.1), for instance weighted ℓ^{2p} sequences whose spatial difference is in ℓ^p, $1 \le p < \infty$, is time independent. For example,

$$\alpha(n,t) = \frac{a}{n^\delta} + O\Big(\frac{1}{n^{\min(2\delta,\delta+1)}}\Big), \quad \beta(n,t) = \frac{b}{n^\delta} + O\Big(\frac{1}{n^{\min(2\delta,\delta+1)}}\Big), \quad n \to \infty,$$

holds for fixed t, provided it holds at the initial time $t = t_0$. Here $a, b \in \mathbb{C}$ and $\delta \ge 0$. This result remains valid for suitable equations in the AL hierarchy, i.e., for certain configurations of summation coefficients $\{c_{j,\pm}\}$. For $\underline{r} = (r_-, r_+) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, they have to satisfy the algebraic constraint

$$\sum_{j=0}^{r_+-1} c_{j,+} + \sum_{j=0}^{r_--1} c_{j,-} = 0.$$

The proof relies on the idea to consider the differential equation in two nested spaces of sequences, the Banach space of all $(\alpha(n), \beta(n))$ with sup norm, and the Banach space with norm $\|.\|_{w,p}$. Since the AL initial value problem has unique smooth solutions, the result follows.

Overview. We conclude with a summary of the principal content of each article.

In [**GHMT07**] (Section 1.1, p. 3) we provide a detailed derivation of all complex-valued algebro-geometric finite-band solutions of the Ablowitz–Ladik hierarchy. The AL hierarchy is constructed using a carefully refined Laurent polynomial ansatz for the zero-curvature matrix $V_{\underline{p}}$ (see also [**GHMT08a**]). This allows to simultaneously treat all equations in the hierarchy. We discuss the stationary AL hierarchy and employ the recursive Laurent polynomial formalism to introduce the basic meromorphic function ϕ on the hyperelliptic curve $\mathcal{K}_p$ of genus p associated with the $\underline{p}$th equation in the stationary AL hierarchy. The nonnegative divisor $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ of degree p (the pole divisor of ϕ) and an auxiliary divisor of degree p are used to prove the trace formulas for α, β. Then we derive the explicit representation of ϕ, of the Baker-Akhiezer vector Ψ, and of the solutions α, β in terms of the Riemann theta function associated with $\mathcal{K}_p$, both in the stationary and time-dependent context. Proofs of elementary results such as basic properties of ϕ and Ψ, the Riccati-type equations for ϕ, and the asymptotic expansions of ϕ and Ψ at the two points of infinity $P_{\infty_\pm}$ and zero $P_{0,\pm}$ are deferred to [**GHMT08a**].

In [**GHMT10**] (Section 1.2, p. 45) we solve the algebro-geometric initial value problem for the AL hierarchy with complex-valued initial data and prove unique solvability globally in time for a set of initial (Dirichlet divisor) data of full measure. To this effect we develop a new algorithm for constructing stationary complex-valued algebro-geometric solutions of the AL hierarchy, which is of independent interest as it solves the inverse algebro-geometric spectral problem for general (non-unitary) AL Lax operators, starting from a suitably chosen set of initial divisors of full measure. Combined with an appropriate first-order system of differential equations with respect to time (a substitute for the Dubrovin-type equations), this yields the construction of global algebro-geometric solutions of the time-dependent AL hierarchy. The treatment of general (non-unitary) Lax operators associated with general coefficients for the AL hierarchy poses a variety of difficulties that, to the best of our knowledge, are successfully overcome here for the first time. The approach described in this paper is not limited to the AL hierarchy but applies universally to constructing algebro-geometric solutions of $(1+1)$-dimensional completely integrable soliton equations of differential-difference type.

In [**GHMT08b**] (Section 1.3, p. 93) we derive the Lax pair $(L, P_{\underline{p}})$ for the AL hierarchy and prove its equivalence with the zero-curvature formulation. We also prove existence of Weyl–Titchmarsh-type solutions for the system $(L, P_{\underline{p}})$. Using Green's function of an $\ell^2(\mathbb{Z})$ realisation of L and high- and low-energy expansions of solutions of the associated Riccati-type equation we recursively derive the infinite sequence of local conservation laws. Variational derivatives for discrete systems are reviewed and the Poisson brackets and Hamiltonians for the AL hierarchy are introduced. Then we show how the AL equations can be recast in terms of the Hamiltonian formalism.

In [**M10**] (Section 1.4, p. 143) a general result on the spatial asymptotical properties of solutions of the AL hierarchy is derived. Namely, the dominant term of suitably decaying solutions, for instance weighted ℓ^{2p} sequences whose spatial difference is in ℓ^p, $1 \le p < \infty$, is time independent. In addition, we show that two arbitrary bounded solutions of the AL hierarchy which are asymptotically close at the initial time stay close. This implies that a solution of the AL hierarchy will stay close to a given background solution which is one of the main technical ingredients for the inverse scattering transform.

2. Inverse scattering transform for the Toda hierarchy

Let us begin with a brief introduction to the Toda hierarchy following [**T00**], [**GHMT08**].

The Toda hierarchy. The Toda lattice describes the dynamics of a lattice of one-dimensional particles interacting with exponential forces,

$$x_{tt}(n,t) = e^{x(n-1,t)-x(n,t)} - e^{x(n,t)-x(n+1,t)}, \quad (n,t) \in \mathbb{Z} \times \mathbb{R},$$

where $x(n,t)$ denotes the displacement of the n-th particle from its equilibrium position at time t. For example, it models a nonlinear one-dimensional crystal in solid state physics. The lattice was introduced by Toda [**T67a**], [**T67b**] in 1967 in the wake of the numerical experiment by Fermi, Pasta, and Ulam (FPU) from 1953, who analysed the behaviour of oscillations in certain nonlinear lattices and observed, instead of the expected equipartition of energy among the modes, that the system returned periodically to its initial state. Flaschka proved integrability of the Toda lattice in 1974 by establishing a Lax pair. He used the variable transform

$$a(n,t) = \frac{1}{2}\exp\Big(\frac{1}{2}\big(x(n,t) - x(n+1,t)\big)\Big), \quad b(n,t) = -\frac{1}{2}x_t(n,t),$$

and obtained the more familiar form of the Toda lattice,

$$\begin{aligned} a_t &= a\big(b^+ - b\big), \\ b_t &= 2\big(a^2 - (a^-)^2\big), \end{aligned} \tag{2.1}$$

for complex-valued sequences $a = a(n,t)$, $b = b(n,t)$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$. Then (2.1) is equivalent to the Lax equation

$$L_t(t) - [P_2(t), L(t)] = 0,$$

where L and P_2 are difference expressions of the form

$$L = aS^+ + a^-S^- + b, \quad P_2 = aS^+ - a^-S^-.$$

Here $S^\pm$ denote the shift operators, $(S^\pm f)(n) = f(n \pm 1)$ for $n \in \mathbb{Z}$.

To derive the Toda hierarchy we will employ the Lax formalism (which is a more feasible approach here in contradistinction to the Ablowitz–Ladik hierarchy) and construct a hierarchy of finite, skew-symmetric operators P_{2p+2} such that the Lax equation

$$L_t(t) - [P_{2p+2}(t), L(t)] = 0, \tag{2.2}$$

holds. More precisely, we seek an operator $P_{2p+2}(t)$ of order at most $2p+2$ such that the commutator with $L(t)$ is a symmetric difference operator of order at most two. Let us assume that a, b are C^1-functions in the time variable satisfying $a(n,t) \neq 0$ for all $(n,t) \in \mathbb{Z} \times \mathbb{R}$. Define polynomial functions g_ℓ, h_ℓ of a, b and certain of its shifts recursively by

$$\begin{aligned} &g_0 = 1, \ h_0 = c_1, \\ &2g_{\ell+1} - h_\ell - h_\ell^- - 2bg_\ell = 0, \quad \ell \in \mathbb{N}_0, \\ &h_{\ell+1} - h_{\ell+1}^- - 2(a^2 g_\ell^+ - (a^-)^2 g_\ell^-) - b(h_\ell - h_\ell^-) = 0, \quad \ell \in \mathbb{N}_0. \end{aligned} \tag{2.3}$$

Here c_1 is a given constant. By solving the difference equation for h_ℓ an arbitrary summation constant c_ℓ is introduced on each level in the recursion. Now we define the Lax operator by

$$P_{2p+2} = -L^{p+1} + \sum_{\ell=0}^{p} (2ag_{p-\ell}S^+ - h_{p-\ell})L^\ell + g_{p+1}. \tag{2.4}$$

Using the recursion (2.3), the commutator of P_{2p+2} and L reads[22]

$$\begin{aligned} [P_{2p+2}, L] = {}& a\big(h_p^+ + h_p - g_{p+1}^+ - g_{p+1} + 2b^+ g_p^+\big)S^+ \\ & 2\big(b(h_p - g_{p+1}) + a^2 g_p^+ - (a^-)^2 g_p^- + b^2 g_p\big) \\ & a^-\big(h_p + h_p^- - g_{p+1} - g_{p+1}^- + 2bg_p\big)S^-. \end{aligned} \tag{2.5}$$

We postulate the Lax equation (2.2) which results in the equations

$$a_t = a(g_{p+1}^+ - g_{p+1}), \quad b_t = h_{p+1} - h_{p+1}^-.$$

Varying $p \in \mathbb{N}_0$, the collection of evolution equations

$$\mathrm{TL}_p(a,b) = \begin{pmatrix} a_t - a(g_{p+1}^+ - g_{p+1}) \\ b_t - h_{p+1} + h_{p+1}^- \end{pmatrix} = 0, \quad t \in \mathbb{R}, \tag{2.6}$$

then defines the *time-dependent Toda hierarchy*. Explicitly, the first few equations are given by

$$\begin{aligned} \mathrm{TL}_0(a,b) &= \begin{pmatrix} a_t - a(b^+ - b) \\ b_t - 2(a^2 - (a^-)^2) \end{pmatrix}, \\ \mathrm{TL}_1(a,b) &= \begin{pmatrix} a_t - a((a^+)^2 - (a^-)^2 + (b^+)^2 - b^2) \\ b_t - 2a^2(b^+ + b) + 2(a^-)^2(b + b^-) \end{pmatrix} - c_1 \begin{pmatrix} a(b^+ - b) \\ 2(a^2 - (a^-)^2) \end{pmatrix}, \\ \mathrm{TL}_2(a,b) &= \begin{pmatrix} a_t - a((a^+)^2(b^{++} + 2b^+) + a^2(2b^+ + b) + (b^+)^3 - a^2(b^+ + 2b) - (a^-)^2(2b + b^-) - b^3) \\ b_t + 2((a^-)^4 + (a^{--}a^-)^2 - a^2(a^2 + (a^+)^2 + (b^+)^2 + bb^+ + b^2) + (a^-)^2(b^2 + b^-b + (b^-)^2)) \end{pmatrix} \\ &\quad - c_1 \begin{pmatrix} a((a^+)^2 - (a^-)^2 + (b^+)^2 - b^2) \\ 2a^2(b^+ + b) - 2(a^-)^2(b + b^-) \end{pmatrix} - c_2 \begin{pmatrix} a(b^+ - b) \\ 2(a^2 - (a^-)^2) \end{pmatrix}, \quad \text{etc.} \end{aligned}$$

Recall that the Lax equation (2.2) also implies existence of a unitary propagator $U_r(t,s)$ such that the family of operators $L(t)$, $t \in \mathbb{R}$, is unitarily equivalent, $L(t) = U_r(t,s)L(s)U_r(s,t)$.

---

[22] As hinted above, the quantities P_{2p+2} and $\{g_\ell, h_\ell\}_{\ell=0,\dots,p}$ are constructed such that all higher-order difference operators in the commutator (2.5) vanish.

Alternatively, one can construct the Toda hierarchy via the zero-curvature approach. Introducing the polynomials

$$G_p(z) = \sum_{\ell=0}^{p} g_{p-\ell} z^\ell, \quad H_{p+1}(z) = z^{p+1} + \sum_{\ell=0}^{p} h_{p-\ell} z^\ell - g_{p+1},$$

for $z \in \mathbb{C}$, one defines a pair of 2×2 matrices depending polynomially on z by

$$\begin{aligned} U(z) &= \begin{pmatrix} 0 & 1 \\ -a^-/a & (z-b)/a \end{pmatrix}, \\ V_{p+1}(z) &= \begin{pmatrix} -H^-_{p+1}(z) & 2a^- G^-_p(z) \\ -2a^- G_p(z) & 2(z-b)G_p(z) - H_{p+1}(z) \end{pmatrix}, \end{aligned}$$

and then postulates the zero-curvature equation

$$U_t + UV_{p+1} - V^+_{p+1} U = 0. \tag{2.7}$$

One verifies that both the Lax approach (2.6), as well as the zero-curvature approach (2.7) reduce to the basic equations

$$\begin{aligned} a_t &= -a\big(2(z-b^+)G^+_p - H^+_{p+1} - H_{p+1}\big), \\ b_t &= 2\big((z-b)^2 G_p - (z-b)H_{p+1} + a^2 G^+_p - (a^-)^2 G^-_p\big). \end{aligned} \tag{2.8}$$

Each one of (2.6), (2.7), and (2.8) defines the Toda hierarchy by varying $p \in \mathbb{N}_0$. The class of *algebro-geometric* Toda potentials, by definition, equals the set of solutions a, b of the stationary Toda hierarchy

$$\text{s-TL}_p(a,b) = \begin{pmatrix} g^+_{p+1} - g_{p+1} \\ h_{p+1} - h^-_{p+1} \end{pmatrix} = 0, \quad p \in \mathbb{N}_0.$$

Since $\det U(z,n) = a^-(n)/a(n) \neq 0$ for $n \in \mathbb{Z}$, the stationary zero-curvature equation $UV_{p+1} - V^+_{p+1}U = 0$ yields that $\det(V_{p+1}(z,n))$ is a lattice constant. Hence, using the first line in (2.8) in the stationary case $a_t = 0$, we obtain from

$$\begin{aligned} &\det(yI_2 - V_{p+1}(z,n)) = y^2 + \det(V_{p+1}(z,n)) \\ &= y^2 - H^-_{p+1}(z,n)\big(2(z-b(n))G_p(z,n) - H_{p+1}(z,n)\big) + 4(a^-(n))^2 G^-_p(z,n) G_p(z,n) \\ &= y^2 + H^-_{p+1}(z,n)^2 + 4(a^-(n))^2 G^-_p(z,n) G_p(z,n) = y^2 - R_{2p+2}(z) \end{aligned}$$

an n-independent monic polynomial R_{2p+2}, which we write as

$$R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m) \quad \text{for some } \{E_m\}_{m=0,\dots,2p+1} \subset \mathbb{C}.$$

The characteristic equation of $V_{p+1}(z,n)$ thus naturally leads to the introduction of the hyperelliptic curve $\mathcal{K}_p$ of genus $p \in \mathbb{N}_0$ defined by

$$\mathcal{K}_p \colon \mathcal{F}_p(z,y) = y^2 - \prod_{m=0}^{2p+1} (z - E_m) = 0. \tag{2.9}$$

The investigation of the stationary and time-dependent Toda hierarchy, starting from the fundamental meromorphic function ϕ on $\mathcal{K}_p$, and the construction of the algebro-geometric solutions and their theta function representations now follow by similar considerations as for the Ablowitz–Ladik case. We refer to (the introduction in) [**GHMT08**] for further details and, in particular, for a recount of the historical development leading up to soliton and algebro-geometric solutions. Note that the Kac–van Moerbeke hierarchy, whose first equation reads

$$\rho_t - \rho\big((\rho^+)^2 - (\rho^-)^2\big) = 0,$$

can be recast as a special case of the Toda hierarchy, see Michor and Teschl [**MT09**].

It was already shown by Toda in [**T67b**] that the Korteweg–de Vries (KdV) equation emerges in a certain scaling limit from the Toda lattice, and the development of the respective theories has been intertwined ever since. For an introduction to the KdV hierarchy and its algebro-geometric solutions see Gesztesy and Holden [**GH03**].

Real-valued algebro-geometric solutions. In the following we will mainly be interested in perturbations of real-valued algebro-geometric solutions[23] of the Toda hierarchy, so we review a few facts next. To obtain real-valued coefficients, one needs to impose certain symmetry constraints on $\mathcal{K}_p$ and additional constraints on the data involved in the theta function representation of a, b. For real-valued coefficients a, b, the Lax difference expression $L = aS^+ + a^-S^- + b$ is formally self-adjoint and leads to the reality constraint

$$E_0 < E_1 < \cdots < E_{2p+1}. \tag{2.10}$$

The ℓ^2-realisation of L with real-valued coefficients a, b is the *Jacobi operator*; we will denote it by H,

$$\begin{array}{rccl} H(t): & \ell^2(\mathbb{Z}) & \to & \ell^2(\mathbb{Z}) \\ & f(n) & \mapsto & a(n,t)f(n+1) + a(n-1,t)f(n-1) + b(n,t)f(n). \end{array} \tag{2.11}$$

The spectrum of the Jacobi operator H with bounded real-valued algebro-geometric coefficients consists of $p+1$ bands, $\sigma(H) = \bigcup_{j=0}^{p}[E_{2j}, E_{2j+1}]$. (Hence such algebro-geometric solutions are also called *finite-gap* or p-gap solutions following the conventional terminology.) One can show that all real-valued and bounded algebro-geometric Jacobi coefficients arise in the following manner[24]. The initial position of the Dirichlet data $\hat{\mu}_j(n_0) \in \mathcal{K}_p$ must be chosen in real position with its projections lying in the closure of the spectral gaps of H, that is,

$$\mu_j(n_0) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, p. \tag{2.12}$$

In particular, as n varies in $\mathbb{Z}$, the motion of the projection $\mu_j(n)$ of $\hat{\mu}_j(n) \in \mathcal{K}_p$ remains confined to the interval $[E_{2j-1}, E_{2j}]$. Since the initial divisor data $\hat{\mu}_j(n_0)$, with the projections $\mu_j(n_0)$ constrained by (2.12) for $j = 1, \dots, p$, are independent of each other, the set of all initial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ corresponds topologically to a product of p circles. Thus, the corresponding isospectral set of all bounded algebro-geometric Jacobi coefficients a^2, b, corresponding to a fixed curve $\mathcal{K}_p$, constrained by (2.10), can be identified with the p-dimensional real torus $\mathbb{T}^p$. Effective coordinates on this torus uniquely characterising a^2, b are the Dirichlet data $\underline{\hat{\mu}}(n_0) = (\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0))$, or equivalently, Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ in real position constrained by (2.12). The theta function formulas for a, b then provide a concrete representation of the elements of this isospectral torus $\mathbb{T}^p$. The coefficients a, b, in general, will be quasi-periodic with respect to $n \in \mathbb{Z}$.

Real-valued Jacobi coefficients a, b associated with $\mathcal{K}_p$ constrained by (2.10) can also be constructed by misplacing one or several initial values $\hat{\mu}_j(n_0)$ in the "wrong" spectral gap closure $(-\infty, E_0]$. This results in additional connected but noncompact components of isospectral and singular, respectively, unbounded Jacobi coefficients a^2, b.

Inverse scattering transform. Our main interest in Part 2 is the inverse scattering transform for the Toda hierarchy relative to different real-valued algebro-geometric backgrounds, which amounts to solving the initial value problem for the Toda hierarchy with initial data a perturbation of real-valued algebro-geometric solutions. Gardner, Greene, Kruskal, and Miura [**GGKM67**] discovered in 1967 that the inverse scattering

[23] See [**BGHT98**] or [**T00**] for specifics of the construction of real-valued algebro-geometric solutions, or the rudimentary summary in [**EMT09b**, Sec. 3] on page 191.

[24] See [**GHMT08**, Rem. 1.22].

method allowed one to solve initial value problems for the KdV equation for sufficiently fast decaying initial data. A non-linear analog of the Fourier transform, this method roughly consists of solving the scattering problem for the KdV Lax operator, deriving the time evolution of the scattering data, and reconstructing the potential function from the (time dependent) scattering data via the inverse scattering problem. In the early 1970s, Zakharov, Shabat, and Ablowitz extended the inverse scattering transform to a wide class of nonlinear PDEs, followed from 1974 onwards by an extension to periodic and certain classes of quasi-periodic and almost periodic KdV finite-band solutions, developed by pioneers such as Dubrovin, Its, Kac, Kriechever, Marchenko, Matveev, McKean, and van Moerbeke.

We are interested in the Toda hierarchy with steplike quasi-periodic finite-gap initial conditions. More precisely, we will assume that the solution $a(t), b(t)$ of the Toda hierarchy is asymptotically close to (in general) different real-valued algebro-geometric quasi-periodic finite-gap potentials $a^\pm(t), b^\pm(t)$ in the sense that

$$\sum_{n=0}^{\pm\infty}(1+|n|)\big(|a(n,t)-a^\pm(n,t)|+|b(n,t)-b^\pm(n,t)|\big)<\infty, \tag{2.13}$$

for one (and hence for any[25]) $t\in\mathbb{R}$. Let H and $H^\pm$ denote the Jacobi operators associated with the sequences a, b and $a^\pm, b^\pm$. For the Toda lattice in the classical case, where the solution is asymptotically equal to the (same) constant solution $a^\pm(n)\equiv 1/2$, $b^\pm(n)\equiv 0$, the inverse scattering method is of course well understood and covered in several monographs, e.g., Faddeev and Takhtajan [**FT87**], Toda [**T89**], or Teschl [**T00**].

The main theoretical pillow for the inverse scattering transform is the direct/inverse scattering problem for the Toda Lax operator. The aim is to find a complete characterisation, that is, necessary and sufficient conditions on the spectral data, which allow to solve the direct and inverse spectral problems in the respective class of the coefficients for the operator. Originally, scattering theory for Jacobi operators with constant background was developed on an informal level by Kac, Case, and Geronimo [**CK73**], [**C73**], [**CG80**], the first rigorous results were given by Guseinov [**G76**] with further extensions by Teschl [**T00b**], [**T00**]. Scattering theory relative to a simple step in the background, $a^\pm(n)\equiv 1/2$, $b^\pm(n)=\operatorname{sgn}(n)b$, $|b|<1$, was treated by Egorova [**E02**], first results for steplike periodic backgrounds of period 2 with a special choice for the respective spectra followed in Bazargan and Egorova [**BE03**]. Periodic backgrounds of period $N\geq 2$ without steps were studied by Khanmamedov [**K03**], [**K05**]. Volberg and Yuditskii [**VY02**] treat the case of almost periodic operators where H has a homogeneous Cantor type spectrum. The case of periodic backgrounds was completely solved by Egorova et al. in [**EMT06**] as a special case of quasi-periodic backgrounds. In this paper we solve the direct/inverse scattering problem relative to a quasi-periodic finite-gap background operator ($H^+=H^-$) and give minimal scattering data which determine the perturbed operator uniquely for finite second moments (see also Michor [**M05**]). The case with (quasi-)periodic background showed new phenomena not present in the constant background case. For example, the eigenvalues and the reflection coefficients can no longer be described independently if one wants to stay in this class. Algebraic constraints on the location of the eigenvalues ensuring single-valuedness of the transmission coefficient and hence solvability of the inverse scattering problem were found by Teschl [**T07**].

Subsequently, we studied in [**EMT07b**] steplike quasi-periodic finite-gap initial conditions, where H^+ and H^- belong to the same isospectral class, $\sigma(H^+)=\sigma(H^-)$, but are associated with possibly different Dirichlet data $\{\hat\mu_j^\pm\}$. Our motivation was

[25] By [**EMT09a**, Lem. 3.2] on page 181.

the investigation of solitons on (quasi-)periodic backgrounds. While solitons on quasi-periodic backgrounds are well studied objects, not much about their stability was known. In fact, as pointed out by Kamvissis and Teschl [**KT07**], [**KT**], the general believe that the stability problem for solitons on quasi-periodic backgrounds is similar to the one for solitons on a constant background is wrong. This is related to the fact that solitons on such backgrounds give rise to different spatial asymptotics which naturally leads to the type of operators studied in [**EMT07b**]. These results form the basis for the investigation of solitons on quasi-periodic backgrounds via the inverse scattering transform in [**EMT09b**], which we will discuss below.

General steplike quasi-periodic finite-gap initial conditions with no restriction on the mutual location of the spectra of H^+ and H^- are treated in [**EMT08**] on pages 155–178. We briefly give an overview of the steps involved in solving the direct/inverse scattering problem with a fixed finite moment.

Step 1. Transformation operators associated with both sides (in the sense of (2.13)) are constructed, which establish the connection between the unperturbed background operators $H^{\pm}$ and the perturbed operator H in (2.13).

Step 2. We determine the decay speed of the kernels $K_{\pm}(n, \cdot)$ of the transformation operators which depends on the moment of perturbation and derive formulas connecting these kernels to the given potentials $a^{\pm}, b^{\pm}$. The analytical properties of the Jost solutions, which clearly depend on the moment of perturbation, are investigated.

Step 3. The spectrum of the perturbed operator is determined. We derive the properties of the scattering matrix, whose entries consist of the transmission and reflection coefficients, and in particular, its unitary property and asymptotical behaviour at infinity. This also involves computing the relation between the coefficients of the scattering matrix and the norms of the left and right eigenfunctions.

Step 4. We derive the left and right Marchenko equations, which are the main equations of the inverse problem and connect the transformation operators to the set $\mathcal{S}$ of scattering data (consisting of the scattering matrix, the discrete spectrum, and normalising constants). The kernels $F_{\pm}$ of the Marchenko equations only depend on the scattering data $\mathcal{S}$.

Step 5. The decaying properties of $F_{\pm}$ for the given moment are determined.
The solution of the inverse scattering problem then consists of the following steps.

Step 6. Now we are given a set $\mathcal{S}$ of the same structure as the set of scattering data and with properties as described in Steps 3 and 5. This implies that we are given functions $F_{\pm}$ which satisfy the prescribed decaying properties. These properties are sufficient to prove that the left and right Marchenko equations are uniquely solvable with respect to the kernels $K_{\pm}(n, \cdot)$ of the transformation operators. Using the formulas derived in Step 2, the kernels give rise to two potentials. Both restored potentials can in general be well controlled on the associated half-axis; on the opposite half-axis, they cannot be analysed. To ensure that the solution lies within the class, one has to show that both potentials have the same finite moments as the initial potential.

Step 7. This is technically the most challenging step. We have to show that the two restored potentials actually coincide and form the unique potential of the Jacobi operator, for which the given set $\mathcal{S}$ is the set of scattering data.

The behaviour of the scattering data at the edge of the continuous spectrum plays a key role in the proof of the uniqueness theorem alluded to in Step 7. For a discussion of the delicate issue of continuity of the reflection coefficients see for example [**EMT12**]. There we show, in particular, that the reflection coefficient is continuous at the edge of the spectrum in the resonance case for finite first moments with constant backgrounds. For finite second moments of perturbation the behaviour of the reflection coefficient was described by Deift and Trubowitz [**DT79**] for Schödinger operators. They derived the characteristic properties of the scattering data and completely solved the direct/inverse scattering problems in the respective class of Schrödinger operators. But the approach used in [**DT79**] was not applicable for the largest class of potentials for which direct/inverse scattering can be studied within the class, namely for finite first moments. There the description of the characteristic properties of the scattering data turned out to be a much more complicated problem. Marchenko solved it in 1977 (see [**M86**]), and the condition on the scattering data at the edge of the continuous spectrum is now referred to as Marchenko condition. Marchenko's approach for solving the inverse scattering problem became the classic method and was successfully generalised to several other types of operators and potentials, such as asymptotically periodic, finite-gap non-periodic, or steplike potentials.

The solution of the direct and inverse scattering problem can now be directly applied to solve the associated initial value problem for the Toda hierarchy via the inverse scattering transform. Since the Lax equation implies unitary equivalence of $H(t)$ for all $t \in \mathbb{R}$ and hence time-independence of the spectral data, this amounts to computing the time evolution of the entries of the scattering matrix and of the norming constants. For the initial value problem relative to a quasi-periodic background this is done in [**EMT07a**]. In [**MT07**] we derive the connection between the transmission coefficient and Krein's spectral shift theory [**K62**] and use it to compute the conserved quantities for the Toda hierarchy with quasi-periodic background. The initial value problem with steplike quasi-periodic finite-gap initial data is solved in [**EMT09a**], page 179. Since we treat the entire Toda hierarchy, our results also cover the Kac–van Moerbeke hierarchy as a special case, [**MT09**].

Stability of solutions. The direct/inverse scattering problem and the time evolution of the reflection coefficient can be applied to derive the long-time asymptotics of the soliton equation under consideration. One rewrites the scattering problem as a Riemann–Hilbert problem and to control the solution for large times, uses the fact that time t enters the problem through the jump matrix $v_t(z)$ only in the form of a multiplier $e^{it\phi}$ of the reflection coefficient. Hence it suffices to derive the asymptotics of the jump matrix in order to describe the long time behaviour of solutions. This can be done following the nonlinear steepest decent method of Deift and Zhou [**DZ93**].

The classical result by Zabusky and Kruskal [**ZK63**] states that a small initial perturbation of the constant solution of a soliton equation eventually splits into a number of stable solitons (originating from the discrete spectrum of the underlying Lax operator) and a small oscillatory tail which decays like $t^{-1/2}$ (arising from the continuous spectrum). Therefore solitons constitute the stable part of arbitrary short-range initial conditions.

For the Toda lattice, the case of decaying initial data relative to a constant background is well understood by now[26]. Less is known about the stability of steplike constant solutions. Here the mutual location of the spectra of the two background operators is crucial, as illustrated in Figure 1 for the solution $a(n,t)$ of the Toda lattice at a frozen time t in the n/t-plane. The left image depicts the *Toda shock* case with initial data $a(n,0) = 1/2$ and $b(n,0) = \mathrm{sgn}(n)c$ for a constant $c > 1$. The two spectra of the background operators do not overlap and one can distinguish three regions: the region where $|n/t| \gg 1$, the shock has not been felt, and $a(n,t)$ is exponentially close to $1/2$; a transitional region, and a region of apparent periodicity. In the last region, Venakides, Deift, and Oba [**VDO91**] showed (using the Dyson formula) that the lattice motion converges to a periodic motion of spatial period 2. The case $c < -1$ corresponds to the *strong Toda rarefaction* problem and was investigated by Deift,

[26]See the lecture notes by Deift [**D98**] or the expository article by Krüger and Teschl [**KT09b**]. The precise asymptotic form was first given by Novokshenov and Habibullin [**NH81**]. Kamvissis [**K93**] derived the long time asymptotics for the Toda lattice when no solitons are present, Krüger and Teschl [**KT09a**] treated the soliton region.

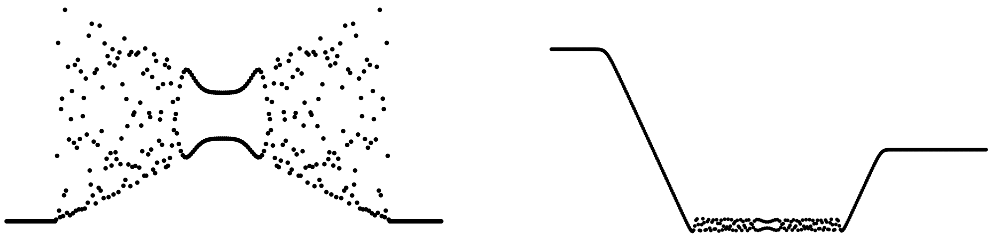

FIGURE 1. Numerically computed solutions of the Toda shock (left figure) and the steplike case with spectra of multiplicity one and two (right figure).

Kamvissis, Kriecherbauer, and Zhou [**DKKZ96**] using the Riemann–Hilbert factorisation method. The right image in Figure 1 illustrates the general case where spectrum of multiplicity two is present, i.e., the two spectra of the background operators overlap, with $a(n,t)$ and $b(n,t)$ asymptotically equal to $1/2$ and 0 at $+\infty$ respectively 0.7 and 1 at $-\infty$. The spectra of multiplicity one manifest itself as the two slopes, with the oscillation part corresponding to the spectrum of multiplicity two. Results in this direction were obtained by Boutet de Monvel and Egorova [**BE00**], Boutet de Monvel, Egorova, and Khruslov [**BEK97**], and Guseinov and Khanmamedov [**GK99**], but a unified treatment is still missing to date.

The investigation of the stability of periodic solutions began only recently, see Kamvissis and Teschl [**KT07**], [**KT**]. In contrast to the issue of stability, solitons on a (quasi-)periodic background have a long tradition. They are used to model localised excitements on a phonon, lattice, or magnetic field background and consequently, solitons travelling on a periodic background and periodic solutions are well understood. However, the asymptotic state is more complicated as it was generally believed. Associated with every soliton is a phase shift which we compute in [**EMT09b**], page 189, and the phase shifts of all solitons do not necessarily add up to zero in general. Hence there must be an additional feature making up for the overall phase shift, for even if no solitons are present, the asymptotic state is not just the periodic background. Kamvissis and Teschl showed that the oscillation part does not decay as in the constant background case, but instead appears as a modulation of the quasi-periodic solution which undergoes a continuous phase transition in the isospectral class of the quasi-periodic background solution. More precisely, let p be the genus of the hyperelliptic curve associated with the unperturbed solution. The n/t-plane contains $p+2$ areas where the perturbed solution is close to a finite-gap solution on the same isospectral torus. In between there are $p+1$ regions where the perturbed solution is asymptotically close to a modulated lattice which undergoes a continuous phase transition in the Jacobian variety and which interpolates between these isospectral solutions. The soliton part can be understood by adding or removing the solitons using the commutation method (Darboux-type transformations) for the underlying Jacobi operator, see [**EMT09b**], page 189. We explicitly compute the phase shift in the Jacobian variety caused by a soliton relative to a quasi-periodic finite-gap background and describe the effect of one commutation step on the scattering data. Alternatively, solitons can also be directly included in the Riemann–Hilbert problem using pole conditions, see Krüger and Teschl [**KT09c**].

Overview. We finish with a brief summary of each article in Part 2.

In [**EMT08**] (Section 2.1, p. 155), we develop direct and inverse scattering theory for Jacobi operators with steplike coefficients which are asymptotically close to (in general) different quasi-periodic finite-gap coefficients as $n \to \pm\infty$. We give a complete characterisation of the scattering data, which allow unique solvability of the inverse scattering problem in the class of perturbations with finite second moment.

The associated inverse scattering transform for the entire Toda hierarchy is derived in [**EMT09a**] (Section 2.2, p. 179). We first show that arbitrary bounded solutions will stay close to a given background solution which implies that a short-range perturbation of a steplike finite-gap solution will stay short-range for all time. This result constitutes the main technical ingredient for the inverse scattering transform as it shows that the time-dependent scattering data satisfy the hypothesis necessary for the Gel'fand–Levitan–Marchenko theory. We compute the time dependence of the scattering data and discuss its dynamics.

In [**EMT09b**] (Section 2.3, p. 189) we investigate soliton solutions of the Toda hierarchy on a quasi-periodic finite-gap background by means of the double commutation method. In particular, the phase shift in the Jacobian variety caused by a soliton on a quasi-periodic finite-gap background is computed. We finish with a full description of the effect of the double commutation method on the scattering data and establish the inverse scattering transform in this setting.

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Bibliography

[ABP07] M. J. Ablowitz, G. Biondini, and B. Prinari, *Inverse scattering transform for the integrable discrete nonlinear Schrödinger equation with nonvanishing boundary conditions*, Inverse Problems **23**, 1711–1758 (2007).

[AC91] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[AKNS74] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, *The inverse scattering transform - Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53**, 249–315 (1974).

[AL75] M. J. Ablowitz and J. F. Ladik, *Nonlinear differential-difference equations*, J. Math. Phys. **16**, 598–603 (1975).

[AL76a] ———, *Nonlinear differential-difference equations and Fourier analysis*, J. Math. Phys. **17**, 1011–1018 (1976).

[AL76b] ———, *A nonlinear difference scheme and inverse scattering*, Studies Appl. Math. **55**, 213–229 (1976).

[AL76c] ———, *On the solution of a class of nonlinear partial difference equations*, Studies Appl. Math. **57**, 1–12 (1977).

[APT04] M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems*, London Mathematical Society Lecture Note Series, Vol. 302, Cambridge University Press, Cambridge, 2004.

[A79] S. I. Al'ber, *Investigation of equations of Korteweg-de Vries type by the method of recurrence relations*, J. London Math. Soc. (2), **19**, 467–480, 1979.

[A81] ———, *On stationary problems for equations of Korteweg–de Vries type*, Commun. Pure Appl. Math. **34**, 259–272, 1981.

[A89] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed., Springer, New York, 1989.

[BE03] J. Bazargan and I. Egorova, *Jacobi operator with step-like asymptotically periodic coefficients*, Mat. Fiz. Anal. Geom. **10**, 425–442 (2003).

[BC84] R. Beals and R. Coifman, *Scattering and inverse scattering for first order systems*, Comm. in Pure and Applied Math. **37**, 39–90 (1984).

[BE00] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16**, No. 4, 955–977 (2000).

[BEK97] A. Boutet de Monvel, I. Egorova, and E. Khruslov, *Soliton asymptotics of the Cauchy problem solution for the Toda lattice*, Inverse Problems **13**, No. 2, 223–237 (1997).

[BGHT98] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl, *Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac–van Moerbeke Hierarchies*, Mem. Amer. Math. Soc. **135-641** (1998).

[BE91] A. Bunse–Gerstner, L. Elsner, *Schur parameter pencils for the solution of the unitary eigenproblem*, Linear Algebra Appl. **154/156**, 741–778 (1991).

[CMV03] M. J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Lin. Algebra Appl. **362**, 29–56 (2003).

[CK73] K. M. Case and M. Kac, *A discrete version of the inverse scattering problem*, J. Math. Phys. **14**, 594–603 (1973).

[C73] K. M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **14**, 2166–2175 (1973).

[CG80] K. M. Case and J. Geronimo, *Scattering theory and polynomials orthogonal on the real line*, Trans. Amer. Math. Soc. **258**, 467–494 (1980).

[D98] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, Courant Lecture Notes **3**, Amer. Math. Soc., Rhode Island, 1998.

[DKKZ96] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, No. 1, 35–83 (1996).

[DT79] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32**, 121–251 (1979).

[DZ93] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann–Hilbert problems*, Ann. of Math. (2) **137**, 295–368 (1993).

[E02] I. Egorova, *The scattering problem for step-like Jacobi operator*, Mat. Fiz. Anal. Geom. **9**, No. 2, 188–205 (2002).

[EMT06] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with quasi-periodic background*, Comm. Math. Phys. **264**, No. 3, 811–842 (2006).

[EMT07a] ———, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817–1827 (2007).

[EMT07b] ———, *Scattering theory for Jacobi operators with steplike quasi-periodic background*, Inverse Problems **23**, 905–918 (2007).

[EMT08] ———, *Scattering theory for Jacobi operators with general steplike quasi-periodic background*, Zh. Mat. Fiz. Anal. Geom. **4**, No. 1, 33–62 (2008).

[EMT09a] ———, *Inverse scattering transform for the Toda hierarchy with steplike finite-gap backgrounds*, J. Math. Physics **50**, 103522 (2009).

[EMT09b] ———, *Soliton solutions of the Toda hierarchy on quasi-periodic backgrounds revisited*, Math. Nach. **282**, No. 4, 526–539 (2009).

[EMT12] ———, *Scattering theory with finite-gap backgrounds: transformation operator and characteristic properties of scattering data*, Preprint, 2012. arXiv:1201.5768.

[FT87] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin, 1987.

[GGKM67] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *A method for solving the Korteweg-de Vries equation*, Phys. Rev. Letters **19**, 1095–1097 (1967).

[GH03] F. Gesztesy, H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions. Volume I:* $(1+1)$*-Dimensional Continuous Models*, Cambridge Studies in Advanced Mathematics **79**, Cambridge University Press, Cambridge, 2003.

[GHMT07] F. Gesztesy, H. Holden, J. Michor, and G. Teschl, *Algebro-geometric finite-band solutions of the Ablowitz–Ladik hierarchy*, Int. Math. Res. Notices **2007**, No. 20, Art. ID rnm082, 55 pp (2007).

[GHMT08] ———, *Soliton Equations and Their Algebro-Geometric Solutions. Volume II:* $(1+1)$*-Dimensional Discrete Models*, Cambridge Studies in Advanced Mathematics **114**, Cambridge University Press, Cambridge, 2008.

[GHMT08a] ———, *The Ablowitz–Ladik hierarchy revisited*, in Methods of Spectral Analysis in Math. Physics, J. Janas et al. (ed.), 139–190, Oper. Theory Adv. Appl. **186**, Birkhäuser, Basel, 2008.

[GHMT08b] ———, *Local conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy*, Stud. Appl. Math. **120**, No. 4, 361–423 (2008).

[GHMT10] ———, *The algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy*, Discrete Contin. Dyn. Syst. **26**, No. 1, 151–196 (2010).

[G76] G. S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl. **17**, 1684–1688 (1976).

[GK99] I. M. Guseinov and A. Kh. Khanmamedov, *The* $t \to \infty$ *asymptotic regime of the Cauchy problem solution for the Toda chain with threshold-type initial data*, Theoret. and Math. Phys. **119**, 739–749 (1999).

[K93] S. Kamvissis, *On the long time behavior of the doubly infinite Toda lattice under initial data decaying at infinity*, Comm. Math. Phys. **153**, No. 3, 479–519 (1993).

[KT07] S. Kamvissis and G. Teschl, *Stability of periodic soliton equations under short range perturbations*, Phys. Lett. A, **364**, No. 6, 480–483 (2007).

[KT] ———, *Stability of the periodic Toda lattice under short range perturbations*, Preprint, arXiv:0705.0346.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge Univ. Press, Cambridge, 1995.

[K03] A. K. Khanmamedov, *Transformation operators for the perturbed Hill difference equation and one of their applications*, Siberian Math. J. **44**, No. 4, 729–738 (2003).

[K05] ———, *Direct and inverse scattering problems for the perturbed Hill difference equation*, Sb. Math. **196**, No. 9-10, 1529–1552 (2005).

[K62] M. G. Krein, *Perturbation determinants and a formula for the traces of unitary and self-adjoint operators*, Soviet. Math. Dokl. **3**, 707–710 (1962).

[KT09a] H. Krüger and G. Teschl, *Long-time asymptotics for the Toda lattice in the soliton region*, Math. Z. 262, 585–602 (2009)

[KT09b] ———, *Long-time asymptotics of the Toda lattice for decaying initial data revisited*, Rev. Math. Phys. **21**, 61–109 (2009).

[KT09c] ———, *Stability of the periodic Toda lattice in the soliton region*, Int. Math. Res. Not. **2009**, Art. ID rnp077, 36pp (2009).

[M86] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[M05] J. Michor, *Scattering Theory for Jacobi Operators and Applications to Completely Integrable Systems*, Ph.D. thesis, University of Vienna, Austria, 2005.

[M10] ———, *On the spatial asymptotics of solutions of the Ablowitz–Ladik hierarchy*, Proc. Amer. Math. Soc. **138**, 4249–4258 (2010).

[MT04] J. Michor and G. Teschl, *Reconstructing Jacobi matrices from three spectra*, in Spectral Methods for Operators of Mathematical Physics, J. Janas et al. (eds.), 151–154, Oper. Theory Adv. Appl. **154**, Birkhäuser, Basel, 2004.

[MT07] ———, *Trace formulas for Jacobi operators in connection with scattering theory for quasi-periodic background*, in Operator Theory, Analysis, and Mathematical Physics, J. Janas, et al. (eds.), 69–76, Oper. Theory Adv. Appl. **174**, Birkhäuser, Basel, 2007.

[MT09] ———, *On the equivalence of different Lax pairs for the Kac–van Moerbeke hierarchy*, in Modern Analysis and Applications, V. Adamyan et al. (ed.), 437–445, Oper. Theory Adv. Appl. **191**, Birkhäuser, Basel, 2009.

[N05] I. Nenciu, *Lax pairs for the Ablowitz–Ladik system via orthogonal polynomials on the unit circle*, Int. Math. Res. Notices **11**, 647–686 (2005).

[N06] ———, *CMV matrices in random matrix theory and integrable systems: a survey*, J. Phys. A **39**, 8811–8822 (2006).

[NH81] V. Yu. Novokshenov and I. T. Habibullin, *Nonlinear differential-difference schemes integrable by the method of the inverse scattering problem. Asymptotics of the solution for $t \to \infty$*, Sov. Math. Doklady **23**, No. 2, 304–307 (1981).

[PF92] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer, Berlin, 1992.

[S07] B. Simon, *CMV matrices: Five years after*, J. Comp. Appl. Math. **208**, 120–154 (2007).

[T00] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

[T00b] ———, *On the initial value problem for the Toda and Kac-van Moerbeke hierarchies*, in Differential Equations and Mathematical Physics, R. Weikard and G. Weinstein (eds.), 375–384, AMS/IP Studies in Advanced Mathematics 16, Amer. Math. Soc., Providence, 2000.

[T07] ———, *Algebro-geometric constraints on solitons with respect to quasi-periodic backgrounds*, Bull. London Math. Soc. **39**, No. 4, 677–684 (2007).

[T10] ———, *On the spatial asymptotics of solutions of the Toda lattice*, Discrete Contin. Dyn. Sys. **27**, No. 3, 1233–1239 (2010).

[T67a] M. Toda, *Vibration of a chain with nonlinear interactions*, J. Phys. Soc. Jpn **23**, 501–506 (1967).

[T67b] ———, *Wave propagation in anharmonic lattices*, J. Phys. Soc. Jpn **22**, 431–436 (1967).

[T89] ———, *Theory of Nonlinear Lattices*, 2nd enl. ed., Springer, Berlin, 1989.

[VDO91] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. Pure Appl. Math. **44** No. 8-9, 1171–1242 (1991).

[VY02] A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi matrices with the spectrum on an interval, a finite system of intervals or a Cantor set of positive length*, Comm. Math. Phys. **226**, 567–605 (2002).

[W93] D. S. Watkins, *Some perspectives on the eigenvalue problem*, SIAM Rev. **35**, 430–471 (1993).

[ZK63] N. J. Zabusky and M. D. Kruskal, *Interaction of solitons in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett. **15**, 240–243 (1963).

[ZS72] V. E. Zakharov, A. B. Shabat, *Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media*, Soviet Phys. JETP **34**, 62–69 (1972).

Part 1

The Ablowitz–Ladik hierarchy and its algebro-geometric solutions

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ALGEBRO-GEOMETRIC FINITE-BAND SOLUTIONS OF THE ABLOWITZ–LADIK HIERARCHY

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To Walter Thirring, on the occasion of his 80th birthday, theoretical and mathematical physicist extraordinaire.

Abstract. We provide a detailed derivation of all complex-valued algebro-geometric finite-band solutions of the Ablowitz–Ladik hierarchy. In addition, we survey a recursive construction of the Ablowitz–Ladik hierarchy and its zero-curvature and Lax formalism.

1. Introduction

In the mid-seventies, Ablowitz and Ladik, in a series of papers [3]–[6] (see also [1], [2, Sect. 3.2.2], [7, Ch. 3], [22]), used inverse scattering methods to analyze certain integrable differential-difference systems. One of their integrable variants of such systems included a discretization of the celebrated AKNS-ZS system, the pair of coupled nonlinear differential difference equations,

$$\begin{aligned} &-i\alpha_t - (1-\alpha\beta)(\alpha^- + \alpha^+) + 2\alpha = 0, \\ &-i\beta_t + (1-\alpha\beta)(\beta^- + \beta^+) - 2\beta = 0 \end{aligned} \tag{1.1}$$

with $\alpha = \alpha(n,t)$, $\beta = \beta(n,t)$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$. Here we used the notation $f^{\pm}(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, for complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}}$. In particular, Ablowitz and Ladik [4] (see also [7, Ch. 3]) showed that in the focusing case, where $\beta = -\overline{\alpha}$, and in the defocusing case, where $\beta = \overline{\alpha}$, (1.1) yields the discrete analog of the nonlinear Schrödinger equation

$$-i\alpha_t - (1 \pm |\alpha|^2)(\alpha^- + \alpha^+) + 2\alpha = 0. \tag{1.2}$$

We will refer to (1.1) as the Ablowitz–Ladik system. The principal theme of this paper will be to derive the algebro-geometric finite-band solutions of the Ablowitz–Ladik (AL) hierarchy, a completely integrable sequence of systems of nonlinear evolution equations on the lattice $\mathbb{Z}$ whose first nonlinear member is the Ablowitz–Ladik system (1.1).

Since the mid-seventies there has been an enormous amount of activity in the area of integrable differential-difference equations. Two principal directions of research are responsible for this development: Originally, the development was driven by the theory of completely integrable systems and its applications to fields such as nonlinear optics, and more recently, it gained additional momentum due to its intimate connections with the theory of orthogonal polynomials. In the following

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we first briefly recall the development in connection with integrable systems and subsequently turn to the one influenced by research on orthogonal polynomials.

The first systematic discussion of the Ablowitz–Ladik hierarchy appears to be due to Schilling [51] (cf. also [58], [62], [65]); infinitely many conservation laws are derived, for instance, by Ding, Sun, and Xu [26]; the bi-Hamiltonian structure of the AL hierarchy is considered by Ercolani and Lozano [28]; connections between the AL hierarchy and the motion of a piecewise linear curve have been established by Doliwa and Santini [27]; Bäcklund and Darboux transformations were studied by Geng [31] and Vekslerchik [63]; the Hirota bilinear formalism, AL τ-functions, etc., were considered by Vekslerchik [62]. The initial value problem for half-infinite AL systems was discussed by Common [24], for an application of the inverse scattering method to (1.2) we refer to Vekslerchik and Konotop [64]. This just scratches the surface of these developments and the interested reader will find much more material in the references cited in these papers and the ones discussed below.

Algebro-geometric (and periodic) solutions of the AL system (1.1) have briefly been studied by Ahmad and Chowdhury [8], [9], Bogolyubov, Prikarpatskii, and Samoilenko [17], Bogolyubov and Prikarpatskii [18], Chow, Conte, and Xu [23], Geng, Dai, and Cao [32], and Vaninsky [60]. In an effort to analyze models describing oscillations in nonlinear dispersive wave systems, Miller, Ercolani, Krichever, and Levermore [47] (see also [46]) gave a detailed analysis of algebro-geometric solutions of the AL system (1.1). Introducing

$$U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix}, \quad V(z) = i \begin{pmatrix} z - 1 - \alpha\beta^- & \alpha - \alpha^- z^{-1} \\ \beta^- z - \beta & 1 + \alpha^-\beta - z^{-1} \end{pmatrix} \tag{1.3}$$

with $z \in \mathbb{C} \setminus \{0\}$ a spectral parameter, the authors in [47] relied on the fact that the Ablowitz–Ladik system (1.1) is equivalent to the zero-curvature equations

$$U_t + UV - V^+U = 0, \tag{1.4}$$

the latter being the compatibility relation for the spatial and temporal linear problems

$$\Phi = U\Phi^-, \quad \Phi_t^- = V\Phi^-. \tag{1.5}$$

Here we extended the notation $f^\pm(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, to $\mathbb{C}^2$-valued and 2×2-matrix-valued sequences with complex-valued entries.

Miller, Ercolani, Krichever, and Levermore [47] then performed a thorough analysis of the solutions $\Phi = \Phi(z, n, t)$ associated with the pair (U, V) and derived the theta function representations of α, β satisfying the AL system (1.1). In the particular focusing and defocusing cases they also discussed periodic and quasi-periodic solutions α with respect to $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Vekslerchik [61] also studied finite-genus solutions for the AL hierarchy by establishing connections with Fay's identity for theta functions.

The connection between the Ablowitz–Ladik system (1.1) and orthogonal polynomials comes about as follows: Let $\{\alpha(n)\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers subject to the condition $|\alpha(n)| < 1$, $n \in \mathbb{N}$, and define the transfer matrix

$$T(z) = \begin{pmatrix} z & \alpha \\ \overline{\alpha} z & 1 \end{pmatrix}, \quad z \in \mathbb{T}, \tag{1.6}$$

with spectral parameter z on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Consider the system of difference equations

$$\Phi(z, n) = T(z, n)\Phi(z, n - 1), \quad (z, n) \in \mathbb{T} \times \mathbb{N}, \tag{1.7}$$

with initial condition $\Phi(z,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where

$$\Phi(z,n) = \begin{pmatrix} \varphi(z,n) \\ z^n \overline{\varphi}(1/z,n) \end{pmatrix}, \quad (z,n) \in \mathbb{T} \times \mathbb{N}_0. \tag{1.8}$$

(Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.) Then $\varphi(\,\cdot\,,n)$ are monic polynomials of degree n first introduced by Szegő in the 1920's in his seminal work on the asymptotic distribution of eigenvalues of sections of Toeplitz forms [55], [56] (see also [57, Ch. XI]). Szegő's point of departure was the trigonometric moment problem and hence the theory of orthogonal polynomials on the unit circle. Indeed, given a probability measure $d\sigma$ supported on an infinite set on the unit circle, one is interested in finding monic polynomials $\chi(\,\cdot\,,n)$ of degree $n \in \mathbb{N}_0$ in $z = e^{i\theta}$, $\theta \in [0,2\pi]$, such that

$$\int_0^{2\pi} d\sigma(e^{i\theta})\, \overline{\chi(e^{i\theta},m)}\chi(e^{i\theta},n) = w(n)^{-2}\delta_{m,n}, \quad m,n \in \mathbb{N}_0, \tag{1.9}$$

where $w(0)^2 = 1$, $w(n)^2 = \prod_{j=1}^n \big(1 - |\alpha(j)|^2\big)^{-1}$, $n \in \mathbb{N}$. Szegő showed that the corresponding polynomials (1.8) with φ replaced by χ satisfy the recurrence formula (1.7). Early work in this area includes important contributions by Akhiezer, Geronimus, Krein, Tomčuk, Verblunsky, Widom, and others, and is summarized in the books by Akhiezer [10], Geronimus [37], Szegő [57], and especially in the recent two-volume treatise by Simon [52].

Unaware of the paper [47], Geronimo and Johnson [35] studied (1.7) in the case where the coefficients α are random variables. Under appropriate ergodicity assumptions on α and the hypothesis of a vanishing Lyapunov exponent on prescribed spectral arcs on the unit circle $\mathbb{T}$, Geronimo and Johnson [35] (cf. also [34], [36]) developed the corresponding spectral theory associated with (1.7) and the unitary operator it generates in $\ell^2(\mathbb{Z})$. In this sense the discussion in [35] is a purely stationary one and connections with a zero-curvature formalism, theta function representations, and integrable hierarchies are not made in [35] (but in this context we refer to the discussion concerning [33] in the next paragraph). More recently, the defocusing case with periodic and quasi-periodic coefficients was studied in great detail by Deift [25], Golinskii and Nevai [43], Killip and Nenciu [44], Li [45], Nenciu [49], [50], and Simon [52, Ch. 11], [53], [54].

An important extension of (1.7) was developed by Baxter in a series of papers on Toeplitz forms [11]–[14] in 1960–63. In these papers the transfer matrix T in (1.6) is replaced by the more general (complexified) transfer matrix $U(z) = \left(\begin{smallmatrix} z & \alpha \\ \beta z & 1 \end{smallmatrix}\right)$ in (1.5), that is, precisely the matrix U responsible for the spatial part in the Ablowitz–Ladik system in its zero-curvature formulation (1.3)–(1.5). Here $\alpha = \{\alpha(n)\}_{n\in\mathbb{N}}$, $\beta = \{\beta(n)\}_{n\in\mathbb{N}}$ are subject to the condition $\alpha(n)\beta(n) \neq 1$, $n \in \mathbb{N}$. Studying the following extension of (1.7),

$$\Psi(z,n) = U(z,n)\Psi(z,n-1), \quad (z,n) \in \mathbb{T} \times \mathbb{N}, \tag{1.10}$$

Baxter was led to biorthogonal polynomials on the unit circle with respect to a complex-valued measure on $\mathbb{T}$. In this context of biorthogonal Laurent polynomials we refer to [16] and [20]. Reference [16], in particular, deals with isomonodromic tau functions and is applicable to generalized integrable lattices of the Toda-type. Baxter's U matrix in (1.5) led to a new hierarchy of nonlinear difference equations, called the Szegő–Baxter (SB) hierarchy in [33], in honor of these two pioneers of orthogonal polynomials on the unit circle. The latter reference also contains an in depth study of algebro-geometric solutions of (1.7).

In addition to these recent developments on the AL system and the AL hierarchy, we offer a variety of results in this paper apparently not covered before. These include:

• An elementary, yet effective recursive construction of the AL hierarchy using Laurent polynomials.

• Explicit formulas for Lax pairs for the AL hierarchy.

• The detailed connection between the AL hierarchy and a "complexified" version of transfer matrices first introduced by Baxter.

• A unified treatment of stationary algebro-geometric finite-band solutions and their theta function representations of the entire AL hierarchy.

• A unified treatment of algebro-geometric solutions and their theta function representations of the time-dependent AL hierarchy by solving the $\underline{r}$th AL flow with initial data given by stationary algebro-geometric finite-band solutions.

The structure of this paper is as follows: In Section 2 we describe our zero-curvature formalism for the Ablowitz–Ladik (AL) hierarchy. Extending a recursive polynomial approach discussed in great detail in [38] in the continuous case and in [19], [39, Ch. 4], [59, Chs. 6, 12] in the discrete context to the case of Laurent polynomials with respect to the spectral parameter, we derive the AL hierarchy of systems of nonlinear evolution equations whose first nonlinear member is the Ablowitz–Ladik system (1.1). Section 3 is devoted to a detailed study of the stationary AL hierarchy. We employ the recursive Laurent polynomial formalism of Section 2 to describe nonnegative divisors of degree p on a hyperelliptic curve $\mathcal{K}_p$ of genus p associated with the $\underline{p}$th system in the stationary AL hierarchy. By means of a fundamental meromorphic function ϕ on $\mathcal{K}_p$ (an analog of the Weyl–Titchmarsh function for the system (1.7)) we then proceed to derive the theta function representations of the associated Baker–Akhiezer vector and all stationary algebro-geometric finite-band solutions of the AL hierarchy. The corresponding time-dependent results for the AL hierarchy are presented in detail in Section 4. Appendix A collects relevant material on hyperelliptic curves and their theta functions and introduces the terminology freely used in Sections 3 and 4. Appendix B is of a technical nature and summarizes expansions of various key quantities related to the Laurent polynomial recursion formalism as the spectral parameter tends to zero and to infinity.

2. The Ablowitz–Ladik Hierarchy, Recursion Relations, Zero-Curvature Pairs, and Hyperelliptic Curves

In this section we summarize the construction of the Ablowitz–Ladik hierarchy employing a Laurent polynomial recursion formalism and derive the associated sequence of Ablowitz–Ladik zero-curvature pairs (we also hint at Lax pairs). Moreover, we discuss the Burchnall–Chaundy Laurent polynomial in connection with the stationary Ablowitz–Ladik hierarchy and the underlying hyperelliptic curve. For a detailed treatment of this material we refer to [39], [40].

We denote by $\mathbb{C}^{\mathbb{Z}}$ the set of complex-valued sequences indexed by $\mathbb{Z}$.

Throughout this section we suppose the following hypothesis.

Hypothesis 2.1. *In the stationary case we assume that α, β satisfy*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0,1\}, \; n \in \mathbb{Z}. \tag{2.1}$$

In the time-dependent case we assume that α, β satisfy

$$\begin{aligned} &\alpha(\,\cdot\,, t), \beta(\,\cdot\,, t) \in \mathbb{C}^{\mathbb{Z}}, \; t \in \mathbb{R}, \quad \alpha(n, \,\cdot\,), \beta(n, \,\cdot\,) \in C^1(\mathbb{R}), \; n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\}, \; (n,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{2.2}$$

We denote by $S^{\pm}$ the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ according to

$$(S^{\pm} f)(n) = f(n \pm 1), \quad n \in \mathbb{Z}. \tag{2.3}$$

Moreover, we will frequently use the notation

$$f^{\pm} = S^{\pm} f, \quad f \in \mathbb{C}^{\mathbb{Z}}. \tag{2.4}$$

To construct the Ablowitz–Ladik hierarchy one typically introduces appropriate zero-curvature pairs of 2×2 matrices, denoted by $U(z)$ and $V_{\underline{p}}(z)$, $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2$ (with z a certain spectral parameter to be discussed later), defined recursively in the following. We take the shortest route to the construction of $V_{\underline{p}}$ and hence to that of the Ablowitz–Ladik hierarchy by starting from the recursion relation (2.5)–(2.12) below.

Define sequences $\{f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ recursively by

$$g_{0,+} = \tfrac{1}{2} c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^+, \quad h_{0,+} = c_{0,+}\beta, \tag{2.5}$$

$$g_{\ell+1,+} - g_{\ell+1,+}^- = \alpha h_{\ell,+}^- + \beta f_{\ell,+}, \quad \ell \in \mathbb{N}_0, \tag{2.6}$$

$$f_{\ell+1,+}^- = f_{\ell,+} - \alpha(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell \in \mathbb{N}_0, \tag{2.7}$$

$$h_{\ell+1,+} = h_{\ell,+}^- + \beta(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell \in \mathbb{N}_0, \tag{2.8}$$

and

$$g_{0,-} = \tfrac{1}{2} c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^+, \tag{2.9}$$

$$g_{\ell+1,-} - g_{\ell+1,-}^- = \alpha h_{\ell,-} + \beta f_{\ell,-}^-, \quad \ell \in \mathbb{N}_0, \tag{2.10}$$

$$f_{\ell+1,-} = f_{\ell,-}^- + \alpha(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell \in \mathbb{N}_0, \tag{2.11}$$

$$h_{\ell+1,-}^- = h_{\ell,-} - \beta(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell \in \mathbb{N}_0. \tag{2.12}$$

Here $c_{0,\pm} \in \mathbb{C}$ are given constants. For later use we also introduce

$$f_{-1,\pm} = h_{-1,\pm} = 0. \tag{2.13}$$

Remark 2.2. The sequences $\{f_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,+}\}_{\ell\in\mathbb{N}_0}$ can be computed recursively as follows: Assume that $f_{\ell,+}$, $g_{\ell,+}$, and $h_{\ell,+}$ are known. Equation (2.6) is a first-order difference equation in $g_{\ell+1,+}$ that can be solved directly and yields a local lattice function that is determined up to a new constant denoted by $c_{\ell+1,+} \in \mathbb{C}$. Relations (2.7) and (2.8) then determine $f_{\ell+1,+}$ and $h_{\ell+1,+}$, etc. The sequences $\{f_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,-}\}_{\ell\in\mathbb{N}_0}$ are determined similarly.

Upon setting

$$\gamma = 1 - \alpha\beta, \tag{2.14}$$

one explicitly obtains

$$\begin{aligned}
&f_{0,+} = c_{0,+}(-\alpha^+), \quad f_{1,+} = c_{0,+}\big(-\gamma^+\alpha^{++} + (\alpha^+)^2\beta\big) + c_{1,+}(-\alpha^+),\\
&g_{0,+} = \tfrac{1}{2}c_{0,+}, \quad g_{1,+} = c_{0,+}(-\alpha^+\beta) + \tfrac{1}{2}c_{1,+},\\
&h_{0,+} = c_{0,+}\beta, \quad h_{1,+} = c_{0,+}\big(\gamma\beta^- - \alpha^+\beta^2\big) + c_{1,+}\beta,\\
&f_{0,-} = c_{0,-}\alpha, \quad f_{1,-} = c_{0,-}\big(\gamma\alpha^- - \alpha^2\beta^+\big) + c_{1,-}\alpha,\\
&g_{0,-} = \tfrac{1}{2}c_{0,-}, \quad g_{1,-} = c_{0,-}(-\alpha\beta^+) + \tfrac{1}{2}c_{1,-},\\
&h_{0,-} = c_{0,-}(-\beta^+), \quad h_{1,-} = c_{0,-}\big(-\gamma^+\beta^{++} + \alpha(\beta^+)^2\big) + c_{1,-}(-\beta^+), \text{ etc.}
\end{aligned} \tag{2.15}$$

Here $\{c_{\ell,\pm}\}_{\ell\in\mathbb{N}}$ denote summation constants which naturally arise when solving the difference equations for $g_{\ell,\pm}$ in (2.6), (2.10). Subsequently, it will also be useful to work with the corresponding homogeneous coefficients $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, and $\hat h_{\ell,\pm}$, defined by the vanishing of all summation constants $c_{k,\pm}$ for $k = 1,\dots,\ell$, and choosing $c_{0,\pm} = 1$,

$$\hat f_{0,+} = -\alpha^+, \quad \hat f_{0,-} = \alpha, \quad \hat f_{\ell,\pm} = f_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}, \tag{2.16}$$

$$\hat g_{0,\pm} = \tfrac{1}{2}, \quad \hat g_{\ell,\pm} = g_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}, \tag{2.17}$$

$$\hat h_{0,+} = \beta, \quad \hat h_{0,-} = -\beta^+, \quad \hat h_{\ell,\pm} = h_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}. \tag{2.18}$$

By induction one infers that

$$f_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat f_{k,\pm}, \quad g_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat g_{k,\pm}, \quad h_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat h_{k,\pm}. \tag{2.19}$$

In a slight abuse of notation we will occasionally stress the dependence of $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ on α,β by writing $f_{\ell,\pm}(\alpha,\beta)$, $g_{\ell,\pm}(\alpha,\beta)$, and $h_{\ell,\pm}(\alpha,\beta)$.

One can show (cf. [40]) that all homogeneous elements $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, and $\hat h_{\ell,\pm}$, $\ell\in\mathbb{N}_0$, are polynomials in α,β, and some of their shifts.

Remark 2.3. As an efficient tool to later distinguish between nonhomogeneous and homogeneous quantities $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, and $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, $\hat h_{\ell,\pm}$, respectively, we now introduce the notion of degree as follows. Denote

$$f^{(r)} = S^{(r)}f, \quad f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad S^{(r)} = \begin{cases} (S^+)^r, & r\ge 0,\\ (S^-)^{-r}, & r<0, \end{cases} \quad r\in\mathbb{Z}, \tag{2.20}$$

and define

$$\deg\big(\alpha^{(r)}\big) = r, \quad \deg\big(\beta^{(r)}\big) = -r, \quad r\in\mathbb{Z}. \tag{2.21}$$

This then results in

$$\begin{aligned}
&\deg\big(\hat f^{(r)}_{\ell,+}\big) = \ell+1+r, \quad \deg\big(\hat f^{(r)}_{\ell,-}\big) = -\ell+r, \quad \deg\big(\hat g^{(r)}_{\ell,\pm}\big) = \pm\ell,\\
&\deg\big(\hat h^{(r)}_{\ell,+}\big) = \ell-r, \quad \deg\big(\hat h^{(r)}_{\ell,-}\big) = -\ell-1-r, \quad \ell\in\mathbb{N}_0,\ r\in\mathbb{Z}.
\end{aligned} \tag{2.22}$$

We also note the following useful result (cf. [40]): Assume (2.1), then,

$$\begin{aligned}
&g_{\ell,+} - g^-_{\ell,+} = \alpha h_{\ell,+} + \beta f^-_{\ell,+}, \quad \ell\in\mathbb{N}_0,\\
&g_{\ell,-} - g^-_{\ell,-} = \alpha h^-_{\ell,-} + \beta f_{\ell,-}, \quad \ell\in\mathbb{N}_0.
\end{aligned} \tag{2.23}$$

Moreover, we note the following symmetries,

$$\hat f_{\ell,\pm}(c_{0,\pm},\alpha,\beta) = \hat h_{\ell,\mp}(c_{0,\mp},\beta,\alpha), \quad \hat g_{\ell,\pm}(c_{0,\pm},\alpha,\beta) = \hat g_{\ell,\mp}(c_{0,\mp},\beta,\alpha), \quad \ell\in\mathbb{N}_0. \tag{2.24}$$

Next we relate the homogeneous coefficients $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$ to certain matrix elements of L, where L will later be identified as the Lax difference expression associated with the Ablowitz–Ladik hierarchy. For this purpose it is useful to introduce the standard basis $\{\delta_m\}_{m\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ by

$$\delta_m = \{\delta_{m,n}\}_{n\in\mathbb{Z}},\ m\in\mathbb{Z}, \quad \delta_{m,n} = \begin{cases} 1, & m=n, \\ 0, & m\neq n. \end{cases} \tag{2.25}$$

The scalar product in $\ell^2(\mathbb{Z})$, denoted by $(\,\cdot\,,\,\cdot\,)$, is defined by

$$(f,g) = \sum_{n\in\mathbb{Z}} \overline{f(n)}g(n), \quad f,g\in\ell^2(\mathbb{Z}). \tag{2.26}$$

In the standard basis just defined, we introduce the difference expression L by

$$L = \left(\begin{array}{cccccccc} \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & -\alpha(0)\rho(-1) & -\beta(-1)\alpha(0) & -\alpha(1)\rho(0) & \rho(0)\rho(1) & & & \text{\Large 0} \\ & \rho(-1)\rho(0) & \beta(-1)\rho(0) & -\beta(0)\alpha(1) & \beta(0)\rho(1) & 0 & & \\ & & 0 & -\alpha(2)\rho(1) & -\beta(1)\alpha(2) & -\alpha(3)\rho(2) & \rho(2)\rho(3) & \\ & \text{\Large 0} & & \rho(1)\rho(2) & \beta(1)\rho(2) & -\beta(2)\alpha(3) & \beta(2)\rho(3) & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right) \tag{2.27}$$

$$= \rho^-\rho\,\delta_{\text{even}}\,S^{--} + (\beta^-\rho\,\delta_{\text{even}} - \alpha^+\rho\,\delta_{\text{odd}})S^- - \beta\alpha^+$$
$$+ (\beta\rho^+\,\delta_{\text{even}} - \alpha^{++}\rho^+\,\delta_{\text{odd}})S^+ + \rho^+\rho^{++}\,\delta_{\text{odd}}\,S^{++},$$

where

$$\delta_{\text{even}} = \chi_{_{2\mathbb{Z}}}, \quad \delta_{\text{odd}} = 1-\delta_{\text{even}} = \chi_{_{2\mathbb{Z}+1}}. \tag{2.28}$$

In particular, terms of the form $-\beta(n)\alpha(n+1)$ represent the diagonal (n,n)-entries, $n\in\mathbb{Z}$, in the infinite matrix (2.27). In addition, we used the abbreviation

$$\rho = \gamma^{1/2} = (1-\alpha\beta)^{1/2}. \tag{2.29}$$

Next, we introduce the unitary operator $U_{\tilde{\varepsilon}}$ in $\ell^2(\mathbb{Z})$ by

$$U_{\tilde{\varepsilon}} = \big(\tilde{\varepsilon}(n)\delta_{m,n}\big)_{(m,n)\in\mathbb{Z}^2}, \quad \tilde{\varepsilon}(n)\in\{1,-1\},\ n\in\mathbb{Z}, \tag{2.30}$$

and the sequence $\varepsilon = \{\varepsilon(n)\}_{n\in\mathbb{Z}}\in\mathbb{C}^{\mathbb{Z}}$ by

$$\varepsilon(n) = \tilde{\varepsilon}(n-1)\tilde{\varepsilon}(n),\ n\in\mathbb{Z}. \tag{2.31}$$

A straightforward computation then shows that

$$U_{\tilde{\varepsilon}} L U_{\tilde{\varepsilon}}^{-1} = \widetilde{L}_{\varepsilon}, \tag{2.32}$$

where $\widetilde{L}_{\varepsilon}$ is associated with the sequences $\tilde{\alpha}=\alpha$, $\tilde{\beta}=\beta$, and $\tilde{\rho}=\varepsilon\rho$. Moreover, the recursion formalism in (2.5)–(2.12) yields coefficients which are polynomials in α, β and some of their shifts and hence depends only quadratically on ρ. As a result, the choice of square root of $\rho(n)$, $n\in\mathbb{Z}$, in (2.29) is immaterial when introducing the AL hierarchy via the Lax equations (2.72).

The half-lattice (i.e., semi-infinite) version of L was rediscovered by Cantero, Moral, and Velázquez [21] in 2003 in the special case where $\beta=\overline{\alpha}$ (see also Simon [52]–[54] who coined the term CMV matrix in this context).

The next result details the connections between L and the recursion coefficients $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$:

Lemma 2.4. *Let $n \in \mathbb{Z}$. Then the homogeneous coefficients $\{\hat f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{\hat g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{\hat h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ satisfy the following relations:*

$$
\begin{aligned}
&\hat f_{\ell,+}(n) = \alpha(n)(\delta_n, L^{\ell+1}\delta_n) + \rho(n) \begin{cases} (\delta_{n-1}, L^{\ell+1}\delta_n), & n \text{ even}, \\ (\delta_n, L^{\ell+1}\delta_{n-1}), & n \text{ odd}, \end{cases} \quad \ell \in \mathbb{N}_0, \\
&\hat f_{\ell,-}(n) = \alpha(n)(\delta_n, L^{-\ell}\delta_n) + \rho(n) \begin{cases} (\delta_{n-1}, L^{-\ell}\delta_n), & n \text{ even}, \\ (\delta_n, L^{-\ell}\delta_{n-1}), & n \text{ odd}, \end{cases} \quad \ell \in \mathbb{N}_0, \\
&\hat g_{0,\pm} = 1/2, \quad \hat g_{\ell,\pm}(n) = (\delta_n, L^{\pm\ell}\delta_n), \quad \ell \in \mathbb{N}, \\
&\hat h_{\ell,+}(n) = \beta(n)(\delta_n, L^{\ell}\delta_n) + \rho(n) \begin{cases} (\delta_n, L^{\ell}\delta_{n-1}), & n \text{ even}, \\ (\delta_{n-1}, L^{\ell}\delta_n), & n \text{ odd}, \end{cases} \quad \ell \in \mathbb{N}_0, \\
&\hat h_{\ell,-}(n) = \beta(n)(\delta_n, L^{-\ell-1}\delta_n) + \rho(n) \begin{cases} (\delta_n, L^{-\ell-1}\delta_{n-1}), & n \text{ even}, \\ (\delta_{n-1}, L^{-\ell-1}\delta_n), & n \text{ odd}, \end{cases} \quad \ell \in \mathbb{N}_0.
\end{aligned} \tag{2.33}
$$

For the proof of Lemma 2.4 and some of its applications in connection with conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy we refer to [42].

Next we define the 2×2 zero-curvature matrices

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix} \tag{2.34}$$

and

$$V_{\underline{p}}(z) = i \begin{pmatrix} G^-_{\underline{p}}(z) & -F^-_{\underline{p}}(z) \\ H^-_{\underline{p}}(z) & -K^-_{\underline{p}}(z) \end{pmatrix}, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2, \tag{2.35}$$

for appropriate Laurent polynomials $F_{\underline{p}}(z)$, $G_{\underline{p}}(z)$, $H_{\underline{p}}(z)$, and $K_{\underline{p}}(z)$ in the spectral parameter $z \in \mathbb{C}\setminus\{0\}$ to be determined shortly. By postulating the stationary zero-curvature relation,

$$0 = UV_{\underline{p}} - V^+_{\underline{p}}U, \tag{2.36}$$

one concludes that (2.36) is equivalent with the following relations

$$z(G^-_{\underline{p}} - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H^-_{\underline{p}} = 0, \tag{2.37}$$

$$z\beta F^-_{\underline{p}} + \alpha H_{\underline{p}} - K_{\underline{p}} + K^-_{\underline{p}} = 0, \tag{2.38}$$

$$-F_{\underline{p}} + zF^-_{\underline{p}} + \alpha(G_{\underline{p}} + K^-_{\underline{p}}) = 0, \tag{2.39}$$

$$z\beta(G^-_{\underline{p}} + K_{\underline{p}}) - zH_{\underline{p}} + H^-_{\underline{p}} = 0. \tag{2.40}$$

In order to make the connection between the zero-curvature formalism and the recursion relations (2.5)–(2.12), we now define Laurent polynomials $F_{\underline{p}}$, $G_{\underline{p}}$, $H_{\underline{p}}$, and $K_{\underline{p}}$ by[1]

$$F_{\underline{p}}(z) = \sum_{\ell=1}^{p_-} f_{p_- -\ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+ -1} f_{p_+ -1-\ell,+} z^{\ell}, \tag{2.41}$$

$$G_{\underline{p}}(z) = \sum_{\ell=1}^{p_-} g_{p_- -\ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+} g_{p_+ -\ell,+} z^{\ell}, \tag{2.42}$$

[1]In this paper, a sum is interpreted as zero whenever the upper limit in the sum is strictly less than its lower limit.

$$H_{\underline{p}}(z) = \sum_{\ell=0}^{p_- -1} h_{p_- -1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} h_{p_+-\ell,+} z^{\ell}, \tag{2.43}$$

$$K_{\underline{p}}(z) = \sum_{\ell=0}^{p_-} g_{p_- -\ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} g_{p_+-\ell,+} z^{\ell} = G_{\underline{p}}(z) + g_{p_-,-} - g_{p_+,+}. \tag{2.44}$$

The corresponding homogeneous quantities are defined by ($\ell \in \mathbb{N}_0$)

$$\begin{aligned}
&\widehat{F}_{0,\mp}(z) = 0, \quad \widehat{F}_{\ell,-}(z) = \sum_{k=1}^{\ell} \hat{f}_{\ell-k,-} z^{-k}, \quad \widehat{F}_{\ell,+}(z) = \sum_{k=0}^{\ell-1} \hat{f}_{\ell-1-k,+} z^{k}, \\
&\widehat{G}_{0,-}(z) = 0, \quad \widehat{G}_{\ell,-}(z) = \sum_{k=1}^{\ell} \hat{g}_{\ell-k,-} z^{-k}, \\
&\widehat{G}_{0,+}(z) = \frac{1}{2}, \quad \widehat{G}_{\ell,+}(z) = \sum_{k=0}^{\ell} \hat{g}_{\ell-k,+} z^{k}, \\
&\widehat{H}_{0,\mp}(z) = 0, \quad \widehat{H}_{\ell,-}(z) = \sum_{k=0}^{\ell-1} \hat{h}_{\ell-1-k,-} z^{-k}, \quad \widehat{H}_{\ell,+}(z) = \sum_{k=1}^{\ell} \hat{h}_{\ell-k,+} z^{k}, \\
&\widehat{K}_{0,-}(z) = \frac{1}{2}, \quad \widehat{K}_{\ell,-}(z) = \sum_{k=0}^{\ell} \hat{g}_{\ell-k,-} z^{-k} = \widehat{G}_{\ell,-}(z) + \hat{g}_{\ell,-}, \\
&\widehat{K}_{0,+}(z) = 0, \quad \widehat{K}_{\ell,+}(z) = \sum_{k=1}^{\ell} \hat{g}_{\ell-k,+} z^{k} = \widehat{G}_{\ell,+}(z) - \hat{g}_{\ell,+}.
\end{aligned} \tag{2.45}$$

The stationary zero-curvature relation (2.36), $0 = UV_{\underline{p}} - V_{\underline{p}}^+ U$, is then equivalent to

$$-\alpha(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- = 0, \tag{2.46}$$

$$\beta(g_{p_+,+}^- + g_{p_-,-}) + h_{p_+-1,+}^- - h_{p_--1,-} = 0. \tag{2.47}$$

Thus, varying $p_\pm \in \mathbb{N}_0$, equations (2.46) and (2.47) give rise to the stationary Ablowitz–Ladik (AL) hierarchy which we introduce as follows

$$\begin{aligned}
\text{s-AL}_{\underline{p}}(\alpha, \beta) = \begin{pmatrix} -\alpha(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- \\ \beta(g_{p_+,+}^- + g_{p_-,-}) + h_{p_+-1,+}^- - h_{p_--1,-} \end{pmatrix} = 0, \\
\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2.
\end{aligned} \tag{2.48}$$

Explicitly (recalling $\gamma = 1 - \alpha\beta$ and taking $p_- = p_+$ for simplicity),

$$\begin{aligned}
&\text{s-AL}_{(0,0)}(\alpha, \beta) = \begin{pmatrix} -c_{(0,0)}\alpha \\ c_{(0,0)}\beta \end{pmatrix} = 0, \\
&\text{s-AL}_{(1,1)}(\alpha, \beta) = \begin{pmatrix} -\gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0, \\
&\text{s-AL}_{(2,2)}(\alpha, \beta) = \begin{pmatrix} -\gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix}
\end{aligned}$$

$$+\begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \text{ etc.}, \tag{2.49}$$

represent the first few equations of the stationary Ablowitz–Ladik hierarchy. Here we introduced

$$c_{\underline{p}} = (c_{p_-,-} + c_{p_+,+})/2, \quad p_\pm \in \mathbb{N}_0. \tag{2.50}$$

By definition, the set of solutions of (2.48), with $p_\pm$ ranging in $\mathbb{N}_0$ and $c_{\ell,\pm} \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, represents the class of algebro-geometric Ablowitz–Ladik solutions.

In the following we will frequently assume that α, β satisfy the $\underline{p}$th stationary AL system $\text{s-AL}_{\underline{p}}(\alpha, \beta) = 0$, supposing a particular choice of summation constants $c_{\ell,\pm} \in \mathbb{C}$, $\ell = 0, \dots, p_\pm$, $p_\pm \in \mathbb{N}_0$, has been made.

In accordance with our notation introduced in (2.16)–(2.18) and (2.45), the corresponding homogeneous stationary Ablowitz–Ladik equations are defined by

$$\text{s-}\widehat{\text{AL}}_{\underline{p}}(\alpha, \beta) = \text{s-AL}_{\underline{p}}(\alpha, \beta)\big|_{c_{0,\pm}=1,\, c_{\ell,\pm}=0,\, \ell=1,\dots,p_\pm}, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2. \tag{2.51}$$

In addition, one can show (cf. [40, Lemma 2.2]) that $g_{p_+,+} = g_{p_-,-}$ up to a lattice constant which can be set equal to zero without loss of generality. Thus, we will henceforth assume that

$$g_{p_+,+} = g_{p_-,-}, \tag{2.52}$$

which in turn implies that

$$K_{\underline{p}} = G_{\underline{p}} \tag{2.53}$$

and hence renders $V_{\underline{p}}$ in (2.35) traceless in the stationary context. (We note that equations (2.52) and (2.53) cease to be valid in the time-dependent context, though.)

Remark 2.5. (*i*) The particular choice $c_{0,+} = c_{0,-} = 1$ yields the stationary Ablowitz–Ladik equation. Scaling $c_{0,\pm}$ with the same constant then amounts to scaling $V_{\underline{p}}$ with this constant which drops out in the stationary zero-curvature equation (2.36).
(*ii*) Different ratios between $c_{0,+}$ and $c_{0,-}$ will lead to different stationary hierarchies. In particular, the choice $c_{0,+} = 2$, $c_{0,-} = \cdots = c_{p_--1,-} = 0$, $c_{p_-,-} \neq 0$, yields the stationary Baxter–Szegő hierarchy considered in detail in [33]. However, in this case some parts from the recursion relation for the negative coefficients still remain. In fact, (2.12) reduces to $g_{p_-,-} - g_{p_-,-}^- = \alpha h_{p_--1,-}$, $h_{p_--1,-} = 0$ and thus requires $g_{p_-,-}$ to be a constant in (2.48) and (2.64). Moreover, $f_{p_--1,-} = 0$ in (2.48) in this case.

Next, taking into account (2.53), one infers that the expression $R_{\underline{p}}$, defined as

$$R_{\underline{p}} = G_{\underline{p}}^2 - F_{\underline{p}} H_{\underline{p}}, \tag{2.54}$$

is a lattice constant, that is, $R_{\underline{p}} - R_{\underline{p}}^- = 0$, by taking determinants in the stationary zero-curvature equation (2.36). Hence, $R_{\underline{p}}(z)$ only depends on z, and assuming in addition to (2.1) that

$$c_{0,\pm} \in \mathbb{C} \setminus \{0\}, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \setminus \{(0,0)\}, \tag{2.55}$$

one may write $R_{\underline{p}}$ as[2]

$$R_{\underline{p}}(z) = (c_{0,+}^2/4)z^{-2p_-}\prod_{m=0}^{2p+1}(z-E_m), \quad \{E_m\}_{m=0}^{2p+1}\subset\mathbb{C}\setminus\{0\}, \quad p=p_-+p_+-1\in\mathbb{N}_0. \tag{2.56}$$

Moreover, multiplying (2.54) by z^{2p_-} and taking $z\to 0$ yields

$$\prod_{m=0}^{2p+1}E_m = \frac{c_{0,-}^2}{c_{0,+}^2}. \tag{2.57}$$

Relation (2.54) allows one to introduce a hyperelliptic curve $\mathcal{K}_p$ of (arithmetic) genus $p=p_-+p_+-1$ (possibly with a singular affine part), where

$$\mathcal{K}_p\colon \mathcal{F}_p(z,y) = y^2 - 4c_{0,+}^{-2}z^{2p_-}R_{\underline{p}}(z) = y^2 - \prod_{m=0}^{2p+1}(z-E_m) = 0, \quad p=p_-+p_+-1. \tag{2.58}$$

Next we turn to the time-dependent Ablowitz–Ladik hierarchy. For that purpose the coefficients α and β are now considered as functions of both the lattice point and time. For each system in the hierarchy, that is, for each $p_\pm$, we introduce a deformation (time) parameter $t_{\underline{p}}\in\mathbb{R}$ in α,β, replacing $\alpha(n),\beta(n)$ by $\alpha(n,t_{\underline{p}}),\beta(n,t_{\underline{p}})$. Moreover, the definitions (2.34), (2.35), and (2.41)–(2.44) of $U, V_{\underline{p}}$, and $F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}, K_{\underline{p}}$, respectively, still apply. Imposing the zero-curvature relation

$$U_{t_{\underline{p}}} + UV_{\underline{p}} - V_{\underline{p}}^+U = 0, \quad \underline{p}\in\mathbb{N}_0^2, \tag{2.59}$$

then results in the equations

$$\alpha_{t_{\underline{p}}} = i\big(zF_{\underline{p}}^- + \alpha(G_{\underline{p}}+K_{\underline{p}}^-) - F_{\underline{p}}\big), \tag{2.60}$$

$$\beta_{t_{\underline{p}}} = -i\big(\beta(G_{\underline{p}}^- + K_{\underline{p}}) - H_{\underline{p}} + z^{-1}H_{\underline{p}}^-\big), \tag{2.61}$$

$$0 = z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^-, \tag{2.62}$$

$$0 = z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} + K_{\underline{p}}^- - K_{\underline{p}}. \tag{2.63}$$

Varying $\underline{p}\in\mathbb{N}_0^2$, the collection of evolution equations

$$\mathrm{AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{\underline{p}}} - \alpha(g_{p_+,+}+g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- \\ -i\beta_{t_{\underline{p}}} + \beta(g_{p_+,+}^- + g_{p_-,-}) - h_{p_--1,-} + h_{p_+-1,+}^- \end{pmatrix} = 0, \quad t_{\underline{p}}\in\mathbb{R},\ \underline{p}=(p_-,p_+)\in\mathbb{N}_0^2, \tag{2.64}$$

then defines the time-dependent Ablowitz–Ladik hierarchy. Explicitly, taking $p_- = p_+$ for simplicity,

$$\begin{aligned}
&\mathrm{AL}_{(0,0)}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{(0,0)}} - c_{(0,0)}\alpha \\ -i\beta_{t_{(0,0)}} + c_{(0,0)}\beta \end{pmatrix} = 0,\\
&\mathrm{AL}_{(1,1)}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{(1,1)}} - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ -i\beta_{t_{(1,1)}} + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0,\\
&\mathrm{AL}_{(2,2)}(\alpha,\beta)
\end{aligned} \tag{2.65}$$

[2]We use the convention that a product is to be interpreted equal to 1 whenever the upper limit of the product is strictly less than its lower limit.

$$= \begin{pmatrix} -i\alpha_{t_{(2,2)}} - \gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ -i\beta_{t_{(2,2)}} + \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix}$$
$$+ \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \text{ etc.,}$$

represent the first few equations of the time-dependent Ablowitz–Ladik hierarchy. Here we recall the definition of $c_{\underline{p}}$ in (2.50).

By (2.64), (2.6), and (2.10), the time derivative of $\gamma = 1 - \alpha\beta$ is given by

$$\gamma_{t_{\underline{p}}} = i\gamma\big((g_{p_+,+} - g^-_{p_+,+}) - (g_{p_-,-} - g^-_{p_-,-})\big). \tag{2.66}$$

Remark 2.6. From (2.37)–(2.40) and the explicit computations of the coefficients $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$, one concludes that the zero-curvature equation (2.59) and hence the Ablowitz–Ladik hierarchy is invariant under the scaling transformation

$$\alpha \to \alpha_c = \{c\,\alpha(n)\}_{n\in\mathbb{Z}}, \quad \beta \to \beta_c = \{\beta(n)/c\}_{n\in\mathbb{Z}}, \quad c \in \mathbb{C}\setminus\{0\}. \tag{2.67}$$

Moreover, $R_{\underline{p}} = G_{\underline{p}}^2 - H_{\underline{p}}F_{\underline{p}}$ and hence $\{E_m\}_{m=0}^{2p+1}$ are invariant under this transformation. Furthermore, choosing $c = e^{ic_{\underline{p}}t}$, one verifies that it is no restriction to assume $c_{\underline{p}} = 0$. This also indicates that stationary solutions α, β can only be constructed up to a multiplicative constant (compare Theorem 3.7, in particular, (3.64), (3.65)).

Finally, we briefly hint at explicit expressions of the Lax pair for the Ablowitz–Ladik hierarchy. More details will be presented in [42]. First we need some notation. Let T be a bounded operator in $\ell^2(\mathbb{Z})$. Given the standard basis (2.25) in $\ell^2(\mathbb{Z})$, we represent T by

$$T = \big(T(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T(m,n) = (\delta_m, T\,\delta_n), \quad (m,n)\in\mathbb{Z}^2. \tag{2.68}$$

Moreover, we introduce the upper and lower triangular parts $T_\pm$ of T by

$$T_\pm = \big(T_\pm(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T_\pm(m,n) = \begin{cases} T(m,n), & \pm(n-m) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.69}$$

Next, consider the finite difference expression $P_{\underline{p}}$ defined by

$$\begin{aligned} P_{\underline{p}} = \frac{i}{2}\sum_{\ell=1}^{p_+} c_{p_+-\ell,+}\big((L^\ell)_+ - (L^\ell)_-\big) - \frac{i}{2}\sum_{\ell=1}^{p_-} c_{p_--\ell,-}\big((L^{-\ell})_+ - (L^{-\ell})_-\big) \\ - \frac{i}{2}c_{\underline{p}}\,Q_d, \quad \underline{p} = (p_-,p_+)\in\mathbb{N}_0^2, \end{aligned} \tag{2.70}$$

with L given by (2.27), $c_{\underline{p}} = (c_{p_-,-} + c_{p_+,+})/2$, and Q_d denoting the doubly infinite diagonal matrix

$$Q_d = \big((-1)^k\delta_{k,\ell}\big)_{k,\ell\in\mathbb{Z}}. \tag{2.71}$$

Then one can show that $(L, P_{\underline{p}})$ represents the Lax pair for the Ablowitz–Ladik equations (2.64) for $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0,0\}$. In particular, the hierarchy of nonlinear Ablowitz–Ladik evolution equations (2.64) then can alternatively be derived by imposing the Lax commutator equations

$$L_{t_{\underline{p}}}(t_{\underline{p}}) - [P_{\underline{p}}(t_{\underline{p}}), L(t_{\underline{p}})] = 0, \quad t_{\underline{p}}\in\mathbb{R},\ \underline{p}\in\mathbb{N}_0^2. \tag{2.72}$$

For additional representations of $P_{\underline{p}}$ in terms of L and a particular factorization of L we refer to [42]. The Ablowitz–Ladik Lax pair in the special defocusing case, where $\beta = \overline{\alpha}$, in the finite-dimensional context, was recently derived by Nenciu [50].

In the special stationary case, where $P_{\underline{p}}$ and L commute, $[P_{\underline{p}}, L] = 0$, they satisfy an algebraic relationship of the type

$$\begin{aligned} &P_{\underline{p}}^2 + R_{\underline{p}}(L) = P_{\underline{p}}^2 + (c_{0,+}^2/4)L^{-2p_-} \prod_{m=0}^{2p+1}(L - E_m) = 0, \\ &R_{\underline{p}}(z) = (c_{0,+}^2/4)z^{-2p_-} \prod_{m=0}^{2p+1}(z - E_m), \quad p = p_- + p_+ - 1. \end{aligned} \tag{2.73}$$

Thus, the expression $P_{\underline{p}}^2 + R_{\underline{p}}(L)$ in (2.73) represents the Burchnall–Chaundy Laurent polynomial of the Lax pair $(L, P_{\underline{p}})$.

3. The Stationary Ablowitz–Ladik Formalism

This section is devoted to a detailed study of the stationary Ablowitz–Ladik hierarchy and its algebro-geometric solutions. Our principal tools are derived from combining the Laurent polynomial recursion formalism introduced in Section 2 and a fundamental meromorphic function ϕ on a hyperelliptic curve $\mathcal{K}_p$. With the help of ϕ we study the Baker–Akhiezer vector Ψ, trace formulas, and theta function representations of ϕ, Ψ, α, and β. For proofs of the elementary results of the stationary formalism we refer to [39], [40].

Unless explicitly stated otherwise, we suppose throughout this section that

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0, 1\}, \ n \in \mathbb{Z}, \tag{3.1}$$

and assume (2.5)–(2.13), (2.34)–(2.36), (2.41)–(2.44), (2.48), (2.53), (2.54), (2.56), keeping $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0,0\}$ fixed.

We recall the hyperelliptic curve

$$\begin{aligned} &\mathcal{K}_p \colon \mathcal{F}_p(z, y) = y^2 - 4c_{0,+}^{-2} z^{2p_-} R_{\underline{p}}(z) = y^2 - \prod_{m=0}^{2p+1}(z - E_m) = 0, \\ &R_{\underline{p}}(z) = \left(\frac{c_{0,+}}{2z^{p_-}}\right)^2 \prod_{m=0}^{2p+1}(z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C} \setminus \{0\}, \ p = p_- + p_+ - 1, \end{aligned} \tag{3.2}$$

as introduced in (2.58). Throughout this section we assume the affine part of $\mathcal{K}_p$ to be nonsingular, that is, we suppose that

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 2p + 1. \tag{3.3}$$

$\mathcal{K}_p$ is compactified by joining two points $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_p$. Points P on $\mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z, y) = 0$. The complex structure on $\mathcal{K}_p$ is then defined in the usual way, see Appendix A. Hence, $\mathcal{K}_p$ becomes a two-sheeted hyperelliptic Riemann surface of genus p in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_p$ (i.e., by fixing $E_0, \dots, E_{2p+1}$), the summation constants $c_{1,\pm}, \dots, c_{p_\pm,\pm}$ in $f_{p_\pm,\pm}$, $g_{p_\pm,\pm}$, and $h_{p_\pm,\pm}$ (and hence in the corresponding stationary s-AL$_{\underline{p}}$ equations) are uniquely determined as is clear from (B.8) which establishes the summation constants $c_{\ell,\pm}$ as symmetric functions of $E_0^{\pm 1}, \dots, E_{2p+1}^{\pm 1}$.

For notational simplicity we will usually tacitly assume that $p \in \mathbb{N}$ and hence $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0,0),(0,1),(1,0)\}$.

We denote by $\{\mu_j(n)\}_{j=1,\dots,p}$ and $\{\nu_j(n)\}_{j=1,\dots,p}$ the zeros of $(\,\cdot\,)^{p_-} F_{\underline{p}}(\,\cdot\,,n)$ and $(\,\cdot\,)^{p_- -1} H_{\underline{p}}(\,\cdot\,,n)$, respectively. Thus, we may write

$$F_{\underline{p}}(z) = -c_{0,+}\alpha^+ z^{-p_-} \prod_{j=1}^{p} (z-\mu_j), \tag{3.4}$$

$$H_{\underline{p}}(z) = c_{0,+}\beta z^{-p_- +1} \prod_{j=1}^{p} (z-\nu_j), \tag{3.5}$$

and we recall that (cf. (2.54))

$$R_{\underline{p}} - G_{\underline{p}}^2 = -F_{\underline{p}} H_{\underline{p}}. \tag{3.6}$$

The next step is crucial; it permits us to "lift" the zeros μ_j and ν_j of $(\,\cdot\,)^{p_-} F_{\underline{p}}$ and $(\,\cdot\,)^{p_- -1} H_{\underline{p}}$ from the complex plane $\mathbb{C}$ to the curve $\mathcal{K}_p$. From (3.6) one infers that

$$R_{\underline{p}}(z) - G_{\underline{p}}(z)^2 = 0, \quad z \in \{\mu_j, \nu_k\}_{j,k=1,\dots,p}. \tag{3.7}$$

We now introduce $\{\hat{\mu}_j\}_{j=1,\dots,p} \subset \mathcal{K}_p$ and $\{\hat{\nu}_j\}_{j=1,\dots,p} \subset \mathcal{K}_p$ by

$$\hat{\mu}_j(n) = (\mu_j(n), (2/c_{0,+})\mu_j(n)^{p_-} G_{\underline{p}}(\mu_j(n),n)), \quad j=1,\dots,p,\ n \in \mathbb{Z}, \tag{3.8}$$

and

$$\hat{\nu}_j(n) = (\nu_j(n), -(2/c_{0,+})\nu_j(n)^{p_-} G_{\underline{p}}(\nu_j(n),n)), \quad j=1,\dots,p,\ n \in \mathbb{Z}. \tag{3.9}$$

We also introduce the points $P_{0,\pm}$ by

$$P_{0,\pm} = (0, \pm(c_{0,-}/c_{0,+})) \in \mathcal{K}_p, \quad \frac{c_{0,-}^2}{c_{0,+}^2} = \prod_{m=0}^{2p+1} E_m. \tag{3.10}$$

We emphasize that $P_{0,\pm}$ and $P_{\infty_\pm}$ are not necessarily on the same sheet of $\mathcal{K}_p$.

Next we introduce the fundamental meromorphic function on $\mathcal{K}_p$ by

$$\begin{aligned} \phi(P,n) &= \frac{(c_{0,+}/2)z^{-p_-}y + G_{\underline{p}}(z,n)}{F_{\underline{p}}(z,n)} && (3.11)\\ &= \frac{-H_{\underline{p}}(z,n)}{(c_{0,+}/2)z^{-p_-}y - G_{\underline{p}}(z,n)}, && (3.12)\\ &\qquad P=(z,y) \in \mathcal{K}_p,\ n \in \mathbb{Z}, \end{aligned}$$

with divisor $(\phi(\,\cdot\,,n))$ of $\phi(\,\cdot\,,n)$ given by

$$(\phi(\,\cdot\,,n)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}, \tag{3.13}$$

using (3.4) and (3.5). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_p\} \in \operatorname{Sym}^p(\mathcal{K}_p). \tag{3.14}$$

Given $\phi(\,\cdot\,,n)$, the meromorphic stationary Baker–Akhiezer vector $\Psi(\,\cdot\,,n,n_0)$ on $\mathcal{K}_p$ is then defined by

$$\Psi(P,n,n_0) = \begin{pmatrix} \psi_1(P,n,n_0) \\ \psi_2(P,n,n_0) \end{pmatrix},$$

$$\psi_1(P,n,n_0) = \begin{cases} \prod_{n'=n_0+1}^{n} \big(z+\alpha(n')\phi^-(P,n')\big), & n \geq n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z+\alpha(n')\phi^-(P,n')\big)^{-1}, & n \leq n_0-1, \end{cases} \tag{3.15}$$

$$\psi_2(P,n,n_0) = \phi(P,n_0) \begin{cases} \prod_{n'=n_0+1}^{n} \big(z\beta(n')\phi^-(P,n')^{-1}+1\big), & n \geq n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z\beta(n')\phi^-(P,n')^{-1}+1\big)^{-1}, & n \leq n_0-1. \end{cases} \tag{3.16}$$

Basic properties of ϕ and Ψ are summarized in the following result.

Lemma 3.1 ([40])**.** *Suppose that α, β satisfy* (3.1) *and the $\underline{p}$th stationary Ablowitz–Ladik system* (2.48)*. Moreover, assume* (3.2) *and* (3.3) *and let $P=(z,y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n,n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{3.17}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \tag{3.18}$$

$$\phi(P)+\phi(P^*) = 2\frac{G_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \tag{3.19}$$

$$\phi(P)-\phi(P^*) = c_{0,+}z^{-p_-}\frac{y(P)}{F_{\underline{p}}(z)}. \tag{3.20}$$

The vector Ψ satisfies

$$U(z)\Psi^-(P) = \Psi(P), \tag{3.21}$$

$$V_{\underline{p}}(z)\Psi^-(P) = -(i/2)c_{0,+}z^{-p_-}y\Psi^-(P), \tag{3.22}$$

$$\psi_2(P,n,n_0) = \phi(P,n)\psi_1(P,n,n_0), \tag{3.23}$$

$$\psi_1(P,n,n_0)\psi_1(P^*,n,n_0) = z^{n-n_0}\frac{F_{\underline{p}}(z,n)}{F_{\underline{p}}(z,n_0)}\Gamma(n,n_0), \tag{3.24}$$

$$\psi_2(P,n,n_0)\psi_2(P^*,n,n_0) = z^{n-n_0}\frac{H_{\underline{p}}(z,n)}{F_{\underline{p}}(z,n_0)}\Gamma(n,n_0), \tag{3.25}$$

$$\begin{aligned} &\psi_1(P,n,n_0)\psi_2(P^*,n,n_0)+\psi_1(P^*,n,n_0)\psi_2(P,n,n_0) \\ &\quad = 2z^{n-n_0}\frac{G_{\underline{p}}(z,n)}{F_{\underline{p}}(z,n_0)}\Gamma(n,n_0), \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\psi_1(P,n,n_0)\psi_2(P^*,n,n_0)-\psi_1(P^*,n,n_0)\psi_2(P,n,n_0) \\ &\quad = -c_{0,+}z^{n-n_0-p_-}\frac{y}{F_{\underline{p}}(z,n_0)}\Gamma(n,n_0), \end{aligned} \tag{3.27}$$

where we used the abbreviation

$$\Gamma(n,n_0) = \begin{cases} \prod_{n'=n_0+1}^{n} \gamma(n'), & n \geq n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \gamma(n')^{-1}, & n \leq n_0-1. \end{cases} \tag{3.28}$$

Combining the Laurent polynomial recursion approach of Section 2 with (3.4) and (3.5) readily yields trace formulas for $f_{\ell,\pm}$ and $h_{\ell,\pm}$ in terms of symmetric functions of the zeros μ_j and ν_k of $(\,\cdot\,)^{p_-}F_{\underline{p}}$ and $(\,\cdot\,)^{p_- -1}H_{\underline{p}}$, respectively. For simplicity we just record the simplest cases.

Lemma 3.2 ([40])**.** *Suppose that α, β satisfy* (3.1) *and the $\underline{p}$th stationary Ablowitz–Ladik system* (2.48)*. Then,*

$$\frac{\alpha}{\alpha^+} = (-1)^{p+1}\frac{c_{0,+}}{c_{0,-}}\prod_{j=1}^{p}\mu_j, \tag{3.29}$$

$$\frac{\beta^+}{\beta} = (-1)^{p+1}\frac{c_{0,+}}{c_{0,-}}\prod_{j=1}^{p}\nu_j, \tag{3.30}$$

$$\sum_{j=1}^{p}\mu_j = \alpha^+\beta - \gamma^+\frac{\alpha^{++}}{\alpha^+} - \frac{c_{1,+}}{c_{0,+}}, \tag{3.31}$$

$$\sum_{j=1}^{p}\nu_j = \alpha^+\beta - \gamma\frac{\beta^-}{\beta} - \frac{c_{1,+}}{c_{0,+}}. \tag{3.32}$$

Next we turn to asymptotic properties of ϕ and Ψ in a neighborhood of $P_{\infty_\pm}$ and $P_{0,\pm}$.

Lemma 3.3 ([40])**.** *Suppose that α, β satisfy* (3.1) *and the $\underline{p}$th stationary Ablowitz–Ladik system* (2.48)*. Moreover, let $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n, n_0) \in \mathbb{Z}^2$. Then ϕ has the asymptotic behavior*

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \beta + \beta^-\gamma\zeta + O(\zeta^2), & P \to P_{\infty_+}, \\ -(\alpha^+)^{-1}\zeta^{-1} + (\alpha^+)^{-2}\alpha^{++}\gamma^+ + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.33}$$

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} \alpha^{-1} - \alpha^{-2}\alpha^-\gamma\zeta + O(\zeta^2), & P \to P_{0,+}, \\ -\beta^+\zeta - \beta^{++}\gamma^+\zeta^2 + O(\zeta^3), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.34}$$

The components of the Baker–Akhiezer vector Ψ have the asymptotic behavior

$$\psi_1(P, n, n_0) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{n_0-n}(1 + O(\zeta)), & P \to P_{\infty_+}, \\ \frac{\alpha^+(n)}{\alpha^+(n_0)}\Gamma(n, n_0) + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.35}$$

$$\psi_1(P, n, n_0) \underset{\zeta\to 0}{=} \begin{cases} \frac{\alpha(n)}{\alpha(n_0)} + O(\zeta), & P \to P_{0,+}, \\ \zeta^{n-n_0}\Gamma(n, n_0)(1 + O(\zeta)), & P \to P_{0,-}, \end{cases} \quad \zeta = z, \tag{3.36}$$

$$\psi_2(P, n, n_0) \underset{\zeta\to 0}{=} \begin{cases} \beta(n)\zeta^{n_0-n}(1 + O(\zeta)), & P \to P_{\infty_+}, \\ -\frac{1}{\alpha^+(n_0)}\Gamma(n, n_0)\zeta^{-1}(1 + O(\zeta)), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{3.37}$$

$$\psi_2(P, n, n_0) \underset{\zeta\to 0}{=} \begin{cases} \frac{1}{\alpha(n_0)} + O(\zeta), & P \to P_{0,+}, \\ -\beta^+(n)\Gamma(n, n_0)\zeta^{n+1-n_0}(1 + O(\zeta)), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{3.38}$$

The divisors (ψ_j) of ψ_j, $j = 1, 2$, are given by

$$(\psi_1(\,\cdot\,, n, n_0)) = \mathcal{D}_{\underline{\hat{\mu}}(n)} - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}), \tag{3.39}$$

$$(\psi_2(\,\cdot\,, n, n_0)) = \mathcal{D}_{\underline{\hat{\nu}}(n)} - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}) + \mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_-}}. \tag{3.40}$$

Since nonspecial divisors play a fundamental role in this section and the next, we now take a closer look at them.

Lemma 3.4 ([40])**.** *Suppose that α, β satisfy* (3.1) *and the $\underline{p}$th stationary Ablowitz–Ladik system* (2.48)*. Moreover, assume* (3.2) *and* (3.3) *and let $n \in \mathbb{Z}$. Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}$, and $\mathcal{D}_{\underline{\hat{\nu}}}$, $\underline{\hat{\nu}} = \{\underline{\hat{\nu}}_1, \dots, \underline{\hat{\nu}}_p\}$, be the pole and zero divisors of degree p, respectively, associated with α, β, and ϕ defined according to* (3.8) *and* (3.9)*, that is,*

$$\begin{aligned} \hat{\mu}_j(n) &= (\mu_j(n), (2/c_{0,+})\mu_j(n)^{p_-} G_{\underline{p}}(\mu_j(n), n)), \quad j = 1, \dots, p, \\ \hat{\nu}_j(n) &= (\nu_j(n), -(2/c_{0,+})\nu_j(n)^{p_-} G_{\underline{p}}(\nu_j(n), n)), \quad j = 1, \dots, p. \end{aligned} \tag{3.41}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ are nonspecial for all $n \in \mathbb{Z}$.

Next, we shall provide an explicit representation of ϕ, Ψ, α, and β in terms of the Riemann theta function associated with $\mathcal{K}_p$. We freely employ the notation established in Appendix A. (We recall our tacit assumption $p \in \mathbb{N}$ to avoid the trivial case $p = 0$.)

Let θ denote the Riemann theta function associated with $\mathcal{K}_p$ and introduce a fixed homology basis $\{a_j, b_j\}_{j=1,\dots,p}$ on $\mathcal{K}_p$. Choosing as a convenient fixed base point one of the branch points, $Q_0 = (E_{m_0}, 0)$, the Abel maps $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ are defined by (A.20) and (A.21) and the Riemann vector Ξ_{Q_0} is given by (A.31). Let $\omega^{(3)}_{P_+,P_-}$ be the normal differential of the third kind holomorphic on $\mathcal{K}_p \setminus \{P_+, P_-\}$ with simple poles at $P_\pm$ and residues ± 1, respectively. In particular, one obtains for $\omega^{(3)}_{P_{0,-},P_{\infty_\pm}}$,

$$\omega^{(3)}_{P_{0,-},P_{\infty_\pm}} = \bigg(\frac{y + y_{0,-}}{z} \mp \prod_{j=1}^{p}(z - \lambda_{\pm,j})\bigg)\frac{dz}{2y}, \quad P_{0,-} = (0, y_{0,-}), \tag{3.42}$$

where the constants $\{\lambda_{\pm,j}\}_{j=1}^{p} \subset \mathbb{C}$ are uniquely determined by employing the normalization

$$\int_{a_j} \omega^{(3)}_{P_{0,-},P_{\infty_\pm}} = 0, \quad j = 1, \dots, p. \tag{3.43}$$

The explicit formula (3.42) then implies the following asymptotic expansions (using the local coordinate $\zeta = z$ near $P_{0,\pm}$ and $\zeta = 1/z$ near $P_{\infty_\pm}$),

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ \ln(\zeta) \end{Bmatrix} + \omega_0^{0,\pm}(P_{0,-}, P_{\infty_-}) + O(\zeta) \text{ as } P \to P_{0,\pm}, \tag{3.44}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ -\ln(\zeta) \end{Bmatrix} + \omega_0^{\infty_\pm}(P_{0,-}, P_{\infty_-}) + O(\zeta) \text{ as } P \to P_{\infty_\pm}, \tag{3.45}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_+}} \underset{\zeta\to 0}{=} \begin{Bmatrix} 0 \\ \ln(\zeta) \end{Bmatrix} + \omega_0^{0,\pm}(P_{0,-}, P_{\infty_+}) + O(\zeta) \text{ as } P \to P_{0,\pm}, \tag{3.46}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_+}} \underset{\zeta\to 0}{=} \begin{Bmatrix} -\ln(\zeta) \\ 0 \end{Bmatrix} + \omega_0^{\infty_\pm}(P_{0,-}, P_{\infty_+}) + O(\zeta) \text{ as } P \to P_{\infty_\pm}. \tag{3.47}$$

Lemma 3.5. *With* $\omega_0^{\infty_\sigma}(P_{0,-},P_{\infty_\pm})$ *and* $\omega_0^{0,\sigma'}(P_{0,-},P_{\infty_\pm})$, $\sigma,\sigma'\in\{+,-\}$, *defined as in* (3.44)–(3.47) *one has*

$$\begin{aligned}\exp\big(&\omega_0^{0,-}(P_{0,-},P_{\infty_\pm})-\omega_0^{\infty_+}(P_{0,-},P_{\infty_\pm})\\&-\omega_0^{\infty_-}(P_{0,-},P_{\infty_\pm})+\omega_0^{0,+}(P_{0,-},P_{\infty_\pm})\big)=1.\end{aligned}\tag{3.48}$$

Proof. Pick $Q_{1,\pm}=(z_1,\pm y_1)\in\mathcal{K}_p\setminus\{P_{\infty_\pm}\}$ in a neighborhood of $P_{\infty_\pm}$ and $Q_{2,\pm}=(z_2,\pm y_2)\in\mathcal{K}_p\setminus\{P_{0,\pm}\}$ in a neighborhood of $P_{0,\pm}$. Without loss of generality one may assume that P_{∞_+} and $P_{0,+}$ lie on the same sheet. Then by (3.42),

$$\begin{aligned}&\int_{Q_0}^{Q_{2,-}}\omega^{(3)}_{P_{0,-},P_{\infty_-}}-\int_{Q_0}^{Q_{1,+}}\omega^{(3)}_{P_{0,-},P_{\infty_-}}-\int_{Q_0}^{Q_{1,-}}\omega^{(3)}_{P_{0,-},P_{\infty_-}}+\int_{Q_0}^{Q_{2,+}}\omega^{(3)}_{P_{0,-},P_{\infty_-}}\\&\quad=\int_{Q_0}^{Q_{2,+}}\frac{dz}{z}-\int_{Q_0}^{Q_{1,+}}\frac{dz}{z}=\ln(z_2)-\ln(z_1)+2\pi ik,\end{aligned}\tag{3.49}$$

for some $k\in\mathbb{Z}$. On the other hand, by (3.44)–(3.47) one obtains

$$\begin{aligned}&\int_{Q_0}^{Q_{2,-}}\omega^{(3)}_{P_{0,-},P_{\infty_-}}-\int_{Q_0}^{Q_{1,+}}\omega^{(3)}_{P_{0,-},P_{\infty_-}}-\int_{Q_0}^{Q_{1,-}}\omega^{(3)}_{P_{0,-},P_{\infty_-}}+\int_{Q_0}^{Q_{2,+}}\omega^{(3)}_{P_{0,-},P_{\infty_-}}\\&\quad=\ln(z_2)+\ln(1/z_1)+\omega_0^{0,-}(P_{0,-},P_{\infty_-})-\omega_0^{\infty_+}(P_{0,-},P_{\infty_-})\\&\qquad-\omega_0^{\infty_-}(P_{0,-},P_{\infty_-})+\omega_0^{0,+}(P_{0,-},P_{\infty_-})+O(z_2)+O(1/z_1),\end{aligned}\tag{3.50}$$

and hence the part of (3.48) concerning $\omega^{(3)}_{P_{0,-},P_{\infty_-}}$ follows. The corresponding result for $\omega^{(3)}_{P_{0,-},P_{\infty_+}}$ is proved analogously. □

In the following it will be convenient to use the abbreviation

$$\begin{aligned}&\underline{z}(P,\underline{Q})=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}),\\&P\in\mathcal{K}_p,\ \underline{Q}=\{Q_1,\dots,Q_p\}\in\mathrm{Sym}^p(\mathcal{K}_p).\end{aligned}\tag{3.51}$$

We note that $\underline{z}(\,\cdot\,,\underline{Q})$ is independent of the choice of base point Q_0.

For later use we state the following result.

Lemma 3.6. *The following relations hold:*

$$\underline{z}(P_{\infty_+},\underline{\hat{\mu}}^+)=\underline{z}(P_{\infty_-},\underline{\hat{\nu}})=\underline{z}(P_{0,-},\underline{\hat{\mu}})=\underline{z}(P_{0,+},\underline{\hat{\nu}}^+),\tag{3.52}$$
$$\underline{z}(P_{\infty_+},\underline{\hat{\nu}}^+)=\underline{z}(P_{0,-},\underline{\hat{\nu}}),\quad \underline{z}(P_{0,+},\underline{\hat{\mu}}^+)=\underline{z}(P_{\infty_-},\underline{\hat{\mu}}).\tag{3.53}$$

Proof. We indicate the proof of some of the relations to be used in (3.69) and (3.70). Let $\underline{\hat{\lambda}}$ denote either $\underline{\hat{\mu}}$ or $\underline{\hat{\nu}}$. Then,

$$\begin{aligned}\underline{z}(P_{0,+},\underline{\hat{\lambda}}^+)&=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{0,+})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^+})\\&=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{0,+})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}})+\underline{A}_{P_{0,-}}(P_{\infty_+})\\&=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{\infty_-})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}})\\&=\underline{z}(P_{\infty_-},\underline{\hat{\lambda}}),\end{aligned}\tag{3.54}$$

$$\begin{aligned}\underline{z}(P_{\infty_+},\underline{\hat{\lambda}}^+)&=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{\infty_+})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^+})\\&=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{\infty_+})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}})+\underline{A}_{P_{0,-}}(P_{\infty_+})\\&=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{0,-})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}})\end{aligned}$$

$$= \underline{z}(P_{0,-}, \underline{\hat{\lambda}}), \text{ etc.} \tag{3.55}$$

Here we used $\underline{A}_{Q_0}(P^*) = -\underline{A}_{Q_0}(P)$, $P \in \mathcal{K}_p$, since Q_0 is a branch point of $\mathcal{K}_p$, and $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+})$. The latter equality immediately follows from (3.39) in the case $\underline{\hat{\lambda}} = \underline{\hat{\mu}}$ and from combining (3.13) and (3.40) in the case $\underline{\hat{\lambda}} = \underline{\hat{\nu}}$. $\square$

Given these preparations, the theta function representations of ϕ, ψ_1, ψ_2, α, and β then read as follows.

Theorem 3.7. *Suppose that α, β satisfy* (3.1) *and the pth stationary Ablowitz–Ladik system* (2.48). *Moreover, assume hypothesis* (3.2) *and* (3.3), *and let $P \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ and $(n, n_0) \in \mathbb{Z}^2$. Then for each $n \in \mathbb{Z}$, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ are nonspecial. Moreover,*

$$\phi(P,n) = C(n)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))} \exp\bigg(\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}\bigg), \tag{3.56}$$

$$\psi_1(P,n,n_0) = C(n,n_0)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))} \exp\bigg((n-n_0)\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_+}}\bigg), \tag{3.57}$$

$$\begin{aligned}\psi_2(P,n,n_0) &= C(n)C(n,n_0)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))} \\ &\quad\times \exp\bigg(\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}} + (n-n_0)\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_+}}\bigg),\end{aligned} \tag{3.58}$$

where

$$\begin{aligned}C(n) &= (-1)^{n-n_0}\exp\big((n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-}))\big) \\ &\quad\times \frac{1}{\alpha(n_0)}\exp\big(-\omega_0^{0,+}(P_{0,-},P_{\infty_-})\big)\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n_0)))},\end{aligned} \tag{3.59}$$

$$C(n,n_0) = \exp\big(-(n-n_0)\omega_0^{\infty_+}(P_{0,-},P_{\infty_+})\big)\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))}. \tag{3.60}$$

The Abel map linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \tag{3.61}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0)}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \tag{3.62}$$

in addition,

$$\begin{aligned}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) - \underline{A}_{Q_0}(P_{0,-}) + \underline{A}_{Q_0}(P_{\infty_-}) \\ &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) + \underline{A}_{P_{0,-}}(P_{\infty_-}).\end{aligned} \tag{3.63}$$

Finally, α, β are of the form

$$\begin{aligned}\alpha(n) &= \alpha(n_0)(-1)^{n-n_0}\exp\big(-(n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-}))\big) \\ &\quad\times \frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n_0)))\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n)))},\end{aligned} \tag{3.64}$$

$$\begin{aligned}\beta(n) &= \beta(n_0)(-1)^{n-n_0}\exp\big((n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-}))\big) \\ &\quad\times \frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))},\end{aligned} \tag{3.65}$$

$$\begin{aligned}\alpha(n)\beta(n) &= \exp\big(\omega_0^{\infty_+}(P_{0,-},P_{\infty_-}) - \omega_0^{0,+}(P_{0,-},P_{\infty_-})\big)\\ &\quad\times \frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))}, \qquad (3.66)\\ \Gamma(n,n_0) &= \exp\big((n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_+}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_+}))\big)\\ &\quad\times \frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n)))}. \qquad (3.67)\end{aligned}$$

Proof. Applying Abel's theorem (cf. Theorem A.1, (A.29)) to (3.13) proves (3.63), and applying it to (3.39), (3.40) results in (3.61) and (3.62). By Lemma 3.4, $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ are nonspecial. By equation (3.13) and Theorem A.3, $\phi(P,n)\exp\big(-\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}\big)$ must be of the type

$$\phi(P,n)\exp\bigg(-\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}\bigg) = C(n)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))} \qquad (3.68)$$

for some constant $C(n)$. A comparison of (3.68) and the asymptotic relations (3.33) then yields, with the help of (3.44), (3.45) and (3.52), (3.53), the following expressions for α and β:

$$\begin{aligned}(\alpha^+)^{-1} &= C^+ e^{\omega_0^{0,+}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}^+))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}^+))}\\ &= C^+ e^{\omega_0^{0,+}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}))}\\ &= -C e^{\omega_0^{\infty_-}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}))}. \qquad (3.69)\end{aligned}$$

Similarly one obtains

$$\begin{aligned}\beta^+ &= C^+ e^{\omega_0^{\infty_+}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}^+))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}^+))}\\ &= C^+ e^{\omega_0^{\infty_+}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}))}\\ &= -C e^{\omega_0^{0,-}(P_{0,-},P_{\infty_-})}\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}))}. \qquad (3.70)\end{aligned}$$

Here we used (3.61) and (3.62), more precisely,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+}), \quad \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}) + \underline{A}_{P_{0,-}}(P_{\infty_+}). \qquad (3.71)$$

Thus, one concludes

$$C(n+1) = -\exp\big[\omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-})\big]C(n), \quad n\in\mathbb{Z}, \qquad (3.72)$$

and

$$C(n+1) = -\exp\big[\omega_0^{\infty_-}(P_{0,-},P_{\infty_-}) - \omega_0^{0,+}(P_{0,-},P_{\infty_-})\big]C(n), \quad n\in\mathbb{Z}, \qquad (3.73)$$

which is consistent with (3.48). The first-order difference equation (3.72) then implies

$$C(n) = (-1)^{(n-n_0)}\exp\big[(n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-}))\big]C(n_0), \quad n,n_0\in\mathbb{Z}. \qquad (3.74)$$

Thus one infers (3.64) and (3.65). Moreover, (3.74) and taking $n = n_0$ in the first line in (3.69) yield (3.59). Dividing the first line in (3.70) by the first line in (3.69) then proves (3.66).

By (3.39) and Theorem A.3, $\psi_1(P, n, n_0)$ must be of the type (3.57). A comparison of (3.15), (3.33), and (3.57) as $P \to P_{\infty_+}$ (with local coordinate $\zeta = 1/z$) then yields

$$\psi_1(P, n, n_0) \underset{\zeta\to 0}{=} \zeta^{n_0 - n}(1 + O(\zeta)) \tag{3.75}$$

and

$$\begin{aligned}\psi_1(P, n, n_0) \underset{\zeta\to 0}{=} &\; C(n, n_0)\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n_0)))} \\ &\times \exp\big[(n - n_0)\omega_0^{\infty_+}(P_{0,-}, P_{\infty_+})\big]\zeta^{n_0 - n}(1 + O(\zeta))\end{aligned} \tag{3.76}$$

proving (3.60). Equation (3.58) is clear from (3.23), (3.56), and (3.57).

Finally, a comparison of (3.36) and (3.57) as $P \to P_{0,-}$ (with local coordinate $\zeta = z$) yields

$$\psi_1(P, n, n_0) \underset{\zeta\to 0}{=} \Gamma(n, n_0)\zeta^{n - n_0}(1 + O(\zeta)) \tag{3.77}$$

$$\begin{aligned}&\underset{\zeta\to 0}{=} C(n, n_0)\frac{\theta(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(n_0)))} \exp\big((n - n_0)\omega_0^{0,-}(P_{0,-}, P_{\infty_+})\big) \\ &\quad\times \zeta^{n - n_0}(1 + O(\zeta))\end{aligned} \tag{3.78}$$

and hence

$$\begin{aligned}\Gamma(n, n_0) &= C(n, n_0)\frac{\theta(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(n_0)))} \exp\big((n - n_0)\omega_0^{0,-}(P_{0,-}, P_{\infty_+})\big) \\ &= \exp\big((n - n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty_+}) - \omega_0^{\infty_+}(P_{0,-}, P_{\infty_+}))\big) \\ &\quad\times \frac{\theta(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(n)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(n_0)))\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n)))}\end{aligned} \tag{3.79}$$

using (3.60). ☐

We note that the apparent n_0-dependence of $C(n)$ in the right-hand side of (3.59) actually drops out to ensure the n_0-independence of ϕ in (3.56).

The theta function representations (3.64), (3.65) for α, β and that for Γ in (3.67) also show that $\gamma(n) \notin \{0, 1\}$ for all $n \in \mathbb{Z}$, and hence condition (3.1) is satisfied for the stationary algebro-geometric AL solutions discussed in this section, provided the associated divisors $\mathcal{D}_{\underline{\hat{\mu}}}(n)$ and $\mathcal{D}_{\underline{\hat{\nu}}}(n)$ stay away from $P_{\infty_\pm}, P_{0,\pm}$ for all $n \in \mathbb{Z}$.

The stationary algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy with complex-valued initial data, that is, the construction of α and β by starting from a set of initial data (nonspecial divisors) of full measure, will be presented in [41].

4. The Time-Dependent Ablowitz–Ladik Formalism

In this section we extend the algebro-geometric analysis of Section 3 to the time-dependent Ablowitz–Ladik hierarchy. For proofs of the elementary results of the time-dependent formalism we refer to [39], [40].

For most of this section we assume the following hypothesis.

Hypothesis 4.1. *(i) Suppose that* α, β *satisfy*

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,),\, \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\},\ (n,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{4.1}$$

(ii) Assume that the hyperelliptic curve $\mathcal{K}_p$ *satisfies* (3.2) *and* (3.3).

The basic problem in the analysis of algebro-geometric solutions of the Ablowitz–Ladik hierarchy consists of solving the time-dependent $\underline{r}$th Ablowitz–Ladik flow with initial data a stationary solution of the $\underline{p}$th system in the hierarchy. More precisely, given $\underline{p} \in \mathbb{N}_0^2 \backslash \{(0,0\}$ we consider a solution $\alpha^{(0)}$, $\beta^{(0)}$ of the $\underline{p}$th stationary Ablowitz–Ladik system s-$\mathrm{AL}_{\underline{p}}(\alpha^{(0)}, \beta^{(0)}) = 0$, $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \backslash \{(0,0\}$, associated with the hyperelliptic curve $\mathcal{K}_p$ and a corresponding set of summation constants $\{c_{\ell,\pm}\}_{\ell=1,\dots,p_\pm} \subset \mathbb{C}$. Next, let $\underline{r} = (r_-, r_+) \in \mathbb{N}_0^2$; we intend to construct a solution α, β of the Ablowitz–Ladik flow $\mathrm{AL}_{\underline{r}}(\alpha, \beta) = 0$ with $\alpha(t_{0,\underline{r}}) = \alpha^{(0)}$, $\beta(t_{0,\underline{r}}) = \beta^{(0)}$ for some $t_{0,\underline{r}} \in \mathbb{R}$. To emphasize that the summation constants in the definitions of the stationary and the time-dependent Ablowitz–Ladik equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde{V}_{\underline{r}}$, $\widetilde{F}_{\underline{r}}$, $\widetilde{G}_{\underline{r}}$, $\widetilde{H}_{\underline{r}}$, $\widetilde{K}_{\underline{r}}$, $\tilde{f}_{s,\pm}$, $\tilde{g}_{s,\pm}$, $\tilde{h}_{s,\pm}$, $\tilde{c}_{s,\pm}$, in order to distinguish them from $V_{\underline{p}}$, $F_{\underline{p}}$, $G_{\underline{p}}$, $H_{\underline{p}}$, $K_{\underline{p}}$, $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, $c_{\ell,\pm}$, in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual $\underline{r}$th Ablowitz–Ladik flow by a separate time variable $t_{\underline{r}} \in \mathbb{R}$.

Summing up, we are looking for solutions α, β of the time-dependent algebro-geometric initial value problem

$$\begin{aligned} &\widetilde{\mathrm{AL}}_{\underline{r}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{\underline{r}}} - \alpha(\tilde{g}_{r_+,+} + \tilde{g}^-_{r_-,-}) + \tilde{f}_{r_+-1,+} - \tilde{f}^-_{r_--1,-} \\ -i\beta_{t_{\underline{r}}} + \beta(\tilde{g}^-_{r_+,+} + \tilde{g}_{r_-,-}) - \tilde{h}_{r_--1,-} + \tilde{h}^-_{r_+-1,+} \end{pmatrix} = 0, \\ &(\alpha,\beta)\big|_{t=t_{0,\underline{r}}} = \big(\alpha^{(0)}, \beta^{(0)}\big), \end{aligned} \tag{4.2}$$

$$\text{s-AL}_{\underline{p}}\left(\alpha^{(0)}, \beta^{(0)}\right) = \begin{pmatrix} -\alpha^{(0)}(g_{p_+,+} + g^-_{p_-,-}) + f_{p_+-1,+} - f^-_{p_--1,-} \\ \beta^{(0)}(g^-_{p_+,+} + g_{p_-,-}) - h_{p_--1,-} + h^-_{p_+-1,+} \end{pmatrix} = 0 \tag{4.3}$$

for some $t_{0,\underline{r}} \in \mathbb{R}$, where $\alpha = \alpha(n, t_{\underline{r}})$, $\beta = \beta(n, t_{\underline{r}})$ satisfy (4.1) and a fixed curve $\mathcal{K}_p$ is associated with the stationary solutions $\alpha^{(0)}, \beta^{(0)}$ in (4.3). Here,

$$\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \setminus \{(0,0)\}, \quad \underline{r} = (r_-, r_+) \in \mathbb{N}_0^2, \quad p = p_- + p_+ - 1. \tag{4.4}$$

In terms of the zero-curvature formulation this amounts to solving

$$U_{t_{\underline{r}}}(z, t_{\underline{r}}) + U(z, t_{\underline{r}})\widetilde{V}_{\underline{r}}(z, t_{\underline{r}}) - \widetilde{V}_{\underline{r}}^+(z, t_{\underline{r}})U(z, t_{\underline{r}}) = 0, \tag{4.5}$$

$$U(z, t_{0,\underline{r}})V_{\underline{p}}(z, t_{0,\underline{r}}) - V_{\underline{p}}^+(z, t_{0,\underline{r}})U(z, t_{0,\underline{r}}) = 0. \tag{4.6}$$

One can show (cf. [41]) that the stationary Ablowitz–Ladik system (4.6) is actually satisfied for all times $t_{\underline{r}} \in \mathbb{R}$. Thus, we impose

$$U_{t_{\underline{r}}}(z, t_{\underline{r}}) + U(z, t_{\underline{r}})\widetilde{V}_{\underline{r}}(z, t_{\underline{r}}) - \widetilde{V}_{\underline{r}}^+(z, t_{\underline{r}})U(z, t_{\underline{r}}) = 0, \tag{4.7}$$

$$U(z, t_{\underline{r}})V_{\underline{p}}(z, t_{\underline{r}}) - V_{\underline{p}}^+(z, t_{\underline{r}})U(z, t_{\underline{r}}) = 0, \tag{4.8}$$

instead of (4.5) and (4.6). For further reference, we recall the relevant quantities here (cf. (2.34), (2.35), (2.41)–(2.43), (2.53)):

$$\begin{aligned} &U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \\ &V_{\underline{p}}(z) = i \begin{pmatrix} G_{\underline{p}}^-(z) & -F_{\underline{p}}^-(z) \\ H_{\underline{p}}^-(z) & -G_{\underline{p}}^-(z) \end{pmatrix}, \quad \widetilde{V}_{\underline{r}}(z) = i \begin{pmatrix} \widetilde{G}_{\underline{r}}^-(z) & -\widetilde{F}_{\underline{r}}^-(z) \\ \widetilde{H}_{\underline{r}}^-(z) & -\widetilde{K}_{\underline{r}}^-(z) \end{pmatrix}, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} F_{\underline{p}}(z) &= \sum_{\ell=1}^{p_-} f_{p_- -\ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+ -1} f_{p_+ -1-\ell,+} z^{\ell} = -c_{0,+}\alpha^+ z^{-p_-} \prod_{j=1}^{p} (z-\mu_j), \\ G_{\underline{p}}(z) &= \sum_{\ell=1}^{p_-} g_{p_- -\ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+} g_{p_+ -\ell,+} z^{\ell}, \\ H_{\underline{p}}(z) &= \sum_{\ell=0}^{p_- -1} h_{p_- -1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} h_{p_+ -\ell,+} z^{\ell} = c_{0,+}\beta z^{-p_- +1} \prod_{j=1}^{p} (z-\nu_j), \\ \widetilde{F}_{\underline{r}}(z) &= \sum_{s=1}^{r_-} \tilde{f}_{r_- -s,-} z^{-s} + \sum_{s=0}^{r_+ -1} \tilde{f}_{r_+ -1-s,+} z^{s}, \\ \widetilde{G}_{\underline{r}}(z) &= \sum_{s=1}^{r_-} \tilde{g}_{r_- -s,-} z^{-s} + \sum_{s=0}^{r_+} \tilde{g}_{r_+ -s,+} z^{s}, \\ \widetilde{H}_{\underline{r}}(z) &= \sum_{s=0}^{r_- -1} \tilde{h}_{r_- -1-s,-} z^{-s} + \sum_{s=1}^{r_+} \tilde{h}_{r_+ -s,+} z^{s}, \\ \widetilde{K}_{\underline{r}}(z) &= \sum_{s=0}^{r_-} \tilde{g}_{r_- -s,-} z^{-s} + \sum_{s=1}^{r_+} \tilde{g}_{r_+ -s,+} z^{s} = \widetilde{G}_{\underline{r}}(z) + \tilde{g}_{r_-,-} - \tilde{g}_{r_+,+} \end{aligned} \tag{4.10}$$

for fixed $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, $\underline{r} \in \mathbb{N}_0^2$. Here $f_{\ell,\pm}$, $\tilde{f}_{s,\pm}$, $g_{\ell,\pm}$, $\tilde{g}_{s,\pm}$, $h_{\ell,\pm}$, and $\tilde{h}_{s,\pm}$ are defined as in (2.5)–(2.12) with appropriate sets of summation constants $c_{\ell,\pm}$, $\ell \in \mathbb{N}_0$, and $\tilde{c}_{k,\pm}$, $k \in \mathbb{N}_0$. Explicitly, (4.7) and (4.8) are equivalent to (cf. (2.37)–(2.40), (2.60)–(2.63)),

$$\alpha_{t_{\underline{r}}} = i\big(z\widetilde{F}_{\underline{r}}^- + \alpha(\widetilde{G}_{\underline{r}} + \widetilde{K}_{\underline{r}}^-) - \widetilde{F}_{\underline{r}}\big), \tag{4.11}$$

$$\beta_{t_{\underline{r}}} = -i\big(\beta(\widetilde{G}_{\underline{r}}^- + \widetilde{K}_{\underline{r}}) - \widetilde{H}_{\underline{r}} + z^{-1}\widetilde{H}_{\underline{r}}^-\big), \tag{4.12}$$

$$0 = z(\widetilde{G}_{\underline{r}}^- - \widetilde{G}_{\underline{r}}) + z\beta\widetilde{F}_{\underline{r}} + \alpha\widetilde{H}_{\underline{r}}^-, \tag{4.13}$$

$$0 = z\beta\widetilde{F}_{\underline{r}}^- + \alpha\widetilde{H}_{\underline{r}} + \widetilde{K}_{\underline{r}}^- - \widetilde{K}_{\underline{r}}, \tag{4.14}$$

$$0 = z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^-, \tag{4.15}$$

$$0 = z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} - G_{\underline{p}} + G_{\underline{p}}^-, \tag{4.16}$$

$$0 = -F_{\underline{p}} + zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + G_{\underline{p}}^-), \tag{4.17}$$

$$0 = z\beta(G_{\underline{p}} + G_{\underline{p}}^-) - zH_{\underline{p}} + H_{\underline{p}}^-, \tag{4.18}$$

respectively. In particular, (2.54) holds in the present $t_{\underline{r}}$-dependent setting, that is,

$$G_{\underline{p}}^2 - F_{\underline{p}}H_{\underline{p}} = R_{\underline{p}}. \tag{4.19}$$

As in the stationary context (3.8), (3.9) we introduce

$$\hat{\mu}_j(n,t_{\underline{r}}) = (\mu_j(n,t_{\underline{r}}), (2/c_{0,+})\mu_j(n,t_{\underline{r}})^{p_-} G_{\underline{p}}(\mu_j(n,t_{\underline{r}}), n, t_{\underline{r}})) \in \mathcal{K}_p, \\ j=1,\dots,p,\ (n,t_{\underline{r}}) \in \mathbb{Z}\times\mathbb{R}, \tag{4.20}$$

and

$$\hat{\nu}_j(n,t_{\underline{r}}) = (\nu_j(n,t_{\underline{r}}), -(2/c_{0,+})\nu_j(n,t_{\underline{r}})^{p_-} G_{\underline{p}}(\nu_j(n,t_{\underline{r}}), n, t_{\underline{r}})) \in \mathcal{K}_p, \\ j=1,\dots,p,\ (n,t_{\underline{r}}) \in \mathbb{Z}\times\mathbb{R}, \tag{4.21}$$

and note that the regularity assumptions (4.1) on α, β imply continuity of μ_j and ν_k with respect to $t_{\underline{r}} \in \mathbb{R}$ (away from collisions of these zeros, μ_j and ν_k are of course C^∞).

In analogy to (3.11), (3.12), one defines the following meromorphic function $\phi(\,\cdot\,, n, t_{\underline{r}})$ on $\mathcal{K}_p$,

$$\phi(P,n,t_{\underline{r}}) = \frac{(c_{0,+}/2)z^{-p_-}y + G_{\underline{p}}(z,n,t_{\underline{r}})}{F_{\underline{p}}(z,n,t_{\underline{r}})} \tag{4.22}$$

$$= \frac{-H_{\underline{p}}(z,n,t_{\underline{r}})}{(c_{0,+}/2)z^{-p_-}y - G_{\underline{p}}(z,n,t_{\underline{r}})}, \tag{4.23}$$

$$P=(z,y)\in\mathcal{K}_p,\ (n,t_{\underline{r}})\in\mathbb{Z}\times\mathbb{R},$$

with divisor $(\phi(\,\cdot\,,n,t_{\underline{r}}))$ of $\phi(\,\cdot\,,n,t_{\underline{r}})$ given by

$$(\phi(\,\cdot\,,n,t_{\underline{r}})) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n,t_{\underline{r}})} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n,t_{\underline{r}})}. \tag{4.24}$$

The time-dependent Baker–Akhiezer vector is then defined in terms of ϕ by

$$\Psi(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \begin{pmatrix} \psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) \\ \psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) \end{pmatrix}, \tag{4.25}$$

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \exp\bigg(i\int_{t_{0,\underline{r}}}^{t_{\underline{r}}} ds\big(\widetilde{G}_{\underline{r}}(z,n_0,s) - \widetilde{F}_{\underline{r}}(z,n_0,s)\phi(P,n_0,s)\big)\bigg) \\ \times \begin{cases} \prod_{n'=n_0+1}^{n} \big(z+\alpha(n',t_{\underline{r}})\phi^-(P,n',t_{\underline{r}})\big), & n \ge n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z+\alpha(n',t_{\underline{r}})\phi^-(P,n',t_{\underline{r}})\big)^{-1}, & n \le n_0-1, \end{cases} \tag{4.26}$$

$$\psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \exp\bigg(i\int_{t_{0,\underline{r}}}^{t_{\underline{r}}} ds\big(\widetilde{G}_{\underline{r}}(z,n_0,s) - \widetilde{F}_{\underline{r}}(z,n_0,s)\phi(P,n_0,s)\big)\bigg) \\ \times \phi(P,n_0,t_{\underline{r}}) \begin{cases} \prod_{n'=n_0+1}^{n} \big(z\beta(n',t_{\underline{r}})\phi^-(P,n',t_{\underline{r}})^{-1}+1\big), & n \ge n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z\beta(n',t_{\underline{r}})\phi^-(P,n',t_{\underline{r}})^{-1}+1\big)^{-1}, & n \le n_0-1, \end{cases} \tag{4.27}$$

$$P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,t_{\underline{r}})\in\mathbb{Z}\times\mathbb{R}.$$

One observes that

$$\psi_1(P,n,n_0,t_{\underline{r}},\tilde{t}_{\underline{r}}) = \psi_1(P,n_0,n_0,t_{\underline{r}},\tilde{t}_{\underline{r}})\psi_1(P,n,n_0,t_{\underline{r}},t_{\underline{r}}), \\ P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,n_0,t_{\underline{r}},\tilde{t}_{\underline{r}})\in\mathbb{Z}^2\times\mathbb{R}^2. \tag{4.28}$$

The following lemma records basic properties of ϕ and Ψ in analogy to the stationary case discussed in Lemma 3.1.

Lemma 4.2 ([40])**.** *Assume Hypothesis* 4.1 *and suppose that* (4.7), (4.8) *hold. In addition, let* $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$, $(n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) \in \mathbb{Z}^2 \times \mathbb{R}^2$. *Then* ϕ *satisfies*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{4.29}$$

$$\phi_{t_{\underline{r}}}(P) = i\widetilde{F}_{\underline{r}}\phi^2(P) - i\big(\widetilde{G}_{\underline{r}}(z) + \widetilde{K}_{\underline{r}}(z)\big)\phi(P) + i\widetilde{H}_{\underline{r}}(z), \tag{4.30}$$

$$\phi(P)\phi(P^*) = \frac{H_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \tag{4.31}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \tag{4.32}$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p_-}\frac{y(P)}{F_{\underline{p}}(z)}. \tag{4.33}$$

Moreover, assuming $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, *then* Ψ *satisfies*

$$\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) = \phi(P, n, t_{\underline{r}})\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}), \tag{4.34}$$

$$U(z)\Psi^-(P) = \Psi(P), \tag{4.35}$$

$$V_{\underline{p}}(z)\Psi^-(P) = -(i/2)c_{0,+}z^{-p_-}y\Psi^-(P), \tag{4.36}$$

$$\Psi_{t_{\underline{r}}}(P) = \widetilde{V}_{\underline{r}}^+(z)\Psi(P), \tag{4.37}$$

$$\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_1(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) = z^{n-n_0}\frac{F_{\underline{p}}(z, n, t_{\underline{r}})}{F_{\underline{p}}(z, n_0, t_{0,\underline{r}})}\Gamma(n, n_0, t_{\underline{r}}), \tag{4.38}$$

$$\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) = z^{n-n_0}\frac{H_{\underline{p}}(z, n, t_{\underline{r}})}{F_{\underline{p}}(z, n_0, t_{0,\underline{r}})}\Gamma(n, n_0, t_{\underline{r}}), \tag{4.39}$$

$$\begin{aligned}&\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) + \psi_1(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\\ &\quad = 2z^{n-n_0}\frac{G_{\underline{p}}(z, n, t_{\underline{r}})}{F_{\underline{p}}(z, n_0, t_{0,\underline{r}})}\Gamma(n, n_0, t_{\underline{r}}),\end{aligned} \tag{4.40}$$

$$\begin{aligned}&\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) - \psi_1(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\\ &\quad = -c_{0,+}z^{n-n_0-p_-}\frac{y}{F_{\underline{p}}(z, n_0, t_{0,\underline{r}})}\Gamma(n, n_0, t_{\underline{r}}),\end{aligned} \tag{4.41}$$

where

$$\Gamma(n, n_0, t_{\underline{r}}) = \begin{cases}\prod_{n'=n_0+1}^{n}\gamma(n', t_{\underline{r}}), & n \geq n_0 + 1,\\ 1, & n = n_0,\\ \prod_{n'=n+1}^{n_0}\gamma(n', t_{\underline{r}})^{-1}, & n \leq n_0 - 1.\end{cases} \tag{4.42}$$

In addition, as long as the zeros $\mu_j(n_0, s)$ *of* $(\,\cdot\,)^{p_-}F_{\underline{p}}(\,\cdot\,, n_0, s)$ *are all simple and distinct from zero for* $s \in \mathcal{I}_\mu$, $\mathcal{I}_\mu \subseteq \mathbb{R}$ *an open interval,* $\Psi(\,\cdot\,, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})$ *is meromorphic on* $\mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ *for* $(n, t_{\underline{r}}, t_{0,\underline{r}}) \in \mathbb{Z} \times \mathcal{I}_\mu^2$.

Next we consider the $t_{\underline{r}}$-dependence of $F_{\underline{p}}$, $G_{\underline{p}}$, and $H_{\underline{p}}$.

Lemma 4.3 ([40])**.** *Assume Hypothesis* 4.1 *and suppose that* (4.7), (4.8) *hold. In addition, let* $(z, n, t_{\underline{r}}) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{R}$. *Then,*

$$F_{\underline{p},t_{\underline{r}}} = -2iG_{\underline{p}}\widetilde{F}_{\underline{r}} + i\big(\widetilde{G}_{\underline{r}} + \widetilde{K}_{\underline{r}}\big)F_{\underline{p}}, \tag{4.43}$$

$$G_{\underline{p},t_{\underline{r}}} = iF_{\underline{p}}\widetilde{H}_{\underline{r}} - iH_{\underline{p}}\widetilde{F}_{\underline{r}}, \tag{4.44}$$

$$H_{\underline{p},t_{\underline{r}}} = 2iG_{\underline{p}}\widetilde{H}_{\underline{r}} - i\big(\widetilde{G}_{\underline{r}} + \widetilde{K}_{\underline{r}}\big)H_{\underline{p}}. \tag{4.45}$$

In particular, (4.43)–(4.45) *are equivalent to*

$$V_{\underline{p},t_{\underline{r}}} = \big[\widetilde{V}_{\underline{r}}, V_{\underline{p}}\big]. \tag{4.46}$$

Next we turn to the Dubrovin equations for the time variation of the zeros μ_j of $(\,\cdot\,)^{p_-}F_{\underline{p}}$ and ν_j of $(\,\cdot\,)^{p_- -1}H_{\underline{p}}$ governed by the $\widetilde{\mathrm{AL}}_{\underline{r}}$ flow.

Lemma 4.4 ([40])**.** *Assume Hypothesis* 4.1 *and suppose that* (4.7), (4.8) *hold on* $\mathbb{Z}\times\mathcal{I}_\mu$ *with* $\mathcal{I}_\mu\subseteq\mathbb{R}$ *an open interval. In addition, assume that the zeros* μ_j, $j=1,\dots,p$, *of* $(\,\cdot\,)^{p_-}F_{\underline{p}}(\,\cdot\,)$ *remain distinct and nonzero on* $\mathbb{Z}\times\mathcal{I}_\mu$*. Then* $\{\hat\mu_j\}_{j=1,\dots,p}$, *defined in* (4.20), *satisfies the following first-order system of differential equations on* $\mathbb{Z}\times\mathcal{I}_\mu$,

$$\mu_{j,t_{\underline{r}}} = -i\widetilde{F}_{\underline{r}}(\mu_j)y(\hat\mu_j)(\alpha^+)^{-1}\prod_{\substack{k=1\\k\neq j}}^{p}(\mu_j-\mu_k)^{-1}, \quad j=1,\dots,p, \tag{4.47}$$

with

$$\hat\mu_j(n,\cdot)\in C^\infty(\mathcal{I}_\mu,\mathcal{K}_p), \quad j=1,\dots,p,\ n\in\mathbb{Z}. \tag{4.48}$$

For the zeros ν_j, $j=1,\dots,p$, *of* $(\,\cdot\,)^{p_- -1}H_{\underline{p}}(\,\cdot\,)$, *identical statements hold with* μ_j *and* $\mathcal{I}_\mu$ *replaced by* ν_j *and* $\mathcal{I}_\nu$, *etc.* (*with* $\mathcal{I}_\nu\subseteq\mathbb{R}$ *an open interval*)*. In particular,* $\{\hat\nu_j\}_{j=1,\dots,p}$, *defined in* (4.21), *satisfies the first-order system on* $\mathbb{Z}\times\mathcal{I}_\nu$,

$$\nu_{j,t_{\underline{r}}} = i\widetilde{H}_{\underline{r}}(\nu_j)y(\hat\nu_j)(\beta\nu_j)^{-1}\prod_{\substack{k=1\\k\neq j}}^{p}(\nu_j-\nu_k)^{-1}, \quad j=1,\dots,p, \tag{4.49}$$

with

$$\hat\nu_j(n,\cdot)\in C^\infty(\mathcal{I}_\nu,\mathcal{K}_p), \quad j=1,\dots,p,\ n\in\mathbb{Z}. \tag{4.50}$$

When attempting to solve the Dubrovin systems (4.47) and (4.49), they must be augmented with appropriate divisors $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,\underline{r}})}\in\operatorname{Sym}^p\mathcal{K}_p$, $t_{0,\underline{r}}\in\mathcal{I}_\mu$, and $\mathcal{D}_{\underline{\hat\nu}(n_0,t_{0,\underline{r}})}\in\operatorname{Sym}^p\mathcal{K}_p$, $t_{0,\underline{r}}\in\mathcal{I}_\nu$, as initial conditions. The algebro-geometric initial value problem for the AL hierarchy with appropriate initial divisors will be discussed in detail in [41].

Next, we turn to the asymptotic expansions of ϕ and Ψ in a neighborhood of $P_{\infty_\pm}$ and $P_{0,\pm}$. Since this is a bit more involved we provide some details.

Lemma 4.5. *Assume Hypothesis* 4.1 *and suppose that* (4.7), (4.8) *hold. Moreover, let* $P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\}$, $(n,n_0,t_{\underline{r}},t_{0,\underline{r}})\in\mathbb{Z}^2\times\mathbb{R}^2$*. Then* ϕ *has the asymptotic behavior*

$$\phi(P) \underset{\zeta\to0}{=} \begin{cases}\beta+\beta^-\gamma\zeta+O(\zeta^2), & P\to P_{\infty_+},\\ -(\alpha^+)^{-1}\zeta^{-1}+(\alpha^+)^{-2}\alpha^{++}\gamma^++O(\zeta), & P\to P_{\infty_-},\end{cases} \quad \zeta=1/z, \tag{4.51}$$

$$\phi(P) \underset{\zeta\to0}{=} \begin{cases}\alpha^{-1}-\alpha^{-2}\alpha^-\gamma\zeta+O(\zeta^2), & P\to P_{0,+},\\ -\beta^+\zeta-\beta^{++}\gamma^+\zeta^2+O(\zeta^3), & P\to P_{0,-},\end{cases} \quad \zeta=z. \tag{4.52}$$

The component ψ_1 *of the Baker–Akhiezer vector* Ψ *has the asymptotic behavior*

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) \underset{\zeta\to0}{=} \exp\bigg(\pm\frac{i}{2}(t_{\underline{r}}-t_{0,\underline{r}})\sum_{s=0}^{r_+}\tilde c_{r_+-s,+}\zeta^{-s}\bigg)(1+O(\zeta))$$

$$\times \begin{cases} \zeta^{n_0-n}, & P \to P_{\infty_+}, \\ \Gamma(n,n_0,t_{\underline{r}})\frac{\alpha^+(n,t_{\underline{r}})}{\alpha^+(n_0,t_{0,\underline{r}})} & \\ \times \exp\Big(i\int_{t_{0,\underline{r}}}^{t_{\underline{r}}} ds\big(\tilde g_{r_+,+}(n_0,s)-\tilde g_{r_-,-}(n_0,s)\big)\Big), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{4.53}$$

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) \underset{\zeta\to 0}{=} \exp\bigg(\pm\frac{i}{2}(t_{\underline{r}}-t_{0,\underline{r}})\sum_{s=0}^{r_-}\tilde c_{r_--s,-}\zeta^{-s}\bigg)(1+O(\zeta))$$

$$\times \begin{cases} \frac{\alpha(n,t_{\underline{r}})}{\alpha(n_0,t_{0,\underline{r}})}, & P \to P_{0,+}, \\ \Gamma(n,n_0,t_{\underline{r}})\zeta^{n-n_0} & \\ \times \exp\Big(i\int_{t_{0,\underline{r}}}^{t_{\underline{r}}} ds\big(\tilde g_{r_+,+}(n_0,s)-\tilde g_{r_-,-}(n_0,s)\big)\Big), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{4.54}$$

Proof. Since by the definition of ϕ in (4.22) the time parameter $t_{\underline{r}}$ can be viewed as an additional but fixed parameter, the asymptotic behavior of ϕ remains the same as in Lemma 3.3. Similarly, also the asymptotic behavior of $\psi_1(P,n,n_0,t_{\underline{r}},t_{\underline{r}})$ is derived in an identical fashion to that in Lemma 3.3. This proves (4.53) and (4.54) for $t_{0,\underline{r}} = t_{\underline{r}}$, that is,

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{\underline{r}}) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{n_0-n}(1+O(\zeta)), & P \to P_{\infty_+}, \\ \Gamma(n,n_0,t_{\underline{r}})\frac{\alpha^+(n,t_{\underline{r}})}{\alpha^+(n_0,t_{\underline{r}})} + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \tag{4.55}$$

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{\underline{r}}) \underset{\zeta\to 0}{=} \begin{cases} \frac{\alpha(n,t_{\underline{r}})}{\alpha(n_0,t_{\underline{r}})} + O(\zeta), & P \to P_{0,+}, \\ \Gamma(n,n_0,t_{\underline{r}})\zeta^{n-n_0}(1+O(\zeta)), & P \to P_{0,-}, \end{cases} \quad \zeta = z. \tag{4.56}$$

Remaining to be investigated is

$$\psi_1(P,n_0,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \exp\bigg(i\int_{t_{0,\underline{r}}}^{t_{\underline{r}}} dt\big(\widetilde G_{\underline{r}}(z,n_0,t) - \widetilde F_{\underline{r}}(z,n_0,t)\phi(P,n_0,t)\big)\bigg). \tag{4.57}$$

The asymptotic expansion of the integrand is derived using Theorem B.1. Focusing on the homogeneous coefficients first, one computes as $P \to P_{\infty_\pm}$,

$$\begin{aligned} \widehat G_{s,+} - \widehat F_{s,+}\phi &= \widehat G_{s,+} - \widehat F_{s,+}\frac{G_{\underline{p}} + (c_{0,+}/2)z^{-p_-}y}{F_{\underline{p}}} \\ &= \widehat G_{s,+} - \widehat F_{s,+}\bigg(\frac{2z^{p_-}}{c_{0,+}}\frac{G_{\underline{p}}}{y}+1\bigg)\bigg(\frac{2z^{p_-}}{c_{0,+}}\frac{F_{\underline{p}}}{y}\bigg)^{-1} \\ &\underset{\zeta\to 0}{=} \pm\frac{1}{2}\zeta^{-s} + \frac{\hat g_{0,+} \mp \frac{1}{2}}{\hat f_{0,+}}\hat f_{s,+} + O(\zeta), \quad P \to P_{\infty_\pm},\ \zeta = 1/z. \end{aligned} \tag{4.58}$$

Since

$$\widetilde F_{\underline{r}} \underset{\zeta\to 0}{=} \sum_{s=0}^{r_+}\tilde c_{r_+-s,+}\widehat F_{s,+} + O(\zeta), \quad \widetilde G_{\underline{r}} \underset{\zeta\to 0}{=} \sum_{s=0}^{r_+}\tilde c_{r_+-s,+}\widehat G_{s,+} + O(\zeta), \tag{4.59}$$

one infers from (4.51)

$$\widetilde{G}_{\underline{r}} - \widetilde{F}_{\underline{r}}\phi \underset{\zeta\to 0}{=} \frac{1}{2}\sum_{s=0}^{r_+} \tilde{c}_{r_+-s,+}\zeta^{-s} + O(\zeta), \quad P \to P_{\infty_+},\ \zeta = 1/z. \tag{4.60}$$

Insertion of (4.60) into (4.57) then proves (4.53) as $P \to P_{\infty_+}$.

As $P \to P_{\infty_-}$, we need one additional term in the asymptotic expansion of $\widetilde{F}_{\underline{r}}$, that is, we will use

$$\widetilde{F}_{\underline{r}} \underset{\zeta\to 0}{=} \sum_{s=0}^{r_+} \tilde{c}_{r_+-s,+}\widehat{F}_{s,+} + \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-}\hat{f}_{s-1,-}\zeta + O(\zeta^2). \tag{4.61}$$

This then yields

$$\widetilde{G}_{\underline{r}} - \widetilde{F}_{\underline{r}}\phi \underset{\zeta\to 0}{=} -\frac{1}{2}\sum_{s=0}^{r_+} \tilde{c}_{r_+-s,+}\zeta^{-s} - (\alpha^+)^{-1}(\tilde{f}_{r_+,+} - \tilde{f}_{r_--1,-}) + O(\zeta). \tag{4.62}$$

Invoking (2.7) and (4.2) one concludes that

$$\tilde{f}_{r_--1,-} - \tilde{f}_{r_+,+} = -i\alpha^+_{t_{\underline{r}}} + \alpha^+(\tilde{g}_{r_+,+} - \tilde{g}_{r_-,-}) \tag{4.63}$$

and hence

$$\widetilde{G}_{\underline{r}} - \widetilde{F}_{\underline{r}}\phi \underset{\zeta\to 0}{=} -\frac{1}{2}\sum_{s=0}^{r_+} \tilde{c}_{r_+-s,+}\zeta^{-s} - \frac{i\alpha^+_{t_{\underline{r}}}}{\alpha^+} + \tilde{g}_{r_+,+} - \tilde{g}_{r_-,-} + O(\zeta). \tag{4.64}$$

Insertion of (4.64) into (4.57) then proves (4.53) as $P \to P_{\infty_-}$.

Using Theorem B.1 again, one obtains in the same manner as $P \to P_{0,\pm}$,

$$\widehat{G}_{s,-} - \widehat{F}_{s,-}\phi \underset{\zeta\to 0}{=} \pm\frac{1}{2}\zeta^{-s} - \hat{g}_{s,-} + \frac{\hat{g}_{0,-} \pm \frac{1}{2}}{\hat{f}_{0,-}}\hat{f}_{s,-} + O(\zeta). \tag{4.65}$$

Since

$$\widetilde{F}_{\underline{r}} \underset{\zeta\to 0}{=} \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-}\widehat{F}_{s,-} + \tilde{f}_{r_+-1,+} + O(\zeta), \quad P \to P_{0,\pm},\ \zeta = z, \tag{4.66}$$

$$\widetilde{G}_{\underline{r}} \underset{\zeta\to 0}{=} \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-}\widehat{G}_{s,-} + \tilde{g}_{r_+,+} + O(\zeta), \quad P \to P_{0,\pm},\ \zeta = z, \tag{4.67}$$

(4.65)–(4.67) yield

$$\widetilde{G}_{\underline{r}} - \widetilde{F}_{\underline{r}}\phi \underset{\zeta\to 0}{=} \pm\frac{1}{2}\sum_{s=0}^{r_-} \tilde{c}_{r_--s,-}\zeta^{-s} + \tilde{g}_{r_+,+} - \tilde{g}_{r_-,-} - \frac{\hat{g}_{0,-} \pm \frac{1}{2}}{\hat{f}_{0,-}}(\tilde{f}_{r_+-1,+} - \tilde{f}_{r_-,-}) + O(\zeta), \tag{4.68}$$

where we again used (4.52), (2.19), and (4.2). As $P \to P_{0,-}$, one thus obtains

$$\widetilde{G}_{\underline{r}} - \widetilde{F}_{\underline{r}}\phi \underset{\zeta\to 0}{=} -\frac{1}{2}\sum_{s=0}^{r_-} \tilde{c}_{r_--s,-}\zeta^{-s} + \tilde{g}_{r_+,+} - \tilde{g}_{r_-,-}, \quad P \to P_{0,-},\ \zeta = z. \tag{4.69}$$

Insertion of (4.69) into (4.57) then proves (4.54) as $P \to P_{0,-}$.

As $P \to P_{0,+}$, one obtains

$$\widetilde{G}_{\underline{r}} - \widetilde{F}_{\underline{r}}\phi \underset{\zeta\to 0}{=} \frac{1}{2}\sum_{s=0}^{r_-} \tilde{c}_{r_--s,-}\zeta^{-s} + \tilde{g}_{r_+,+} - \tilde{g}_{r_-,-} - \frac{1}{\alpha}(\tilde{f}_{r_+-1,+} - \tilde{f}_{r_-,-}) + O(\zeta),$$

$$\underset{\zeta\to 0}{=} \frac{1}{2}\sum_{s=0}^{r_-}\tilde c_{r_--s,-}\zeta^{-s} - \frac{i\alpha_{t_{\underline{r}}}}{\alpha} + O(\zeta), \quad P\to P_{0,+},\ \zeta=z, \tag{4.70}$$

using $\tilde f_{r_-,-} = \tilde f^-_{r_--1,-} + \alpha(\tilde g_{r_-,-} - \tilde g^-_{r_-,-})$ (cf. (2.11)) and (4.2). Insertion of (4.70) into (4.57) then proves (4.54) as $P\to P_{0,+}$. □

Next, we note that Lemmas 3.2 and 3.4 on trace formulas and nonspecial divisors in the stationary context immediately extend to the present time-dependent situation since $t_{\underline{r}}\in\mathbb{R}$ just plays the role of a parameter. We thus omit the details.

Finally, we turn to the principal result of this section, the representation of ϕ, Ψ, α, and β in terms of the Riemann theta function associated with $\mathcal{K}_p$, assuming $p\in\mathbb{N}$ for the remainder of this section.

In addition to (3.42)–(3.48), let $\omega^{(2)}_{P_{\infty_\pm},q}$ and $\omega^{(2)}_{P_{0,\pm},q}$ be the normalized differentials of the second kind with a unique pole at $P_{\infty_\pm}$ and $P_{0,\pm}$, respectively, and principal parts

$$\omega^{(2)}_{P_{\infty_\pm},q} \underset{\zeta\to 0}{=} \big(\zeta^{-2-q}+O(1)\big)d\zeta, \quad P\to P_{\infty_\pm},\ \zeta=1/z,\ q\in\mathbb{N}_0, \tag{4.71}$$

$$\omega^{(2)}_{P_{0,\pm},q} \underset{\zeta\to 0}{=} \big(\zeta^{-2-q}+O(1)\big)d\zeta, \quad P\to P_{0,\pm},\ \zeta=z,\ q\in\mathbb{N}_0, \tag{4.72}$$

with vanishing a-periods,

$$\int_{a_j}\omega^{(2)}_{P_{\infty_\pm},q} = \int_{a_j}\omega^{(2)}_{P_{0,\pm},q} = 0, \quad j=1,\dots,p. \tag{4.73}$$

Moreover, we define

$$\begin{aligned}\widetilde\Omega^{(2)}_{\underline{r}} = \frac{i}{2}\bigg(&\sum_{s=1}^{r_-} s\tilde c_{r_--s,-}\big(\omega^{(2)}_{P_{0,+},s-1}-\omega^{(2)}_{P_{0,-},s-1}\big)\\ &+\sum_{s=1}^{r_+} s\tilde c_{r_+-s,+}\big(\omega^{(2)}_{P_{\infty_+},s-1}-\omega^{(2)}_{P_{\infty_-},s-1}\big)\bigg),\end{aligned} \tag{4.74}$$

where $\tilde c_{\ell,\pm}$ are the summation constants in $\widetilde F_{\underline{r}}$. The corresponding vector of b-periods of $\widetilde\Omega^{(2)}_{\underline{r}}/(2\pi i)$ is then denoted by

$$\underline{\widetilde U}^{(2)}_{\underline{r}} = \big(\widetilde U^{(2)}_{\underline{r},1},\dots,\widetilde U^{(2)}_{\underline{r},p}\big), \quad \widetilde U^{(2)}_{\underline{r},j} = \frac{1}{2\pi i}\int_{b_j}\widetilde\Omega^{(2)}_{\underline{r}}, \quad j=1,\dots,p. \tag{4.75}$$

Finally, we abbreviate

$$\widetilde\Omega^{\infty_\pm}_{\underline{r}} = \lim_{P\to P_{\infty_\pm}}\bigg(\int_{Q_0}^P\widetilde\Omega^{(2)}_{\underline{r}} \pm \frac{i}{2}\sum_{s=0}^{r_+}\tilde c_{r_+-s,+}\zeta^{-s}\bigg), \tag{4.76}$$

$$\widetilde\Omega^{0,\pm}_{\underline{r}} = \lim_{P\to P_{0,\pm}}\bigg(\int_{Q_0}^P\widetilde\Omega^{(2)}_{\underline{r}} \pm \frac{i}{2}\sum_{s=0}^{r_-}\tilde c_{r_--s,-}\zeta^{-s}\bigg). \tag{4.77}$$

Theorem 4.6. *Assume Hypothesis* 4.1 *and suppose that* (4.7), (4.8) *hold. In addition, let $P\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\}$ and $(n,n_0,t_{\underline{r}},t_{0,\underline{r}})\in\mathbb{Z}^2\times\mathbb{R}^2$. Then for each $(n,t_{\underline{r}})\in\mathbb{Z}\times\mathbb{R}$, $\mathcal{D}_{\underline{\hat\mu}(n,t_{\underline{r}})}$ and $\mathcal{D}_{\underline{\hat\nu}(n,t_{\underline{r}})}$ are nonspecial. Moreover,*

$$\phi(P,n,t_{\underline{r}}) = C(n,t_{\underline{r}})\frac{\theta(\underline{z}(P,\underline{\hat\nu}(n,t_{\underline{r}})))}{\theta(\underline{z}(P,\underline{\hat\mu}(n,t_{\underline{r}})))}\exp\bigg(\int_{Q_0}^P\omega^{(3)}_{P_{0,-},P_{\infty_-}}\bigg), \tag{4.78}$$

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = C(n,n_0,t_{\underline{r}},t_{0,\underline{r}})\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_{\underline{r}})))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})))} \times \exp\bigg((n-n_0)\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_+}} - (t_{\underline{r}}-t_{0,\underline{r}})\int_{Q_0}^P \widetilde{\Omega}^{(2)}_{\underline{r}}\bigg), \tag{4.79}$$

$$\psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = C(n,t_{\underline{r}})C(n,n_0,t_{\underline{r}},t_{0,\underline{r}})\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n,t_{\underline{r}})))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})))} \times \exp\bigg(\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}} + (n-n_0)\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_+}} - (t_{\underline{r}}-t_{0,\underline{r}})\int_{Q_0}^P \widetilde{\Omega}^{(2)}_{\underline{r}}\bigg), \tag{4.80}$$

where

$$C(n,t_{\underline{r}}) = \frac{(-1)^{n-n_0}}{\alpha(n_0,t_{\underline{r}})}\exp\big((n-n_0)(\omega_0^{0,-}(P_{0,-},P_{\infty_-}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_-}))\big) \times \exp\big(-\omega_0^{0,+}(P_{0,-},P_{\infty_-})\big)\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n_0,t_{\underline{r}})))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n_0,t_{\underline{r}})))}, \tag{4.81}$$

$$C(n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n,t_{\underline{r}})))} \times \exp\big((t_{\underline{r}}-t_{0,\underline{r}})\widetilde{\Omega}^{\infty_+}_{\underline{r}} - (n-n_0)\omega_0^{\infty_+}(P_{0,-},P_{\infty_+})\big). \tag{4.82}$$

The Abel map linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n,t_{\underline{r}})}$ *in the sense that*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + (t_{\underline{r}}-t_{0,\underline{r}})\underline{\widetilde{U}}^{(2)}_{\underline{r}}, \tag{4.83}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n,t_{\underline{r}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0,t_{0,\underline{r}})}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + (t_{\underline{r}}-t_{0,\underline{r}})\underline{\widetilde{U}}^{(2)}_{\underline{r}}. \tag{4.84}$$

Finally, α, β *are of the form*

$$\alpha(n,t_{\underline{r}}) = \alpha(n_0,t_{0,\underline{r}})\exp\big((n-n_0)(\omega_0^{0,+}(P_{0,-},P_{\infty_+}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_+}))\big) \times \exp\big((t_{\underline{r}}-t_{0,\underline{r}})(\widetilde{\Omega}^{\infty_+}_{\underline{r}} - \widetilde{\Omega}^{0,+}_{\underline{r}})\big)\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n,t_{\underline{r}})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n,t_{\underline{r}})))}, \tag{4.85}$$

$$\beta(n,t_{\underline{r}}) = \frac{1}{\alpha(n_0,t_{0,\underline{r}})}\exp\big((n-n_0)(\omega_0^{0,+}(P_{0,-},P_{\infty_+}) - \omega_0^{\infty_+}(P_{0,-},P_{\infty_+}))\big) \times \exp\big(\omega_0^{\infty_+}(P_{0,-},P_{\infty_-}) - \omega_0^{0,+}(P_{0,-},P_{\infty_-})\big) \times \exp\big(-(t_{\underline{r}}-t_{0,\underline{r}})(\widetilde{\Omega}^{\infty_+}_{\underline{r}} - \widetilde{\Omega}^{0,+}_{\underline{r}})\big)\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n,t_{\underline{r}})))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n_0,t_{0,\underline{r}})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n,t_{\underline{r}})))}, \tag{4.86}$$

and

$$\alpha(n,t_{\underline{r}})\beta(n,t_{\underline{r}}) = \exp\big(\omega_0^{\infty_+}(P_{0,-},P_{\infty_-}) - \omega_0^{0,+}(P_{0,-},P_{\infty_-})\big) \times \frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(n,t_{\underline{r}})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(n,t_{\underline{r}})))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(n,t_{\underline{r}})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n,t_{\underline{r}})))}. \tag{4.87}$$

Proof. As in Theorem 3.7 one concludes that $\phi(P,n,t_{\underline{r}})$ is of the form (4.78) and that for $t_{0,\underline{r}} = t_{\underline{r}}$, $\psi_1(P,n,n_0,t_{\underline{r}},t_{\underline{r}})$ is of the form

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{\underline{r}}) = C(n,n_0,t_{\underline{r}},t_{\underline{r}})\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_{\underline{r}})))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_{\underline{r}})))}\exp\bigg((n-n_0)\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_+}}\bigg). \tag{4.88}$$

To discuss $\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}})$ we recall (4.28), that is,

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \psi_1(P,n,n_0,t_{\underline{r}},t_{\underline{r}})\psi_1(P,n_0,n_0,t_{\underline{r}},t_{0,\underline{r}}), \tag{4.89}$$

and hence remaining to be studied is

$$\psi_1(P,n_0,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \exp\bigg(i\int_{t_{0,\underline{r}}}^{t_{\underline{r}}} ds\big(\widetilde{G}_{\underline{r}}(z,n_0,s) - \widetilde{F}_{\underline{r}}(z,n_0,s)\phi(P,n_0,s)\big)\bigg). \tag{4.90}$$

Introducing $\hat{\psi}_1(P)$ on $\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-}\}$ by

$$\begin{aligned}\hat{\psi}_1(P,n_0,t_{\underline{r}},t_{0,\underline{r}}) &= C(n_0,n_0,t_{\underline{r}},t_{0,\underline{r}})\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_{\underline{r}})))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})))}\\ &\quad\times\exp\bigg(-(t_{\underline{r}}-t_{0,\underline{r}})\int_{Q_0}^{P}\widetilde{\Omega}_{\underline{r}}^{(2)}\bigg),\end{aligned} \tag{4.91}$$

we intend to prove that

$$\begin{aligned}&\psi_1(P,n_0,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \hat{\psi}_1(P,n_0,t_{\underline{r}},t_{0,\underline{r}}),\\ &P\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-}\},\ n_0\in\mathbb{Z},\ t_{\underline{r}},t_{0,\underline{r}}\in\mathbb{R},\end{aligned} \tag{4.92}$$

for an appropriate choice of the normalization constant $C(n_0,n_0,t_{\underline{r}},t_{0,\underline{r}})$ in (4.91). We start by noting that a comparison of (4.53), (4.54) and (4.71), (4.72), (4.74), (4.79) shows that ψ_1 and $\hat{\psi}_1$ have the same essential singularities at $P_{\infty_\pm}$ and $P_{0,\pm}$. Thus, we turn to the local behavior of ψ_1 and $\hat{\psi}_1$. By (4.91), $\hat{\psi}_1$ has zeros and poles at $\underline{\hat{\mu}}(n_0,t_{\underline{r}})$ and $\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})$, respectively. Similarly, by (4.90), ψ_1 can have zeros and poles only at poles of $\phi(P,n_0,s)$, $s\in[t_{0,\underline{r}},t_{\underline{r}}]$ (resp., $s\in[t_{\underline{r}},t_{0,\underline{r}}]$). In the following we temporarily restrict $t_{0,\underline{r}}$ and $t_{\underline{r}}$ to a sufficiently small nonempty interval $I\subseteq\mathbb{R}$ and pick $n_0\in\mathbb{Z}$ such that for all $s\in I$, $\mu_j(n_0,s)\neq\mu_k(n_0,s)$ for all $j\neq k$, $j,k=1,\dots,p$. One computes

$$\begin{aligned}&i\widetilde{G}_{\underline{r}}(z,n_0,s) - i\widetilde{F}_{\underline{r}}(z,n_0,s)\phi(P,n_0,s)\\ &\quad= i\widetilde{G}_{\underline{r}}(z,n_0,s) - i\widetilde{F}_{\underline{r}}(z,n_0,s)\frac{(c_{0,+}/2)z^{-p_-}y + G_{\underline{p}}(z,n_0,s)}{F_{\underline{p}}(z,n_0,s)}\\ &\quad\underset{P\to\hat{\mu}_j(n_0,s)}{=} \frac{i\widetilde{F}_{\underline{r}}(\mu_j(n_0,s),n_0,s)y(\hat{\mu}_j(n_0,s))}{\alpha^+(n_0,s)(z-\mu_j(n_0,s))\prod_{\substack{k=1\\k\neq j}}^{p}(\mu_j(n_0,s)-\mu_k(n_0,s))} + O(1)\\ &\quad\underset{P\to\hat{\mu}_j(n_0,s)}{=} \frac{\partial}{\partial s}\ln\big(\mu_j(n_0,s)-z\big) + O(1).\end{aligned} \tag{4.93}$$

Restricting P to a sufficiently small neighborhood $\mathcal{U}_j(n_0)$ of $\{\hat{\mu}_j(n_0,s)\in\mathcal{K}_p\,|\,s\in[t_{0,\underline{r}},t_{\underline{r}}]\subseteq I\}$ such that $\hat{\mu}_k(n_0,s)\notin\mathcal{U}_j(n_0)$ for all $s\in[t_{0,\underline{r}},t_{\underline{r}}]\subseteq I$ and all $k\in\{1,\dots,p\}\setminus\{j\}$, (4.91) and (4.93) imply

$$\begin{aligned}&\psi_1(P,n_0,n_0,t_{\underline{r}},t_{0,\underline{r}})\\ &\quad=\begin{cases}(\mu_j(n_0,t_{\underline{r}})-z)O(1) & \text{as } P\to\hat{\mu}_j(n_0,t_{\underline{r}})\neq\hat{\mu}_j(n_0,t_{0,\underline{r}}),\\ O(1) & \text{as } P\to\hat{\mu}_j(n_0,t_{\underline{r}})=\hat{\mu}_j(n_0,t_{0,\underline{r}}),\\ (\mu_j(n_0,t_{0,\underline{r}})-z)^{-1}O(1) & \text{as } P\to\hat{\mu}_j(n_0,t_{0,\underline{r}})\neq\hat{\mu}_j(n_0,t_{\underline{r}}),\end{cases}\\ &\qquad\qquad P=(z,y)\in\mathcal{K}_p,\end{aligned} \tag{4.94}$$

with $O(1) \neq 0$. Thus, ψ_1 and $\hat{\psi}_1$ have the same local behavior and identical essential singularities at $P_{\infty_\pm}$ and $P_{0,\pm}$. By Lemma A.4, ψ_1 and $\hat{\psi}_1$ coincide up to a multiple constant (which may depend on $n_0, t_{\underline{r}}, t_{0,\underline{r}}$). This proves (4.92) for $t_{0,\underline{r}}, t_{\underline{r}} \in I$ and for n_0 as restricted above. By continuity with respect to divisors this extends to all $n_0 \in \mathbb{Z}$ since by hypothesis $\mathcal{D}_{\underline{\hat{\mu}}(n,s)}$ remain nonspecial for all $(n,s) \in \mathbb{Z} \times \mathbb{R}$. Moreover, since by (4.90), for fixed P and n_0, $\psi_1(P, n_0, n_0, ., t_{0,\underline{r}})$ is entire in $t_{\underline{r}}$ (and this argument is symmetric in $t_{\underline{r}}$ and $t_{0,\underline{r}}$), (4.92) holds for all $t_{\underline{r}}, t_{0,\underline{r}} \in \mathbb{R}$ (for an appropriate choice of $C(n_0, n_0, t_{\underline{r}}, t_{0,\underline{r}})$). Together with (4.89), this proves (4.79) for all $(n, t_{\underline{r}}), (n_0, t_{0,\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$. The expression (4.80) for ψ_2 then immediately follows from (4.78) and (4.79).

To determine the constant $C(n, n_0, t_{\underline{r}}, t_{0,\underline{r}})$ one compares the asymptotic expansions of $\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})$ for $P \to P_{\infty_+}$ in (4.53) and (4.79)

$$
\begin{aligned}
C(n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) = \exp\big((t_{\underline{r}} - t_{0,\underline{r}})\widetilde{\Omega}_{\underline{r}}^{\infty_+} - (n - n_0)\omega_0^{\infty_+}(P_{0,-}, P_{\infty_+})\big) \\
\times \frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n_0, t_{0,\underline{r}})))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n, t_{\underline{r}})))}.
\end{aligned}
\tag{4.95}
$$

Remaining to be computed are the expressions for α and β. Comparing the asymptotic expansions of $\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})$ for $P \to P_{0,+}$ in (4.54) and (4.79) shows

$$
\begin{aligned}
\frac{\alpha(n, t_{\underline{r}})}{\alpha(n_0, t_{0,\underline{r}})} &= C(n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) \exp\big((n - n_0)(\omega_0^{0,+}(P_{0,-}, P_{\infty_+}) - (t_{\underline{r}} - t_{0,\underline{r}})\widetilde{\Omega}_{\underline{r}}^{0,+}\big) \\
&\quad \times \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n, t_{\underline{r}})))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n_0, t_{0,\underline{r}})))}
\end{aligned}
\tag{4.96}
$$

and inserting $C(n, n_0, t_{\underline{r}}, t_{0,\underline{r}})$ proves (4.85). Equation (4.81) for $C(n, t_{\underline{r}})$ follows as in the stationary case since $t_{\underline{r}}$ can be viewed as an additional but fixed parameter. By the first line of (3.69),

$$
\alpha(n, t_{\underline{r}}) = \frac{1}{C(n, t_{\underline{r}})} \exp\big(-\omega_0^{0,+}(P_{0,-}, P_{\infty_-})\big) \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n, t_{\underline{r}})))}{\theta(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(n, t_{\underline{r}})))}.
\tag{4.97}
$$

Inserting the result (4.85) for $\alpha(n, t_{\underline{r}})$ into (4.97) then yields (using Lemma 3.6)

$$
\begin{aligned}
C(n, t_{\underline{r}}) &= \frac{1}{\alpha(n_0, t_{0,\underline{r}})} \frac{\theta(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(n_0, t_{0,\underline{r}})))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n_0, t_{0,\underline{r}})))} \exp\big((t_{\underline{r}} - t_{0,\underline{r}})(\widetilde{\Omega}_{\underline{r}}^{0,+} - \widetilde{\Omega}_{\underline{r}}^{\infty_+})\big) \\
&\quad \times \exp\big((n - n_0)(\omega_0^{\infty_+}(P_{0,-}, P_{\infty_+}) - \omega_0^{0,+}(P_{0,-}, P_{\infty_+})) - \omega_0^{0,+}(P_{0,-}, P_{\infty_-})\big).
\end{aligned}
\tag{4.98}
$$

Also, since the first line of (3.70) holds,

$$
\beta(n, t_{\underline{r}}) = C(n, t_{\underline{r}}) \exp\big(\omega_0^{\infty_+}(P_{0,-}, P_{\infty_-})\big) \frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(n, t_{\underline{r}})))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(n, t_{\underline{r}})))},
\tag{4.99}
$$

an insertion of (4.98) into (4.99), observing Lemma 3.6, yields equation (4.86) for $\beta(n, t_{\underline{r}})$. Finally, multiplying (4.97) and (4.99) proves (4.87).

Single-valuedness of $\psi_1(\,\cdot\,, n_0, n_0, t_{\underline{r}}, t_{0,\underline{r}})$ on $\mathcal{K}_p$ implies

$$
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{\underline{r}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}) + i(t_{\underline{r}} - t_{0,\underline{r}})\underline{\widetilde{U}}_{\underline{r}}^{(2)}.
\tag{4.100}
$$

Inserting (4.100) into (3.61),

$$
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{\underline{r}})}) + \underline{A}_{P_{0,-}}(P_{\infty_+})(n - n_0),
\tag{4.101}
$$

one obtains the result (4.83). □

Again we note that the apparent n_0-dependence of $C(n, t_{\underline{r}})$ in the right-hand side of (4.81) actually drops out to ensure the n_0-independence of ϕ in (4.78).

The theta function representations (4.85) and (4.86) for α and β, and the one for $\Gamma(\,\cdot\,, \,\cdot\,, t_{\underline{r}})$ analogous to that in (4.87) also show that $\gamma(n, t_{\underline{r}}) \notin \{0, 1\}$ for all $(n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$. Hence, condition (4.1) is satisfied for the time-dependent algebro-geometric AL solutions discussed in this section, provided the associated divisors $\mathcal{D}_{\underline{\hat{\mu}}}(n, t_{\underline{r}})$ and $\mathcal{D}_{\underline{\hat{\nu}}}(n, t_{\underline{r}})$ stay away from $P_{\infty_\pm}, P_{0,\pm}$ for all $(n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$.

The time-dependent algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy with complex-valued initial data, that is, the construction of α and β by starting from a set of initial data (nonspecial divisors) of full measure, will be presented in [41].

Appendix A. Hyperelliptic Curves and Their Theta Functions

We provide a very brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [29], [30], and [48], as well as monographs dedicated to integrable systems such as [15, Ch. 2], [38, App. A, B].

Fix $p \in \mathbb{N}$. The hyperelliptic curve $\mathcal{K}_p$ of genus p used in Sections 2 and 3 is defined by

$$\mathcal{K}_p \colon \mathcal{F}_p(z, y) = y^2 - R_{2p+2}(z) = 0, \quad R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \tag{A.1}$$

$$\{E_m\}_{m=0,\dots,2p+1} \subset \mathbb{C}, \quad E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 0, \dots, 2p+1. \tag{A.2}$$

The curve (A.1) is compactified by adding the points P_{∞_+} and P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, at infinity. One then introduces an appropriate set of $p+1$ nonintersecting cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$ and denotes

$$\mathcal{C} = \bigcup_{j \in \{1, \dots, p+1\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Defining the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \tag{A.4}$$

and introducing the holomorphic function

$$R_{2p+2}(\,\cdot\,)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2p+1} (z - E_m) \right)^{1/2} \tag{A.5}$$

on Π with an appropriate choice of the square root branch in (A.5), one considers

$$\mathcal{M}_p = \{(z, \sigma R_{2p+2}(z)^{1/2}) \,|\, z \in \mathbb{C},\ \sigma \in \{\pm 1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \tag{A.6}$$

by extending $R_{2p+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_p$ is then the set $\mathcal{M}_p$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_p$ crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_p)$ of $\mathcal{K}_p$ is given by

$$\mathcal{B}(\mathcal{K}_p) = \{(E_m, 0)\}_{m=0,\dots,2p+1} \tag{A.7}$$

and finite points P on $\mathcal{K}_p$ are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z, y) = y^2 - R_{2p+2}(z) = 0$. Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_p \setminus (\mathcal{B}(\mathcal{K}_p) \cup \{P_{\infty_+}, P_{\infty_-}\})$ are given by $\zeta_{P_0} = z - z_0$, near $P_{\infty_\pm}$ by $\zeta_{P_{\infty_\pm}} = 1/z$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_p)$ by $\zeta_{(E_{m_0},0)} = (z - E_{m_0})^{1/2}$. The Riemann surface $\mathcal{K}_p$ defined in this manner has topological genus p. Moreover, we introduce the holomorphic sheet exchange map (involution)

$$* \colon \mathcal{K}_p \to \mathcal{K}_p, \quad P = (z, y) \mapsto P^* = (z, -y), \ P_{\infty_\pm} \mapsto P^*_{\infty_\pm} = P_{\infty_\mp}. \tag{A.8}$$

One verifies that dz/y is a holomorphic differential on $\mathcal{K}_p$ with zeros of order $p-1$ at $P_{\infty_\pm}$ and hence

$$\eta_j = \frac{z^{j-1} dz}{y}, \quad j = 1, \dots, p, \tag{A.9}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_p$. Introducing the invertible matrix C in $\mathbb{C}^p$,

$$\begin{aligned} &C = (C_{j,k})_{j,k=1,\dots,p}, \quad C_{j,k} = \int_{a_k} \eta_j, \\ &\underline{c}(k) = (c_1(k), \dots, c_p(k)), \quad c_j(k) = C^{-1}_{j,k}, \ j, k = 1, \dots, p, \end{aligned} \tag{A.10}$$

the corresponding basis of normalized holomorphic differentials ω_j, $j = 1, \dots, p$ on $\mathcal{K}_p$ is given by

$$\omega_j = \sum_{\ell=1}^p c_j(\ell) \eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j, k = 1, \dots, p. \tag{A.11}$$

Here $\{a_j, b_j\}_{j=1,\dots,p}$ is a homology basis for $\mathcal{K}_p$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k}, \ a_j \circ a_k = 0, \ b_j \circ b_k = 0, \quad j, k = 1, \dots, p. \tag{A.12}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,p}$ we also recall the canonical dissection of $\mathcal{K}_p$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_p$ of the fundamental polygon $\partial \widehat{\mathcal{K}}_p$ given by

$$\partial \widehat{\mathcal{K}}_p = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_p^{-1} b_p^{-1}. \tag{A.13}$$

Let $\mathcal{M}(\mathcal{K}_p)$ and $\mathcal{M}^1(\mathcal{K}_p)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_p$. Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind, $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_p)$, are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a-periods vanish, that is, $\int_{a_j} \omega^{(2)} = 0$, $j = 1, \dots, p$. Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_p$ not of the first or second kind is said to be of the third kind. A differential of the third kind $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_p)$ is usually normalized by the vanishing of its a-periods, that is, $\int_{a_j} \omega^{(3)} = 0$, $j = 1, \dots, p$. A normal differential of the third kind $\omega^{(3)}_{P_1,P_2}$ associated with two points P_1, $P_2 \in \widehat{\mathcal{K}}_p$, $P_1 \neq P_2$, by definition, has simple poles at P_j with residues $(-1)^{j+1}$, $j = 1, 2$ and vanishing a-periods.

Next, define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,p}$ by

$$\tau_{j,\ell} = \int_{b_\ell} \omega_j, \quad j, \ell = 1, \dots, p. \tag{A.14}$$

Then

$$\operatorname{Im}(\tau) > 0 \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j, \ell = 1, \dots, p. \tag{A.15}$$

Associated with τ one introduces the period lattice

$$L_p = \{\underline{z} \in \mathbb{C}^p \,|\, \underline{z} = \underline{m} + \underline{n}\tau,\ \underline{m}, \underline{n} \in \mathbb{Z}^p\} \tag{A.16}$$

and the Riemann theta function associated with $\mathcal{K}_p$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,p}$,

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^p} \exp\big(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \underline{n}\tau)\big), \quad \underline{z} \in \mathbb{C}^p, \tag{A.17}$$

where $(\underline{u}, \underline{v}) = \overline{\underline{u}}\,\underline{v}^\top = \sum_{j=1}^p \overline{u_j}\, v_j$ denotes the scalar product in $\mathbb{C}^p$. It has the fundamental properties

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_p) = \theta(\underline{z}), \tag{A.18}$$

$$\theta(\underline{z} + \underline{m} + \underline{n}\tau) = \exp\big(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \underline{n}\tau)\big)\theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^p. \tag{A.19}$$

Next, fix a base point $Q_0 \in \mathcal{K}_p \setminus \{P_{0,\pm}, P_{\infty\pm}\}$, denote by $J(\mathcal{K}_p) = \mathbb{C}^p/L_p$ the Jacobi variety of $\mathcal{K}_p$, and define the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_p \to J(\mathcal{K}_p), \quad \underline{A}_{Q_0}(P) = \bigg(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_p\bigg) \pmod{L_p}, \quad P \in \mathcal{K}_p. \tag{A.20}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0} \colon \operatorname{Div}(\mathcal{K}_p) \to J(\mathcal{K}_p), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P\in\mathcal{K}_p} \mathcal{D}(P)\underline{A}_{Q_0}(P), \tag{A.21}$$

where $\operatorname{Div}(\mathcal{K}_p)$ denotes the set of divisors on $\mathcal{K}_p$. Here $\mathcal{D}\colon \mathcal{K}_p \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_p$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_p$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_p)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.)

In connection with divisors on $\mathcal{K}_p$ we shall employ the following (additive) notation,

$$\begin{aligned} &\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\ &\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_p, \quad Q_0 \in \mathcal{K}_p,\ m \in \mathbb{N}, \end{aligned} \tag{A.22}$$

where for any $Q \in \mathcal{K}_p$,

$$\mathcal{D}_Q \colon \mathcal{K}_p \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_p \setminus \{Q\}, \end{cases} \tag{A.23}$$

and $\operatorname{Sym}^n \mathcal{K}_p$ denotes the nth symmetric product of $\mathcal{K}_p$. In particular, $\operatorname{Sym}^m \mathcal{K}_p$ can be identified with the set of nonnegative divisors $0 \leq \mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$ of degree m.

For $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_p)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_p) \,|\, \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ \deg((\omega)) = 2(p-1),\ f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\},\ \omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}, \tag{A.24}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P\in\mathcal{K}_p} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_p) \,|\, f = 0 \text{ or } (f) \geq \mathcal{D}\}, \quad r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \tag{A.25}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_p) \,|\, \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}, \quad i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}), \tag{A.26}$$

with $i(\mathcal{D})$ the index of speciality of $\mathcal{D}$, one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$, $\omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}$. Then,*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad p \in \mathbb{N}_0. \tag{A.27}$$

The Riemann–Roch theorem reads

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - p + 1, \quad p \in \mathbb{N}_0. \tag{A.28}$$

By Abel's theorem, $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$, $p \in \mathbb{N}$ is principal if and only if

$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \tag{A.29}$$

Finally, assume $p \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_p) \to J(\mathcal{K}_p)$ is surjective (Jacobi's inversion theorem).

Theorem A.2. *Let $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^p \mathcal{K}_p$, $\underline{Q} = \{Q_1, \dots, Q_p\}$. Then,*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s \tag{A.30}$$

if and only if $\{Q_1, \dots, Q_p\}$ contains s pairings of the type $\{P, P^*\}$. (This includes, of course, branch points for which $P = P^*$.) One has $s \leq p/2$.

Denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,p}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{p} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, p. \tag{A.31}$$

Theorem A.3. *Let $\underline{Q} = \{Q_1, \dots, Q_p\} \in \operatorname{Sym}^p \mathcal{K}_p$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*

$$\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_p\}. \tag{A.32}$$

Lemma A.4. [19, Lemmas 5.4 and 6.1] *Let $(n, t_{\underline{r}}), (n_0, t_{0,\underline{r}}) \in \Omega$ for some $\Omega \subseteq \mathbb{Z} \times \mathbb{R}$. Assume $\psi(\,\cdot\,, n, t_{\underline{r}})$ to be meromorphic on $\mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ with possible essential singularities at $P_{\infty_\pm}$, $P_{0,\pm}$ such that $\tilde{\psi}(\,\cdot\,, n, t_{\underline{r}})$ defined by*

$$\tilde{\psi}(P, n, t_{\underline{r}}) = \psi(P, n, t_{\underline{r}}) \exp\bigg((t_{\underline{r}} - t_{0,\underline{r}}) \int_{Q_0}^{P} \widetilde{\Omega}_{\underline{r}}^{(2)}\bigg) \tag{A.33}$$

is meromorphic on $\mathcal{K}_p$ and its divisor satisfies

$$(\tilde{\psi}(\,\cdot\,, n, t_{\underline{r}})) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n_0, t_{0,\underline{r}})} + (n - n_0)\big(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}\big) \tag{A.34}$$

for some positive divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0, t_{0,\underline{r}})}$ of degree p. Here $\widetilde{\Omega}_{\underline{r}}^{(2)}$ is defined in (4.74) *and the path of integration is chosen identical to that in the Abel maps*[3] (A.20) *and* (A.21)*. Define a divisor $\mathcal{D}_0(n, t_{\underline{r}})$ by*

$$(\tilde{\psi}(\,\cdot\,, n, t_{\underline{r}})) = \mathcal{D}_0(n, t_{\underline{r}}) - \mathcal{D}_{\underline{\hat{\mu}}(n_0, t_{0,\underline{r}})} + (n - n_0)\big(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty_+}}\big). \tag{A.35}$$

[3] This is to avoid multi-valued expressions and hence the use of the multiplicative Riemann–Roch theorem in the proof of Lemma A.4.

Then

$$\mathcal{D}_0(n,t_{\underline{r}}) \in \operatorname{Sym}^p \mathcal{K}_p, \quad \mathcal{D}_0(n,t_{\underline{r}}) > 0, \quad \deg(\mathcal{D}_0(n,t_{\underline{r}})) = p. \tag{A.36}$$

Moreover, if $\mathcal{D}_0(n,t_{\underline{r}})$ *is nonspecial for all* $(n,t_{\underline{r}}) \in \Omega$, *that is, if*

$$i(\mathcal{D}_0(n,t_{\underline{r}})) = 0, \quad (n,t_{\underline{r}}) \in \Omega, \tag{A.37}$$

then $\psi(\,\cdot\,,n,t_{\underline{r}})$ *is unique up to a constant multiple* (*which may depend on the parameters* $(n,t_{\underline{r}}),(n_0,t_{0,\underline{r}}) \in \Omega$).

Appendix B. Asymptotic Spectral Parameter Expansions

In this appendix we consider asymptotic spectral parameter expansions of $F_{\underline{p}}/y$, $G_{\underline{p}}/y$, and $H_{\underline{p}}/y$, the resulting recursion relations for the homogeneous coefficients $\hat{f}_\ell$, $\hat{g}_\ell$, and $\hat{h}_\ell$, their connection with the nonhomogeneous coefficients f_ℓ, g_ℓ, and h_ℓ, and the connection between $c_{\ell,\pm}$ and $c_\ell(\underline{E}^{\pm 1})$ (cf. (B.3)). For detailed proofs of the material in this section we refer to [39], [40]. We will employ the notation

$$\underline{E}^{\pm 1} = \big(E_0^{\pm 1}, \dots, E_{2p+1}^{\pm 1}\big). \tag{B.1}$$

We start with the following elementary results (consequences of the binomial expansion) assuming $\eta \in \mathbb{C}$ such that $|\eta| < \min\{|E_0|^{-1}, \dots, |E_{2p+1}|^{-1}\}$:

$$\left(\prod_{m=0}^{2p+1} \big(1 - E_m \eta\big)\right)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{E}) \eta^k, \tag{B.2}$$

where

$$\begin{aligned} &c_0(\underline{E}) = 1, \\ &c_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{2p+1}=0 \\ j_0+\dots+j_{2p+1}=k}}^{k} \frac{(2j_0)! \cdots (2j_{2p+1})!\, E_0^{j_0} \cdots E_{2p+1}^{j_{2p+1}}}{2^{2k} (j_0!)^2 \cdots (j_{2p+1}!)^2 (2j_0 - 1) \cdots (2j_{2p+1} - 1)}, \quad k \in \mathbb{N}. \end{aligned} \tag{B.3}$$

The first few coefficients explicitly are given by

$$c_0(\underline{E}) = 1,\; c_1(\underline{E}) = -\frac{1}{2} \sum_{m=0}^{2p+1} E_m,\; c_2(\underline{E}) = \frac{1}{4} \sum_{\substack{m_1,m_2=0 \\ m_1<m_2}}^{2p+1} E_{m_1} E_{m_2} - \frac{1}{8} \sum_{m=0}^{2p+1} E_m^2, \quad \text{etc.} \tag{B.4}$$

Next we turn to asymptotic expansions. We recall the convention $y(P) = \mp\zeta^{-p-1} + O(\zeta^{-p})$ near $P_{\infty_\pm}$ (where $\zeta = 1/z$) and $y(P) = \pm(c_{0,-}/c_{0,+}) + O(\zeta)$ near $P_{0,\pm}$ (where $\zeta = z$).

Theorem B.1 ([40]). *Assume* (3.1), s-$\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$, *and suppose* $P = (z,y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$. *Then* $z^{p_-} F_{\underline{p}}/y$, $z^{p_-} G_{\underline{p}}/y$, *and* $z^{p_-} H_{\underline{p}}/y$ *have the following convergent expansions as* $P \to P_{\infty_\pm}$, *respectively,* $P \to P_{0,\pm}$,

$$\frac{z^{p_-}}{c_{0,+}} \frac{F_{\underline{p}}(z)}{y} = \begin{cases} \mp \sum_{\ell=0}^{\infty} \hat{f}_{\ell,+} \zeta^{\ell+1}, & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm \sum_{\ell=0}^{\infty} \hat{f}_{\ell,-} \zeta^{\ell}, & P \to P_{0,\pm}, \quad \zeta = z, \end{cases} \tag{B.5}$$

$$\frac{z^{p_-}}{c_{0,+}} \frac{G_{\underline{p}}(z)}{y} = \begin{cases} \mp \sum_{\ell=0}^{\infty} \hat{g}_{\ell,+} \zeta^{\ell}, & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm \sum_{\ell=0}^{\infty} \hat{g}_{\ell,-} \zeta^{\ell}, & P \to P_{0,\pm}, \quad \zeta = z, \end{cases} \tag{B.6}$$

$$\frac{z^{p_-}}{c_{0,+}}\frac{H_{\underline{p}}(z)}{y} = \begin{cases} \mp\sum_{\ell=0}^{\infty}\hat{h}_{\ell,+}\zeta^{\ell}, & P\to P_{\infty\pm}, \quad \zeta=1/z, \\ \pm\sum_{\ell=0}^{\infty}\hat{h}_{\ell,-}\zeta^{\ell+1}, & P\to P_{0,\pm}, \quad \zeta=z, \end{cases} \tag{B.7}$$

where $\zeta = 1/z$ *(resp.,* $\zeta = z$*) is the local coordinate near* $P_{\infty_\pm}$ *(resp.,* $P_{0,\pm}$*) and* $\hat{f}_{\ell,\pm}$*,* $\hat{g}_{\ell,\pm}$*, and* $\hat{h}_{\ell,\pm}$ *are the homogeneous versions of the coefficients* $f_{\ell,\pm}$*,* $g_{\ell,\pm}$*, and* $h_{\ell,\pm}$ *introduced in* (2.16)*–*(2.18)*.*

Moreover, the E_m*-dependent summation constants* $c_{\ell,\pm}$*,* $\ell = 0,\dots,p_\pm$*, in* $F_{\underline{p}}$*,* $G_{\underline{p}}$*, and* $H_{\underline{p}}$ *are given by*

$$c_{\ell,\pm} = c_{0,\pm}c_\ell(\underline{E}^{\pm 1}), \quad \ell = 0,\dots,p_\pm. \tag{B.8}$$

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References

[1] M. J. Ablowitz. Nonlinear evolution equations – continuous and discrete. *SIAM Rev.* 19:663–684, 1977.
[2] M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering.* Cambridge University Press, Cambridge, 1991.
[3] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations. *J. Math. Phys.* 16:598–603, 1975.
[4] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.* 17:1011–1018, 1976.
[5] M. J. Ablowitz and J. F. Ladik. A nonlinear difference scheme and inverse scattering. *Studies Appl. Math.* 55:213–229, 1976.
[6] M. J. Ablowitz and J. F. Ladik. On the solution of a class of nonlinear partial difference equations. *Studies Appl. Math.* 57:1–12, 1977.
[7] M. J. Ablowitz, B. Prinari, and A. D. Trubatch. *Discrete and Continuous Nonlinear Schrödinger Systems.* London Mathematical Society Lecture Note Series, Vol. 302, Cambridge University Press, Cambridge, 2004.
[8] S. Ahmad and A. Roy Chowdhury. On the quasi-periodic solutions to the discrete non-linear Schrödinger equation. *J. Phys. A* 20:293–303, 1987.
[9] S. Ahmad and A. Roy Chowdhury. The quasi-periodic solutions to the discrete non-linear Schrödinger equation. *J. Math. Phys.* 28:134–1137, 1987.
[10] N. I. Akhiezer. *The Classical Moment Problem.* Oliver & Boyd., Edinburgh, 1965.
[11] G. Baxter. Polynomials defined by a difference system. *Bull. Amer. Math. Soc.* 66:187–190, 1960.
[12] G. Baxter. Polynomials defined by a difference system. *J. Math. Anal. Appl.* 2:223–263, 1961.
[13] G. Baxter. A convergence equivalence related to polynomials orthogonal on the unit circle. *Trans. Amer. Math. Soc.* 99:471–487, 1961.
[14] G. Baxter. A norm inequality for a "finite-section" Wiener-Hopf equation. *Illinois J. Math.* 7:97–103, 1963.
[15] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev. *Algebro-Geometric Approach to Nonlinear Integrable Equations.* Springer, Berlin, 1994.
[16] M. Bertola and M. Gekhtman. Biorthogonal Laurent polynomials, Töplitz determinants, minimal Toda orbits and isomonodromic tau functions. *Constr. Approx.*, 26:383–430, 2007.
[17] N. N. Bogolyubov, A. K. Prikarpatskii, and V. G. Samoilenko. Discrete periodic problem for the modified nonlinear Korteweg–de Vries equation. *Sov. Phys. Dokl.* 26:490–492, 1981.
[18] N. N. Bogolyubov and A. K. Prikarpatskii. The inverse periodic problem for a discrete approximation of a nonlinear Schrödinger equation. *Sov. Phys. Dokl.* 27:113–116, 1982.
[19] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl. Algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac-van Moerbeke hierarchy. *Mem. Amer. Math. Soc.* no 641, 135:1–79, 1998.

[20] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal Rational Functions*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 1999.

[21] M. J. Cantero, L. Moral, and L. Velázquez. Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. *Lin. Algebra Appl.* 362:29–56, 2003.

[22] S.-C. Chiu and J. F. Ladik. Generating exactly soluble nonlinear discrete evolution equations by a generalized Wronskian technique. *J. Math. Phys.* 18:690–700, 1977.

[23] K. W. Chow, R. Conte, and N. Xu. Analytic doubly periodic wave patterns for the integrable discrete nonlinear Schrödinger (Ablowitz–Ladik) model. *Physics Letters A* 349:422–429, 2006.

[24] A. K. Common. A solution of the initial value problem for half-infinite integrable systems. *Inverse Problems* 8:393–408, 1992.

[25] P. Deift. Riemann–Hilbert methods in the theory of orthogonal polynomials. *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday. Ergodic Schrödinger Operators, Singular Spectrum, Orthogonal Polynomials, and Inverse Spectral Theory*, F. Gesztesy, P. Deift, C. Galvez, P. Perry, and W. Schlag (eds.), Proceedings of Symposia in Pure Mathematics, Vol. 76/2, Amer. Math. Soc., Providence, RI, 2007, pp. 715–740.

[26] H.-Y. Ding, Y.-P. Sun, and X.-X. Xu. A hierarchy of nonlinear lattice soliton equations, its integrable coupling systems and infinitely many conservation laws. *Chaos, Solitons and Fractals* 30:227–234, 2006.

[27] A. Doliwa and P. M. Santini. Integrable dynamics of a discrete curve and the Ablowitz–Ladik hierarchy. *J. Math. Phys.* 36:1259–1273, 1995.

[28] N. M. Ercolani and G. I. Lozano. A bi-Hamiltonian structure for the integrable, discrete non-linear Schrödinger system. *Physica D* 218:105–121, 2006.

[29] H. M. Farkas and I. Kra. *Riemann Surfaces*. Springer, New York, 2nd ed., 1992.

[30] J. Fay. *Theta Functions on Riemann Surfaces*. Lecture Notes in Mathematics, Vol. 352, Springer, New York, 1973.

[31] X. Geng. Darboux transformation of the discrete Ablowitz–Ladik eigenvalue problem. *Acta Math. Sci.* 9:21–26, 1989.

[32] X. Geng, H. H. Dai, and C. Cao. Algebro-geometric constructions of the discrete Ablowitz–Ladik flows and applications. *J. Math. Phys.* 44:4573–4588, 2003.

[33] J. S. Geronimo, F. Gesztesy, and H. Holden. Algebro-geometric solutions of the Baxter–Szegő difference equation. *Commun. Math. Phys.* 258:149–177, 2005.

[34] J. S. Geronimo and R. Johnson. Rotation number associated with difference equations satisfied by polynomials orthogonal on the unit circle. *J. Differential Equations* 132:140–178, 1996.

[35] J. S. Geronimo and R. Johnson. An inverse problem associated with polynomials orthogonal on the unit circle. *Commun. Math. Phys.* 193:125–150, 1998.

[36] J. S. Geronimo and A. Teplyaev. A difference equation arising from the trigonometric moment problem having random reflection coefficients–an operator theoretic approach. *J. Funct. Anal.* 123:12–45, 1994.

[37] Ya. L. Geronimus. *Orthogonal Polynomials*. Consultants Bureau, New York, 1961.

[38] F. Gesztesy and H. Holden. *Soliton Equations and Their Algebro-Geometric Solutions. Volume I:* $(1+1)$*-Dimensional Continuous Models*. Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge University Press, Cambridge, 2003.

[39] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. *Soliton Equations and Their Algebro-Geometric Solutions. Volume II:* $(1+1)$*-Dimensional Discrete Models.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, in preparation.

[40] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. The Ablowitz–Ladik hierarchy revisited. To appear in Operator Theory, Advances and Applications, Birkhäuser, Basel; arXiv:nlin/0702058.

[41] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. The algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy, arXiv:nlin/07063370.

[42] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. Local conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy, in preparation.

[43] L. Golinskii and P. Nevai. Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle. *Commun. Math. Phys.* 223:223–259, 2001.

[44] R. Killip and I. Nenciu. CMV: The unitary analogue of Jacobi matrices. *Commun. Pure Appl. Math.* 59:1–41, 2006.
[45] L.-C. Li. Some remarks on CMV matrices and dressing orbits. *Int. Math. Res. Notices* 40:2437–2446, 2005.
[46] P. D. Miller. Macroscopic behavior in the Ablowitz–Ladik equations. Nonlinear Evolution Equations & Dynamical Systems, V. G. Makhankov, A. R. Bishop, and D. D. Holm (eds.), World Scientific, Singapore, 1995, pp. 158–167.
[47] P. D. Miller, N. M. Ercolani, I. M. Krichever, and C. D. Levermore. Finite genus solutions to the Ablowitz–Ladik equations. *Comm. Pure Appl. Math.* 48:1369–1440, 1995.
[48] D. Mumford. *Tata Lectures on Theta II.* Birkhäuser, Boston, 1984.
[49] I. Nenciu. Lax pairs for the Ablowitz–Ladik system via orthogonal polynomials on the unit circle. *Int. Math. Res. Notices* 11:647–686, 2005.
[50] I. Nenciu. CMV matrices in random matrix theory and integrable systems: a survey. *J. Phys. A* 39:8811–8822, 2006.
[51] R. J. Schilling. A systematic approach to the soliton equations of a discrete eigenvalue problem. *J. Math. Phys.* 30:1487–1501, 1989.
[52] B. Simon. *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Part 2: Spectral Theory*, AMS Colloquium Publication Series, Vol. 54, Amer. Math. Soc., Providence, R.I., 2005.
[53] B. Simon. OPUC on one foot. *Bull. Amer. Math. Soc.* 42:431–460, 2005.
[54] B. Simon. CMV matrices: Five years after. *J. Comp. Appl. Math.* 208:120–154, 2007.
[55] G. Szegő. Beiträge zur Theorie der Toeplitzschen Formen I. *Math. Z.* 6:167–202, 1920.
[56] G. Szegő. Beiträge zur Theorie der Toeplitzschen Formen II. *Math. Z.* 9:167–190, 1921.
[57] G. Szegő. *Orthogonal Polynomials.* Amer Math. Soc. Colloq. Publ. Vol. 23, Amer. Math. Soc., Providence, R.I., 1978.
[58] K. M. Tamizhmani and Wen-Xiu Ma. Master symmetries from Lax operators for certain lattice soliton hierarchies. *J. Phys. Soc. Japan* 69:351–361, 2000.
[59] G. Teschl. *Jacobi Operators and Completely Integrable Nonlinear Lattices.* Math. Surveys Monographs, Vol. 72, Amer. Math. Soc., Providence, R.I., 2000.
[60] K. L. Vaninsky. An additional Gibbs' state for the cubic Schrödinger equation on the circle. *Comm. Pure Appl. Math.* 54:537–582, 2001.
[61] V. E. Vekslerchik. Finite genus solutions for the Ablowitz–Ladik hierarchy. *J. Phys. A* 32:4983–4994, 1999.
[62] V. E. Vekslerchik. Functional representation of the Ablowitz–Ladik hierarchy. II. *J. Nonlin. Math. Phys.* 9:157–180, 2002.
[63] V. E. Vekslerchik. Implementation of the Bäcklund transformations for the Ablowitz–Ladik hierarchy. *J. Phys. A* 39:6933–6953, 2006.
[64] V. E. Vekslerchik and V. V. Konotop. Discrete nonlinear Schrödinger equation under nonvanishing boundary conditions. *Inverse Problems* 8:889–909, 1992.
[65] Y. Zeng and S. Rauch-Wojciechowski. Restricted flows of the Ablowitz–Ladik hierarchy and their continuous limits. *J. Phys. A* 28:113–134, 1995.

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THE ALGEBRO-GEOMETRIC INITIAL VALUE PROBLEM FOR THE ABLOWITZ–LADIK HIERARCHY

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Dedicated with great pleasure to Percy Deift on the occasion of his 60th birthday

Abstract. We discuss the algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy with complex-valued initial data and prove unique solvability globally in time for a set of initial (Dirichlet divisor) data of full measure. To this effect we develop a new algorithm for constructing stationary complex-valued algebro-geometric solutions of the Ablowitz–Ladik hierarchy, which is of independent interest as it solves the inverse algebro-geometric spectral problem for general (non-unitary) Ablowitz–Ladik Lax operators, starting from a suitably chosen set of initial divisors of full measure. Combined with an appropriate first-order system of differential equations with respect to time (a substitute for the well-known Dubrovin-type equations), this yields the construction of global algebro-geometric solutions of the time-dependent Ablowitz–Ladik hierarchy.

The treatment of general (non-unitary) Lax operators associated with general coefficients for the Ablowitz–Ladik hierarchy poses a variety of difficulties that, to the best of our knowledge, are successfully overcome here for the first time. Our approach is not confined to the Ablowitz–Ladik hierarchy but applies generally to $(1+1)$-dimensional completely integrable soliton equations of differential-difference type.

1. Introduction

The principal aim of this paper is an explicit construction of unique global solutions of the algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy for a general class of initial data. However, to put this circle of ideas into a proper perspective, we first very briefly recall the origins of this subject: In the mid-seventies, Ablowitz and Ladik, in a series of papers [3]–[6] (see also [1], [2, Sect. 3.2.2], [7, Ch. 3]), used inverse scattering methods to analyze certain integrable differential-difference systems. One of their integrable variants of such systems included a discretization of the celebrated AKNS-ZS system, the pair of coupled nonlinear differential-difference equations,

$$
\begin{aligned}
&-i\alpha_t - (1-\alpha\beta)(\alpha^- + \alpha^+) + 2\alpha = 0,\\
&-i\beta_t + (1-\alpha\beta)(\beta^- + \beta^+) - 2\beta = 0
\end{aligned}
\tag{1.1}
$$

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with $\alpha = \alpha(n,t)$, $\beta = \beta(n,t)$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$. Here we used the notation $f^{\pm}(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, for complex-valued sequences $f = \{f(n)\}_{n \in \mathbb{Z}}$. In particular, Ablowitz and Ladik [4] (see also [7, Ch. 3]) showed that in the focusing case, where $\beta = -\overline{\alpha}$, and in the defocusing case, where $\beta = \overline{\alpha}$, (1.1) yields the discrete analog of the nonlinear Schrödinger equation

$$-i\alpha_t - (1 \pm |\alpha|^2)(\alpha^- + \alpha^+) + 2\alpha = 0. \tag{1.2}$$

Since then there has been an enormous activity in this area and we refer, for instance, to [7, Ch. 3], [27], [31], [32], [33], [36], [37], [40], [43], [44], [45], [50] and the extensive literature cited therein, for developments leading up to current research in this particular area of completely integrable differential-difference systems. Particularly relevant to this paper are algebro-geometric (and periodic) solutions of the AL system (1.1) and its associated hierarchy of integrable equations. The first systematic and detailed treatment of algebro-geometric solutions of the AL system (1.1) was performed by Miller, Ercolani, Krichever, and Levermore [40] (see also [9], [10], [14], [15], [39], [52]). Algebro-geometric solutions of the AL hierarchy were discussed in great detail in [32] (see also [25], [26], [53]). The initial value problem for the half-infinite discrete linear Schrödinger equation and the Schur flow were discussed by Common [17] (see also [18]) using a continued fraction approach. The corresponding nonabelian cases on a finite interval were studied by Gekhtman [24]. In addition to these developments within integrable systems and their applications to fields such as nonlinear optics, the study of AL systems recently gained considerable momentum due to its connections with the theory of orthogonal polynomials. Especially, the particular defocusing case $\beta = \overline{\alpha}$ and the associated CMV matrices and orthogonal polynomials on the unit circle attracted great interest. In this context we refer the interested reader to the two-volume treatise by Simon [47] (see also [48]) and the survey by Deift [19] and the detailed reference lists therein.

Returning to the principal scope of this paper, we intend to describe a solution of the following problem: Given $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, $\underline{r} \in \mathbb{N}_0^2$, assume $\alpha^{(0)}, \beta^{(0)}$ to be solutions of the $\underline{p}$th stationary Ablowitz–Ladik system s-AL$_{\underline{p}}(\alpha, \beta) = 0$ associated with a prescribed hyperelliptic curve $\mathcal{K}_p$ of genus $p = p_- + p_+ - 1$ (with nonsingular affine part). We want to construct a unique global solution $\alpha = \alpha(t_{\underline{r}}), \beta = \beta(t_{\underline{r}})$ of the $\underline{r}$th AL flow AL$_{\underline{r}}(\alpha, \beta) = 0$ with $\alpha(t_{0,\underline{r}}) = \alpha^{(0)}, \beta(t_{0,\underline{r}}) = \beta^{(0)}$ for some $t_{0,\underline{r}} \in \mathbb{R}$. Thus, we seek the unique global solution of the initial value problem

$$\begin{aligned} &\mathrm{AL}_{\underline{r}}(\alpha, \beta) = 0, \\ &(\alpha, \beta)\big|_{t_{\underline{r}} = t_{0,\underline{r}}} = \big(\alpha^{(0)}, \beta^{(0)}\big), \end{aligned} \tag{1.3}$$

$$\text{s-AL}_{\underline{p}}\big(\alpha^{(0)}, \beta^{(0)}\big) = 0, \tag{1.4}$$

where $\alpha = \alpha(n, t_{\underline{r}}), \beta = \beta(n, t_{\underline{r}})$ satisfy the conditions in (2.2).

Given the particularly familiar case of real-valued algebro-geometric solutions of the Toda hierarchy (see, e.g., [16], [29, Sect. 1.3], [51, Sect. 8.3] and the extensive literature cited therein), the actual solution of this algebro-geometric initial value problem, naively, might consist of the following two-step procedure:[1]

[1]We freely use the notation of divisors of degree p as introduced in Appendix A (see also the beginning of Section 3).

(i) An algorithm that constructs admissible (cf. Section 4) nonspecial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)} \in \operatorname{Sym}^p \mathcal{K}_p$ for all $n \in \mathbb{Z}$, starting from a nonspecial initial Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \operatorname{Sym}^p \mathcal{K}_p$. "Trace formulas" of the type (3.30) and (3.31) (the latter requires prior construction of the Neumann divisor $\mathcal{D}_{\underline{\hat{\nu}}}$ from the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$, though, cf. (4.3)) should then construct the stationary solutions $\alpha^{(0)}, \beta^{(0)}$ of s-$\mathrm{AL}_{\underline{p}}(\alpha, \beta) = 0$.

(ii) The first-order Dubrovin-type system of differential equations (5.43), augmented by the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})} = \mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ (cf. step (i)) together with the analogous "trace formulas" (3.30), (3.31) in the time-dependent context should then yield unique global solutions $\alpha = \alpha(t_{\underline{r}}), \beta = \beta(t_{\underline{r}})$ of the $\underline{r}$th AL flow $\mathrm{AL}_{\underline{r}}(\alpha, \beta) = 0$ satisfying $\alpha(t_{0,\underline{r}}) = \alpha^{(0)}, \beta(t_{0,\underline{r}}) = \beta^{(0)}$.

However, this approach can be expected to work only if the Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})} \in \operatorname{Sym}^p \mathcal{K}_p$, $\underline{\hat{\mu}}(n, t_{\underline{r}}) = (\hat{\mu}_1(n, t_{\underline{r}}), \dots, \hat{\mu}_p(n, t_{\underline{r}}))$, yield pairwise distinct Dirichlet eigenvalues $\mu_j(n, t_{\underline{r}})$, $j = 1, \dots, p$, for fixed $(n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$, such that formula (5.43) for the time-derivatives $\mu_{j,t_{\underline{r}}}$, $j = 1, \dots, p$, is well-defined. Analogous considerations apply to the Neumann divisors $\mathcal{D}_{\underline{\hat{\nu}}} \in \operatorname{Sym}^p \mathcal{K}_p$.

Unfortunately, this scenario of pairwise distinct Dirichlet eigenvalues is not realistic and "collisions" between them can occur at certain values of $(n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$. Thus, the stationary algorithm in step (i) as well as the Dubrovin-type first-order system of differential equations (5.43) in step (ii) above breaks down at such values of $(n, t_{\underline{r}})$. A priori, one has no control over such collisions, especially, it is not possible to identify initial conditions $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}$ at some $(n_0, t_{0,\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$, which avoid collisions for all $(n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$. We solve this problem head on by explicitly permitting collisions in the stationary as well as the time-dependent context from the outset. In the stationary context, we introduce an appropriate algorithm alluded to in step (i) by employing a general interpolation formalism (cf. Appendix B) for polynomials, going beyond the usual Lagrange interpolation formulas. In the time-dependent context we replace the first-order system of Dubrovin-type equations (5.43), augmented with the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}$, by a different first-order system of differential equations (6.15), (6.23), and (6.24) with initial conditions (6.25) which focuses on symmetric functions of $\mu_1(n, t_{\underline{r}}), \dots, \mu_p(n, t_{\underline{r}})$ rather than individual Dirichlet eigenvalues $\mu_j(n, t_{\underline{r}})$, $j = 1, \dots, p$. In this manner it will be shown that collisions of Dirichlet eigenvalues no longer pose a problem.

In addition, there is an additional complication: In general, it cannot be guaranteed that $\mu_j(n, t_{\underline{r}})$ and $\nu_j(n, t_{\underline{r}})$, $j = 1, \dots, p$, stay finite and nonzero for all $(n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$. We solve this particular problem in the stationary as well as the time-dependent case by properly restricting the initial Dirichlet and Neumann divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}, \mathcal{D}_{\underline{\hat{\nu}}(n_0,t_{0,\underline{r}})} \in \operatorname{Sym}^p \mathcal{K}_p$ to a dense set of full measure.

Summing up, we offer a new algorithm to solve the inverse algebro-geometric spectral problem for general Ablowitz–Ladik Lax operators, starting from a properly chosen dense set of initial divisors of full measure. Combined with an appropriate first-order system of differential equations with respect to time (a substitute for the Dubrovin-type equations), this yields the construction of global algebro-geometric solutions of the time-dependent Ablowitz–Ladik hierarchy.

We emphasize that the approach described in this paper is not limited to the Ablowitz–Ladik hierarchy but applies universally to constructing algebro-geometric solutions of (1+1)-dimensional integrable soliton equations. In particular, it applies to the Toda lattice hierarchy as discussed in [30]. Moreover, the principal idea of replacing Dubrovin-type equations by a first-order system of the type (6.15), (6.23), and (6.24) is also relevant in the context of general (non-self-adjoint) Lax operators for the continuous models in $(1+1)$-dimensions. In particular, the models studied in detail in [28] can be revisited from this point of view. (However, the fact that the set in (6.67) is of measure zero relies on the fact that n varies in the countable set $\mathbb{Z}$ and hence is not applicable to continuous models in $1+1$-dimensions.) We also note that while the periodic case with complex-valued α, β is of course included in our analysis, we throughout consider the more general algebro-geometric case (in which α, β need not be quasi-periodic).

Finally we briefly describe the content of each section. Section 2 presents a quick summary of the basics of the Ablowitz–Ladik hierarchy, its recursive construction, Lax pairs, and zero-curvature equations. The stationary algebro-geometric Ablowitz–Ladik hierarchy solutions, the underlying hyperelliptic curve, trace formulas, etc., are the subject of Section 3. A new algorithm solving the algebro-geometric inverse spectral problem for general Ablowitz–Ladik Lax operators is presented in Section 4. In Section 5 we briefly summarize the properties of algebro-geometric time-dependent solutions of the Ablowitz–Ladik hierarchy and formulate the algebro-geometric initial value problem. Uniqueness and existence of global solutions of the algebro-geometric initial value problem as well as their explicit construction are then presented in our final and principal Section 6. Appendix A reviews the basics of hyperelliptic Riemann surfaces of the Ablowitz–Ladik-type and sets the stage of much of the notation used in this paper. Finally, various interpolation formulas of fundamental importance to our stationary inverse spectral algorithm developed in Section 4 are summarized in Appendix B. These appendices support our intention to make this paper reasonably self-contained.

2. The Ablowitz–Ladik Hierarchy in a Nutshell

We briefly review the recursive construction of the Ablowitz–Ladik hierarchy and zero-curvature equations following [31] and [33].

Throughout this section we suppose the following hypothesis.

Hypothesis 2.1. *In the stationary case we assume that α, β satisfy*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0,1\},\ n \in \mathbb{Z}. \tag{2.1}$$

In the time-dependent case we assume that α, β satisfy

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,), \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\},\ (n,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{2.2}$$

Here $\mathbb{C}^{\mathbb{Z}}$ denotes the set of complex-valued sequences indexed by $\mathbb{Z}$. For a discussion of assumptions (2.1) and (2.2) we refer to Remark 3.4.

We denote by $S^{\pm}$ the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ according to

$$(S^{\pm} f)(n) = f(n \pm 1), \quad n \in \mathbb{Z}. \tag{2.3}$$

Moreover, we will frequently use the notation

$$f^{\pm} = S^{\pm} f, \quad f \in \mathbb{C}^{\mathbb{Z}}. \tag{2.4}$$

Next, we define sequences $\{f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ recursively by

$$g_{0,+} = \tfrac{1}{2}c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^+, \quad h_{0,+} = c_{0,+}\beta, \tag{2.5}$$

$$g_{\ell+1,+} - g_{\ell+1,+}^- = \alpha h_{\ell,+}^- + \beta f_{\ell,+}, \quad \ell \in \mathbb{N}_0, \tag{2.6}$$

$$f_{\ell+1,+}^- = f_{\ell,+} - \alpha(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell \in \mathbb{N}_0, \tag{2.7}$$

$$h_{\ell+1,+} = h_{\ell,+}^- + \beta(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell \in \mathbb{N}_0, \tag{2.8}$$

and

$$g_{0,-} = \tfrac{1}{2}c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^+, \tag{2.9}$$

$$g_{\ell+1,-} - g_{\ell+1,-}^- = \alpha h_{\ell,-} + \beta f_{\ell,-}^-, \quad \ell \in \mathbb{N}_0, \tag{2.10}$$

$$f_{\ell+1,-} = f_{\ell,-}^- + \alpha(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell \in \mathbb{N}_0, \tag{2.11}$$

$$h_{\ell+1,-}^- = h_{\ell,-} - \beta(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell \in \mathbb{N}_0. \tag{2.12}$$

Here $c_{0,\pm} \in \mathbb{C}$ are given constants. For later use we also introduce

$$f_{-1,\pm} = h_{-1,\pm} = 0. \tag{2.13}$$

Remark 2.2. The sequences $\{f_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,+}\}_{\ell\in\mathbb{N}_0}$ can be computed recursively as follows: Assume that $f_{\ell,+}$, $g_{\ell,+}$, and $h_{\ell,+}$ are known. Equation (2.6) is a first-order difference equation in $g_{\ell+1,+}$ that can be solved directly and yields a local lattice function that is determined up to a new constant denoted by $c_{\ell+1,+} \in \mathbb{C}$. Relations (2.7) and (2.8) then determine $f_{\ell+1,+}$ and $h_{\ell+1,+}$, etc. The sequences $\{f_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,-}\}_{\ell\in\mathbb{N}_0}$ are determined similarly.

Upon setting

$$\gamma = 1 - \alpha\beta, \tag{2.14}$$

one explicitly obtains

$$\begin{aligned}
&f_{0,+} = c_{0,+}(-\alpha^+), \quad f_{1,+} = c_{0,+}\big(-\gamma^+\alpha^{++} + (\alpha^+)^2\beta\big) + c_{1,+}(-\alpha^+),\\
&g_{0,+} = \tfrac{1}{2}c_{0,+}, \quad g_{1,+} = c_{0,+}(-\alpha^+\beta) + \tfrac{1}{2}c_{1,+},\\
&h_{0,+} = c_{0,+}\beta, \quad h_{1,+} = c_{0,+}\big(\gamma\beta^- - \alpha^+\beta^2\big) + c_{1,+}\beta,\\
&f_{0,-} = c_{0,-}\alpha, \quad f_{1,-} = c_{0,-}\big(\gamma\alpha^- - \alpha^2\beta^+\big) + c_{1,-}\alpha,\\
&g_{0,-} = \tfrac{1}{2}c_{0,-}, \quad g_{1,-} = c_{0,-}(-\alpha\beta^+) + \tfrac{1}{2}c_{1,-},\\
&h_{0,-} = c_{0,-}(-\beta^+), \quad h_{1,-} = c_{0,-}\big(-\gamma^+\beta^{++} + \alpha(\beta^+)^2\big) + c_{1,-}(-\beta^+), \text{ etc.}
\end{aligned} \tag{2.15}$$

Here $\{c_{\ell,\pm}\}_{\ell\in\mathbb{N}}$ denote summation constants which naturally arise when solving the difference equations for $g_{\ell,\pm}$ in (2.6), (2.10). Subsequently, it will also be useful to work with the corresponding homogeneous coefficients $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$, defined by the vanishing of all summation constants $c_{k,\pm}$ for $k = 1,\dots,\ell$, and choosing $c_{0,\pm} = 1$,

$$\hat{f}_{0,+} = -\alpha^+, \quad \hat{f}_{0,-} = \alpha, \quad \hat{f}_{\ell,\pm} = f_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{2.16}$$

$$\hat{g}_{0,\pm} = \tfrac{1}{2}, \quad \hat{g}_{\ell,\pm} = g_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{2.17}$$

$$\hat{h}_{0,+} = \beta, \quad \hat{h}_{0,-} = -\beta^+, \quad \hat{h}_{\ell,\pm} = h_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{2.18}$$

By induction one infers that

$$f_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{f}_{k,\pm}, \quad g_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{g}_{k,\pm}, \quad h_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat{h}_{k,\pm}. \tag{2.19}$$

Next we define the 2×2 zero-curvature matrices

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix} \tag{2.20}$$

and

$$V_{\underline{p}}(z) = i \begin{pmatrix} G_{\underline{p}}^-(z) & -F_{\underline{p}}^-(z) \\ H_{\underline{p}}^-(z) & -K_{\underline{p}}^-(z) \end{pmatrix}, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2, \tag{2.21}$$

for appropriate Laurent polynomials $F_{\underline{p}}(z)$, $G_{\underline{p}}(z)$, $H_{\underline{p}}(z)$, and $K_{\underline{p}}(z)$ in the spectral parameter $z \in \mathbb{C}\setminus\{0\}$ to be determined shortly. By postulating the stationary zero-curvature relation,

$$0 = UV_{\underline{p}} - V_{\underline{p}}^+ U, \tag{2.22}$$

one concludes that (2.22) is equivalent to the following relations

$$z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^- = 0, \tag{2.23}$$

$$z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} - K_{\underline{p}} + K_{\underline{p}}^- = 0, \tag{2.24}$$

$$-F_{\underline{p}} + zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + K_{\underline{p}}^-) = 0, \tag{2.25}$$

$$z\beta(G_{\underline{p}}^- + K_{\underline{p}}) - zH_{\underline{p}} + H_{\underline{p}}^- = 0. \tag{2.26}$$

In order to make the connection between the zero-curvature formalism and the recursion relations (2.5)–(2.12), we now define Laurent polynomials $F_{\underline{p}}$, $G_{\underline{p}}$, $H_{\underline{p}}$, and $K_{\underline{p}}$ by[2]

$$F_{\underline{p}}(z) = \sum_{\ell=1}^{p_-} f_{p_- -\ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+ -1} f_{p_+ -1-\ell,+} z^{\ell}, \tag{2.27}$$

$$G_{\underline{p}}(z) = \sum_{\ell=1}^{p_-} g_{p_- -\ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+} g_{p_+ -\ell,+} z^{\ell}, \tag{2.28}$$

$$H_{\underline{p}}(z) = \sum_{\ell=0}^{p_- -1} h_{p_- -1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} h_{p_+ -\ell,+} z^{\ell}, \tag{2.29}$$

$$K_{\underline{p}}(z) = \sum_{\ell=0}^{p_-} g_{p_- -\ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} g_{p_+ -\ell,+} z^{\ell} = G_{\underline{p}}(z) + g_{p_-,-} - g_{p_+,+}. \tag{2.30}$$

The stationary zero-curvature relation (2.22), $0 = UV_{\underline{p}} - V_{\underline{p}}^+ U$, is then equivalent to

$$-\alpha(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- = 0, \tag{2.31}$$

$$\beta(g_{p_+,+}^- + g_{p_-,-}) + h_{p_+-1,+}^- - h_{p_--1,-} = 0. \tag{2.32}$$

Thus, varying $p_\pm \in \mathbb{N}_0$, equations (2.31) and (2.32) give rise to the stationary Ablowitz–Ladik (AL) hierarchy which we introduce as follows

$$\begin{gathered} \text{s-AL}_{\underline{p}}(\alpha, \beta) = \begin{pmatrix} -\alpha(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- \\ \beta(g_{p_+,+}^- + g_{p_-,-}) + h_{p_+-1,+}^- - h_{p_--1,-} \end{pmatrix} = 0, \\ \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2. \end{gathered} \tag{2.33}$$

[2] In this paper, a sum is interpreted as zero whenever the upper limit in the sum is strictly less than its lower limit.

Explicitly (recalling $\gamma = 1 - \alpha\beta$ and taking $p_- = p_+$ for simplicity),

$$\begin{aligned}
\text{s-AL}_{(0,0)}(\alpha,\beta) &= \begin{pmatrix} -c_{(0,0)}\alpha \\ c_{(0,0)}\beta \end{pmatrix} = 0, \\
\text{s-AL}_{(1,1)}(\alpha,\beta) &= \begin{pmatrix} -\gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0, \\
\text{s-AL}_{(2,2)}(\alpha,\beta) &= \begin{pmatrix} -\gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix} \\
&\quad + \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \quad \text{etc.},
\end{aligned} \tag{2.34}$$

represent the first few equations of the stationary Ablowitz–Ladik hierarchy. Here we introduced

$$c_{\underline{p}} = (c_{p_-,-} + c_{p_+,+})/2, \quad p_\pm \in \mathbb{N}_0. \tag{2.35}$$

By definition, the set of solutions of (2.33), with $\underline{p}$ ranging in $\mathbb{N}_0^2$ and $c_{\ell,\pm} \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, represents the class of algebro-geometric Ablowitz–Ladik solutions.

In the following we will frequently assume that α, β satisfy the $\underline{p}$th stationary Ablowitz–Ladik system supposing a particular choice of summation constants $c_{\ell,\pm} \in \mathbb{C}$, $\ell = 0, \dots, p_\pm$, $p_\pm \in \mathbb{N}_0$, has been made.

In accordance with our notation introduced in (2.16)–(2.18), the corresponding homogeneous stationary Ablowitz–Ladik equations are defined by

$$\text{s-}\widehat{\text{AL}}_{\underline{p}}(\alpha,\beta) = \text{s-AL}_{\underline{p}}(\alpha,\beta)\big|_{c_{0,\pm}=1,\, c_{\ell,\pm}=0,\, \ell=1,\dots,p_\pm}, \quad \underline{p} = (p_-,p_+) \in \mathbb{N}_0^2. \tag{2.36}$$

One can show (cf. [31, Lemma 2.2]) that $g_{p_+,+} = g_{p_-,-}$ up to a lattice constant which can be set equal to zero without loss of generality. Thus, we will henceforth assume that

$$g_{p_+,+} = g_{p_-,-}, \tag{2.37}$$

which in turn implies that

$$K_{\underline{p}} = G_{\underline{p}} \tag{2.38}$$

and hence renders $V_{\underline{p}}$ in (2.21) traceless in the stationary context. (We note that equations (2.37) and (2.38) cease to be valid in the time-dependent context, though.)

Next, still assuming (2.1) and taking into account (2.38), one infers by taking determinants in the stationary zero-curvature equation (2.22) that the quantity

$$R_{\underline{p}} = G_{\underline{p}}^2 - F_{\underline{p}} H_{\underline{p}} \tag{2.39}$$

is a lattice constant, that is, $R_{\underline{p}} - R_{\underline{p}}^- = 0$. Hence, $R_{\underline{p}}(z)$ only depends on z, and assuming in addition to (2.1) that

$$c_{0,\pm} \in \mathbb{C} \setminus \{0\}, \tag{2.40}$$

one may write $R_{\underline{p}}$ as[3]

$$R_{\underline{p}}(z) = (c_{0,+}^2/4)z^{-2p_-}\prod_{m=0}^{2p+1}(z-E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}\setminus\{0\}, \tag{2.41}$$

where

$$p = p_- + p_+ - 1. \tag{2.42}$$

In addition, we note that the summation constants $c_{1,\pm},\dots,c_{p_\pm,\pm}$ in (2.33) can be expressed as symmetric functions in the zeros $E_0,\dots,E_{2p+1}$ of the associated Laurent polynomial $R_{\underline{p}}$ in (2.41). In fact, one can prove (cf. [31]) that

$$c_{\ell,\pm} = c_{0,\pm}c_\ell\big(\underline{E}^{\pm 1}\big), \quad \ell = 0,\dots,p_\pm, \tag{2.43}$$

where

$$\begin{aligned}
&c_0\big(\underline{E}^{\pm1}\big) = 1,\\
&c_k\big(\underline{E}^{\pm1}\big)\\
&= -\sum_{\substack{j_0,\dots,j_{2p+1}=0\\ j_0+\cdots+j_{2p+1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{2p+1})!}{2^{2k}(j_0!)^2\cdots(j_{2p+1}!)^2(2j_0-1)\cdots(2j_{2p+1}-1)} E_0^{\pm j_0}\cdots E_{2p+1}^{\pm j_{2p+1}},\\
&\hspace{25em} k\in\mathbb{N},
\end{aligned} \tag{2.44}$$

are symmetric functions of $\underline{E}^{\pm1} = (E_0^{\pm1},\dots,E_{2p+1}^{\pm1})$.

Next we turn to the time-dependent Ablowitz–Ladik hierarchy. For that purpose the coefficients α and β are now considered as functions of both the lattice point and time. For each system in the hierarchy, that is, for each $\underline{p}\in\mathbb{N}_0^2$, we introduce a deformation (time) parameter $t_{\underline{p}}\in\mathbb{R}$ in α,β, replacing $\alpha(n),\beta(n)$ by $\alpha(n,t_{\underline{p}}),\beta(n,t_{\underline{p}})$. Moreover, the definitions (2.20), (2.21), and (2.27)–(2.30) of $U, V_{\underline{p}}$, and $F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}, K_{\underline{p}}$, respectively, still apply. Imposing the zero-curvature relation

$$U_{t_{\underline{p}}} + UV_{\underline{p}} - V_{\underline{p}}^+U = 0, \quad \underline{p}\in\mathbb{N}_0^2, \tag{2.45}$$

then results in the equations

$$\alpha_{t_{\underline{p}}} = i\big(zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + K_{\underline{p}}^-) - F_{\underline{p}}\big), \tag{2.46}$$

$$\beta_{t_{\underline{p}}} = -i\big(\beta(G_{\underline{p}}^- + K_{\underline{p}}) - H_{\underline{p}} + z^{-1}H_{\underline{p}}^-\big), \tag{2.47}$$

$$0 = z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^-, \tag{2.48}$$

$$0 = z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} + K_{\underline{p}}^- - K_{\underline{p}}. \tag{2.49}$$

Varying $\underline{p}\in\mathbb{N}_0^2$, the collection of evolution equations

$$\begin{aligned}
\mathrm{AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{\underline{p}}} - \alpha(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- \\ -i\beta_{t_{\underline{p}}} + \beta(g_{p_+,+}^- + g_{p_-,-}) - h_{p_--1,-} + h_{p_+-1,+}^- \end{pmatrix} = 0,\\
t_{\underline{p}}\in\mathbb{R},\ \underline{p} = (p_-,p_+)\in\mathbb{N}_0,
\end{aligned} \tag{2.50}$$

then defines the time-dependent Ablowitz–Ladik hierarchy. Explicitly,

$$\mathrm{AL}_{(0,0)}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{(0,0)}} - c_{(0,0)}\alpha \\ -i\beta_{t_{(0,0)}} + c_{(0,0)}\beta \end{pmatrix} = 0,$$

[3] We use the convention that a product is to be interpreted equal to 1 whenever the upper limit of the product is strictly less than its lower limit.

$$\mathrm{AL}_{(1,1)}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{(1,1)}} - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ -i\beta_{t_{(1,1)}} + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0,$$

$$\mathrm{AL}_{(2,2)}(\alpha,\beta) \tag{2.51}$$

$$= \begin{pmatrix} -i\alpha_{t_{(2,2)}} - \gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ -i\beta_{t_{(2,2)}} + \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix}$$

$$+ \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \text{ etc.,}$$

represent the first few equations of the time-dependent Ablowitz–Ladik hierarchy. Here we recall the definition of $c_{\underline{p}}$ in (2.35).

The special case $\underline{p} = (1,1)$, $c_{0,\pm} = 1$, and $c_{(1,1)} = -2$, that is,

$$\begin{pmatrix} -i\alpha_{t_1} - \gamma(\alpha^- + \alpha^+) + 2\alpha \\ -i\beta_{t_1} + \gamma(\beta^- + \beta^+) - 2\beta \end{pmatrix} = 0, \tag{2.52}$$

represents *the* Ablowitz–Ladik system (1.1).

By (2.50), (2.6), and (2.10), the time derivative of $\gamma = 1 - \alpha\beta$ is given by

$$\gamma_{t_{\underline{p}}} = i\gamma\big((g_{p_+,+} - g^-_{p_+,+}) - (g_{p_-,-} - g^-_{p_-,-})\big). \tag{2.53}$$

The corresponding homogeneous equations are then defined by

$$\widehat{\mathrm{AL}}_{\underline{p}}(\alpha,\beta) = \mathrm{AL}_{\underline{p}}(\alpha,\beta)\big|_{c_{0,\pm}=1,\, c_{\ell,\pm}=0,\, \ell=1,\dots,p_\pm} = 0, \quad \underline{p} = (p_-,p_+) \in \mathbb{N}_0^2. \tag{2.54}$$

Remark 2.3. From (2.23)–(2.26) and the explicit computations of the coefficients $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$, one concludes that the zero-curvature equation (2.45) and hence the Ablowitz–Ladik hierarchy is invariant under the scaling transformation

$$\alpha \to \alpha_c = \{c\,\alpha(n)\}_{n\in\mathbb{Z}}, \quad \beta \to \beta_c = \{\beta(n)/c\}_{n\in\mathbb{Z}}, \quad c \in \mathbb{C}\setminus\{0\}. \tag{2.55}$$

In particular, solutions α, β of the stationary and time-dependent AL equations are determined only up to a multiplicative constant.

Remark 2.4. (i) The special choices $\beta = \pm\overline{\alpha}$, $c_{0,\pm} = 1$ lead to the discrete nonlinear Schrödinger hierarchy. In particular, choosing $c_{(1,1)} = -2$ yields the discrete nonlinear Schrödinger equation in its usual form (see, e.g., [7, Ch. 3] and the references cited therein),

$$-i\alpha_t - (1 \mp |\alpha|^2)(\alpha^- + \alpha^+) + 2\alpha = 0, \tag{2.56}$$

as its first nonlinear element. The choice $\beta = \overline{\alpha}$ is called the *defocusing* case, $\beta = -\overline{\alpha}$ represents the *focusing* case of the discrete nonlinear Schrödinger hierarchy.
(ii) The alternative choice $\beta = \overline{\alpha}$, $c_{0,\pm} = \mp i$, leads to the hierarchy of Schur flows. In particular, choosing $c_{(1,1)} = 0$ yields

$$\alpha_t - (1 - |\alpha|^2)(\alpha^+ - \alpha^-) = 0 \tag{2.57}$$

as the first nonlinear element of this hierarchy (cf. [11], [22], [23], [34], [41], [49]).

Finally, we briefly recall the Lax pair $(L, P_{\underline{p}})$ for the Ablowitz–Ladik hierarchy and refer to [33] for detailed discussions of this topic.

In the standard basis $\{\delta_m\}_{m\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ defined by

$$\delta_m = \{\delta_{m,n}\}_{n\in\mathbb{Z}},\ m\in\mathbb{Z}, \quad \delta_{m,n} = \begin{cases} 1, & m=n, \\ 0, & m\neq n, \end{cases} \tag{2.58}$$

the underlying Lax difference expression L is given by

$$L = \left(\begin{array}{cccccccc} \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & -\alpha(0)\rho(-1) & -\beta(-1)\alpha(0) & -\alpha(1)\rho(0) & \rho(0)\rho(1) & & \text{\Large 0} & \\ & \rho(-1)\rho(0) & \beta(-1)\rho(0) & -\beta(0)\alpha(1) & \beta(0)\rho(1) & 0 & & \\ & & 0 & -\alpha(2)\rho(1) & -\beta(1)\alpha(2) & -\alpha(3)\rho(2) & \rho(2)\rho(3) & \\ & & & \rho(1)\rho(2) & \beta(1)\rho(2) & -\beta(2)\alpha(3) & \beta(2)\rho(3) & 0 \\ & \text{\Large 0} & & & \ddots & \ddots & \ddots & \ddots \quad \ddots \end{array}\right) \tag{2.59}$$

$$\begin{aligned} &= \rho^-\rho\,\delta_{\text{even}}\,S^{--} + (\beta^-\rho\,\delta_{\text{even}} - \alpha^+\rho\,\delta_{\text{odd}})S^- - \beta\alpha^+ \\ &\quad + (\beta\rho^+\,\delta_{\text{even}} - \alpha^{++}\rho^+\,\delta_{\text{odd}})S^+ + \rho^+\rho^{++}\,\delta_{\text{odd}}\,S^{++}, \end{aligned} \tag{2.60}$$

where δ_{even} and δ_{odd} denote the characteristic functions of the even and odd integers,

$$\delta_{\text{even}} = \chi_{2\mathbb{Z}}, \quad \delta_{\text{odd}} = 1 - \delta_{\text{even}} = \chi_{2\mathbb{Z}+1}. \tag{2.61}$$

In particular, terms of the form $-\beta(n)\alpha(n+1)$ represent the diagonal (n,n)-entries, $n\in\mathbb{Z}$, in the infinite matrix (2.59). In addition, we used the abbreviation

$$\rho = \gamma^{1/2} = (1-\alpha\beta)^{1/2}. \tag{2.62}$$

Next, let T be a bounded operator in the Hilbert space $\ell^2(\mathbb{Z})$ (with scalar product denoted by $(\,\cdot\,,\,\cdot\,)$). Given the standard basis (2.58) in $\ell^2(\mathbb{Z})$, we represent T by

$$T = \big(T(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T(m,n) = (\delta_m, T\,\delta_n), \quad (m,n)\in\mathbb{Z}^2. \tag{2.63}$$

Actually, for our purpose below, it is sufficient that T is an N-diagonal matrix for some $N\in\mathbb{N}$. Moreover, we introduce the upper and lower triangular parts $T_\pm$ of T by

$$T_\pm = \big(T_\pm(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T_\pm(m,n) = \begin{cases} T(m,n), & \pm(n-m)>0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.64}$$

Then, the finite difference expression $P_{\underline{p}}$ is given by

$$\begin{aligned} P_{\underline{p}} &= \frac{i}{2}\sum_{\ell=1}^{p_+} c_{p_+-\ell,+}\big((L^\ell)_+ - (L^\ell)_-\big) - \frac{i}{2}\sum_{\ell=1}^{p_-} c_{p_--\ell,-}\big((L^{-\ell})_+ - (L^{-\ell})_-\big) \\ &\quad - \frac{i}{2}c_{\underline{p}}\,Q_d, \quad \underline{p} = (p_-,p_+)\in\mathbb{N}_0^2, \end{aligned} \tag{2.65}$$

with Q_d denoting the doubly infinite diagonal matrix

$$Q_d = \big((-1)^k\delta_{k,\ell}\big)_{k,\ell\in\mathbb{Z}} \tag{2.66}$$

and $c_{\underline{p}} = (c_{p_-,-} + c_{p_+,+})/2$. The commutator relations $[P_{\underline{p}}, L] = 0$ and $L_{t_{\underline{p}}} - [P_{\underline{p}}, L] = 0$ are then equivalent to the stationary and time-dependent Ablowitz–Ladik equations (2.33) and (2.50), respectively.

3. Properties of Stationary Algebro-Geometric Solutions of the Ablowitz–Ladik Hierarchy

In this section we present a quick review of properties of algebro-geometric solutions of the stationary Ablowitz–Ladik hierarchy. We refer to [31] and [32] for detailed presentations.

We recall the hyperelliptic curve $\mathcal{K}_p$ of genus p, where

$$\begin{aligned} &\mathcal{K}_p\colon \mathcal{F}_p(z,y) = y^2 - 4c_{0,+}^{-2} z^{2p_-} R_{\underline{p}}(z) = y^2 - \prod_{m=0}^{2p+1} (z - E_m) = 0, \\ &R_{\underline{p}}(z) = \left(\frac{c_{0,+}}{2z^{p_-}}\right)^2 \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}\setminus\{0\},\ p = p_- + p_+ - 1. \end{aligned} \tag{3.1}$$

Throughout this section we make the assumption:

Hypothesis 3.1. *Suppose that*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}} \text{ and } \alpha(n)\beta(n) \notin \{0,1\} \text{ for all } n \in \mathbb{Z}. \tag{3.2}$$

In addition, assume that the affine part of the hyperelliptic curve $\mathcal{K}_p$ *in* (3.1) *is nonsingular, that is, suppose that*

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 2p+1. \tag{3.3}$$

The curve $\mathcal{K}_p$ is compactified by joining two points $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_p$. Points P on $\mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z,y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z,y) = 0$. The complex structure on $\mathcal{K}_p$ is then defined in the usual way, see Appendix A. Hence, $\mathcal{K}_p$ becomes a two-sheeted hyperelliptic Riemann surface of genus p in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_p$ (i.e., by fixing $E_0, \dots, E_{2p+1}$), the summation constants $c_{1,\pm}, \dots, c_{p_\pm,\pm}$ in $f_{p_\pm,\pm}$, $g_{p_\pm,\pm}$, and $h_{p_\pm,\pm}$ (and hence in the corresponding stationary s-AL$_{\underline{p}}$ equations) are uniquely determined as is clear from (2.44) which establishes the summation constants $c_{\ell,\pm}$ as symmetric functions of $E_0^{\pm 1}, \dots, E_{2p+1}^{\pm 1}$.

For notational simplicity we will usually tacitly assume that $p \in \mathbb{N}$.

We denote by $\{\mu_j(n)\}_{j=1,\dots,p}$ and $\{\nu_j(n)\}_{j=1,\dots,p}$ the zeros of $(\,\cdot\,)^{p_-} F_{\underline{p}}(\,\cdot\,,n)$ and $(\,\cdot\,)^{p_- -1} H_{\underline{p}}(\,\cdot\,,n)$, respectively. Thus, we may write

$$F_{\underline{p}}(z) = -c_{0,+}\alpha^+ z^{-p_-} \prod_{j=1}^{p} (z - \mu_j), \tag{3.4}$$

$$H_{\underline{p}}(z) = c_{0,+}\beta z^{-p_- +1} \prod_{j=1}^{p} (z - \nu_j), \tag{3.5}$$

and we recall that (cf. (2.39))

$$R_{\underline{p}} - G_{\underline{p}}^2 = -F_{\underline{p}} H_{\underline{p}}. \tag{3.6}$$

The next step is crucial; it permits us to "lift" the zeros μ_j and ν_j from the complex plane $\mathbb{C}$ to the curve $\mathcal{K}_p$. From (3.6) one infers that

$$R_{\underline{p}}(z) - G_{\underline{p}}(z)^2 = 0, \quad z \in \{\mu_j, \nu_k\}_{j,k=1,\dots,p}. \tag{3.7}$$

We now introduce $\{\hat{\mu}_j\}_{j=1,\dots,p} \subset \mathcal{K}_p$ and $\{\hat{\nu}_j\}_{j=1,\dots,p} \subset \mathcal{K}_p$ by

$$\hat{\mu}_j(n) = (\mu_j(n), (2/c_{0,+})\mu_j(n)^{p_-} G_{\underline{p}}(\mu_j(n), n)), \quad j = 1, \dots, p, \ n \in \mathbb{Z}, \tag{3.8}$$

and

$$\hat{\nu}_j(n) = (\nu_j(n), -(2/c_{0,+})\nu_j(n)^{p_-} G_{\underline{p}}(\nu_j(n), n)), \quad j = 1, \dots, p, \ n \in \mathbb{Z}. \tag{3.9}$$

We also introduce the points $P_{0,\pm}$ by

$$P_{0,\pm} = (0, \pm(c_{0,-}/c_{0,+})) \in \mathcal{K}_p, \quad \frac{c_{0,-}^2}{c_{0,+}^2} = \prod_{m=0}^{2p+1} E_m. \tag{3.10}$$

We emphasize that $P_{0,\pm}$ and $P_{\infty_\pm}$ are not necessarily on the same sheet of $\mathcal{K}_p$. Moreover,

$$y(P) \underset{\zeta\to 0}{=} \begin{cases} \mp\zeta^{-2p}(1 + O(\zeta)), & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm(c_{0,-}/c_{0,+}) + O(\zeta), & P \to P_{0,\pm}, \quad \zeta = z. \end{cases} \tag{3.11}$$

Next we introduce the fundamental meromorphic function on $\mathcal{K}_p$ by

$$\begin{aligned} \phi(P, n) &= \frac{(c_{0,+}/2) z^{-p_-} y + G_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n)} && (3.12) \\ &= \frac{-H_{\underline{p}}(z, n)}{(c_{0,+}/2) z^{-p_-} y - G_{\underline{p}}(z, n)}, && (3.13) \\ &\qquad P = (z, y) \in \mathcal{K}_p, \ n \in \mathbb{Z}, \end{aligned}$$

with divisor $(\phi(\,\cdot\,, n))$ of $\phi(\,\cdot\,, n)$ given by

$$(\phi(\,\cdot\,, n)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}, \tag{3.14}$$

using (3.4) and (3.5). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}, \ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_p\} \in \operatorname{Sym}^p(\mathcal{K}_p). \tag{3.15}$$

For brevity, and in close analogy to the Toda hierarchy, we will frequently refer to $\underline{\hat{\mu}}$ and $\underline{\hat{\nu}}$ as the Dirichlet and Neumann divisors, respectively.

Given $\phi(\,\cdot\,, n)$, the meromorphic stationary Baker–Akhiezer vector $\Psi(\,\cdot\,, n, n_0)$ on $\mathcal{K}_p$ is then defined by

$$\begin{aligned} &\Psi(P, n, n_0) = \begin{pmatrix} \psi_1(P, n, n_0) \\ \psi_2(P, n, n_0) \end{pmatrix}, \\ &\psi_1(P, n, n_0) = \begin{cases} \prod_{n'=n_0+1}^{n} \big(z + \alpha(n')\phi^-(P, n')\big), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n+1}^{n_0} \big(z + \alpha(n')\phi^-(P, n')\big)^{-1}, & n \leq n_0 - 1, \end{cases} && (3.16) \\ &\psi_2(P, n, n_0) = \phi(P, n_0) \begin{cases} \prod_{n'=n_0+1}^{n} \big(z\beta(n')\phi^-(P, n')^{-1} + 1\big), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n+1}^{n_0} \big(z\beta(n')\phi^-(P, n')^{-1} + 1\big)^{-1}, & n \leq n_0 - 1. \end{cases} && (3.17) \end{aligned}$$

Basic properties of ϕ and Ψ are summarized in the following result.

Lemma 3.2 ([31])**.** *Suppose that α, β satisfy* (3.2) *and the $\underline{p}$th stationary Ablowitz–Ladik system* (2.33)*. Moreover, assume* (3.1) *and* (3.3) *and let $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n, n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \tag{3.18}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \tag{3.19}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \tag{3.20}$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p_-}\frac{y(P)}{F_{\underline{p}}(z)}. \tag{3.21}$$

The vector Ψ satisfies

$$U(z)\Psi^-(P) = \Psi(P), \tag{3.22}$$

$$V_{\underline{p}}(z)\Psi^-(P) = -(i/2)c_{0,+}z^{-p_-}y\Psi^-(P), \tag{3.23}$$

$$\psi_2(P, n, n_0) = \phi(P, n)\psi_1(P, n, n_0), \tag{3.24}$$

$$\psi_1(P, n, n_0)\psi_1(P^*, n, n_0) = z^{n-n_0}\frac{F_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n_0)}\Gamma(n, n_0), \tag{3.25}$$

$$\psi_2(P, n, n_0)\psi_2(P^*, n, n_0) = z^{n-n_0}\frac{H_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n_0)}\Gamma(n, n_0), \tag{3.26}$$

$$\begin{aligned}&\psi_1(P, n, n_0)\psi_2(P^*, n, n_0) + \psi_1(P^*, n, n_0)\psi_2(P, n, n_0)\\ &\quad = 2z^{n-n_0}\frac{G_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n_0)}\Gamma(n, n_0),\end{aligned} \tag{3.27}$$

$$\begin{aligned}&\psi_1(P, n, n_0)\psi_2(P^*, n, n_0) - \psi_1(P^*, n, n_0)\psi_2(P, n, n_0)\\ &\quad = -c_{0,+}z^{n-n_0-p_-}\frac{y}{F_{\underline{p}}(z, n_0)}\Gamma(n, n_0),\end{aligned} \tag{3.28}$$

where we used the abbreviation

$$\Gamma(n, n_0) = \begin{cases} \prod_{n'=n_0+1}^{n} \gamma(n') & n \ge n_0 + 1, \\ 1 & n = n_0, \\ \prod_{n'=n+1}^{n_0} \gamma(n')^{-1} & n \le n_0 - 1. \end{cases} \tag{3.29}$$

Combining the Laurent polynomial recursion approach of Section 2 with (3.4) and (3.5) readily yields trace formulas for $f_{\ell,\pm}$ and $h_{\ell,\pm}$ in terms of symmetric functions of the zeros μ_j and ν_k of $(\,\cdot\,)^{p_-}F_{\underline{p}}$ and $(\,\cdot\,)^{p_--1}H_{\underline{p}}$, respectively. For simplicity we just record the simplest cases.

Lemma 3.3 ([31])**.** *Suppose that α, β satisfy* (3.2) *and the $\underline{p}$th stationary Ablowitz–Ladik system* (2.33)*. Then,*

$$\frac{\alpha}{\alpha^+} = (-1)^{p+1}\frac{c_{0,+}}{c_{0,-}}\prod_{j=1}^{p}\mu_j, \tag{3.30}$$

$$\frac{\beta^+}{\beta} = (-1)^{p+1}\frac{c_{0,+}}{c_{0,-}}\prod_{j=1}^{p}\nu_j, \tag{3.31}$$

$$\sum_{j=1}^{p}\mu_j = \alpha^+\beta - \gamma^+\frac{\alpha^{++}}{\alpha^+} - \frac{c_{1,+}}{c_{0,+}}, \tag{3.32}$$

$$\sum_{j=1}^{p}\nu_j = \alpha^+\beta - \gamma\frac{\beta^-}{\beta} - \frac{c_{1,+}}{c_{0,+}}. \tag{3.33}$$

Remark 3.4. The trace formulas in Lemma 3.3 illustrate why we assumed the condition $\alpha(n)\beta(n) \neq 0$ for all $n \in \mathbb{N}$ throughout this paper. Moreover, the following section shows that this condition is intimately connected with admissible divisors $\mathcal{D}_{\underline{\hat{\mu}}}, \mathcal{D}_{\underline{\hat{\nu}}}$ avoiding the exceptional points $P_{\infty_\pm}, P_{0,\pm}$. On the other hand, as is clear from the matrix representation (2.59) of the Lax difference expression L, if $\alpha(n_0)\beta(n_0) = 1$ for some $n_0 \in \mathbb{N}$, and hence $\rho(n_0) = 0$, the infinite matrix L splits into a direct sum of two half-line matrices $L_\pm(n_0)$ (in analogy to the familiar singular case of infinite Jacobi matrices $aS^+ + a^-S^- + b$ on $\mathbb{Z}$ with $a(n_0) = 0$). This explains why we assumed $\alpha(n)\beta(n) \neq 1$ for all $n \in \mathbb{N}$ throughout this paper.

Since nonspecial divisors and the linearization property of the Abel map when applied to $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ will play a fundamental role later on, we also recall the following facts.

Lemma 3.5 ([31], [32])**.** *Suppose that α, β satisfy* (3.2) *and the $\underline{p}$th stationary Ablowitz–Ladik system* (2.33)*. Moreover, assume* (3.1) *and* (3.3) *and let $n \in \mathbb{Z}$. Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}$, and $\mathcal{D}_{\underline{\hat{\nu}}}$, $\underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_p\}$, be the pole and zero divisors of degree p, respectively, associated with α, β, and ϕ defined according to* (3.8) *and* (3.9)*, that is,*

$$\begin{aligned} \hat{\mu}_j(n) &= (\mu_j(n), (2/c_{0,+})\mu_j(n)^{p_-}G_{\underline{p}}(\mu_j(n), n)), \quad j = 1, \dots, p, \\ \hat{\nu}_j(n) &= (\nu_j(n), -(2/c_{0,+})\nu_j(n)^{p_-}G_{\underline{p}}(\nu_j(n), n)), \quad j = 1, \dots, p. \end{aligned} \tag{3.34}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ are nonspecial for all $n \in \mathbb{Z}$. Moreover, the Abel map linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + (n - n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \tag{3.35}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0)}) + (n - n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \tag{3.36}$$

where $Q_0 \in \mathcal{K}_p$ is a given base point. In addition,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) + \underline{A}_{P_{0,-}}(P_{\infty_-}). \tag{3.37}$$

For theta function representations of α and β we refer to [32] and the references cited therein. These theta function representations also show that $\gamma(n) \notin \{0,1\}$ for all $n \in \mathbb{Z}$, that is, the second condition in (3.2) is satisfied for the stationary algebro-geometric AL solutions discussed in this section provided the associated Dirichlet and Neumann divisors are admissible.

4. The Stationary Algorithm

The aim of this section is to derive an algorithm that enables one to construct algebro-geometric solutions for the stationary Ablowitz–Ladik hierarchy for general initial data. Equivalently, we offer a solution of the inverse algebro-geometric spectral problem for general Lax operators L in (2.60), starting with initial divisors in general position.

Up to the end of Section 3 the material was based on the assumption that $\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}$ satisfy the $\underline{p}$th stationary AL system (2.33). Now we embark on the corresponding inverse problem consisting of constructing a solution of (2.33) given certain initial data. More precisely, we seek to construct solutions $\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}$ satisfying the $\underline{p}$th stationary Ablowitz–Ladik system (2.33) starting from a properly restricted set $\mathcal{M}_0$ of admissible nonspecial Dirichlet divisor initial data $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ at some fixed $n_0 \in \mathbb{Z}$,

$$\begin{aligned} &\underline{\hat{\mu}}(n_0) = \{\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)\} \in \mathcal{M}_0, \quad \mathcal{M}_0 \subset \operatorname{Sym}^p(\mathcal{K}_p), \\ &\hat{\mu}_j(n_0) = (\mu_j(n_0), (2/c_{0,+})\mu_j(n_0)^{p_-} G_{\underline{p}}(\mu_j(n_0), n_0)), \quad j = 1, \dots, p. \end{aligned} \tag{4.1}$$

For convenience we will frequently use the phrase that α, β *blow up* in this manuscript whenever one of the divisors $\mathcal{D}_{\underline{\hat{\mu}}}$ or $\mathcal{D}_{\underline{\hat{\nu}}}$ hits one of the points $P_{\infty_\pm}$, $P_{0,\pm}$.

Of course we would like to ensure that the sequences α, β obtained via our algorithm do not blow up. To investigate when this happens, we study the image of our divisors under the Abel map. A key ingredient in our analysis will be (3.35) which yields a linear discrete dynamical system on the Jacobi variety $J(\mathcal{K}_p)$. In particular, we will be led to investigate solutions $\mathcal{D}_{\underline{\hat{\mu}}}$, $\mathcal{D}_{\underline{\hat{\nu}}}$ of the discrete initial value problem

$$\begin{aligned} &\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + (n - n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \\ &\underline{\hat{\mu}}(n_0) = \{\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)\} \in \operatorname{Sym}^p(\mathcal{K}_p), \end{aligned} \tag{4.2}$$

respectively

$$\begin{aligned} &\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + \underline{A}_{P_{0,-}}(P_{\infty_-}) + (n - n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}), \\ &\underline{\hat{\nu}}(n_0) = \{\hat{\nu}_1(n_0), \dots, \hat{\nu}_p(n_0)\} \in \operatorname{Sym}^p(\mathcal{K}_p), \end{aligned} \tag{4.3}$$

where $Q_0 \in \mathcal{K}_p$ is a given base point. Eventually, we will be interested in solutions $\mathcal{D}_{\underline{\hat{\mu}}}$, $\mathcal{D}_{\underline{\hat{\nu}}}$ of (4.2), (4.3) with initial data $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ satisfying (4.1) and $\mathcal{M}_0$ to be specified as in (the proof of) Lemma 4.1.

Before proceeding to develop the stationary Ablowitz–Ladik algorithm, we briefly analyze the dynamics of (4.2).

Lemma 4.1. *Let $n \in \mathbb{Z}$ and suppose that $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is defined via* (4.2) *for some divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \operatorname{Sym}^p(\mathcal{K}_p)$.*
(i) If $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is nonspecial and does not contain any of the points $P_{0,\pm}$, $P_{\infty_\pm}$, and $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ contains one of the points $P_{0,\pm}$, $P_{\infty_\pm}$, then $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ contains $P_{0,-}$ or P_{∞_-} but not P_{∞_+} or $P_{0,+}$.
(ii) If $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is nonspecial and $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ is special, then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ contains at least one of the points P_{∞_+}, P_{∞_-} and one of the points $P_{0,+}$, $P_{0,-}$.
(iii) Item (i) holds if $n+1$ is replaced by $n-1$, P_{∞_+} by P_{∞_-}, and $P_{0,+}$ by $P_{0,-}$.
(iv) Items (i)–(iii) also hold for $\mathcal{D}_{\underline{\hat{\nu}}(n)}$.

Proof. (*i*) Suppose one point in $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ equals P_{∞_+} and denote the remaining ones by $\mathcal{D}_{\underline{\tilde{\mu}}(n+1)}$. Then (4.2) implies that $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\tilde{\mu}}(n+1)}) + \underline{A}_{Q_0}(P_{\infty_+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) + \underline{A}_{P_{0,-}}(P_{\infty_+})$. Since $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is assumed to be nonspecial one concludes $\mathcal{D}_{\underline{\hat{\mu}}(n)} = \mathcal{D}_{\underline{\tilde{\mu}}(n+1)} + \mathcal{D}_{P_{0,-}}$, contradicting our assumption on $\mathcal{D}_{\underline{\hat{\mu}}(n)}$. The statement for $P_{0,+}$ follows similarly; here we choose Q_0 to be a branch point of $\mathcal{K}_p$ such that $\underline{A}_{Q_0}(P^*) =$

$-\underline{A}_{Q_0}(P)$.
(*ii*) Next, we choose Q_0 to be a branch point of $\mathcal{K}_p$. If $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ is special, then it contains a pair of points (Q, Q^*) whose contribution will cancel under the Abel map, that is, $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n+1)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\eta}}(n+1)})$ for some $\mathcal{D}_{\underline{\hat{\eta}}(n+1)} \in \operatorname{Sym}^{p-2}(\mathcal{K}_p)$. Invoking (4.2) then shows that $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\eta}}(n+1)}) + \underline{A}_{Q_0}(P_{\infty_-}) + \underline{A}_{Q_0}(P_{0,-})$. As $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ was assumed to be nonspecial, this shows that $\mathcal{D}_{\underline{\hat{\mu}}(n)} = \mathcal{D}_{\underline{\hat{\eta}}(n+1)} + \mathcal{D}_{P_{\infty_-}} + \mathcal{D}_{P_{0,-}}$, as claimed.
(*iii*) This is proved as in item (*i*).
(*iv*) Since $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ satisfies the same equation as $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ in (4.2) (cf. (3.36)), items (*i*)–(*iii*) also hold for $\mathcal{D}_{\underline{\hat{\nu}}(n)}$. ☐

We also note the following result:

Lemma 4.2. *Let $n \in \mathbb{Z}$ and assume that $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ are nonspecial. Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ contains $P_{0,-}$ if and only if $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ contains P_{∞_-}. Moreover, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ contains P_{∞_+} if and only if $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ contains $P_{0,+}$.*

Proof. Suppose a point in $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ equals $P_{0,-}$ and denote the remaining ones by $\mathcal{D}_{\underline{\tilde{\mu}}(n)}$. By (3.37),

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\tilde{\mu}}(n)}) + \underline{A}_{Q_0}(P_{0,-}) + \underline{A}_{P_{0,-}}(P_{\infty_-}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\tilde{\mu}}(n)}) + \underline{A}_{Q_0}(P_{\infty_-}). \tag{4.4}$$

Since $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ is nonspecial, $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ contains P_{∞_-}, and vice versa. The second statement follows similarly. ☐

Let us call the points P_{∞_+}, P_{∞_-}, $P_{0,+}$, and $P_{0,-}$ *exceptional points*. Then Lemma 4.1 yields the following behavior of $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ assuming one starts with some nonspecial initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ without exceptional points: As n increases, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ stays nonspecial as long as it does not include exceptional points. If an exceptional point appears, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is still nonspecial and contains $P_{0,-}$ or P_{∞_-} at least once (but not $P_{0,+}$ and P_{∞_+}). Further increasing n, all instances of $P_{0,-}$ and P_{∞_-} will be rendered into $P_{0,+}$ and P_{∞_+}, until we have again a nonspecial divisor that has the same number of $P_{0,+}$ and P_{∞_+} as the first one had of $P_{0,-}$ and P_{∞_-}. Generically, one expects the subsequent divisor to be nonspecial without exceptional points again.

Next we show that most initial divisors are well-behaved in the sense that their iterates stay away from $P_{\infty_\pm}$, $P_{0,\pm}$. Since we want to show that this set is of full measure, it will be convenient to identify $\operatorname{Sym}^p(\mathcal{K}_p)$ with the Jacobi variety $J(\mathcal{K}_p)$ via the Abel map and take the Haar measure on $J(\mathcal{K}_p)$. Of course, the Abel map is only injective when restricted to the set of nonspecial divisors, but these are the only ones we are interested in.

Lemma 4.3. *The set $\mathcal{M}_0 \subset \operatorname{Sym}^p(\mathcal{K}_p)$ of initial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ for which $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n)}$, defined via* (4.2) *and* (4.3)*, are admissible* (*i.e., do not contain the points $P_{\infty_\pm}, P_{0,\pm}$*) *and hence are nonspecial for all $n \in \mathbb{Z}$, forms a dense set of full measure in the set $\operatorname{Sym}^p(\mathcal{K}_p)$ of positive divisors of degree p.*

Proof. Let $\mathcal{M}_{\infty,0}$ be the set of divisors in $\operatorname{Sym}^p(\mathcal{K}_p)$ for which (at least) one point is equal to $P_{\infty_\pm}$ or $P_{0,\pm}$. The image $\underline{\alpha}_{Q_0}(\mathcal{M}_{\infty,0})$ of $\mathcal{M}_{\infty,0}$ is then contained in the following set,

$$\underline{\alpha}_{Q_0}(\mathcal{M}_{\infty,0}) \subseteq \bigcup_{P\in\{P_{0,\pm},P_{\infty_\pm}\}} \big(\underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathrm{Sym}^{p-1}(\mathcal{K}_p))\big) \subset J(\mathcal{K}_p). \tag{4.5}$$

Since the (complex) dimension of $\mathrm{Sym}^{p-1}(\mathcal{K}_p)$ is $p-1$, its image must be of measure zero by Sard's theorem (see, e.g., [8, Sect. 3.6]). Similarly, let $\mathcal{M}_{\mathrm{sp}}$ be the set of special divisors, then its image is given by

$$\underline{\alpha}_{Q_0}(\mathcal{M}_{\mathrm{sp}}) = \underline{\alpha}_{Q_0}(\mathrm{Sym}^{p-2}(\mathcal{K}_p)), \tag{4.6}$$

assuming Q_0 to be a branch point. In particular, we conclude that $\underline{\alpha}_{Q_0}(\mathcal{M}_{\mathrm{sp}}) \subset \underline{\alpha}_{Q_0}(\mathcal{M}_{\infty,0})$ and thus $\underline{\alpha}_{Q_0}(\mathcal{M}_{\mathrm{sing}}) = \underline{\alpha}_{Q_0}(\mathcal{M}_{\infty,0})$ has measure zero, where

$$\mathcal{M}_{\mathrm{sing}} = \mathcal{M}_{\infty,0} \cup \mathcal{M}_{\mathrm{sp}}. \tag{4.7}$$

Hence,

$$\mathcal{S}_\mu = \bigcup_{n\in\mathbb{Z}} \big(\underline{\alpha}_{Q_0}(\mathcal{M}_{\mathrm{sing}}) + n\underline{A}_{P_{0,-}}(P_{\infty_+})\big) \quad \text{and} \quad \mathcal{S}_\nu = \mathcal{S}_\mu + \underline{A}_{P_{0,-}}(P_{\infty_-}) \tag{4.8}$$

are of measure zero as well. But the set $\mathcal{S}_\mu \cup \mathcal{S}_\nu$ contains all initial divisors for which $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ or $\mathcal{D}_{\underline{\hat{\nu}}(n)}$ will hit $P_{\infty_\pm}$ or $P_{0,\pm}$, or become special at some $n \in \mathbb{Z}$. We denote by $\mathcal{M}_0$ the inverse image of the complement of the set $\mathcal{S}_\mu \cup \mathcal{S}_\nu$ under the Abel map,

$$\mathcal{M}_0 = \underline{\alpha}_{Q_0}^{-1}\big(\mathrm{Sym}^p(\mathcal{K}_p) \setminus (\mathcal{S}_\mu \cup \mathcal{S}_\nu)\big). \tag{4.9}$$

Since $\mathcal{M}_0$ is of full measure, it is automatically dense in $\mathrm{Sym}^p(\mathcal{K}_p)$. ☐

Next, we describe the stationary Ablowitz–Ladik algorithm. Since this is a somewhat lengthy affair, we will break it up into several steps.

The Stationary Ablowitz–Ladik Algorithm:

We prescribe the following data

(i) The coefficient $\alpha(n_0) \in \mathbb{C}\setminus\{0\}$ and the constant $c_{0,+} \in \mathbb{C}\setminus\{0\}$.

(ii) The set

$$\{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}\setminus\{0\}, \quad E_m \neq E_{m'} \text{ for } m \neq m', \quad m,m' = 0,\dots,2p+1, \tag{4.10}$$

for some fixed $p \in \mathbb{N}$. Given $\{E_m\}_{m=0}^{2p+1}$, we introduce the function

$$R_{\underline{p}}(z) = \left(\frac{c_{0,+}}{2z^{p_-}}\right)^2 \prod_{m=0}^{2p+1} (z - E_m) \tag{4.11}$$

and the hyperelliptic curve $\mathcal{K}_p$ with nonsingular affine part as in (3.1).

(iii) The nonspecial divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathrm{Sym}^p(\mathcal{K}_p), \tag{4.12}$$

where $\underline{\hat{\mu}}(n_0)$ is of the form

$$\underline{\hat{\mu}}(n_0) = \{\underbrace{\hat{\mu}_1(n_0),\dots,\hat{\mu}_1(n_0)}_{p_1(n_0)\text{ times}},\dots,\underbrace{\hat{\mu}_{q(n_0)},\dots,\hat{\mu}_{q(n_0)}}_{p_{q(n_0)}(n_0)\text{ times}}\} \tag{4.13}$$

with

$$\hat{\mu}_k(n_0) = (\mu_k(n_0), y(\hat{\mu}_k(n_0))), \quad \mu_k(n_0) \neq \mu_{k'}(n_0) \text{ for } k \neq k',\ k,k' = 1,\dots,q(n_0), \tag{4.14}$$

and

$$p_k(n_0) \in \mathbb{N}, \ k = 1, \dots, q(n_0), \quad \sum_{k=1}^{q(n_0)} p_k(n_0) = p. \tag{4.15}$$

With $\{E_m\}_{m=0}^{2p+1}$, $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$, $\alpha(n_0)$, and $c_{0,+}$ prescribed, we next introduce the following quantities (for $z \in \mathbb{C} \setminus \{0\}$):

$$\alpha^+(n_0) = \alpha(n_0)\bigg(\prod_{m=0}^{2p+1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n_0)} \mu_k(n_0)^{-p_k(n_0)}, \tag{4.16}$$

$$c_{0,-}^2 = c_{0,+}^2 \prod_{m=0}^{2p+1} E_m, \tag{4.17}$$

$$F_{\underline{p}}(z, n_0) = -c_{0,+}\alpha^+(n_0) z^{-p_-} \prod_{k=1}^{q(n_0)} (z - \mu_k(n_0))^{p_k(n_0)}, \tag{4.18}$$

$$\begin{aligned} G_{\underline{p}}(z, n_0) &= \frac{1}{2}\bigg(\frac{1}{\alpha(n_0)} - \frac{z}{\alpha^+(n_0)}\bigg) F_{\underline{p}}(z, n_0) \\ &\quad - \frac{z}{2\alpha^+(n_0)} F_{\underline{p}}(z, n_0) \sum_{k=1}^{q(n_0)} \sum_{\ell=0}^{p_k(n_0)-1} \frac{\big(d^\ell\big(\zeta^{-1} y(P)\big)/d\zeta^\ell\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n_0)}}{\ell!(p_k(n_0) - \ell - 1)!} \\ &\quad \times \Bigg(\frac{d^{p_k(n_0)-\ell-1}}{d\zeta^{p_k(n_0)-\ell-1}}\Bigg((z-\zeta)^{-1} \prod_{k'=1,\, k'\neq k}^{q(n_0)} (\zeta - \mu_{k'}(n_0))^{-p_{k'}(n_0)}\Bigg)\Bigg)\Bigg|_{\zeta=\mu_k(n_0)}. \end{aligned} \tag{4.19}$$

Here the sign of the square root is chosen according to (4.14).

Next we record a series of facts:

(I) By construction (cf. Lemma B.1),

$$\begin{aligned} &\frac{d^\ell\big(G_{\underline{p}}(z, n_0)^2\big)}{dz^\ell}\bigg|_{z=\mu_k(n_0)} = \frac{d^\ell R_{\underline{p}}(z)}{dz^\ell}\bigg|_{z=\mu_k(n_0)}, \\ &z \in \mathbb{C} \setminus \{0\}, \quad \ell = 0, \dots, p_k(n_0) - 1, \ k = 1, \dots, q(n_0). \end{aligned} \tag{4.20}$$

(II) Since $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ is nonspecial by hypothesis, one concludes that

$$p_k(n_0) \geq 2 \ \text{ implies } \ R_{\underline{p}}(\mu_k(n_0)) \neq 0, \quad k = 1, \dots, q(n_0). \tag{4.21}$$

(III) By (4.19) and (4.20) one infers that $F_{\underline{p}}$ divides $G_{\underline{p}}^2 - R_{\underline{p}}$.

(IV) By (4.11) and (4.19) one verifies that

$$G_{\underline{p}}(z, n_0)^2 - R_{\underline{p}}(z) \underset{z\to\infty}{=} O(z^{2p_+-1}), \tag{4.22}$$

$$G_{\underline{p}}(z, n_0)^2 - R_{\underline{p}}(z) \underset{z\to 0}{=} O(z^{-2p_-+1}). \tag{4.23}$$

By **(III)** and **(IV)** we may write

$$G_{\underline{p}}(z, n_0)^2 - R_{\underline{p}}(z) = F_{\underline{p}}(z, n_0)\check{H}_{q,r}(z, n_0), \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.24}$$

for some $q \in \{0, \dots, p_- - 1\}$, $r \in \{0, \dots, p_+\}$, where $\check{H}_{q,r}(z, n_0)$ is a Laurent polynomial of the form $c_{-q}z^{-q} + \cdots + c_r z^r$. If, in fact, $\check{H}_{0,0} = 0$, then $R_{\underline{p}}(z) = G_{\underline{p}}(z, n_0)^2$ would yield double zeros of $R_{\underline{p}}$, contradicting our basic hypothesis (4.10).

Thus we conclude that in the case $r = q = 0$, $\check{H}_{0,0}$ cannot vanish identically and hence we may break up (4.24) in the following manner

$$\check{\phi}(P, n_0) = \frac{G_{\underline{p}}(z, n_0) + (c_{0,+}/2)z^{-p_-}y}{F_{\underline{p}}(z, n_0)} = \frac{\check{H}_{q,r}(z, n_0)}{G_{\underline{p}}(z, n_0) - (c_{0,+}/2)z^{-p_-}y}, \quad P = (z, y) \in \mathcal{K}_p. \tag{4.25}$$

Next we decompose

$$\check{H}_{q,r}(z, n_0) = Cz^{-q} \prod_{j=1}^{r+q} (z - \nu_j(n_0)), \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.26}$$

where $C \in \mathbb{C} \setminus \{0\}$ and $\{\nu_j(n_0)\}_{j=1}^{r+q} \subset \mathbb{C}$ (if $r = q = 0$ we replace the product in (4.26) by 1). By inspection of the local zeros and poles as well as the behavior near $P_{0,\pm}$, $P_{\infty_\pm}$ of the function $\check{\phi}(\,\cdot\,, n_0)$ using (3.11), its divisor, $\big(\check{\phi}(\,\cdot\,, n_0)\big)$, is given by

$$\big(\check{\phi}(\,\cdot\,, n_0)\big) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n_0)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n_0)}, \tag{4.27}$$

where

$$\underline{\hat{\nu}}(n_0) = \{\underbrace{P_{0,-}, \dots, P_{0,-}}_{p_- -1-q \text{ times}}, \hat{\nu}_1(n_0), \dots, \hat{\nu}_{r+q}(n_0), \underbrace{P_{\infty_+}, \dots, P_{\infty_+}}_{p_+ - r \text{ times}}\}. \tag{4.28}$$

In the following we call a positive divisor of degree p *admissible* if it does not contain any of the points $P_{\infty_\pm}, P_{0,\pm}$.

Hence,

$$\mathcal{D}_{\underline{\hat{\nu}}(n_0)} \text{ is an admissible divisor if and only if } r = p_+ \text{ and } q = p_- - 1. \tag{4.29}$$

We note that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) + \underline{A}_{P_{0,-}}(P_{\infty_-}), \tag{4.30}$$

in accordance with (3.37).

(V) Assuming that (4.22), (4.23) are precisely of order $z^{\pm(2p_\pm - 1)}$, that is, assuming $r = p_+$ and $q = p_- - 1$ in (4.24), we rewrite (4.24) in the more appropriate manner

$$G_{\underline{p}}(z, n_0)^2 - R_{\underline{p}}(z) = F_{\underline{p}}(z, n_0) H_{\underline{p}}(z, n_0), \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.31}$$

(We will later discuss conditions which indeed guarantee that $q = p_- - 1$ and $r = p_+$, cf. (4.29) and the discussion in step **(X)** below.) By construction, $H_{\underline{p}}(\,\cdot\,, n_0)$ is then of the type

$$H_{\underline{p}}(z, n_0) = c_{0,+}\beta(n_0) z^{-p_- + 1} \prod_{k=1}^{\ell(n_0)} (z - \nu_k(n_0))^{s_k(n_0)}, \quad \sum_{k=1}^{\ell(n_0)} s_k(n_0) = p,$$
$$\nu_k(n_0) \neq \nu_{k'}(n_0) \text{ for } k \neq k', \ k, k' = 1, \dots, \ell(n_0), \ z \in \mathbb{C} \setminus \{0\}, \tag{4.32}$$

where we introduced the coefficient $\beta(n_0)$. We define

$$\hat{\nu}_k(n_0) = (\nu_k(n_0), -(2/c_{0,+})\nu_k(n_0)^{p_-} G_{\underline{p}}(\nu_k(n_0), n_0)), \quad k = 1, \dots, \ell(n_0). \tag{4.33}$$

An explicit computation of $\beta(n_0)$ then yields

$$\alpha^+(n_0)\beta(n_0) = -\frac{1}{2} \sum_{k=1}^{q(n_0)} \frac{\big(d^{p_k(n_0)-1}\big(\zeta^{-1}y(P)\big)/d\zeta^{p_k(n_0)-1}\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n_0)}}{(p_k(n_0)-1)!}$$

$$\times \prod_{k'=1,\, k'\neq k}^{q(n_0)} (\mu_k(n_0)-\mu_{k'}(n_0))^{-p_k(n_0)}$$

$$+\frac{1}{2}\bigg(\frac{\alpha^+(n_0)}{\alpha(n_0)}+\sum_{k=1}^{q(n_0)} p_k(n_0)\mu_k(n_0)-\frac{1}{2}\sum_{m=0}^{2p+1} E_m\bigg). \tag{4.34}$$

The result (4.34) is obtained by inserting the expressions (4.18), (4.19), and (4.32) for $F_{\underline{p}}(\,\cdot\,,n_0)$, $G_{\underline{p}}(\,\cdot\,,n_0)$, and $H_{\underline{p}}(\,\cdot\,,n_0)$ into (4.31) and collecting all terms of order z^{2p_+-1}.

(VI) Introduce

$$\beta^+(n_0)=\beta(n_0)\prod_{k=1}^{\ell(n_0)} \nu_k(n_0)^{s_k(n_0)}\bigg(\prod_{m=0}^{2p+1} E_m\bigg)^{-1/2}. \tag{4.35}$$

(VII) Using $G_{\underline{p}}(z,n_0)$, $H_{\underline{p}}(z,n_0)$, $F_{\underline{p}}(z,n_0)$, $\beta(n_0)$, $\alpha^+(n_0)$, and $\beta^+(n_0)$, we next construct the $n_0\pm 1$ terms from the following equations:

$$F_{\underline{p}}^- = \frac{1}{z\gamma}(\alpha^2 H_{\underline{p}} - 2\alpha G_{\underline{p}} + F_{\underline{p}}), \tag{4.36}$$

$$H_{\underline{p}}^- = \frac{z}{\gamma}(\beta^2 F_{\underline{p}} - 2\beta G_{\underline{p}} + H_{\underline{p}}), \tag{4.37}$$

$$G_{\underline{p}}^- = \frac{1}{\gamma}((1+\alpha\beta) G_{\underline{p}} - \alpha H_{\underline{p}} - \beta F_{\underline{p}}), \tag{4.38}$$

respectively,

$$F_{\underline{p}}^+ = \frac{1}{z\gamma^+}((\alpha^+)^2 H_{\underline{p}} + 2\alpha^+ z G_{\underline{p}} + z^2 F_{\underline{p}}), \tag{4.39}$$

$$H_{\underline{p}}^+ = \frac{1}{z\gamma^+}((\beta^+ z)^2 F_{\underline{p}} + 2\beta^+ z G_{\underline{p}} + H_{\underline{p}}), \tag{4.40}$$

$$G_{\underline{p}}^+ = \frac{1}{z\gamma^+}((1+\alpha^+\beta^+) z G_{\underline{p}} + \alpha^+ H_{\underline{p}} + \beta^+ z^2 F_{\underline{p}}). \tag{4.41}$$

Moreover,

$$(G_{\underline{p}}^-)^2 - F_{\underline{p}}^- H_{\underline{p}}^- = R_{\underline{p}}, \quad (G_{\underline{p}}^+)^2 - F_{\underline{p}}^+ H_{\underline{p}}^+ = R_{\underline{p}}. \tag{4.42}$$

Inserting (4.18), (4.19), and (4.32) in (4.36)–(4.38) one verifies

$$F_{\underline{p}}^-(z,n_0) \underset{z\to\infty}{=} -c_{0,+}\alpha(n_0) z^{p_+-1} + O(z^{p_+-2}), \tag{4.43}$$

$$H_{\underline{p}}^-(z,n_0) \underset{z\to\infty}{=} O(z^{p_+}), \tag{4.44}$$

$$F_{\underline{p}}^-(z,n_0) \underset{z\to 0}{=} O(z^{-p_-}), \tag{4.45}$$

$$H_{\underline{p}}^-(z,n_0) \underset{z\to 0}{=} -c_{0,-}\beta(n_0) z^{-p_-+1} + O(z^{-p_-+2}), \tag{4.46}$$

$$G_{\underline{p}}^-(z,n_0) = \tfrac{1}{2}c_{0,-}z^{-p_-} + \cdots + \tfrac{1}{2}c_{0,+}z^{p_+}. \tag{4.47}$$

The last equation implies

$$G_{\underline{p}}(z,n_0-1)^2 - R_{\underline{p}}(z) \underset{z\to\infty}{=} O(z^{2p_+-1}), \tag{4.48}$$

$$G_{\underline{p}}(z,n_0-1)^2 - R_{\underline{p}}(z) \underset{z\to 0}{=} O(z^{-2p_-+1}), \tag{4.49}$$

so we may write

$$G_{\underline{p}}(z,n_0-1)^2 - R_{\underline{p}}(z) = \check F_{s,p_+-1}(z,n_0-1)\check H_{p_--1,r}(z,n_0-1), \quad z\in\mathbb{C}\setminus\{0\}, \tag{4.50}$$

for some $s \in \{1,\dots,p_-\}$, $r \in \{1,\dots,p_+\}$, where

$$\check F_{s,p_+-1}(n_0-1) = c_{-s}z^{-s} + \cdots - c_{0,+}\alpha(n_0)z^{p_+-1},$$
$$\check H_{p_--1,r}(n_0-1) = -c_{0,-}\beta(n_0)z^{-p_-+1} + \cdots + c_r z^r.$$

The right-hand side of (4.50) cannot vanish identically (since otherwise $R_{\underline{p}}(z) = G_{\underline{p}}(z,n_0-1)^2$ would yield double zeros of $R_{\underline{p}}(z)$), and hence,

$$\check\phi(P,n_0-1) = \frac{G_{\underline{p}}(z,n_0-1) + (c_{0,+}/2)z^{-p_-}y}{\check F_{s,p_+-1}(z,n_0-1)} = \frac{\check H_{p_--1,r}(z,n_0-1)}{G_{\underline{p}}(z,n_0-1) - (c_{0,+}/2)z^{-p_-}y},$$
$$P=(z,y)\in\mathcal{K}_p. \qquad (4.51)$$

Next, we decompose

$$\check F_{s,p_+-1}(z,n_0-1) = -c_{0,+}\alpha(n_0)z^{-s}\prod_{j=1}^{p_+-1+s}(z-\mu_j(n_0-1)), \qquad (4.52)$$

$$\check H_{p_--1,r}(z,n_0-1) = Cz^{-p_-+1}\prod_{j=1}^{p_--1+r}(z-\nu_j(n_0-1)), \qquad (4.53)$$

where $C\in\mathbb{C}\setminus\{0\}$ and $\{\mu_j(n_0-1)\}_{j=1}^{p_+-1+s}\subset\mathbb{C}$, $\{\nu_j(n_0-1)\}_{j=1}^{p_--1+r}\subset\mathbb{C}$. The divisor of $\check\phi(\,\cdot\,,n_0-1)$ is then given by

$$\big(\check\phi(\,\cdot\,,n_0-1)\big) = \mathcal{D}_{P_{0,-}\underline{\hat\nu}(n_0-1)} - \mathcal{D}_{P_{\infty_-}\underline{\hat\mu}(n_0-1)}, \qquad (4.54)$$

where

$$\underline{\hat\mu}(n_0-1) = \{\underbrace{P_{0,+},\dots,P_{0,+}}_{p_--s\text{ times}},\hat\mu_1(n_0-1),\dots,\hat\mu_{p_+-1+s}(n_0-1)\}, \qquad (4.55)$$

$$\underline{\hat\nu}(n_0-1) = \{\hat\nu_1(n_0-1),\dots,\hat\nu_{p_--1+r}(n_0-1),\underbrace{P_{\infty_+},\dots,P_{\infty_+}}_{p_+-r\text{ times}}\}. \qquad (4.56)$$

In particular,

$$\mathcal{D}_{\underline{\hat\mu}(n_0-1)} \text{ is an admissible divisor if and only if } s=p_-, \qquad (4.57)$$
$$\mathcal{D}_{\underline{\hat\nu}(n_0-1)} \text{ is an admissible divisor if and only if } r=p_+. \qquad (4.58)$$

(VIII) Assuming that (4.48), (4.49) are precisely of order $z^{\pm(2p_\pm-1)}$, that is, assuming $s=p_-$ and $r=p_+$ in (4.51), we rewrite (4.51) as

$$G_{\underline{p}}(z,n_0-1)^2 - R_{\underline{p}}(z) = F_{\underline{p}}(z,n_0-1)H_{\underline{p}}(z,n_0-1), \quad z\in\mathbb{C}\setminus\{0\}. \qquad (4.59)$$

By construction, $F_{\underline{p}}(\,\cdot\,,n_0-1)$ and $H_{\underline{p}}(\,\cdot\,,n_0-1)$ are then of the type

$$F_{\underline{p}}(z,n_0-1) = -c_{0,+}\alpha(n_0)z^{-p_-}\prod_{k=1}^{q(n_0-1)}(z-\mu_j(n_0-1))^{p_k(n_0-1)},$$
$$\sum_{k=1}^{q(n_0-1)} p_k(n_0-1) = p, \qquad (4.60)$$
$$\mu_k(n_0-1)\neq\mu_{k'}(n_0-1) \text{ for } k\neq k',\ k,k'=1,\dots,q(n_0-1),\ \ z\in\mathbb{C}\setminus\{0\},$$
$$H_{\underline{p}}(z,n_0-1) = c_{0,+}\beta(n_0-1)z^{-p_-+1}\prod_{k=1}^{\ell(n_0-1)}(z-\nu_k(n_0-1))^{s_k(n_0-1)},$$

$$\sum_{k=1}^{\ell(n_0-1)} s_k(n_0-1) = p, \tag{4.61}$$
$$\nu_k(n_0-1) \neq \nu_{k'}(n_0-1) \text{ for } k \neq k',\ k,k' = 1,\dots,\ell(n_0-1),\quad z \in \mathbb{C}\setminus\{0\},$$

where we introduced the coefficient $\beta(n_0-1)$. We define

$$\begin{aligned} \hat{\mu}_k(n_0-1) &= (\mu_k(n_0-1), (2/c_{0,+})\mu_k(n_0-1)^{p_-} G_{\underline{p}}(\mu_k(n_0-1), n_0-1)), \\ &\qquad\qquad k = 1,\dots,q(n_0-1), \\ \hat{\nu}_k(n_0-1) &= (\nu_k(n_0-1), -(2/c_{0,+})\nu_k(n_0-1)^{p_-} G_{\underline{p}}(\nu_k(n_0-1), n_0-1)), \\ &\qquad\qquad k = 1,\dots,\ell(n_0-1). \end{aligned} \tag{4.62}$$

(IX) At this point one can iterate the procedure step by step to construct $F_{\underline{p}}(\,\cdot\,,n)$, $G_{\underline{p}}(\,\cdot\,,n)$, $H_{\underline{p}}(\,\cdot\,,n)$, $\alpha(n)$, $\beta(n)$, $\mu_j(n)$, $\nu_j(n)$, etc., for $n \in (-\infty, n_0] \cap \mathbb{Z}$, subject to the following assumption (cf. (4.57), (4.58)) at each step:

$$\mathcal{D}_{\underline{\hat{\mu}}(n-1)} \text{ is an admissible divisor (and hence } \alpha(n-1) \neq 0) \tag{4.63}$$
$$\text{for all } n \in (-\infty, n_0] \cap \mathbb{Z},$$
$$\mathcal{D}_{\underline{\hat{\nu}}(n-1)} \text{ is an admissible divisor (and hence } \beta(n-1) \neq 0) \tag{4.64}$$
$$\text{for all } n \in (-\infty, n_0] \cap \mathbb{Z}.$$

The formalism is symmetric with respect to n_0 and can equally well be developed for $n \in (-\infty, n_0] \cap \mathbb{Z}$ subject to the analogous assumption

$$\mathcal{D}_{\underline{\hat{\mu}}(n+1)} \text{ is an admissible divisor (and hence } \alpha(n+2) \neq 0) \tag{4.65}$$
$$\text{for all } n \in [n_0, \infty) \cap \mathbb{Z},$$
$$\mathcal{D}_{\underline{\hat{\nu}}(n+1)} \text{ is an admissible divisor (and hence } \beta(n+2) \neq 0) \tag{4.66}$$
$$\text{for all } n \in [n_0, \infty) \cap \mathbb{Z}.$$

(X) Choosing the initial data $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ such that

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathcal{M}_0, \tag{4.67}$$

where $\mathcal{M}_0 \subset \mathrm{Sym}^p(\mathcal{K}_p)$ is the set of admissible initial divisors introduced in Lemma 4.3, then guarantees that assumptions (4.63)–(4.66) are satisfied for all $n \in \mathbb{Z}$.

Equations (4.36)–(4.41) (for arbitrary $n \in \mathbb{Z}$) are equivalent to s-$\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$.

At this stage we have verified the basic hypotheses of Section 3 (i.e., (3.2) and the assumption that α, β satisfy the $\underline{p}$th stationary AL system (2.33)) and hence all results of Section 3 apply.

In summary, we proved the following result:

Theorem 4.4. *Let $n \in \mathbb{Z}$, suppose the set $\{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}$ satisfies $E_m \neq E_{m'}$ for $m \neq m'$, $m, m' = 0,\dots,2p+1$, and introduce the function $R_{\underline{p}}$ and the hyperelliptic curve $\mathcal{K}_p$ as in* (3.1). *Choose $\alpha(n_0) \in \mathbb{C}\setminus\{0\}$, $c_{0,+} \in \mathbb{C}\setminus\{0\}$, and a nonspecial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathcal{M}_0$, where $\mathcal{M}_0 \subset \mathrm{Sym}^p(\mathcal{K}_p)$ is the set of admissible initial divisors introduced in Lemma* 4.3. *Then the stationary* (*complex*) *Ablowitz–Ladik algorithm* *as outlined in steps* $(\boldsymbol{I})$*–*$(\boldsymbol{X})$ *produces solutions* α, β *of the* $\underline{p}$*th stationary Ablowitz–Ladik system,*

$$\text{s-AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -\alpha(g_{p_+,+} + g^-_{p_-,-}) + f_{p_+-1,+} - f^-_{p_--1,-} \\ \beta(g^-_{p_+,+} + g_{p_-,-}) + h^-_{p_+-1,+} - h_{p_--1,-} \end{pmatrix} = 0, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2, \tag{4.68}$$

satisfying (3.2) *and*

$$\alpha(n) = \bigg(\prod_{m=0}^{2p+1} E_m\bigg)^{(n-n_0)/2} \mathcal{A}(n,n_0)\, \alpha(n_0), \tag{4.69}$$

$$\begin{aligned}
\beta(n) = \Bigg(&-\frac{1}{2}\sum_{k=1}^{q(n)} \frac{\big(d^{p_k(n)-1}\big(\zeta^{-1}y(P)\big)/d\zeta^{p_k(n)-1}\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n)}}{(p_k(n)-1)!} \\
&\qquad \times \prod_{k'=1,\, k'\neq k}^{q(n)} (\mu_k(n) - \mu_{k'}(n))^{-p_k(n)} \\
&+ \frac{1}{2}\bigg(\bigg(\prod_{m=0}^{2p+1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n)} \mu_k(n)^{-p_k(n)} + \sum_{k=1}^{q(n)} p_k(n)\mu_k(n) - \frac{1}{2}\sum_{m=0}^{2p+1} E_m\bigg)\Bigg) \\
&\times \bigg(\prod_{m=0}^{2p+1} E_m\bigg)^{-(n+1-n_0)/2} \mathcal{A}(n+1,n_0)^{-1}\, \alpha(n_0)^{-1},
\end{aligned} \tag{4.70}$$

where

$$\mathcal{A}(n,n_0) = \begin{cases} \prod_{n'=n_0}^{n-1} \prod_{k=1}^{q(n')} \mu_k(n')^{-p_k(n')}, & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n}^{n_0-1} \prod_{k=1}^{q(n')} \mu_k(n')^{p_k(n')}, & n \leq n_0 - 1. \end{cases} \tag{4.71}$$

Moreover, Lemmas 3.2–3.5 *apply.*

Finally, we briefly illustrate some aspects of this analysis in the special case $\underline{p} = (1,1)$ (i.e., the case where (3.1) represents an elliptic Riemann surface) in more detail.

Example 4.5. *The case* $\underline{p} = (1,1)$.
In this case one has

$$\begin{aligned}
F_{(1,1)}(z,n) &= -c_{0,+}\alpha(n+1)z^{-1}(z - \mu_1(n)), \\
G_{(1,1)}(z,n) &= \frac{1}{2}\bigg(\frac{1}{\alpha(n)} - \frac{z}{\alpha(n+1)}\bigg)F_{(1,1)}(z,n) + R_{(1,1)}(\hat{\mu}_1(n))^{1/2}, \\
R_{(1,1)}(z) &= \bigg(\frac{c_{0,+}\alpha^+}{z}\bigg)^2 \prod_{m=0}^{3}(z - E_m),
\end{aligned} \tag{4.72}$$

and hence a straightforward calculation shows that

$$\begin{aligned}
G_{(1,1)}(z,n)^2 - R_{(1,1)}(z) &= -c_{0,+}^2\alpha(n+1)\beta(n)z^{-1}(z - \mu_1(n))(z - \nu_1(n)) \\
&= -\frac{c_{0,+}^2}{2z}(z - \mu_1(n))\bigg(\bigg(-\frac{y(\hat{\mu}_1(n))}{\mu_1(n)} + \frac{\widetilde{E}^{1/2}}{\mu_1(n)} + \mu_1(n) - \frac{\widehat{E}^+}{2}\bigg)z
\end{aligned} \tag{4.73}$$

$$- \frac{\widetilde{E}}{\mu_1(n)}\bigg(- \frac{1}{\widetilde{E}^{1/2}}\frac{y(\hat{\mu}_1(n))}{\mu_1(n)} + \frac{\mu_1(n)}{\widetilde{E}^{1/2}} + \frac{1}{\mu_1(n)} - \frac{\widehat{E}^-}{2}\bigg)\bigg),$$

where

$$\widehat{E}^{\pm} = \sum_{m=0}^{3} E_m^{\pm 1}, \quad \widetilde{E} = \prod_{m=0}^{3} E_m. \tag{4.74}$$

Solving for $\nu_1(n)$ one then obtains

$$\nu_1(n) = \frac{\widetilde{E}}{\mu_1(n)} \frac{-\frac{y(\hat{\mu}_1(n))}{\mu_1(n)} + \frac{\widetilde{E}^{1/2}}{\mu_1(n)} + \mu_1(n) - \frac{\widehat{E}^+}{2}}{-\frac{1}{\widetilde{E}^{1/2}}\frac{y(\hat{\mu}_1(n))}{\mu_1(n)} + \frac{\mu_1(n)}{\widetilde{E}^{1/2}} + \frac{1}{\mu_1(n)} - \frac{\widehat{E}^-}{2}}. \tag{4.75}$$

Thus, $\nu_1(n_0)$ could be 0 or ∞ even if $\mu_1(n_0) \neq 0, \infty$.

5. Properties of Algebro-Geometric Solutions of the Time-Dependent Ablowitz–Ladik Hierarchy

In this section we present a quick review of properties of algebro-geometric solutions of the time-dependent Ablowitz–Ladik hierarchy. Again we omit all proofs and refer to [29], [31], and [32] for details.

For most of this section we assume the following hypothesis.

Hypothesis 5.1. (*i*) *Suppose that α, β satisfy*

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,),\ \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\},\ (n,t) \in \mathbb{Z}\times\mathbb{R}. \end{aligned} \tag{5.1}$$

(*ii*) *Assume that the hyperelliptic curve $\mathcal{K}_p$ satisfies* (3.1) *and* (3.3).

In order to briefly analyze algebro-geometric solutions of the time-dependent Ablowitz–Ladik hierarchy we proceed as follows. Given $\underline{p} \in \mathbb{N}_0^2$, consider a complex-valued solution $\alpha^{(0)}, \beta^{(0)}$ of the $\underline{p}$th stationary Ablowitz–Ladik system s-$\mathrm{AL}_{\underline{p}}(a,b) = 0$, associated with $\mathcal{K}_p$ and a given set of summation constants $\{c_{\ell,\pm}\}_{\ell=1,\dots,p_\pm} \subset \mathbb{C}$. Next, let $\underline{r} \in \mathbb{N}_0^2$; we intend to consider solutions $\alpha = \alpha(t_{\underline{r}}), \beta = \beta(t_{\underline{r}})$ of the $\underline{r}$th AL flow $\mathrm{AL}_{\underline{r}}(\alpha,\beta) = 0$ with $\alpha(t_{0,\underline{r}}) = \alpha^{(0)}, \beta(t_{0,\underline{r}}) = \beta^{(0)}$ for some $t_{0,\underline{r}} \in \mathbb{R}$. To emphasize that the summation constants in the definitions of the stationary and the time-dependent Ablowitz–Ladik equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde{V}_{\underline{r}}$, $\widetilde{F}_{\underline{r}}$, $\widetilde{G}_{\underline{r}}$, $\widetilde{H}_{\underline{r}}$, $\widetilde{K}_{\underline{r}}$, $\tilde{f}_{s,\pm}$, $\tilde{g}_{s,\pm}$, $\tilde{h}_{s,\pm}$, $\tilde{c}_{s,\pm}$, in order to distinguish them from $V_{\underline{p}}$, $F_{\underline{p}}$, $G_{\underline{p}}$, $H_{\underline{p}}$, $K_{\underline{p}}$, $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, $c_{\ell,\pm}$, in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual $\underline{r}$th Ablowitz–Ladik flow by a separate time variable $t_{\underline{r}} \in \mathbb{R}$. More precisely, we will review properties of solutions α, β of the time-dependent algebro-geometric initial value problem

$$\begin{aligned} &\widetilde{\mathrm{AL}}_{\underline{r}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{\underline{r}}} - \alpha(\tilde{g}_{r_+,+} + \tilde{g}^-_{r_-,-}) + \tilde{f}_{r_+-1,+} - \tilde{f}^-_{r_- -1,-} \\ -i\beta_{t_{\underline{r}}} + \beta(\tilde{g}^-_{r_+,+} + \tilde{g}_{r_-,-}) - \tilde{h}_{r_- -1,-} + \tilde{h}^-_{r_+-1,+} \end{pmatrix} = 0, \\ &(\alpha,\beta)\big|_{t_{\underline{r}}=t_{0,\underline{r}}} = \big(\alpha^{(0)},\beta^{(0)}\big), \end{aligned} \tag{5.2}$$

$$\text{s-AL}_{\underline{p}}\big(\alpha^{(0)},\beta^{(0)}\big) = \begin{pmatrix} -\alpha^{(0)}(g_{p_+,+} + g^-_{p_-,-}) + f_{p_+-1,+} - f^-_{p_- -1,-} \\ \beta^{(0)}(g^-_{p_+,+} + g_{p_-,-}) - h_{p_- -1,-} + h^-_{p_+-1,+} \end{pmatrix} = 0 \tag{5.3}$$

for some $t_{0,\underline{r}} \in \mathbb{R}$, where $\alpha = \alpha(n, t_{\underline{r}})$, $\beta = \beta(n, t_{\underline{r}})$ satisfy (5.1) and a fixed curve $\mathcal{K}_p$ is associated with the stationary solutions $\alpha^{(0)}, \beta^{(0)}$ in (5.3). Here,

$$\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \setminus \{(0,0)\}, \quad \underline{r} = (r_-, r_+) \in \mathbb{N}_0^2, \quad p = p_- + p_+ - 1. \tag{5.4}$$

In terms of the zero-curvature formulation this amounts to solving

$$U_{t_{\underline{r}}}(z, t_{\underline{r}}) + U(z, t_{\underline{r}})\widetilde{V}_{\underline{r}}(z, t_{\underline{r}}) - \widetilde{V}_{\underline{r}}^{+}(z, t_{\underline{r}})U(z, t_{\underline{r}}) = 0, \tag{5.5}$$

$$U(z, t_{0,\underline{r}})V_{\underline{p}}(z, t_{0,\underline{r}}) - V_{\underline{p}}^{+}(z, t_{0,\underline{r}})U(z, t_{0,\underline{r}}) = 0. \tag{5.6}$$

One can show (cf. Lemma 6.2) that the stationary Ablowitz–Ladik system (5.6) is actually satisfied for all times $t_{\underline{r}} \in \mathbb{R}$. Thus, we impose

$$U_{t_{\underline{r}}} + U\widetilde{V}_{\underline{r}} - \widetilde{V}_{\underline{r}}^{+}U = 0, \tag{5.7}$$

$$UV_{\underline{p}} - V_{\underline{p}}^{+}U = 0, \tag{5.8}$$

instead of (5.5) and (5.6). For further reference, we recall the relevant quantities here (cf. (2.20), (2.21), (2.27)–(2.30), (2.38)):

$$\begin{aligned} &U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \\ &V_{\underline{p}}(z) = i\begin{pmatrix} G_{\underline{p}}^{-}(z) & -F_{\underline{p}}^{-}(z) \\ H_{\underline{p}}^{-}(z) & -G_{\underline{p}}^{-}(z) \end{pmatrix}, \quad \widetilde{V}_{\underline{r}}(z) = i\begin{pmatrix} \widetilde{G}_{\underline{r}}^{-}(z) & -\widetilde{F}_{\underline{r}}^{-}(z) \\ \widetilde{H}_{\underline{r}}^{-}(z) & -\widetilde{K}_{\underline{r}}^{-}(z) \end{pmatrix}, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} F_{\underline{p}}(z) &= \sum_{\ell=1}^{p_-} f_{p_- - \ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+ - 1} f_{p_+ - 1 - \ell,+} z^{\ell} = -c_{0,+}\alpha^{+} z^{-p_-} \prod_{j=1}^{p} (z - \mu_j), \\ G_{\underline{p}}(z) &= \sum_{\ell=1}^{p_-} g_{p_- - \ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+} g_{p_+ - \ell,+} z^{\ell}, \\ H_{\underline{p}}(z) &= \sum_{\ell=0}^{p_- - 1} h_{p_- - 1 - \ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} h_{p_+ - \ell,+} z^{\ell} = c_{0,+}\beta z^{-p_- + 1} \prod_{j=1}^{p} (z - \nu_j), \\ \widetilde{F}_{\underline{r}}(z) &= \sum_{s=1}^{r_-} \tilde{f}_{r_- - s,-} z^{-s} + \sum_{s=0}^{r_+ - 1} \tilde{f}_{r_+ - 1 - s,+} z^{s}, \\ \widetilde{G}_{\underline{r}}(z) &= \sum_{s=1}^{r_-} \tilde{g}_{r_- - s,-} z^{-s} + \sum_{s=0}^{r_+} \tilde{g}_{r_+ - s,+} z^{s}, \\ \widetilde{H}_{\underline{r}}(z) &= \sum_{s=0}^{r_- - 1} \tilde{h}_{r_- - 1 - s,-} z^{-s} + \sum_{s=1}^{r_+} \tilde{h}_{r_+ - s,+} z^{s}, \\ \widetilde{K}_{\underline{r}}(z) &= \sum_{s=0}^{r_-} \tilde{g}_{r_- - s,-} z^{-s} + \sum_{s=1}^{r_+} \tilde{g}_{r_+ - s,+} z^{s} = \widetilde{G}_{\underline{r}}(z) + \tilde{g}_{r_-,-} - \tilde{g}_{r_+,+} \end{aligned} \tag{5.10}$$

for fixed $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, $\underline{r} \in \mathbb{N}_0^2$. Here $f_{\ell,\pm}$, $\tilde{f}_{s,\pm}$, $g_{\ell,\pm}$, $\tilde{g}_{s,\pm}$, $h_{\ell,\pm}$, and $\tilde{h}_{s,\pm}$ are defined as in (2.5)–(2.12) with appropriate sets of summation constants $c_{\ell,\pm}$, $\ell \in \mathbb{N}_0$, and $\tilde{c}_{k,\pm}$, $k \in \mathbb{N}_0$. Explicitly, (5.7) and (5.8) are equivalent to (cf. (2.23)–(2.26), (2.46)–(2.49))

$$\alpha_{t_{\underline{r}}} = i\big(z\widetilde{F}_{\underline{r}}^{-} + \alpha(\widetilde{G}_{\underline{r}} + \widetilde{K}_{\underline{r}}^{-}) - \widetilde{F}_{\underline{r}}\big), \tag{5.11}$$

$$\beta_{t_{\underline{r}}} = -i\big(\beta(\widetilde{G}_{\underline{r}}^- + \widetilde{K}_{\underline{r}}) - \widetilde{H}_{\underline{r}} + z^{-1}\widetilde{H}_{\underline{r}}^-\big), \tag{5.12}$$
$$0 = z(\widetilde{G}_{\underline{r}}^- - \widetilde{G}_{\underline{r}}) + z\beta\widetilde{F}_{\underline{r}} + \alpha\widetilde{H}_{\underline{r}}^-, \tag{5.13}$$
$$0 = z\beta\widetilde{F}_{\underline{r}}^- + \alpha\widetilde{H}_{\underline{r}} + \widetilde{K}_{\underline{r}}^- - \widetilde{K}_{\underline{r}}, \tag{5.14}$$
$$0 = z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^-, \tag{5.15}$$
$$0 = z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} - G_{\underline{p}} + G_{\underline{p}}^-, \tag{5.16}$$
$$0 = -F_{\underline{p}} + zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + G_{\underline{p}}^-), \tag{5.17}$$
$$0 = z\beta(G_{\underline{p}} + G_{\underline{p}}^-) - zH_{\underline{p}} + H_{\underline{p}}^-, \tag{5.18}$$

respectively. In particular, (2.39) holds in the present $t_{\underline{r}}$-dependent setting, that is,

$$G_{\underline{p}}^2 - F_{\underline{p}}H_{\underline{p}} = R_{\underline{p}}. \tag{5.19}$$

As in the stationary context (3.8), (3.9) we introduce

$$\begin{aligned}\hat{\mu}_j(n,t_{\underline{r}}) = (\mu_j(n,t_{\underline{r}}), (2/c_{0,+})\mu_j(n,t_{\underline{r}})^{p_-}G_{\underline{p}}(\mu_j(n,t_{\underline{r}}),n,t_{\underline{r}})) \in \mathcal{K}_p,&\\ j=1,\dots,p,\ (n,t_{\underline{r}}) \in \mathbb{Z}\times\mathbb{R},&\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}\hat{\nu}_j(n,t_{\underline{r}}) = (\nu_j(n,t_{\underline{r}}), -(2/c_{0,+})\nu_j(n,t_{\underline{r}})^{p_-}G_{\underline{p}}(\nu_j(n,t_{\underline{r}}),n,t_{\underline{r}})) \in \mathcal{K}_p,&\\ j=1,\dots,p,\ (n,t_{\underline{r}}) \in \mathbb{Z}\times\mathbb{R},&\end{aligned} \tag{5.21}$$

and note that the regularity assumptions (5.1) on α,β imply continuity of μ_j and ν_k with respect to $t_{\underline{r}} \in \mathbb{R}$ (away from collisions of these zeros, μ_j and ν_k are of course C^∞).

In analogy to (3.12), (3.13), one defines the following meromorphic function $\phi(\,\cdot\,,n,t_{\underline{r}})$ on $\mathcal{K}_p$,

$$\begin{aligned}\phi(P,n,t_{\underline{r}}) &= \frac{(c_{0,+}/2)z^{-p_-}y + G_{\underline{p}}(z,n,t_{\underline{r}})}{F_{\underline{p}}(z,n,t_{\underline{r}})} &(5.22)\\ &= \frac{-H_{\underline{p}}(z,n,t_{\underline{r}})}{(c_{0,+}/2)z^{-p_-}y - G_{\underline{p}}(z,n,t_{\underline{r}})}, &(5.23)\\ &\qquad P=(z,y)\in\mathcal{K}_p,\ (n,t_{\underline{r}})\in\mathbb{Z}\times\mathbb{R},\end{aligned}$$

with divisor $(\phi(\,\cdot\,,n,t_{\underline{r}}))$ of $\phi(\,\cdot\,,n,t_{\underline{r}})$ given by

$$(\phi(\,\cdot\,,n,t_{\underline{r}})) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(n,t_{\underline{r}})} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n,t_{\underline{r}})}. \tag{5.24}$$

The time-dependent Baker–Akhiezer vector is then defined in terms of ϕ by

$$\Psi(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \begin{pmatrix}\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}})\\ \psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}})\end{pmatrix}, \tag{5.25}$$

$$\begin{aligned}\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) &= \exp\bigg(i\int_{t_{0,\underline{r}}}^{t_{\underline{r}}} ds\big(\widetilde{G}_{\underline{r}}(z,n_0,s) - \widetilde{F}_{\underline{r}}(z,n_0,s)\phi(P,n_0,s)\big)\bigg) &(5.26)\\ &\quad\times\begin{cases}\prod_{n'=n_0+1}^{n}\big(z+\alpha(n',t_{\underline{r}})\phi^-(P,n',t_{\underline{r}})\big), & n\ge n_0+1,\\ 1, & n=n_0,\\ \prod_{n'=n+1}^{n_0}\big(z+\alpha(n',t_{\underline{r}})\phi^-(P,n',t_{\underline{r}})\big)^{-1}, & n\le n_0-1,\end{cases}\end{aligned}$$

$$\psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) = \exp\bigg(i\int_{t_{0,\underline{r}}}^{t_{\underline{r}}} ds\big(\widetilde{G}_{\underline{r}}(z,n_0,s) - \widetilde{F}_{\underline{r}}(z,n_0,s)\phi(P,n_0,s)\big)\bigg) \tag{5.27}$$

$$\times\, \phi(P,n_0,t_{\underline{r}}) \begin{cases} \prod_{n'=n_0+1}^{n} \big(z\beta(n',t_{\underline{r}})\phi^-(P,n',t_{\underline{r}})^{-1}+1\big), & n \geq n_0+1, \\ 1, & n=n_0, \\ \prod_{n'=n+1}^{n_0} \big(z\beta(n',t_{\underline{r}})\phi^-(P,n',t_{\underline{r}})^{-1}+1\big)^{-1}, & n \leq n_0-1, \end{cases}$$
$$P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,t_{\underline{r}})\in\mathbb{Z}\times\mathbb{R}.$$

One observes that

$$\begin{aligned} &\psi_1(P,n,n_0,t_{\underline{r}},\tilde{t}_{\underline{r}}) = \psi_1(P,n_0,n_0,t_{\underline{r}},\tilde{t}_{\underline{r}})\psi_1(P,n,n_0,t_{\underline{r}},t_{\underline{r}}), \\ &P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (n,n_0,t_{\underline{r}},\tilde{t}_{\underline{r}})\in\mathbb{Z}^2\times\mathbb{R}^2. \end{aligned} \tag{5.28}$$

The following lemma records basic properties of ϕ and Ψ in analogy to the stationary case discussed in Lemma 3.2.

Lemma 5.2 ([31])**.** *Assume Hypothesis* 5.1 *and suppose that* (5.7), (5.8) *hold. In addition, let* $P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-}\}$, $(n,n_0,t_{\underline{r}},t_{0,\underline{r}})\in\mathbb{Z}^2\times\mathbb{R}^2$. *Then* ϕ *satisfies*

$$\alpha\phi(P)\phi^-(P)-\phi^-(P)+z\phi(P)=z\beta, \tag{5.29}$$

$$\phi_{t_{\underline{r}}}(P)=i\widetilde{F}_{\underline{r}}\phi^2(P)-i\big(\widetilde{G}_{\underline{r}}(z)+\widetilde{K}_{\underline{r}}(z)\big)\phi(P)+i\widetilde{H}_{\underline{r}}(z), \tag{5.30}$$

$$\phi(P)\phi(P^*)=\frac{H_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \tag{5.31}$$

$$\phi(P)+\phi(P^*)=2\frac{G_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \tag{5.32}$$

$$\phi(P)-\phi(P^*)=c_{0,+}z^{-p_-}\frac{y(P)}{F_{\underline{p}}(z)}. \tag{5.33}$$

Moreover, assuming $P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\}$, *then* Ψ *satisfies*

$$\psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}})=\phi(P,n,t_{\underline{r}})\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}), \tag{5.34}$$

$$U(z)\Psi^-(P)=\Psi(P), \tag{5.35}$$

$$V_{\underline{p}}(z)\Psi^-(P)=-(i/2)c_{0,+}z^{-p_-}y\Psi^-(P), \tag{5.36}$$

$$\Psi_{t_{\underline{r}}}(P)=\widetilde{V}_{\underline{r}}^+(z)\Psi(P), \tag{5.37}$$

$$\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}})\psi_1(P^*,n,n_0,t_{\underline{r}},t_{0,\underline{r}})=z^{n-n_0}\frac{F_{\underline{p}}(z,n,t_{\underline{r}})}{F_{\underline{p}}(z,n_0,t_{0,\underline{r}})}\Gamma(n,n_0,t_{\underline{r}}), \tag{5.38}$$

$$\psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}})\psi_2(P^*,n,n_0,t_{\underline{r}},t_{0,\underline{r}})=z^{n-n_0}\frac{H_{\underline{p}}(z,n,t_{\underline{r}})}{F_{\underline{p}}(z,n_0,t_{0,\underline{r}})}\Gamma(n,n_0,t_{\underline{r}}), \tag{5.39}$$

$$\begin{aligned} &\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}})\psi_2(P^*,n,n_0,t_{\underline{r}},t_{0,\underline{r}})+\psi_1(P^*,n,n_0,t_{\underline{r}},t_{0,\underline{r}})\psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) \\ &\quad =2z^{n-n_0}\frac{G_{\underline{p}}(z,n,t_{\underline{r}})}{F_{\underline{p}}(z,n_0,t_{0,\underline{r}})}\Gamma(n,n_0,t_{\underline{r}}), \end{aligned} \tag{5.40}$$

$$\begin{aligned} &\psi_1(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}})\psi_2(P^*,n,n_0,t_{\underline{r}},t_{0,\underline{r}})-\psi_1(P^*,n,n_0,t_{\underline{r}},t_{0,\underline{r}})\psi_2(P,n,n_0,t_{\underline{r}},t_{0,\underline{r}}) \\ &\quad =-c_{0,+}z^{n-n_0-p_-}\frac{y}{F_{\underline{p}}(z,n_0,t_{0,\underline{r}})}\Gamma(n,n_0,t_{\underline{r}}), \end{aligned} \tag{5.41}$$

where

$$\Gamma(n,n_0,t_{\underline{r}}) = \begin{cases} \prod_{n'=n_0+1}^{n} \gamma(n',t_{\underline{r}}) & n \geq n_0+1, \\ 1 & n = n_0, \\ \prod_{n'=n+1}^{n_0} \gamma(n',t_{\underline{r}})^{-1} & n \leq n_0-1. \end{cases} \tag{5.42}$$

In addition, as long as the zeros $\mu_j(n_0,s)$ of $(\,\cdot\,)^{p_-}F_{\underline{p}}(\,\cdot\,,n_0,s)$ are all simple and distinct from zero for $s \in \mathcal{I}_\mu$, $\mathcal{I}_\mu \subseteq \mathbb{R}$ an open interval, $\Psi(\,\cdot\,,n,n_0,t_{\underline{r}},t_{0,\underline{r}})$ is meromorphic on $\mathcal{K}_p \setminus \{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\}$ for $(n,t_{\underline{r}},t_{0,\underline{r}}) \in \mathbb{Z}\times\mathcal{I}_\mu^2$.

The trace formulas recorded in Lemma 3.3 extend to the present time-dependent context without any change as $t_{\underline{r}} \in \mathbb{R}$ can be viewed as a fixed parameter. Further details are thus omitted.

For completeness we next mention the Dubrovin-type equations for the time variation of the zeros μ_j of $(\,\cdot\,)^{p_-}F_{\underline{p}}$ and ν_j of $(\,\cdot\,)^{p_--1}H_{\underline{p}}$ governed by the $\widetilde{\mathrm{AL}}_{\underline{r}}$ flow.

Lemma 5.3 ([31])**.** *Assume Hypothesis* 5.1 *and suppose that* (5.7), (5.8) *hold on $\mathbb{Z}\times\mathcal{I}_\mu$ with $\mathcal{I}_\mu \subseteq \mathbb{R}$ an open interval. In addition, assume that the zeros μ_j, $j = 1,\dots,p$, of $(\,\cdot\,)^{p_-}F_{\underline{p}}(\,\cdot\,)$ remain distinct and nonzero on $\mathbb{Z}\times\mathcal{I}_\mu$. Then $\{\hat\mu_j\}_{j=1,\dots,p}$, defined in* (5.20)*, satisfies the following first-order system of differential equations on $\mathbb{Z}\times\mathcal{I}_\mu$,*

$$\mu_{j,t_{\underline{r}}} = -i\widetilde{F}_{\underline{r}}(\mu_j)y(\hat\mu_j)(\alpha^+)^{-1}\prod_{\substack{k=1\\k\neq j}}^{p}(\mu_j-\mu_k)^{-1}, \quad j=1,\dots,p, \tag{5.43}$$

with

$$\hat\mu_j(n,\cdot) \in C^\infty(\mathcal{I}_\mu,\mathcal{K}_p), \quad j=1,\dots,p,\ n\in\mathbb{Z}. \tag{5.44}$$

For the zeros ν_j, $j=1,\dots,p$, of $(\,\cdot\,)^{p_--1}H_{\underline{p}}(\,\cdot\,)$, identical statements hold with μ_j and $\mathcal{I}_\mu$ replaced by ν_j and $\mathcal{I}_\nu$, etc. (*with $\mathcal{I}_\nu \subseteq \mathbb{R}$ an open interval*)*. In particular, $\{\hat\nu_j\}_{j=1,\dots,p}$, defined in* (5.21)*, satisfies the first-order system on $\mathbb{Z}\times\mathcal{I}_\nu$,*

$$\nu_{j,t_{\underline{r}}} = i\widetilde{H}_{\underline{r}}(\nu_j)y(\hat\nu_j)(\beta\nu_j)^{-1}\prod_{\substack{k=1\\k\neq j}}^{p}(\nu_j-\nu_k)^{-1}, \quad j=1,\dots,p, \tag{5.45}$$

with

$$\hat\nu_j(n,\cdot) \in C^\infty(\mathcal{I}_\nu,\mathcal{K}_p), \quad j=1,\dots,p,\ n\in\mathbb{Z}. \tag{5.46}$$

When attempting to solve the Dubrovin-type systems (5.43) and (5.45), they must be augmented with appropriate divisors $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,\underline{r}})} \in \mathrm{Sym}^p\,\mathcal{K}_p$, $t_{0,\underline{r}} \in \mathcal{I}_\mu$, and $\mathcal{D}_{\underline{\hat\nu}(n_0,t_{0,\underline{r}})} \in \mathrm{Sym}^p\,\mathcal{K}_p$, $t_{0,\underline{r}} \in \mathcal{I}_\nu$, as initial conditions.

For the $t_{\underline{r}}$-dependence of $F_{\underline{p}}$, $G_{\underline{p}}$, and $H_{\underline{p}}$ one obtains the following result.

Lemma 5.4 ([31])**.** *Assume Hypothesis* 5.1 *and suppose that* (5.7), (5.8) *hold. In addition, let $(z,n,t_{\underline{r}}) \in \mathbb{C}\times\mathbb{Z}\times\mathbb{R}$. Then,*

$$F_{\underline{p},t_{\underline{r}}} = -2iG_{\underline{p}}\widetilde{F}_{\underline{r}} + i\big(\widetilde{G}_{\underline{r}}+\widetilde{K}_{\underline{r}}\big)F_{\underline{p}}, \tag{5.47}$$

$$G_{\underline{p},t_{\underline{r}}} = iF_{\underline{p}}\widetilde{H}_{\underline{r}} - iH_{\underline{p}}\widetilde{F}_{\underline{r}}, \tag{5.48}$$

$$H_{\underline{p},t_{\underline{r}}} = 2iG_{\underline{p}}\widetilde{H}_{\underline{r}} - i\big(\widetilde{G}_{\underline{r}}+\widetilde{K}_{\underline{r}}\big)H_{\underline{p}}. \tag{5.49}$$

In particular, (5.47)–(5.49) *are equivalent to*

$$V_{\underline{p},t_{\underline{r}}} = \big[\widetilde{V}_{\underline{r}}, V_{\underline{p}}\big]. \tag{5.50}$$

It will be shown in Section 6 that Lemma 5.4 yields a first-order system of differential equations for $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$, that serves as a pertinent substitute for the Dubrovin equations (5.43) even (in fact, especially) when some of the μ_j coincide.

Lemma 3.5 on nonspecial divisors and the linearization property of the Abel map extend to the present time-dependent setting. For this fact we need to introduce a particular differential of the second kind, $\widetilde{\Omega}^{(2)}_{\underline{r}}$, defined as follows. Let $\omega^{(2)}_{P_{\infty_\pm},q}$ and $\omega^{(2)}_{P_{0,\pm},q}$ be the normalized differentials of the second kind with a unique pole at $P_{\infty_\pm}$ and $P_{0,\pm}$, respectively, and principal parts

$$\omega^{(2)}_{P_{\infty_\pm},q} \underset{\zeta\to 0}{=} \big(\zeta^{-2-q} + O(1)\big)d\zeta, \quad P \to P_{\infty_\pm},\ \zeta = 1/z,\ q \in \mathbb{N}_0, \tag{5.51}$$

$$\omega^{(2)}_{P_{0,\pm},q} \underset{\zeta\to 0}{=} \big(\zeta^{-2-q} + O(1)\big)d\zeta, \quad P \to P_{0,\pm},\ \zeta = z,\ q \in \mathbb{N}_0, \tag{5.52}$$

with vanishing a-periods,

$$\int_{a_j} \omega^{(2)}_{P_{\infty_\pm},q} = \int_{a_j} \omega^{(2)}_{P_{0,\pm},q} = 0, \quad j = 1,\dots,p. \tag{5.53}$$

Moreover, we define

$$\begin{aligned} \widetilde{\Omega}^{(2)}_{\underline{r}} = \frac{i}{2}\bigg(&\sum_{s=1}^{r_-} s\tilde{c}_{r_--s,-}\big(\omega^{(2)}_{P_{0,+},s-1} - \omega^{(2)}_{P_{0,-},s-1}\big) \\ &+ \sum_{s=1}^{r_+} s\tilde{c}_{r_+-s,+}\big(\omega^{(2)}_{P_{\infty_+},s-1} - \omega^{(2)}_{P_{\infty_-},s-1}\big)\bigg), \end{aligned} \tag{5.54}$$

where $\tilde{c}_{\ell,\pm}$ are the summation constants in $\widetilde{F}_{\underline{r}}$. The corresponding vector of b-periods of $\widetilde{\Omega}^{(2)}_{\underline{r}}/(2\pi i)$ is then denoted by

$$\underline{\widetilde{U}}^{(2)}_{\underline{r}} = \big(\widetilde{U}^{(2)}_{\underline{r},1},\dots,\widetilde{U}^{(2)}_{\underline{r},p}\big), \quad \widetilde{U}^{(2)}_{\underline{r},j} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}^{(2)}_{\underline{r}}, \quad j = 1,\dots,p. \tag{5.55}$$

The time-dependent analog of Lemma 3.5 then reads as follows.

Lemma 5.5 ([31], [32])**.** *Assume Hypothesis* 5.1 *and suppose that* (5.7), (5.8) *hold. Moreover, let* $(n,t_{\underline{r}}) \in \mathbb{Z}\times\mathbb{R}$. *Denote by* $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_p\}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}}$, $\underline{\hat{\nu}} = \{\hat{\nu}_1,\dots,\hat{\nu}_p\}$, *the pole and zero divisors of degree* p, *respectively, associated with* α, β, *and* ϕ *defined according to* (5.20) *and* (5.21), *that is,*

$$\hat{\mu}_j(n,t_{\underline{r}}) = (\mu_j(n,t_{\underline{r}}), (2/c_{0,+})\mu_j(n,t_{\underline{r}})^{p_-}G_{\underline{p}}(\mu_j(n,t_{\underline{r}}),n,t_{\underline{r}})), \quad j = 1,\dots,p, \tag{5.56}$$

$$\hat{\nu}_j(n,t_{\underline{r}}) = (\nu_j(n,t_{\underline{r}}), -(2/c_{0,+})\nu_j(n,t_{\underline{r}})^{p_-}G_{\underline{p}}(\nu_j(n,t_{\underline{r}}),n,t_{\underline{r}})), \quad j = 1,\dots,p. \tag{5.57}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n,t_{\underline{r}})}$ *are nonspecial for all* $(n,t_{\underline{r}}) \in \mathbb{Z}\times\mathbb{R}$. *Moreover, the Abel map linearizes the auxiliary divisors* $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(n,t_{\underline{r}})}$ *in the sense that*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + (t_{\underline{r}} - t_{0,\underline{r}})\underline{\widetilde{U}}^{(2)}_{\underline{r}}, \tag{5.58}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n,t_{\underline{r}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(n_0,t_{0,\underline{r}})}) + (n-n_0)\underline{A}_{P_{0,-}}(P_{\infty_+}) + (t_{\underline{r}} - t_{0,\underline{r}})\underline{\widetilde{U}}_{\underline{r}}^{(2)}, \quad (5.59)$$

where $Q_0 \in \mathcal{K}_p$ *is a given base point and* $\underline{\widetilde{U}}_{\underline{r}}^{(2)}$ *is the vector of* b*-periods introduced in* (5.55)*.*

Again we refer to [32] (and the references cited therein) for theta function representations of α and β. These theta function representations also show that $\gamma(n,t_{\underline{r}}) \notin \{0,1\}$ for all $(n,t_{\underline{r}}) \in \mathbb{Z}\times\mathbb{R}$, that is, the last condition in (5.1) is satisfied for the time-dependent algebro-geometric AL solutions discussed in this section provided the associated Dirichlet and Neumann divisors are admissible.

6. The Algebro-Geometric Ablowitz–Ladik Hierarchy Initial Value Problem

In this section we consider the algebro-geometric Ablowitz–Ladik hierarchy initial value problem (5.2), (5.3) with complex-valued initial data. For a generic set of initial data we will prove unique solvability of the initial value problem globally in time.

As mentioned in the introduction, the strategy of basing the solution of the algebro-geometric initial value problem on the Dubrovin-type equations (5.43), (5.45), and the trace formulas for α and β, meets with serious difficulties as the Dirichlet eigenvalues $\hat{\mu}_j$ may collide on $\mathcal{K}_p$ and hence the denominator of (5.43) can blow up. Hence, we will develop an alternative strategy based on the use of elementary symmetric functions of the variables $\{\mu_j\}_{j=1,\dots,p}$ in this section, which can accommodate collisions of $\hat{\mu}_j$. In short, our strategy will consist of the following:

(i) Replace the first-order autonomous Dubrovin-type system (5.43) of differential equations in $t_{\underline{r}}$ for the Dirichlet eigenvalues $\mu_j(n,t_{\underline{r}})$, $j=1,\dots,p$, augmented by appropriate initial conditions, by the first-order autonomous system (6.57), (6.58) for the coefficients $f_{\ell,\pm}$, $h_{\ell,\pm}$, $\ell=1,\dots,p_\pm-1$, and $g_{\ell,\pm}$, $\ell=1,\dots,p_\pm$, with respect to $t_{\underline{r}}$. Solve this first-order autonomous system in some time interval $(t_{0,\underline{r}}-T_0, t_{0,\underline{r}}+T_0)$ under appropriate initial conditions at $(n_0,t_{0,\underline{r}})$ derived from an initial (nonspecial) Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}$.

(ii) Use the stationary algorithm derived in Section 4 to extend the solution of step (i) from $\{n_0\}\times(t_{0,\underline{r}}-T_0, t_{0,\underline{r}}+T_0)$ to $\mathbb{Z}\times(t_{0,\underline{r}}-T_0, t_{0,\underline{r}}+T_0)$ (cf. Lemma 6.2).

(iii) Prove consistency of this approach, that is, show that the discrete algorithm of Section 4 is compatible with the time-dependent Lax and zero-curvature equations in the sense that first solving the autonomous system (6.57), (6.58) and then applying the discrete algorithm, or first applying the discrete algorithm and then solving the autonomous system (6.57), (6.58) yields the same result whenever the same endpoint $(n,t_{\underline{r}})$ is reached (cf. Lemma 6.3 and Theorem 6.4).

(iv) Prove that there is a dense set of initial conditions of full measure for which this strategy yields global solutions of the algebro-geometric Ablowitz–Ladik hierarchy initial value problem.

To set up this formalism we need some preparations. From the outset we make the following assumption.

Hypothesis 6.1. *Suppose that*

$$\alpha,\beta\in\mathbb{C}^{\mathbb{Z}} \text{ and } \alpha(n)\beta(n)\notin\{0,1\} \text{ for all } n\in\mathbb{Z}, \quad (6.1)$$

and assume that α, β *satisfy the* $\underline{p}$*th stationary Ablowitz–Ladik equation* (2.33). *In addition, suppose that the affine part of the hyperelliptic curve* $\mathcal{K}_p$ *in* (3.1) *is nonsingular.*

We introduce a deformation (time) parameter $t_{\underline{r}} \in \mathbb{R}$ in $\alpha = \alpha(t_{\underline{r}})$ and $\beta = \beta(t_{\underline{r}})$ and hence obtain $t_{\underline{r}}$-dependent quantities $f_\ell = f_\ell(t_{\underline{r}})$, $g_\ell = g_\ell(t_{\underline{r}})$, $F_{\underline{p}}(z) = F_{\underline{p}}(z, t_{\underline{r}})$, $G_{\underline{p}}(z) = G_{\underline{p}}(z, t_{\underline{r}})$, etc. At a fixed initial time $t_{0,\underline{r}} \in \mathbb{R}$ we require that

$$(\alpha, \beta)|_{t_{\underline{r}} = t_{0,\underline{r}}} = \big(\alpha^{(0)}, \beta^{(0)}\big), \tag{6.2}$$

where $\alpha^{(0)} = \alpha(\,\cdot\,, t_{0,\underline{r}})$, $\beta^{(0)} = \beta(\,\cdot\,, t_{0,\underline{r}})$ satisfy the $\underline{p}$th stationary Ablowitz–Ladik system (2.33). As discussed in Section 4, in order to guarantee that the stationary solutions (6.2) can be constructed for all $n \in \mathbb{Z}$ one starts from a particular divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0, t_{0,\underline{r}})} \in \mathcal{M}_0, \tag{6.3}$$

where $\underline{\hat{\mu}}(n_0, t_{0,\underline{r}})$ is of the form

$$\begin{aligned} &\underline{\hat{\mu}}(n_0, t_{0,\underline{r}}) \\ &\quad = \{\underbrace{\hat{\mu}_1(n_0, t_{0,\underline{r}}), \dots, \hat{\mu}_1(n_0, t_{0,\underline{r}})}_{p_1(n_0, t_{0,\underline{r}}) \text{ times}}, \dots, \underbrace{\hat{\mu}_{q(n_0,t_{0,\underline{r}})}(n_0, t_{0,\underline{r}}), \dots, \hat{\mu}_{q(n_0,t_{0,\underline{r}})}(n_0, t_{0,\underline{r}})}_{p_{q(n_0,t_{0,\underline{r}})}(n_0,t_{0,\underline{r}}) \text{ times}}\}. \end{aligned} \tag{6.4}$$

Moreover, as in Section 4 we prescribe the data

$$\alpha(n_0, t_{0,\underline{r}}) \in \mathbb{C} \setminus \{0\} \text{ and } c_{0,+} \in \mathbb{C} \setminus \{0\}, \tag{6.5}$$

and of course the hyperelliptic curve $\mathcal{K}_p$ with nonsingular affine part (cf. (4.10)). In addition, we introduce

$$\alpha^+(n_0, t_{0,\underline{r}}) = \alpha(n_0, t_{0,\underline{r}}) \bigg(\prod_{m=0}^{2p+1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n_0,t_{0,\underline{r}})} \mu_k(n_0, t_{0,\underline{r}})^{-p_k(n_0,t_{0,\underline{r}})}, \tag{6.6}$$

$$\begin{aligned} F_{\underline{p}}(z, n_0, t_{0,\underline{r}}) &= \sum_{\ell=1}^{p_-} f_{p_- - \ell, -}(n_0, t_{0,\underline{r}}) z^{-\ell} + \sum_{\ell=0}^{p_+ - 1} f_{p_+ - 1 - \ell, +}(n_0, t_{0,\underline{r}}) z^\ell \\ &= -c_{0,+} \alpha^+(n_0, t_{0,\underline{r}}) z^{-p_-} \prod_{k=1}^{q(n_0,t_{0,\underline{r}})} (z - \mu_k(n_0, t_{0,\underline{r}}))^{p_k(n_0,t_{0,\underline{r}})}, \end{aligned} \tag{6.7}$$

$$\begin{aligned} &G_{\underline{p}}(z, n_0, t_{0,\underline{r}}) = \frac{1}{2}\bigg(\frac{1}{\alpha(n_0, t_{0,\underline{r}})} - \frac{z}{\alpha^+(n_0, t_{0,\underline{r}})}\bigg) F_{\underline{p}}(z, n_0, t_{0,\underline{r}}) \\ &\quad - \frac{z}{2\alpha^+(n_0, t_{0,\underline{r}})} F_{\underline{p}}(z, n_0, t_{0,\underline{r}}) \\ &\quad \times \sum_{k=1}^{q(n_0,t_{0,\underline{r}})} \sum_{\ell=0}^{p_k(n_0,t_{0,\underline{r}})-1} \frac{\big(d^\ell\big(\zeta^{-1} y(P)\big)/d\zeta^\ell\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n_0,t_{0,\underline{r}})}}{\ell!(p_k(n_0, t_{0,\underline{r}}) - \ell - 1)!} \\ &\quad \times \Bigg(\frac{d^{p_k(n_0,t_{0,\underline{r}})-\ell-1}}{d\zeta^{p_k(n_0,t_{0,\underline{r}})-\ell-1}} \Bigg((z-\zeta)^{-1} \\ &\qquad\qquad \times \prod_{k'=1,\, k' \neq k}^{q(n_0,t_{0,\underline{r}})} (\zeta - \mu_{k'}(n_0, t_{0,\underline{r}}))^{-p_{k'}(n_0,t_{0,\underline{r}})}\Bigg)\Bigg)\Bigg|_{\zeta = \mu_k(n_0,t_{0,\underline{r}})}, \end{aligned} \tag{6.8}$$

in analogy to (4.16).

Our aim is to find an autonomous first-order system of ordinary differential equations with respect to $t_{\underline{r}}$ for $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ rather than for μ_j. We divide the differential equation

$$F_{\underline{p},t_{\underline{r}}} = -2iG_{\underline{p}}\widetilde{F}_{\underline{r}} + i(\widetilde{G}_{\underline{r}} + \widetilde{K}_{\underline{r}})F_{\underline{p}} \tag{6.9}$$

by $c_{0,+}z^{-p_-}y$ and rewrite it using Theorem C.1 as

$$\begin{aligned}\sum_{\ell=0}^{\infty}\hat{f}_{\ell,+,t_{\underline{r}}}\zeta^{\ell+1} &= -2i\bigg(\sum_{s=1}^{r_-}\tilde{f}_{r_--s,-}\zeta^s + \sum_{s=0}^{r_+-1}\tilde{f}_{r_+-1-s,+}\zeta^{-s}\bigg)\sum_{\ell=0}^{\infty}\hat{g}_{\ell,+}\zeta^{\ell} \\ &\quad + i\bigg(2\sum_{s=0}^{r_-}\tilde{g}_{r_--s,-}\zeta^s + 2\sum_{s=1}^{r_+}\tilde{g}_{r_+-s,+}\zeta^{-s} - \tilde{g}_{r_-,-} + \tilde{g}_{r_+,+}\bigg)\sum_{\ell=0}^{\infty}\hat{f}_{\ell,+}\zeta^{\ell+1}, \\ &\qquad P\to P_{\infty_-},\ \zeta = 1/z.\end{aligned} \tag{6.10}$$

The coefficients of ζ^{-s}, $s = 0,\dots,r_+-1$, cancel since

$$\sum_{k=0}^{\ell}\tilde{f}_{\ell-k,+}\hat{g}_{k,+} = \sum_{k=0}^{\ell}\tilde{g}_{\ell-k,+}\hat{f}_{k,+}, \quad \ell\in\mathbb{N}_0. \tag{6.11}$$

In (6.11) we used (2.19),

$$\tilde{f}_{\ell,+} = \sum_{k=0}^{\ell}\tilde{c}_{\ell-k,+}\hat{f}_{k,+}, \quad \tilde{g}_{\ell,+} = \sum_{k=0}^{\ell}\tilde{c}_{\ell-k,+}\hat{g}_{k,+}. \tag{6.12}$$

Comparing coefficients in (6.10) then yields [4]

$$\begin{aligned}\hat{f}_{\ell,+,t_{\underline{r}}} &= i\hat{f}_{\ell,+}(\tilde{g}_{r_+,+} - \tilde{g}_{r_-,-}) + 2i\sum_{k=0}^{r_+-1}\big(\tilde{g}_{k,+}\hat{f}_{r_++\ell-k,+} - \tilde{f}_{k,+}\hat{g}_{r_++\ell-k,+}\big) \\ &\quad - 2i\sum_{k=(\ell+1-r_-)\vee 0}^{\ell}\hat{g}_{k,+}\tilde{f}_{r_--1-\ell+k,-} + 2i\sum_{k=(\ell+2-r_-)\vee 0}^{\ell}\hat{f}_{k,+}\tilde{g}_{r_--\ell+k,-}, \quad \ell\in\mathbb{N}_0.\end{aligned} \tag{6.13}$$

By (6.11), the last sum in (6.13) can be rewritten as

$$\begin{aligned}&\sum_{j=0}^{r_+-1}\big(\tilde{g}_{j,+}\hat{f}_{r_++\ell-j,+} - \tilde{f}_{j,+}\hat{g}_{r_++\ell-j,+}\big) \\ &\quad = \bigg(\sum_{j=0}^{r_++\ell} - \sum_{j=r_+}^{r_++\ell}\bigg)\big(\tilde{g}_{j,+}\hat{f}_{r_++\ell-j,+} - \tilde{f}_{j,+}\hat{g}_{r_++\ell-j,+}\big) \\ &\quad = -\sum_{j=r_+}^{r_++\ell}\big(\tilde{g}_{j,+}\hat{f}_{r_++\ell-j,+} - \tilde{f}_{j,+}\hat{g}_{r_++\ell-j,+}\big) \\ &\quad = \sum_{j=0}^{\ell}\big(\hat{g}_{j,+}\tilde{f}_{r_++\ell-j,+} - \hat{f}_{j,+}\tilde{g}_{r_++\ell-j,+}\big).\end{aligned} \tag{6.14}$$

One performs a similar computation for $\hat{f}_{\ell,-,t_{\underline{r}}}$ using Theorem C.1 at $P\to P_{0,+}$. In summary, since $f_{k,\pm} = \sum_{\ell=0}^{k} c_{k-\ell,\pm}\hat{f}_{\ell,\pm}$, (6.13) and (6.14) yield the following

[4] $m\vee n = \max\{m,n\}$.

autonomous first-order system (for fixed $n = n_0$)

$$f_{\ell,\pm,t_{\underline{r}}} = \mathcal{F}_{\ell,\pm}(f_{j,-}, f_{j,+}, g_{j,-}, g_{j,+}), \quad \ell = 0, \dots, p_\pm - 1, \tag{6.15}$$

with initial conditions

$$\begin{aligned} &f_{\ell,\pm}(n_0, t_{0,\underline{r}}), \quad \ell = 0, \dots, p_\pm - 1, \\ &g_{\ell,\pm}(n_0, t_{0,\underline{r}}), \quad \ell = 0, \dots, p_\pm, \end{aligned} \tag{6.16}$$

where $\mathcal{F}_{\ell,\pm}$, $\ell = 0, \dots, p_\pm - 1$, are polynomials in $2p+3$ variables,

$$\begin{aligned} \mathcal{F}_{\ell,\pm} &= i f_{\ell,\pm}(\tilde g_{r_\pm,\pm} - \tilde g_{r_\mp,\mp}) \\ &\quad + 2i \sum_{k=0}^{\ell} \big(f_{k,\pm}(\tilde g_{r_\mp-\ell+k,\mp} - \tilde g_{r_\pm+\ell-k,\pm}) + g_{k,\pm}(\tilde f_{r_\pm+\ell-k,\pm} - \tilde f_{r_\mp-1-\ell+k,\mp}) \big) \\ &\quad + 2i \sum_{k=0}^{\ell} c_{\ell-k,\pm} \times \begin{cases} 0, & 0 \le k < r_\mp - 1, \\ \sum_{j=0}^{k-r_\mp} \hat g_{j,\pm} \tilde f_{r_\mp-1-k+j,\mp} & \\ - \sum_{j=0}^{k+1-r_\mp} \hat f_{j,\pm} \tilde g_{r_\mp-k+j,\mp}, & k \ge r_\mp - 1. \end{cases} \end{aligned} \tag{6.17}$$

Explicitly, one obtains (for simplicity, $r_\pm > 1$)

$$\begin{aligned} \mathcal{F}_{0,\pm} &= i f_{0,\pm}(\tilde g_{r_\mp,\mp} - \tilde g_{r_\pm,\pm}) + 2i g_{0,\pm}(\tilde f_{r_\pm,\pm} - \tilde f_{r_\mp-1,\mp}), \\ \mathcal{F}_{1,\pm} &= 2i f_{0,\pm}(\tilde g_{r_\mp-1,\mp} - \tilde g_{r_\pm+1,\pm}) + i f_{1,\pm}(\tilde g_{r_\mp,\mp} - \tilde g_{r_\pm,\pm}) \\ &\quad + 2i g_{0,\pm}(\tilde f_{r_\pm+1,\pm} - \tilde f_{r_\mp-2,\mp}) + 2i g_{1,\pm}(\tilde f_{r_\pm,\pm} - \tilde f_{r_\mp-1,\mp}), \ \text{ etc.} \end{aligned} \tag{6.18}$$

By (6.6)–(6.8), the initial conditions (6.16) are uniquely determined by the initial divisor $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,\underline{r}})}$ in (6.3) and by the data in (6.5).

Similarly, one transforms

$$G_{\underline{p},t_{\underline{r}}} = i F_{\underline{p}} \widetilde H_{\underline{r}} - i H_{\underline{p}} \widetilde F_{\underline{r}}, \tag{6.19}$$

$$H_{\underline{p},t_{\underline{r}}} = 2i G_{\underline{p}} \widetilde H_{\underline{r}} - i\big(\widetilde G_{\underline{r}} + \widetilde K_{\underline{r}}\big) H_{\underline{p}} \tag{6.20}$$

into (for fixed $n = n_0$) [5]

$$\begin{aligned} \hat g_{0,\pm,t_{\underline{r}}} &= 0, \\ \hat g_{\ell,\pm,t_{\underline{r}}} &= i \sum_{k=0}^{r_\pm-1} \big(\tilde h_{k,\pm} \hat f_{r_\pm-1+\ell-k,\pm} - \tilde f_{k,\pm} \hat h_{r_\pm-1+\ell-k,\pm}\big) \\ &\quad + i \sum_{k=(\ell-r_\mp)\vee 0}^{\ell-1} \big(\hat f_{k,\pm} \tilde h_{r_\mp-\ell+k,\mp} - \hat h_{k,\pm} \tilde f_{r_\mp-\ell+k,\mp}\big) \\ &= i \sum_{k=0}^{\ell-1} \big(\hat h_{k,\pm} \tilde f_{r_\pm-1+\ell-k,\pm} - \hat f_{k,\pm} \tilde h_{r_\pm-1+\ell-k,\pm}\big) \\ &\quad + i \sum_{k=(\ell-r_\mp)\vee 0}^{\ell-1} \big(\hat f_{k,\pm} \tilde h_{r_\mp-\ell+k,\mp} - \hat h_{k,\pm} \tilde f_{r_\mp-\ell+k,\mp}\big), \quad \ell \in \mathbb{N}, \end{aligned} \tag{6.21}$$

$$\hat h_{\ell,\pm,t_{\underline{r}}} = i \hat h_{\ell,\pm}\big(\tilde g_{r_\mp,\mp} - \tilde g_{r_\pm,\pm}\big) + 2i \sum_{k=0}^{r_\pm-1} \big(\tilde h_{k,\pm} \hat g_{r_\pm+\ell-k,\pm} - \tilde g_{k,\pm} \hat h_{r_\pm+\ell-k,\pm}\big)$$

[5] $m \vee n = \max\{m, n\}$.

$$+ 2i \sum_{k=(\ell-r_\mp+1)\vee 0}^{\ell} \hat g_{k,\pm}\tilde h_{r_\mp-1-\ell+k,\mp} - 2i \sum_{k=(\ell-r_\mp)\vee 0}^{\ell} \hat h_{k,\pm}\tilde g_{r_\mp-\ell+k,\mp}$$

$$= i\hat h_{\ell,\pm}\big(\tilde g_{r_\mp,\mp} - \tilde g_{r_\pm,\pm}\big) + 2i\sum_{k=0}^{\ell}\big(\hat h_{k,\pm}\tilde g_{r_\pm+\ell-k,\pm} - \hat g_{k,\pm}\tilde h_{r_\pm+\ell-k,\pm}\big)$$

$$+ 2i \sum_{k=(\ell-r_\mp+1)\vee 0}^{\ell} \hat g_{k,\pm}\tilde h_{r_\mp-1-\ell+k,\mp} - 2i \sum_{k=(\ell-r_\mp)\vee 0}^{\ell} \hat h_{k,\pm}\tilde g_{r_\mp-\ell+k,\mp},$$

$$\ell \in \mathbb{N}_0. \quad (6.22)$$

Summing over ℓ in (6.21), (6.22) then yields the following first-order system

$$g_{\ell,\pm,t_{\underline{r}}} = \mathcal{G}_{\ell,\pm}(f_{k,-}, f_{k,+}, h_{k,-}, h_{k,+}), \quad \ell = 0,\dots,p_\pm, \quad (6.23)$$

$$h_{\ell,\pm,t_{\underline{r}}} = \mathcal{H}_{\ell,\pm}(g_{k,-}, g_{k,+}, h_{k,-}, h_{k,+}), \quad \ell = 0,\dots,p_\pm-1, \quad (6.24)$$

with initial conditions

$$\begin{aligned} &f_{\ell,\pm}(n_0,t_{0,\underline{r}}), \quad \ell=0,\dots,p_\pm-1,\\ &g_{\ell,\pm}(n_0,t_{0,\underline{r}}), \quad \ell=0,\dots,p_\pm,\\ &h_{\ell,\pm}(n_0,t_{0,\underline{r}}), \quad \ell=0,\dots,p_\pm-1, \end{aligned} \quad (6.25)$$

where $\mathcal{G}_{\ell,\pm}$, $\mathcal{H}_{\ell,\pm}$, are polynomials in $2p+2$, $2p+3$ variables

$$\begin{aligned} \mathcal{G}_{\ell,\pm} &= i\sum_{k=0}^{\ell-1}\big(f_{k,\pm}(\tilde h_{r_\mp-\ell+k,\mp} - \tilde h_{r_\pm-1+\ell-k,\pm}) + h_{k,\pm}(\tilde f_{r_\pm-1+\ell-k,\pm} - \tilde f_{r_\mp-\ell+k,\mp})\big)\\ &\quad - i\sum_{k=0}^{\ell-1} c_{\ell-1-k,\pm}\times\begin{cases}0, & 0\le k\le r_\mp,\\ \sum_{j=0}^{k-r_\mp-1}(\hat f_{j,\pm}\tilde h_{r_\mp-k+j,\mp} - \hat h_{j,\pm}\tilde f_{r_\mp-k+j,\mp}), & k>r_\mp,\end{cases} \end{aligned} \quad (6.26)$$

$$\begin{aligned} \mathcal{H}_{\ell,\pm} &= ih_{\ell,\pm}\big(\tilde g_{r_\mp,\mp} - \tilde g_{r_\pm,\pm}\big)\\ &\quad + 2i\sum_{k=0}^{\ell}\big(g_{k,\pm}(\tilde h_{r_\mp-1-\ell+k,\mp} - \tilde h_{r_\pm+\ell-k,\pm}) + h_{k,\pm}(\tilde g_{r_\pm+\ell-k,\pm} - \tilde g_{r_\mp-\ell+k,\mp})\big)\\ &\quad + 2i\sum_{k=0}^{\ell} c_{\ell-k,\pm}\times\begin{cases}0, & 0\le k< r_\mp,\\ -\sum_{j=0}^{k-r_\mp}\hat g_{j,\pm}\tilde h_{r_\mp-1-k+j,\mp} & \\ +\sum_{j=0}^{k-r_\mp-1}\hat h_{j,\pm}\tilde g_{r_\mp-k+j,\mp}, & k\ge r_\mp.\end{cases} \end{aligned} \quad (6.27)$$

Explicitly (assuming $r_\pm > 2$),

$$\begin{aligned} \mathcal{G}_{0,\pm} &= 0,\\ \mathcal{G}_{1,\pm} &= if_{0,\pm}(\tilde h_{r_\mp-1,\mp} - \tilde h_{r_\pm,\pm}) + ih_{0,\pm}(\tilde f_{r_\pm,\pm} - \tilde f_{r_\mp-1,\mp}), \qquad (6.28)\\ \mathcal{G}_{2,\pm} &= if_{0,\pm}(\tilde h_{r_\mp-2,\mp} - \tilde h_{r_\pm+1,\pm}) + if_{1,\pm}(\tilde h_{r_\mp-1,\mp} - \tilde h_{r_\pm,\pm})\\ &\quad + ih_{0,\pm}(\tilde f_{r_\pm+1,\pm} - \tilde f_{r_\mp-2,\mp}) + ih_{1,\pm}(\tilde f_{r_\pm,\pm} - \tilde f_{r_\mp-1,\mp}), \text{ etc.},\\ \mathcal{H}_{0,\pm} &= 2ig_{0,\pm}(\tilde h_{r_\mp-1,\mp} - \tilde h_{r_\pm,\pm}) + ih_{0,\pm}(\tilde g_{r_\pm,\pm} - \tilde g_{r_\mp,\mp}), \qquad (6.29)\\ \mathcal{H}_{1,\pm} &= 2ig_{0,\pm}(\tilde h_{r_\mp-2,\mp} - \tilde h_{r_\pm+1,\pm}) + 2ig_{1,\pm}(\tilde h_{r_\mp-1,\mp} - \tilde h_{r_\pm,\pm})\\ &\quad + 2ih_{0,\pm}(\tilde g_{r_\pm+1,\pm} - \tilde g_{r_\mp-1,\mp}) + ih_{1,\pm}(\tilde g_{r_\pm,\pm} - \tilde g_{r_\mp,\mp}), \text{ etc.} \end{aligned}$$

Again by (6.6)–(6.8), the initial conditions (6.25) are uniquely determined by the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}$ in (6.3) and by the data in (6.5).

Being autonomous with polynomial right-hand sides, there exists a $T_0>0$, such that the first-order initial value problem (6.15), (6.23), (6.24) with initial conditions (6.25) has a unique solution

$$\begin{aligned} &f_{\ell,\pm} = f_{\ell,\pm}(n_0,t_{\underline{r}}), \quad \ell=0,\dots,p_\pm-1,\\ &g_{\ell,\pm} = g_{\ell,\pm}(n_0,t_{\underline{r}}), \quad \ell=0,\dots,p_\pm,\\ &h_{\ell,\pm} = h_{\ell,\pm}(n_0,t_{\underline{r}}), \quad \ell=0,\dots,p_\pm-1,\\ &\text{for all } \ t_{\underline{r}}\in(t_{0,\underline{r}}-T_0,t_{0,\underline{r}}+T_0) \end{aligned} \tag{6.30}$$

(cf., e.g., [54, Sect. III.10]). Given the solution (6.30), we proceed as in Section 4 and introduce the following quantities (where $t_{\underline{r}}\in(t_{0,\underline{r}}-T_0,t_{0,\underline{r}}+T_0)$):

$$\alpha^+(n_0,t_{\underline{r}}) = \alpha(n_0,t_{\underline{r}})\bigg(\prod_{m=0}^{2p+1}E_m\bigg)^{1/2}\prod_{k=1}^{q(n_0,t_{\underline{r}})}\mu_k(n_0,t_{\underline{r}})^{-p_k(n_0,t_{\underline{r}})}, \tag{6.31}$$

$$\begin{aligned} F_{\underline{p}}(z,n_0,t_{\underline{r}}) &= \sum_{\ell=1}^{p_-} f_{p_--\ell,-}(n_0,t_{\underline{r}})z^{-\ell} + \sum_{\ell=0}^{p_+-1} f_{p_+-1-\ell,+}(n_0,t_{\underline{r}})z^{\ell}\\ &= -c_{0,+}\alpha^+(n_0,t_{\underline{r}})z^{-p_-}\prod_{k=1}^{q(n_0,t_{\underline{r}})}(z-\mu_k(n_0,t_{\underline{r}}))^{p_k(n_0,t_{\underline{r}})}, \end{aligned} \tag{6.32}$$

$$\begin{aligned} G_{\underline{p}}(z,n_0,t_{\underline{r}}) &= \frac{1}{2}\bigg(\frac{1}{\alpha(n_0,t_{\underline{r}})}-\frac{z}{\alpha^+(n_0,t_{\underline{r}})}\bigg)F_{\underline{p}}(z,n_0,t_{\underline{r}}) \\ &- \frac{z}{2\alpha^+(n_0,t_{\underline{r}})}F_{\underline{p}}(z,n_0,t_{\underline{r}})\sum_{k=1}^{q(n_0,t_{\underline{r}})}\sum_{\ell=0}^{p_k(n_0,t_{\underline{r}})-1}\frac{\big(d^\ell\big(\zeta^{-1}y(P)\big)/d\zeta^\ell\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n_0,t_{\underline{r}})}}{\ell!(p_k(n_0,t_{\underline{r}})-\ell-1)!}\\ &\times\Bigg(\frac{d^{p_k(n_0,t_{\underline{r}})-\ell-1}}{d\zeta^{p_k(n_0,t_{\underline{r}})-\ell-1}}\Bigg((z-\zeta)^{-1}\prod_{k'=1,\,k'\neq k}^{q(n_0,t_{\underline{r}})}(\zeta-\mu_{k'}(n_0,t_{\underline{r}}))^{-p_{k'}(n_0,t_{\underline{r}})}\Bigg)\Bigg)\Bigg|_{\zeta=\mu_k(n_0,t_{\underline{r}})}. \end{aligned} \tag{6.33}$$

In particular, this leads to the divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{\underline{r}})}\in\operatorname{Sym}^p(\mathcal{K}_p) \tag{6.34}$$

and the sign of y in (6.32) is chosen as usual by

$$\begin{aligned} \hat{\mu}_k(n_0,t_{\underline{r}}) = (\mu_k(n_0,t_{\underline{r}}),(2/c_{0,+})\mu_j(n_0,t_{\underline{r}})^{p_-}G_{\underline{p}}(\mu_k(n_0,t_{\underline{r}}),n_0,t_{\underline{r}})),&\\ k=1,\dots,q(n_0,t_{\underline{r}}),& \end{aligned} \tag{6.35}$$

and

$$\underline{\hat{\mu}}(n_0,t_{\underline{r}}) = \{\underbrace{\mu_1(n_0,t_{\underline{r}}),\dots,\mu_1(n_0,t_{\underline{r}})}_{p_1(n_0,t_{\underline{r}})\text{ times}},\dots,\underbrace{\mu_{q(n_0,t_{\underline{r}})}(n_0,t_{\underline{r}}),\dots,\mu_{q(n_0,t_{\underline{r}})}(n_0,t_{\underline{r}})}_{p_{q(n_0,t_{\underline{r}})}(n_0,t_{\underline{r}})\text{ times}}\} \tag{6.36}$$

with

$$\mu_k(n_0,t_{\underline{r}})\neq\mu_{k'}(n_0,t_{\underline{r}}) \ \text{ for } \ k\neq k', \ k,k'=1,\dots,q(n_0,t_{\underline{r}}), \tag{6.37}$$

and

$$p_k(n_0,t_{\underline{r}})\in\mathbb{N}, \ k=1,\dots,q(n_0,t_{\underline{r}}), \quad \sum_{k=1}^{q(n_0,t_{\underline{r}})}p_k(n_0,t_{\underline{r}})=p. \tag{6.38}$$

By construction (cf. (6.35)), the divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{\underline{r}})}$ is nonspecial for all $t_{\underline{r}} \in (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$.

In exactly the same manner as in (4.19)–(4.21) one then infers that $F_{\underline{p}}(\,\cdot\,, n_0, t_{\underline{r}})$ divides $R_{\underline{p}} - G_{\underline{p}}^2$ (since $t_{\underline{r}}$ is just a fixed parameter).

As in Section 4, the assumption that the Laurent polynomial $F_{\underline{p}}(\,\cdot\,, n_0 - 1, t_{\underline{r}})$ is of full order is implied by the hypothesis that

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{\underline{r}})} \in \mathcal{M}_0 \text{ for all } t_{\underline{r}} \in (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0). \tag{6.39}$$

The explicit formula for $\beta(n_0, t_{\underline{r}})$ then reads (for $t_{\underline{r}} \in (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$)

$$\begin{aligned}
&\alpha^+(n_0, t_{\underline{r}})\beta(n_0, t_{\underline{r}}) \qquad (6.40)\\
&= -\frac{1}{2}\sum_{k=1}^{q(n_0,t_{\underline{r}})} \frac{\big(d^{p_k(n_0,t_{\underline{r}})-1}\big(\zeta^{-1}y(P)\big)/d\zeta^{p_k(n_0,t_{\underline{r}})-1}\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n_0,t_{\underline{r}})}}{(p_k(n_0,t_{\underline{r}})-1)!}\\
&\quad\times \prod_{k'=1,\,k'\neq k}^{q(n_0,t_{\underline{r}})} (\mu_k(n_0,t_{\underline{r}}) - \mu_{k'}(n_0,t_{\underline{r}}))^{-p_k(n_0,t_{\underline{r}})}\\
&\quad+ \frac{1}{2}\bigg(\frac{\alpha^+(n_0,t_{\underline{r}})}{\alpha(n_0,t_{\underline{r}})} + \sum_{k=1}^{q(n_0,t_{\underline{r}})} p_k(n_0,t_{\underline{r}})\mu_k(n_0,t_{\underline{r}}) - \frac{1}{2}\sum_{m=0}^{2p+1} E_m\bigg). \qquad (6.41)
\end{aligned}$$

With (6.21)–(6.41) in place, we can now apply the stationary formalism as summarized in Theorem 4.4, subject to the additional hypothesis (6.39), for each fixed $t_{\underline{r}} \in (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$. This yields, in particular, the quantities

$$F_{\underline{p}},\ G_{\underline{p}},\ H_{\underline{p}},\ \alpha,\ \beta, \text{ and } \underline{\hat{\mu}},\ \underline{\hat{\nu}} \text{ for } (n, t_{\underline{r}}) \in \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0), \tag{6.42}$$

which are of the form (6.32)–(6.41), replacing the fixed $n_0 \in \mathbb{Z}$ by an arbitrary $n \in \mathbb{Z}$. In addition, one has the following result.

Lemma 6.2. *Assume Hypothesis* 6.1 *and condition* (6.39). *Then the following relations are valid on* $\mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$,

$$G_{\underline{p}}^2 - F_{\underline{p}}H_{\underline{p}} = R_{\underline{p}}, \tag{6.43}$$

$$z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^- = 0, \tag{6.44}$$

$$z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} - G_{\underline{p}} + G_{\underline{p}}^- = 0, \tag{6.45}$$

$$-F_{\underline{p}} + zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + G_{\underline{p}}^-) = 0, \tag{6.46}$$

$$z\beta(G_{\underline{p}} + G_{\underline{p}}^-) - zH_{\underline{p}} + H_{\underline{p}}^- = 0, \tag{6.47}$$

and hence the stationary part, (5.8), *of the algebro-geometric initial value problem holds,*

$$UV_{\underline{p}} - V_{\underline{p}}^+U = 0 \ \text{ on } \mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0). \tag{6.48}$$

In particular, Lemmas 3.2*–*3.5 *apply.*

Lemma 6.2 now raises the following important consistency issue: On the one hand, one can solve the initial value problem (6.57), (6.58) at $n = n_0$ in some interval $t_{\underline{r}} \in (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$, and then extend the quantities $F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}$ to all $\mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$ using the stationary algorithm summarized in Theorem 4.4 as just recorded in Lemma 6.2. On the other hand, one can solve the initial value problem (6.57), (6.58) at $n = n_1$, $n_1 \neq n_0$, in some interval

$t_{\underline{r}} \in (t_{0,\underline{r}} - T_1, t_{0,\underline{r}} + T_1)$ with the initial condition obtained by applying the discrete algorithm to the quantities $F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}$ starting at $(n_0, t_{0,\underline{r}})$ and ending at $(n_1, t_{0,\underline{r}})$. Consistency then requires that the two approaches yield the same result at $n = n_1$ for $t_{\underline{r}}$ in some open neighborhood of $t_{0,\underline{r}}$.

Equivalently, and pictorially speaking, envisage a vertical $t_{\underline{r}}$-axis and a horizontal n-axis. Then, consistency demands that first solving the initial value problem (6.57), (6.58) at $n = n_0$ in some $t_{\underline{r}}$-interval around $t_{0,\underline{r}}$ and using the stationary algorithm to extend $F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}$ horizontally to $n = n_1$ and the same $t_{\underline{r}}$-interval around $t_{0,\underline{r}}$, or first applying the stationary algorithm starting at $(n_0, t_{0,\underline{r}})$ to extend $F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}$ horizontally to $(n_1, t_{0,\underline{r}})$ and then solving the initial value problem (6.57), (6.58) at $n = n_1$ in some $t_{\underline{r}}$-interval around $t_{0,\underline{r}}$ should produce the same result at $n = n_1$ in a sufficiently small open $t_{\underline{r}}$ interval around $t_{0,\underline{r}}$.

To settle this consistency issue, we will prove the following result. To this end we find it convenient to replace the initial value problem (6.57), (6.58) by the original $t_{\underline{r}}$-dependent zero-curvature equation (5.7), $U_{t_{\underline{r}}} + U\widetilde{V}_{\underline{r}} - \widetilde{V}_{\underline{r}}^{+}U = 0$ on $\mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$.

Lemma 6.3. *Assume Hypothesis* 6.1 *and condition* (6.39). *Moreover, suppose that* (5.47)–(5.49) *hold on* $\mathbb{C} \times \{n_0\} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$. *Then* (5.47)–(5.49) *hold on* $\mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$, *that is,*

$$\begin{aligned} F_{\underline{p},t_{\underline{r}}}(z,n,t_{\underline{r}}) &= -2iG_{\underline{p}}(z,n,t_{\underline{r}})\widetilde{F}_{\underline{r}}(z,n,t_{\underline{r}}) \\ &\quad + i\big(\widetilde{G}_{\underline{r}}(z,n,t_{\underline{r}}) + \widetilde{K}_{\underline{r}}(z,n,t_{\underline{r}})\big)F_{\underline{p}}(z,n,t_{\underline{r}}), \qquad (6.49)\\ G_{\underline{p},t_{\underline{r}}}(z,n,t_{\underline{r}}) &= iF_{\underline{p}}(z,n,t_{\underline{r}})\widetilde{H}_{\underline{r}}(z,n,t_{\underline{r}}) - iH_{\underline{p}}(z,n,t_{\underline{r}})\widetilde{F}_{\underline{r}}(z,n,t_{\underline{r}}), \qquad (6.50)\\ H_{\underline{p},t_{\underline{r}}}(z,n,t_{\underline{r}}) &= 2iG_{\underline{p}}(z,n,t_{\underline{r}})\widetilde{H}_{\underline{r}}(z,n,t_{\underline{r}}) \\ &\quad - i\big(\widetilde{G}_{\underline{r}}(z,n,t_{\underline{r}}) + \widetilde{K}_{\underline{r}}(z,n,t_{\underline{r}})\big)H_{\underline{p}}(z,n,t_{\underline{r}}), \qquad (6.51)\\ &\quad (z,n,t_{\underline{r}}) \in \mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0). \end{aligned}$$

Moreover,

$$\begin{aligned} \phi_{t_{\underline{r}}}(P,n,t_{\underline{r}}) &= i\widetilde{F}_{\underline{r}}(z,n,t_{\underline{r}})\phi^2(P,n,t_{\underline{r}}) \\ &\quad - i\big(\widetilde{G}_{\underline{r}}(z,n,t_{\underline{r}}) + \widetilde{K}_{\underline{r}}(z,n,t_{\underline{r}})\big)\phi(P,n,t_{\underline{r}}) + i\widetilde{H}_{\underline{r}}(z,n,t_{\underline{r}}), \qquad (6.52)\\ \alpha_{t_{\underline{r}}}(n,t_{\underline{r}}) &= iz\widetilde{F}_{\underline{r}}^{-}(z,n,t_{\underline{r}}) \\ &\quad + i\alpha(n,t_{\underline{r}})\big(\widetilde{G}_{\underline{r}}(z,n,t_{\underline{r}}) + \widetilde{K}_{\underline{r}}^{-}(z,n,t_{\underline{r}})\big) - i\widetilde{F}_{\underline{r}}(z,n,t_{\underline{r}}), \qquad (6.53)\\ \beta_{t_{\underline{r}}}(n,t_{\underline{r}}) &= -i\beta(n,t_{\underline{r}})\big(\widetilde{G}_{\underline{r}}^{-}(z,n,t_{\underline{r}}) + \widetilde{K}_{\underline{r}}(z,n,t_{\underline{r}})\big) \\ &\quad + i\widetilde{H}_{\underline{r}}(z,n,t_{\underline{r}}) - iz^{-1}\widetilde{H}_{\underline{r}}^{-}(z,n,t_{\underline{r}}), \qquad (6.54)\\ &(z,n,t_{\underline{r}}) \in \mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0). \end{aligned}$$

Proof. By Lemma 6.2 we have (5.22), (5.23), (5.29), (5.31)–(5.33), and (6.43)–(6.47) for $(n, t_{\underline{r}}) \in \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$ at our disposal.

Differentiating (5.22) at $n = n_0$ with respect to $t_{\underline{r}}$ and inserting (6.49) and (6.50) at $n = n_0$ then yields (6.52) at $n = n_0$.

We note that the sequences $\tilde{f}_{\ell,\pm}$, $\tilde{g}_{\ell,\pm}$, $\tilde{h}_{\ell,\pm}$ satisfy the recursion relations (2.6)–(2.12) (since the homogeneous sequences satisfy these relations). Hence, to prove

(6.53) and (6.54) at $n = n_0$ it remains to show

$$\begin{aligned} \alpha_{t_{\underline{r}}} &= i\alpha(\tilde{g}_{r_+,+} + \tilde{g}^-_{r_-,-}) + i(\tilde{f}^-_{r_--1,-} - \tilde{f}_{r_+-1,+}), \\ \beta_{t_{\underline{r}}} &= -i\beta(\tilde{g}^-_{r_+,+} + \tilde{g}_{r_-,-}) - i(\tilde{h}^-_{r_+-1,+} - \tilde{h}_{r_--1,-}). \end{aligned} \tag{6.55}$$

But this follows from (6.49), (6.51) at $n = n_0$ (cf. (6.18), (6.29))

$$\begin{aligned} \alpha_{t_{\underline{r}}} &= i\alpha(\tilde{g}_{r_+,+} - \tilde{g}_{r_-,-}) + i(\tilde{f}_{r_-,-} - \tilde{f}_{r_+-1,+}), \\ \beta_{t_{\underline{r}}} &= i\beta(\tilde{g}_{r_+,+} - \tilde{g}_{r_-,-}) + i(\tilde{h}_{r_--1,-} - \tilde{h}_{r_+,+}). \end{aligned}$$

Inserting now (2.11) at $\ell = r_- - 1$ and (2.8) at $\ell = r_+ - 1$ then yields (6.55).

For the step $n = n_0 \mp 1$ we differentiate (4.36)–(4.41) (which are equivalent to (6.43)–(6.47)) and insert (6.49)–(6.51), (5.11)–(5.18) at $n = n_0$. For the case $n > n_0$ we obtain $\alpha^+_{t_{\underline{r}}}$ and $\beta^+_{t_{\underline{r}}}$ from (6.49), (6.51) at $n = n_0$ as before using the other two signs in (6.18), (6.29). Iterating these arguments proves (6.49)–(6.54) for $(z, n, t_{\underline{r}}) \in \mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$. $\square$

We summarize Lemmas 6.2 and 6.3 next.

Theorem 6.4. *Assume Hypothesis* 6.1 *and condition* (6.39). *Moreover, suppose that*

$$\begin{aligned} & f_{\ell,\pm} = f_{\ell,\pm}(n_0, t_{\underline{r}}), \quad \ell = 0, \dots, p_\pm - 1, \\ & g_{\ell,\pm} = g_{\ell,\pm}(n_0, t_{\underline{r}}), \quad \ell = 0, \dots, p_\pm, \\ & h_{\ell,\pm} = h_{\ell,\pm}(n_0, t_{\underline{r}}), \quad \ell = 0, \dots, p_\pm - 1 \\ & \text{for all } t_{\underline{r}} \in (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0), \end{aligned} \tag{6.56}$$

satisfy the autonomous first-order system of ordinary differential equations (*for fixed* $n = n_0$)

$$\begin{aligned} f_{\ell,\pm,t_{\underline{r}}} &= \mathcal{F}_{\ell,\pm}(f_{k,-}, f_{k,+}, g_{k,-}, g_{k,+}), \quad \ell = 0, \dots, p_\pm - 1, \\ g_{\ell,\pm,t_{\underline{r}}} &= \mathcal{G}_{\ell,\pm}(f_{k,-}, f_{k,+}, h_{k,-}, h_{k,+}), \quad \ell = 0, \dots, p_\pm, \\ h_{\ell,\pm,t_{\underline{r}}} &= \mathcal{H}_{\ell,\pm}(g_{k,-}, g_{k,+}, h_{k,-}, h_{k,+}), \quad \ell = 0, \dots, p_\pm - 1, \end{aligned} \tag{6.57}$$

with $\mathcal{F}_{\ell,\pm}$, $\mathcal{G}_{\ell,\pm}$, $\mathcal{H}_{\ell,\pm}$ *given by* (6.17), (6.26), (6.27), *and with initial conditions*

$$\begin{aligned} & f_{\ell,\pm}(n_0, t_{0,\underline{r}}), \quad \ell = 0, \dots, p_\pm - 1, \\ & g_{\ell,\pm}(n_0, t_{0,\underline{r}}), \quad \ell = 0, \dots, p_\pm, \\ & h_{\ell,\pm}(n_0, t_{0,\underline{r}}), \quad \ell = 0, \dots, p_\pm - 1. \end{aligned} \tag{6.58}$$

Then $F_{\underline{p}}$, $G_{\underline{p}}$, *and* $H_{\underline{p}}$ *as constructed in* (6.32)*–*(6.42) *on* $\mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$ *satisfy the zero-curvature equations* (5.7), (5.8), *and* (5.50) *on* $\mathbb{C} \times \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$,

$$U_{t_{\underline{r}}} + U\widetilde{V}_{\underline{r}} - \widetilde{V}^+_{\underline{r}} U = 0, \tag{6.59}$$

$$U V_{\underline{p}} - V^+_{\underline{p}} U = 0, \tag{6.60}$$

$$V_{\underline{p},t_{\underline{r}}} - \big[\widetilde{V}_{\underline{r}}, V_{\underline{p}}\big] = 0 \tag{6.61}$$

with U, $V_{\underline{p}}$, and $\widetilde{V}_{\underline{r}}$ given by (5.9). *In particular, α, β satisfy* (5.1) *and the algebro-geometric initial value problem* (5.2), (5.3) *on $\mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$,*

$$\begin{aligned}
&\widetilde{\mathrm{AL}}_{\underline{r}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{\underline{r}}} - \alpha(\tilde{g}_{r_+,+} + \tilde{g}^-_{r_-,-}) + \tilde{f}_{r_+-1,+} - \tilde{f}^-_{r_- -1,-} \\ -i\beta_{t_{\underline{r}}} + \beta(\tilde{g}^-_{r_+,+} + \tilde{g}_{r_-,-}) - \tilde{h}_{r_- -1,-} + \tilde{h}^-_{r_+-1,+} \end{pmatrix} = 0, \\
&(\alpha,\beta)\big|_{t=t_{0,\underline{r}}} = \big(\alpha^{(0)},\beta^{(0)}\big),
\end{aligned} \tag{6.62}$$

$$\text{s-AL}_{\underline{p}}\left(\alpha^{(0)},\beta^{(0)}\right) = \begin{pmatrix} -\alpha^{(0)}(g_{p_+,+} + g^-_{p_-,-}) + f_{p_- -1,+} - f^-_{p_- -1,-} \\ \beta^{(0)}(g^-_{p_+,+} + g_{p_-,-}) - h_{p_- -1,-} + h^-_{p_+-1,+} \end{pmatrix} = 0. \tag{6.63}$$

In addition, α, β are given by

$$\alpha^+(n,t_{\underline{r}}) = \alpha(n,t_{\underline{r}})\bigg(\prod_{m=0}^{2p+1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n,t_{\underline{r}})} \mu_k(n,t_{\underline{r}})^{-p_k(n,t_{\underline{r}})}, \tag{6.64}$$

$$\begin{aligned}
\alpha^+(n,t_{\underline{r}})\beta(n,t_{\underline{r}}) = {}& -\frac{1}{2}\sum_{k=1}^{q(n,t_{\underline{r}})} \frac{\big(d^{p_k(n,t_{\underline{r}})-1}\big(\zeta^{-1}y(P)\big)/d\zeta^{p_k(n,t_{\underline{r}})-1}\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n,t_{\underline{r}})}}{(p_k(n,t_{\underline{r}})-1)!} \\
&\times \prod_{k'=1,\,k'\neq k}^{q(n,t_{\underline{r}})} (\mu_k(n,t_{\underline{r}}) - \mu_{k'}(n,t_{\underline{r}}))^{-p_k(n,t_{\underline{r}})} \\
&+ \frac{1}{2}\bigg(\bigg(\prod_{m=0}^{2p+1} E_m\bigg)^{1/2} \prod_{k=1}^{q(n,t_{\underline{r}})} \mu_k(n,t_{\underline{r}})^{-p_k(n,t_{\underline{r}})} + \sum_{k=1}^{q(n,t_{\underline{r}})} p_k(n,t_{\underline{r}})\mu_k(n,t_{\underline{r}}) \\
&- \frac{1}{2}\sum_{m=0}^{2p+1} E_m\bigg), \quad (z,n,t_{\underline{r}}) \in \mathbb{Z} \times (t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0).
\end{aligned} \tag{6.65}$$

Moreover, Lemmas 3.2–3.5 *and* 5.2–5.4 *apply.*

As in Lemma 4.3 we now show that also in the time-dependent case, most initial divisors are well-behaved in the sense that the corresponding divisor trajectory stays away from $P_{\infty_\pm}, P_{0,\pm}$ for all $(n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$.

Lemma 6.5. *The set $\mathcal{M}_1$ of initial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})}$ for which $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n,t_{\underline{r}})}$, defined via* (5.58) *and* (5.59), *are admissible* (*i.e., do not contain $P_{\infty_\pm}$, $P_{0,\pm}$*) *and hence are nonspecial for all $(n,t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$, forms a dense set of full measure in the set $\mathrm{Sym}^p(\mathcal{K}_p)$ of nonnegative divisors of degree p.*

Proof. Let $\mathcal{M}_{\mathrm{sing}}$ be as introduced in the proof of Lemma 4.3. Then

$$\begin{aligned}
&\bigcup_{t_{\underline{r}}\in\mathbb{R}} \Big(\underline{\alpha}_{Q_0}(\mathcal{M}_{\mathrm{sing}}) + t_{\underline{r}}\underline{\widetilde{U}}^{(2)}_{\underline{r}}\Big) \\
&\quad\subseteq \bigcup_{P\in\{P_{\infty_\pm},P_{0,\pm}\}} \bigcup_{t_{\underline{r}}\in\mathbb{R}} \Big(\underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathrm{Sym}^{p-1}(\mathcal{K}_p)) + t_{\underline{r}}\underline{\widetilde{U}}^{(2)}_{\underline{r}}\Big)
\end{aligned} \tag{6.66}$$

is of measure zero as well, since it is contained in the image of $\mathbb{R} \times \mathrm{Sym}^{p-1}(\mathcal{K}_p)$ which misses one real dimension in comparison to the $2p$ real dimensions of $J(\mathcal{K}_p)$. But then

$$\bigcup_{(n,t_{\underline{r}})\in\mathbb{Z}\times\mathbb{R}} \Big(\underline{\alpha}_{Q_0}(\mathcal{M}_{\mathrm{sing}}) + n\underline{A}_{P_{0,-}}(P_{\infty_+}) + t_{\underline{r}}\underline{\widetilde{U}}^{(2)}_{\underline{r}}\Big)$$

$$\cup \bigg(\bigcup_{(n,t_{\underline{r}})\in\mathbb{Z}\times\mathbb{R}} \Big(\underline{\alpha}_{Q_0}(\mathcal{M}_{\rm sing}) + n\underline{A}_{P_{0,-}}(P_{\infty_+}) + t_{\underline{r}}\underline{\widetilde{U}}_{\underline{r}}^{(2)} \Big) + \underline{A}_{P_{0,-}}(P_{\infty_-}) \bigg) \tag{6.67}$$

is also of measure zero. Applying $\underline{\alpha}_{Q_0}^{-1}$ to the complement of the set in (6.67) then yields a set $\mathcal{M}_1$ of full measure in $\mathrm{Sym}^p(\mathcal{K}_p)$. In particular, $\mathcal{M}_1$ is necessarily dense in $\mathrm{Sym}^p(\mathcal{K}_p)$. □

Theorem 6.6. *Let $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})} \in \mathcal{M}_1$ be an initial divisor as in Lemma* 6.5. *Then the sequences α, β constructed from $\underline{\hat{\mu}}(n_0, t_{0,\underline{r}})$ as described in Theorem* 6.4 *satisfy Hypothesis* 5.1. *In particular, the solution α, β of the algebro-geometric initial value problem* (6.64), (6.65) *is global in $(n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}$.*

Proof. Starting with $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,\underline{r}})} \in \mathcal{M}_1$, the procedure outlined in this section and summarized in Theorem 6.4 leads to $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n,t_{\underline{r}})}$ for all $(n, t_{\underline{r}}) \in \mathbb{Z}\times(t_{0,\underline{r}} - T_0, t_{0,\underline{r}} + T_0)$ such that (5.58) and (5.59) hold. But if α, β should blow up, then $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}$ or $\mathcal{D}_{\underline{\hat{\nu}}(n,t_{\underline{r}})}$ must hit one of $P_{\infty_\pm}$ or $P_{0,\pm}$, which is excluded by our choice of initial condition. □

We note, however, that in general (i.e., unless one is, e.g., in the special periodic case), $\mathcal{D}_{\underline{\hat{\mu}}(n,t_{\underline{r}})}$ will get arbitrarily close to $P_{\infty_\pm}$, $P_{0,\pm}$ since straight motions on the torus are generically dense (see e.g. [12, Sect. 51] or [35, Sects. 1.4, 1.5]) and hence no uniform bound (and no uniform bound away from zero) on the sequences $\alpha(n, t_{\underline{r}}), \beta(n, t_{\underline{r}})$ exists as $(n, t_{\underline{r}})$ varies in $\mathbb{Z}\times\mathbb{R}$. In particular, these complex-valued algebro-geometric solutions of the Ablowitz–Ladik hierarchy initial value problem, in general, will not be quasi-periodic with respect to n or $t_{\underline{r}}$ (cf. the usual definition of quasi-periodic functions, e.g., in [46, p. 31]).

Appendix A. Hyperelliptic Curves in a Nutshell

We provide a very brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [20], [21], and [42], as well as monographs dedicated to integrable systems such as [13, Ch. 2], [29, App. A, B], [51, App. A].

Fix $p \in \mathbb{N}$. The hyperelliptic curve $\mathcal{K}_p$ of genus p used in Sections 3–6 is defined by

$$\mathcal{K}_p \colon \mathcal{F}_p(z,y) = y^2 - R_{2p+2}(z) = 0, \quad R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \tag{A.1}$$

$$\{E_m\}_{m=0,\dots,2p+1} \subset \mathbb{C}, \quad E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 0, \dots, 2p+1. \tag{A.2}$$

The curve (A.1) is compactified by adding the points P_{∞_+} and P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, at infinity. One then introduces an appropriate set of $p+1$ nonintersecting cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$ and denotes

$$\mathcal{C} = \bigcup_{j\in\{1,\dots,p+1\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Defining the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \tag{A.4}$$

and introducing the holomorphic function

$$R_{2p+2}(\,\cdot\,)^{1/2}\colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2p+1}(z-E_m)\right)^{1/2} \tag{A.5}$$

on Π with an appropriate choice of the square root branch in (A.5), one considers

$$\mathcal{M}_p = \{(z, \sigma R_{2p+2}(z)^{1/2}) \,|\, z \in \mathbb{C},\ \sigma \in \{\pm 1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \tag{A.6}$$

by extending $R_{2p+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_p$ is then the set $\mathcal{M}_p$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_p$ crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_p)$ of $\mathcal{K}_p$ is given by

$$\mathcal{B}(\mathcal{K}_p) = \{(E_m, 0)\}_{m=0,\dots,2p+1} \tag{A.7}$$

and finite points P on $\mathcal{K}_p$ are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z, y) = y^2 - R_{2p+2}(z) = 0$. Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_p \setminus (\mathcal{B}(\mathcal{K}_p) \cup \{P_{\infty_+}, P_{\infty_-}\})$ are given by $\zeta_{P_0} = z - z_0$, near $P_{\infty_\pm}$ by $\zeta_{P_{\infty_\pm}} = 1/z$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_p)$ by $\zeta_{(E_{m_0},0)} = (z - E_{m_0})^{1/2}$. The Riemann surface $\mathcal{K}_p$ defined in this manner has topological genus p. Moreover, we introduce the holomorphic sheet exchange map (involution)

$$*\colon \mathcal{K}_p \to \mathcal{K}_p, \quad P = (z, y) \mapsto P^* = (z, -y),\ P_{\infty_\pm} \mapsto P_{\infty_\pm}^* = P_{\infty_\mp}. \tag{A.8}$$

One verifies that dz/y is a holomorphic differential on $\mathcal{K}_p$ with zeros of order $p-1$ at $P_{\infty_\pm}$ and hence

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j = 1, \dots, p, \tag{A.9}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_p$. Introducing the invertible matrix C in $\mathbb{C}^p$,

$$\begin{aligned} &C = (C_{j,k})_{j,k=1,\dots,p}, \quad C_{j,k} = \int_{a_k} \eta_j, \\ &\underline{c}(k) = (c_1(k), \dots, c_p(k)), \quad c_j(k) = C^{-1}_{j,k},\ \ j,k = 1, \dots, p, \end{aligned} \tag{A.10}$$

the corresponding basis of normalized holomorphic differentials ω_j, $j = 1, \dots, p$, on $\mathcal{K}_p$ is given by

$$\omega_j = \sum_{\ell=1}^{p} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1, \dots, p. \tag{A.11}$$

Here $\{a_j, b_j\}_{j=1,\dots,p}$ is a homology basis for $\mathcal{K}_p$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k},\ a_j \circ a_k = 0,\ b_j \circ b_k = 0, \quad j,k = 1, \dots, p. \tag{A.12}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,p}$ we also recall the canonical dissection of $\mathcal{K}_p$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_p$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_p$ given by

$$\partial\widehat{\mathcal{K}}_p = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_p^{-1} b_p^{-1}. \tag{A.13}$$

Let $\mathcal{M}(\mathcal{K}_p)$ and $\mathcal{M}^1(\mathcal{K}_p)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_p$. Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind,

$\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_p)$, are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a-periods vanish, that is, $\int_{a_j} \omega^{(2)} = 0$, $j = 1, \dots, p$. Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_p$ not of the first or second kind is said to be of the third kind. A differential of the third kind $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_p)$ is usually normalized by the vanishing of its a-periods, that is, $\int_{a_j} \omega^{(3)} = 0$, $j = 1, \dots, p$. A normal differential of the third kind $\omega^{(3)}_{P_1,P_2}$ associated with two points P_1, $P_2 \in \widehat{\mathcal{K}}_p$, $P_1 \neq P_2$, by definition, has simple poles at P_j with residues $(-1)^{j+1}$, $j = 1, 2$ and vanishing a-periods.

Next, define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,p}$ by

$$\tau_{j,\ell} = \int_{b_\ell} \omega_j, \quad j, \ell = 1, \dots, p. \tag{A.14}$$

Then

$$\operatorname{Im}(\tau) > 0 \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j, \ell = 1, \dots, p. \tag{A.15}$$

Associated with τ one introduces the period lattice

$$L_p = \{\underline{z} \in \mathbb{C}^p \mid \underline{z} = \underline{m} + \underline{n}\tau, \ \underline{m}, \underline{n} \in \mathbb{Z}^p\}. \tag{A.16}$$

Next, fix a base point $Q_0 \in \mathcal{K}_p \setminus \{P_{0,\pm}, P_{\infty\pm}\}$, denote by $J(\mathcal{K}_p) = \mathbb{C}^p / L_p$ the Jacobi variety of $\mathcal{K}_p$, and define the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_p \to J(\mathcal{K}_p), \quad \underline{A}_{Q_0}(P) = \bigg(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_p \bigg) \pmod{L_p}, \quad P \in \mathcal{K}_p. \tag{A.17}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0} \colon \operatorname{Div}(\mathcal{K}_p) \to J(\mathcal{K}_p), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_p} \mathcal{D}(P) \underline{A}_{Q_0}(P), \tag{A.18}$$

where $\operatorname{Div}(\mathcal{K}_p)$ denotes the set of divisors on $\mathcal{K}_p$. Here $\mathcal{D} \colon \mathcal{K}_p \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_p$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_p$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_p)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.)

In connection with divisors on $\mathcal{K}_p$ we shall employ the following (additive) notation,

$$\begin{aligned} &\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\ &\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_p, \quad Q_0 \in \mathcal{K}_p, \ m \in \mathbb{N}, \end{aligned} \tag{A.19}$$

where for any $Q \in \mathcal{K}_p$,

$$\mathcal{D}_Q \colon \mathcal{K}_p \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_p \setminus \{Q\}, \end{cases} \tag{A.20}$$

and $\operatorname{Sym}^n \mathcal{K}_p$ denotes the nth symmetric product of $\mathcal{K}_p$. In particular, $\operatorname{Sym}^m \mathcal{K}_p$ can be identified with the set of nonnegative divisors $0 \leq \mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$ of degree m.

For $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_p)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_p) \,|\, \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0, \ \deg((\omega)) = 2(p-1), \ f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}, \ \omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}, \quad \text{(A.21)}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_p} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_p) \,|\, f = 0 \text{ or } (f) \geq \mathcal{D}\}, \quad r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \quad \text{(A.22)}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_p) \,|\, \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}, \quad i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}), \quad \text{(A.23)}$$

with $i(\mathcal{D})$ the index of speciality of $\mathcal{D}$, one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental fact.

Theorem A.1. *Let $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^p \mathcal{K}_p$, $\underline{Q} = \{Q_1, \dots, Q_p\}$. Then,*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s \quad \text{(A.24)}$$

if and only if $\{Q_1, \dots, Q_p\}$ contains s pairings of the type $\{P, P^*\}$. (This includes, of course, branch points for which $P = P^*$.) One has $s \leq p/2$.

Appendix B. Some Interpolation Formulas

In this appendix we recall a useful interpolation formula which goes beyond the standard Lagrange interpolation formula for polynomials in the sense that the zeros of the interpolating polynomial need not be distinct.

Lemma B.1 ([30])**.** *Let $p \in \mathbb{N}$ and S_{p-1} be a polynomial of degree $p-1$. In addition, let F_p be a monic polynomial of degree p of the form*

$$F_p(z) = \prod_{k=1}^{q} (z - \mu_k)^{p_k}, \quad p_j \in \mathbb{N}, \ \mu_j \in \mathbb{C}, \ j = 1, \dots, q, \quad \sum_{k=1}^{q} p_k = p. \quad \text{(B.1)}$$

Then,

$$S_{p-1}(z) = F_p(z) \sum_{k=1}^{q} \sum_{\ell=0}^{p_k - 1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!(p_k - \ell - 1)!} \times \left(\frac{d^{p_k - \ell - 1}}{d\zeta^{p_k - \ell - 1}} \left((z - \zeta)^{-1} \prod_{k'=1,\, k' \neq k}^{q} (\zeta - \mu_{k'})^{-p_{k'}} \right) \right) \bigg|_{\zeta = \mu_k}, \quad z \in \mathbb{C}. \quad \text{(B.2)}$$

In particular, S_{p-1} is uniquely determined by prescribing the p values

$$S_{p-1}(\mu_k), S'_{p-1}(\mu_k), \dots, S_{p-1}^{(p_k - 1)}(\mu_k), \quad k = 1, \dots, q, \quad \text{(B.3)}$$

at the given points $\mu_1 \dots, \mu_q$.

Conversely, prescribing the p complex numbers

$$\alpha_k^{(0)}, \alpha_k^{(1)}, \dots, \alpha_k^{(p_k - 1)}, \quad k = 1, \dots, q, \quad \text{(B.4)}$$

there exists a unique polynomial T_{p-1} of degree $p-1$,

$$T_{p-1}(z) = F_p(z) \sum_{k=1}^{q} \sum_{\ell=0}^{p_k - 1} \frac{\alpha_k^{(\ell)}}{\ell!(p_k - \ell - 1)!} \quad \text{(B.5)}$$

$$\times\left(\frac{d^{p_k-\ell-1}}{d\zeta^{p_k-\ell-1}}\left((z-\zeta)^{-1}\prod_{k'=1,\,k'\neq k}^{q}(\zeta-\mu_{k'})^{-p_{k'}}\right)\right)\bigg|_{\zeta=\mu_k}, \quad z\in\mathbb{C},$$

such that

$$T_{p-1}(\mu_k)=\alpha_k^{(0)},\ T'_{p-1}(\mu_k)=\alpha_k^{(1)},\dots,T_{p-1}^{(p_k-1)}(\mu_k)=\alpha_k^{(p_k-1)}, \quad k=1,\dots,q. \tag{B.6}$$

We briefly mention two special cases of (B.2). First, assume the generic case where all zeros of F_p are distinct, that is,

$$q=p, \quad p_k=1, \quad \mu_k\neq\mu_{k'} \text{ for } k\neq k',\ k,k'=1,\dots,p. \tag{B.7}$$

In this case (B.2) reduces to the classical Lagrange interpolation formula

$$S_{p-1}(z)=F_p(z)\sum_{k=1}^{p}\frac{S_{p-1}(\mu_k)}{((dF_p(\zeta)/d\zeta)|_{\zeta=\mu_k})(z-\mu_k)}, \quad z\in\mathbb{C}. \tag{B.8}$$

Second, we consider the other extreme case where all zeros of F_p coincide, that is,

$$q=1, \quad p_1=p, \quad F_p(z)=(z-\mu_1)^p, \quad z\in\mathbb{C}. \tag{B.9}$$

In this case (B.2) reduces of course to the Taylor expansion of S_{p-1} around $z=\mu_1$,

$$S_{p-1}(z)=\sum_{\ell=0}^{p-1}\frac{S_{p-1}^{(\ell)}(\mu_1)}{\ell!}(z-\mu_1)^{\ell}, \quad z\in\mathbb{C}. \tag{B.10}$$

Appendix C. Asymptotic Spectral Parameter Expansions

In this appendix we consider asymptotic spectral parameter expansions of $F_{\underline{p}}/y$, $G_{\underline{p}}/y$, and $H_{\underline{p}}/y$, the resulting recursion relations for the homogeneous coefficients $\hat f_\ell$, $\hat g_\ell$, and $\hat h_\ell$, their connection with the nonhomogeneous coefficients f_ℓ, g_ℓ, and h_ℓ, and the connection between $c_{\ell,\pm}$ and $c_\ell(\underline{E}^{\pm 1})$ (cf. (C.3)). For detailed proofs of the material in this section we refer to [29], [32]. We will employ the notation

$$\underline{E}^{\pm 1}=\big(E_0^{\pm 1},\dots,E_{2p+1}^{\pm 1}\big). \tag{C.1}$$

We start with the following elementary result (a consequence of the binomial expansion) assuming $\eta\in\mathbb{C}$ such that $|\eta|<\min\{|E_0|^{-1},\dots,|E_{2p+1}|^{-1}\}$:

$$\left(\prod_{m=0}^{2p+1}\big(1-E_m\eta\big)\right)^{1/2}=\sum_{k=0}^{\infty}c_k(\underline{E})\eta^k, \tag{C.2}$$

where

$$c_0(\underline{E})=1,$$
$$c_k(\underline{E})=\sum_{\substack{j_0,\dots,j_{2p+1}=0\\ j_0+\dots+j_{2p+1}=k}}^{k}\frac{(2j_0)!\cdots(2j_{2p+1})!\,E_0^{j_0}\cdots E_{2p+1}^{j_{2p+1}}}{2^{2k}(j_0!)^2\cdots(j_{2p+1}!)^2(2j_0-1)\cdots(2j_{2p+1}-1)}, \quad k\in\mathbb{N}. \tag{C.3}$$

The first few coefficients explicitly are given by

$$c_0(\underline{E})=1,\ c_1(\underline{E})=-\frac{1}{2}\sum_{m=0}^{2p+1}E_m,\ c_2(\underline{E})=\frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{2p+1}E_{m_1}E_{m_2}-\frac{1}{8}\sum_{m=0}^{2p+1}E_m^2, \quad \text{etc.} \tag{C.4}$$

Next we turn to asymptotic expansions. We recall the convention $y(P) = \mp\zeta^{-p-1} + O(\zeta^{-p})$ near $P_{\infty_\pm}$ (where $\zeta = 1/z$) and $y(P) = \pm(c_{0,-}/c_{0,+}) + O(\zeta)$ near $P_{0,\pm}$ (where $\zeta = z$).

Theorem C.1 ([32])**.** *Assume* (3.2), s-AL$_{\underline{p}}(\alpha,\beta) = 0$, *and suppose* $P = (z,y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$. *Then* $z^{p_-}F_{\underline{p}}/y$, $z^{p_-}G_{\underline{p}}/y$, *and* $z^{p_-}H_{\underline{p}}/y$ *have the following convergent expansions as* $P \to P_{\infty_\pm}$, *respectively,* $P \to P_{0,\pm}$,

$$\frac{z^{p_-}}{c_{0,+}}\frac{F_{\underline{p}}(z)}{y} = \begin{cases} \mp\sum_{\ell=0}^{\infty}\hat{f}_{\ell,+}\zeta^{\ell+1}, & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm\sum_{\ell=0}^{\infty}\hat{f}_{\ell,-}\zeta^{\ell}, & P \to P_{0,\pm}, \quad \zeta = z, \end{cases} \tag{C.5}$$

$$\frac{z^{p_-}}{c_{0,+}}\frac{G_{\underline{p}}(z)}{y} = \begin{cases} \mp\sum_{\ell=0}^{\infty}\hat{g}_{\ell,+}\zeta^{\ell}, & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm\sum_{\ell=0}^{\infty}\hat{g}_{\ell,-}\zeta^{\ell}, & P \to P_{0,\pm}, \quad \zeta = z, \end{cases} \tag{C.6}$$

$$\frac{z^{p_-}}{c_{0,+}}\frac{H_{\underline{p}}(z)}{y} = \begin{cases} \mp\sum_{\ell=0}^{\infty}\hat{h}_{\ell,+}\zeta^{\ell}, & P \to P_{\infty_\pm}, \quad \zeta = 1/z, \\ \pm\sum_{\ell=0}^{\infty}\hat{h}_{\ell,-}\zeta^{\ell+1}, & P \to P_{0,\pm}, \quad \zeta = z, \end{cases} \tag{C.7}$$

where $\zeta = 1/z$ *(resp.,* $\zeta = z$*) is the local coordinate near* $P_{\infty_\pm}$ *(resp.,* $P_{0,\pm}$*) and* $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, *and* $\hat{h}_{\ell,\pm}$ *are the homogeneous versions of the coefficients* $f_{\ell,\pm}$, $g_{\ell,\pm}$, *and* $h_{\ell,\pm}$ *introduced in* (2.16)–(2.18).

Moreover, the E_m*-dependent summation constants* $c_{\ell,\pm}$, $\ell = 0,\dots,p_\pm$, *in* $F_{\underline{p}}$, $G_{\underline{p}}$, *and* $H_{\underline{p}}$ *are given by*

$$c_{\ell,\pm} = c_{0,\pm}c_\ell(\underline{E}^{\pm 1}), \quad \ell = 0,\dots,p_\pm. \tag{C.8}$$

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References

[1] M. J. Ablowitz. Nonlinear evolution equations – continuous and discrete. *SIAM Rev.* 19:663–684, 1977.

[2] M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering.* Cambridge University Press, Cambridge, 1991.

[3] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations. *J. Math. Phys.* 16:598–603, 1975.

[4] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.* 17:1011–1018, 1976.

[5] M. J. Ablowitz and J. F. Ladik. A nonlinear difference scheme and inverse scattering. *Studies Appl. Math.* 55:213–229, 1976.

[6] M. J. Ablowitz and J. F. Ladik. On the solution of a class of nonlinear partial difference equations. *Studies Appl. Math.* 57:1–12, 1977.

[7] M. J. Ablowitz, B. Prinari, and A. D. Trubatch. *Discrete and Continuous Nonlinear Schrödinger Systems.* London Mathematical Society Lecture Note Series, Vol. 302, Cambridge University Press, Cambridge, 2004.

[8] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications.* 2nd ed., Springer, New York, 1988.

[9] S. Ahmad and A. Roy Chowdhury. On the quasi-periodic solutions to the discrete non-linear Schrödinger equation. *J. Phys. A* 20:293–303, 1987.

[10] S. Ahmad and A. Roy Chowdhury. The quasiperiodic solutions to the discrete nonlinear Schrödinger equation. *J. Math. Phys.* 28:134–137, 1987.

[11] G. S. Ammar and W. B. Gragg. Schur flows for orthogonal Hessenberg matrices. *Hamiltonian and Gradient Flows, Algorithms and Control*, A. Bloch (ed.), Fields Inst. Commun., Vol. 3, Amer. Math. Soc., Providence, RI, 1994, pp. 27–34.

[12] V. I. Arnold. *Mathematical Methods of Classical Mechanics.* 2nd ed., Springer, New York, 1989.

[13] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev. *Algebro-Geometric Approach to Nonlinear Integrable Equations.* Springer, Berlin, 1994.

[14] N. N. Bogolyubov, A. K. Prikarpatskii, and V. G. Samoilenko. Discrete periodic problem for the modified nonlinear Korteweg–de Vries equation. *Sov. Phys. Dokl.* 26:490–492, 1981.

[15] N. N. Bogolyubov and A. K. Prikarpatskii. The inverse periodic problem for a discrete approximation of a nonlinear Schrödinger equation. *Sov. Phys. Dokl.* 27:113–116, 1982.

[16] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl. Algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac-van Moerbeke hierarchy. *Mem. Amer. Math. Soc.* no. 641, 135:1–79, 1998.

[17] A. K. Common. A solution of the initial value problem for half-infinite integrable lattice systems. *Inverse Probl.* 8:393–408, 1992.

[18] A. K. Common and S. T. Hafez. Continued-fraction solutions to the Riccati equation and integrable lattice systems. *J. Phys. A* 23:455–466, 1990.

[19] P. Deift. Riemann–Hilbert methods in the theory of orthogonal polynomials. *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday. Ergodic Schrödinger Operators, Singular Spectrum, Orthogonal Polynomials, and Inverse Spectral Theory*, F. Gesztesy, P. Deift, C. Galvez, P. Perry, and W. Schlag (eds.), Proceedings of Symposia in Pure Mathematics, Vol. 76/2, Amer. Math. Soc., Providence, RI, 2007, pp. 715–740.

[20] H. M. Farkas and I. Kra. *Riemann Surfaces.* Springer, New York, 2nd ed., 1992.

[21] J. Fay. *Theta Functions on Riemann Surfaces.* Lecture Notes in Mathematics, Vol. 352, Springer, New York, 1973.

[22] L. Faybusovich and M. Gekhtman. On Schur flows. *J. Phys. A* 32:4671–4680, 1999.

[23] L. Faybusovich and M. Gekhtman. Elementary Toda orbits and integrable lattices. *Inverse Probl.* 41:2905–2921, 2000.

[24] M. Gekhtman. Non-abelian nonlinear lattice equations on finite interval. *J. Phys. A* 26:6303–6317, 1993.

[25] X. Geng, H. H. Dai, and C. Cao. Algebro-geometric constructions of the discrete Ablowitz–Ladik flows and applications. *J. Math. Phys.* 44:4573–4588, 2003.

[26] X. Geng, H. H. Dai, and J. Zhu. Decompositions of the discrete Ablowitz–Ladik hierarchy. *Stud. Appl. Math.* 118:281–312, 2007.

[27] J. S. Geronimo, F. Gesztesy, and H. Holden. Algebro-geometric solutions of the Baxter–Szegő difference equation. *Comm. Math. Phys.* 258:149–177, 2005.

[28] F. Gesztesy and H. Holden. *Soliton Equations and Their Algebro-Geometric Solutions. Volume I:* $(1+1)$*-Dimensional Continuous Models.* Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge University Press, Cambridge, 2003.

[29] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. *Soliton Equations and Their Algebro-Geometric Solutions. Volume II:* $(1+1)$*-Dimensional Discrete Models.* Cambridge Studies in Advanced Mathematics, Vol. 114, Cambridge University Press, Cambridge, 2008.

[30] F. Gesztesy, H. Holden, and G. Teschl. The algebro-geometric Toda hierarchy initial value problem for complex-valued initial data. *Rev. Mat. Iberoamericana*, **24**, 117–182 (2008).

[31] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. The Ablowitz–Ladik hierarchy revisited. In *Methods of Spectral Analysis in Mathematical Physics*, Conference on Operator Theory, Analysis and Mathematical Physics (OTAMP) 2006, Lund, Sweden, J. Janas, P. Kurasov, A. Laptev, S. Naboko, and G. Stolz (eds.), Operator Theory: Advances and Applications, Vol. **186**, Birkhäuser, Basel, 2008, pp. 139–190.

[32] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. Algebro-geometric finite-band solutions of the Ablowitz–Ladik hierarchy. *Int. Math. Res. Notices*, 2007:1–55, rnm082.

[33] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. Local conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy, Stud. Appl. Math. **120**, 361–423 (2008).

[34] L. Golinskii. Schur flows and orthogonal polynomials on the unit circle. *Sbornik Math.* 197:1145–1165, 2006.

[35] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge Univ. Press, Cambridge, 1995.
[36] R. Killip and I. Nenciu. CMV: The unitary analogue of Jacobi matrices. *Comm. Pure Appl. Math.* 60:1148–1188, 2007.
[37] L.-C. Li. Some remarks on CMV matrices and dressing orbits. *Int. Math. Res. Notices* 2005, No. 40, 2437–2446.
[38] A. I. Markushevich. *Theory of Functions of a Complex Variable.* 2nd. ed., Chelsea, New York, 1985.
[39] P. D. Miller. Macroscopic behavior in the Ablowitz–Ladik equations. Nonlinear Evolution Equations & Dynamical Systems, V. G. Makhankov, A. R. Bishop, and D. D. Holm (eds.), World Scientific, Singapore, 1995, pp. 158–167.
[40] P. D. Miller, N. M. Ercolani, I. M. Krichever, and C. D. Levermore. Finite genus solutions to the Ablowitz–Ladik equations. *Comm. Pure Appl. Math.* 48:1369–1440, 1995.
[41] A. Mukaihira and Y. Nakamura. Schur flow for orthogonal polynomials on the unit circle and its integrable discretization. *J. Comput. Appl. Math.* 139:75–94, 2002.
[42] D. Mumford. *Tata Lectures on Theta II.* Birkhäuser, Boston, 1984.
[43] I. Nenciu. Lax pairs for the Ablowitz–Ladik system via orthogonal polynomials on the unit circle. *Int. Math. Res. Notices* 11:647–686, 2005.
[44] I. Nenciu. CMV matrices in random matrix theory and integrable systems: a survey. *J. Phys. A* 39:8811–8822, 2006.
[45] R. J. Schilling. A systematic approach to the soliton equations of a discrete eigenvalue problem. *J. Math. Phys.* 30:1487–1501, 1989.
[46] L. Pastur and A. Figotin. *Spectra of Random and Almost-Periodic Operators.* Springer, Berlin, 1992.
[47] B. Simon. *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Part 2: Spectral Theory*, AMS Colloquium Publication Series, Vol. 54, Amer. Math. Soc., Providence, Providence, R.I., 2005.
[48] B. Simon. CMV matrices: Five years after. *J. Comp. Appl. Math.* 208:120–154, 2007.
[49] B. Simon. Zeros of OPUC and long time asymptotics of Schur and related flows. *Inverse Probl. Imaging* 1:189–215, 2007.
[50] Yu. B. Suris. *The Problem of Integrable Discretization: Hamiltonian Approach.* Birkhäuser, Basel, 2003.
[51] G. Teschl. *Jacobi Operators and Completely Integrable Nonlinear Lattices.* Math. Surveys Monographs, Vol. 72, Amer. Math. Soc., Providence, R.I., 2000.
[52] K. L. Vaninsky. An additional Gibbs' state for the cubic Schrödinger equation on the circle. *Comm. Pure Appl. Math.* 54:537–582, 2001.
[53] V. E. Vekslerchik. Finite genus solutions for the Ablowitz–Ladik hierarchy. *J. Phys. A* 32:4983–4994, 1999.
[54] W. Walter. *Ordinary Differential Equations.* Springer, New York, 1998.

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LOCAL CONSERVATION LAWS AND THE HAMILTONIAN FORMALISM FOR THE ABLOWITZ–LADIK HIERARCHY

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Abstract. We derive a systematic and recursive approach to local conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik (AL) hierarchy. Our methods rely on a recursive approach to the AL hierarchy using Laurent polynomials and on asymptotic expansions of the Green's function of the AL Lax operator, a five-diagonal finite difference operator.

1. Introduction

The principal aim of this paper is to provide a systematic and recursive approach to local conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik (AL) hierarchy of integrable differential-difference equations.

Consider sequences $\{\alpha(n,t),\beta(n,t)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ satisfying some additional assumptions to be specified later, parametrized by the deformation (time) parameter $t \in \mathbb{R}$, that are solutions of the Ablowitz–Ladik equations

$$\begin{pmatrix} -i\alpha_t - (1-\alpha\beta)(\alpha^- + \alpha^+) + 2\alpha \\ -i\beta_t + (1-\alpha\beta)(\beta^- + \beta^+) - 2\beta \end{pmatrix} = 0. \tag{1.1}$$

Here $c^\pm$ denote shifts, that is, $c^\pm(n) = c(n\pm 1)$, $n \in \mathbb{Z}$. Then clearly

$$\partial_t \sum_{n\in\mathbb{Z}} \alpha^+(n,t)\beta(n,t) = \partial_t \sum_{n\in\mathbb{Z}} \alpha(n,t)\beta^+(n,t) = 0. \tag{1.2}$$

Indeed, one can show the existence of an infinite sequence $\{\rho_{j,\pm}\}_{j\in\mathbb{N}}$ of polynomials of α,β and certain shifts thereof, with the property that the lattice sum is time-independent,

$$\partial_t \sum_{n\in\mathbb{Z}} \rho_{j,\pm}(n,t) = 0, \quad j \in \mathbb{N}. \tag{1.3}$$

This result is obtained by deriving local conservation laws of the type

$$\partial_t \rho_{j,\pm} + (S^+ - I)J_{j,\pm} = 0, \quad j \in \mathbb{N}, \tag{1.4}$$

for certain polynomials $J_{j,\pm}$ of α,β and certain shifts thereof. The polynomials $J_{j,\pm}$ will be constructed via an explicit recursion relation. For a detailed discussion of these results we refer to Theorem 5.7 and Remarks 5.8 and 5.9.

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The above analysis extends to the full Ablowitz–Ladik hierarchy as follows. The $\underline{p}$th equation, $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), in the AL hierarchy is given by

$$\operatorname{AL}_{\underline{p}}(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_{\underline{p}}} - \alpha(g_{p_+,+} + g^-_{p_-,-}) + f_{p_+-1,+} - f^-_{p_--1,-} \\ -i\beta_{t_{\underline{p}}} + \beta(g^-_{p_+,+} + g_{p_-,-}) - h_{p_--1,-} + h^-_{p_+-1,+} \end{pmatrix} = 0, \quad t_{\underline{p}} \in \mathbb{R}, \ \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2, \tag{1.5}$$

where $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ are carefully designed polynomial expressions of α, β and certain shifts thereof. Recursively, they are given by (2.5)–(2.12). On each level in the recursion an arbitrary constant $c_{\ell,\pm} \in \mathbb{C}$ is introduced. In the homogeneous case, where all these constants c_ℓ, $\ell \in \mathbb{N}$, are set equal to zero, a hat $\hat{}$ is added in the notation, that is, $\hat{f}_{\ell,\pm}, \hat{g}_{\ell,\pm}, \hat{h}_{\ell,\pm}$, etc., denote the corresponding homogeneous quantities. The homogeneous coefficients $\hat{f}_{\ell,\pm}, \hat{g}_{\ell,\pm}, \hat{h}_{\ell,\pm}$ can also be expressed explicitly in terms of appropriate matrix elements of powers of the AL Lax finite difference expression L defined in (3.3), (3.5) and the finite difference expressions D and E in (3.14), as described in Lemma 3.1. The conserved densities $\rho_{j,\pm}$ are independent of the equation in the hierarchy while the currents $J_{\underline{p},j,\pm}$ depend on $\underline{p}$; thus one finds (cf. Theorem 5.7)

$$\partial_{t_{\underline{p}}} \rho_{j,\pm} + (S^+ - I) J_{\underline{p},j,\pm} = 0, \quad t_{\underline{p}} \in \mathbb{R}, \ j \in \mathbb{N}, \ \underline{p} \in \mathbb{N}_0^2. \tag{1.6}$$

For $\alpha, \beta \in \ell^1(\mathbb{Z})$ it then follows that

$$\frac{d}{dt_{\underline{p}}} \sum_{n\in\mathbb{Z}} \rho_{j,\pm}(n, t_{\underline{p}}) = 0, \quad t_{\underline{p}} \in \mathbb{R}, \ j \in \mathbb{N}, \ \underline{p} \in \mathbb{N}_0^2. \tag{1.7}$$

By showing that $\rho_{j,\pm}$ equals $\hat{g}_{j,\pm}$ up to a first-order difference expression (cf. Lemma 4.4), and by investigating the time-dependence of $\gamma = 1 - \alpha\beta$, one concludes (cf. Remark 5.8) that

$$\frac{d}{dt_{\underline{p}}} \sum_{n\in\mathbb{Z}} \ln(\gamma(n, t_{\underline{p}})) = 0, \quad \frac{d}{dt_{\underline{p}}} \sum_{n\in\mathbb{Z}} \hat{g}_{j,\pm}(n, t_{\underline{p}}) = 0, \quad t_{\underline{p}} \in \mathbb{R}, \ j \in \mathbb{N}, \ \underline{p} \in \mathbb{N}_0^2, \tag{1.8}$$

represent the two infinite sequences of AL conservation laws. Our approach to (1.6) is based on a careful analysis of asymptotic expansions of the Green's function (as the spectral parameter tends to zero and to infinity) for the operator realization $\breve{L}$ in $\ell^2(\mathbb{Z})$ corresponding to the Lax difference expression L in (3.3), (3.5).

In addition, we provide a detailed study of the Hamiltonian formalism for the AL hierarchy. In particular, the $\underline{p}$th equation in the AL hierarchy can be written as (cf. Theorem 6.5)

$$\operatorname{AL}_{\underline{p}}(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_{\underline{p}}} \\ -i\beta_{t_{\underline{p}}} \end{pmatrix} + \mathcal{D}\nabla\mathcal{H}_{\underline{p}} = 0, \quad \underline{p} \in \mathbb{N}_0^2, \tag{1.9}$$

where the Hamiltonians $\mathcal{H}_{\underline{p}}$ are given by

$$\mathcal{H}_{\underline{p}} = \sum_{\ell=1}^{p_+} c_{p_+-\ell,+} \widehat{\mathcal{H}}_{\ell,+} + \sum_{\ell=1}^{p_-} c_{p_--\ell,-} \widehat{\mathcal{H}}_{\ell,-} + c_{\underline{p}} \widehat{\mathcal{H}}_0, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2, \tag{1.10}$$

$$\widehat{\mathcal{H}}_0 = \sum_{n\in\mathbb{Z}} \ln(\gamma(n)), \qquad \widehat{\mathcal{H}}_{p_\pm,\pm} = \frac{1}{p_\pm} \sum_{n\in\mathbb{Z}} \hat{g}_{p_\pm,\pm}(n), \quad p_\pm \in \mathbb{N}. \tag{1.11}$$

Here $\mathcal{D} = (1-\alpha\beta)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Furthermore, we show that any $\mathcal{H}_{\underline{r}}$ is conserved by the Hamiltonian flows in (1.9) (cf. Theorem 6.6), that is,

$$\frac{d\mathcal{H}_{\underline{r}}}{dt_{\underline{p}}} = 0, \quad \underline{p}, \underline{r} \in \mathbb{N}_0^2. \tag{1.12}$$

Moreover, for general sequences α, β (i.e., not assuming that they satisfy an equation in the AL hierarchy), we show in Theorem 6.7 that

$$\{\widehat{\mathcal{H}}_{\underline{p}}, \widehat{\mathcal{H}}_{\underline{r}}\} = 0, \quad \underline{p}, \underline{r} \in \mathbb{N}_0^2, \tag{1.13}$$

for a suitably defined Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ (see (6.16)), that is, $\mathcal{H}_{\underline{p}}$ and $\mathcal{H}_{\underline{r}}$ are in involution for all $\underline{p}, \underline{r} \in \mathbb{N}_0^2$.

The Ablowitz–Ladik hierarchy has been extensively discussed in the completely integrable system literature (cf., e.g., [3]–[6], [1], [2, Sect. 3.2.2], [7, Ch. 3], [12], [13], [34], [36], [37], [38], [41], [45], [47], [48] and the references cited therein) and in recent years especially due to its close connections with the theory of orthogonal polynomials, a field that underwent a remarkable resurgency in recent years (cf. [42], [43], [44] and the literature quoted therein). Rather than repeating some of the AL hierarchy history and its relevance to the theory of orthogonal polynomials at this place, we refer to the detailed introductions of [25], [26], [27] and the extensive bibliography listed therein. Here we just mention references intimately connected with the topics discussed in this paper: Infinitely many conservation laws are discussed, for instance, by Ablowitz and Ladik [4], Ablowitz, Prinari, and Trubatch [7, Ch. 3], Ding, Sun, and Xu [14], Zhang and Chen [50], and Zhang, Ning, Bi, and Chen [52]; the bi-Hamiltonian structure of the AL hierarchy is considered by Ercolani and Lozano [15], Hydon [30], and Lozano [33], multi-Hamiltonian structures for the defocusing AL hierarchy were studied by Gekhtman and Nenciu [18], Zeng and Rauch-Wojciechowski [49], and Zhang and Chen [51]; Poisson brackets for orthogonal polynomials on the unit circle relevant to the case of the defocusing AL hierarchy (where $\beta = \overline{\alpha}$) have been studied by Cantero and Simon [10], Killip and Nenciu [31], and Nenciu [39]; Lenard recursions and Hamiltonian structures were discussed in Geng and Dai [19] and Geng, Dai, and Zhu [20].

Next we briefly describe the structure of this paper: Section 2 recalls the recursive construction of the AL hierarchy as discussed in detail in [25] (see also [26], [27]). In Section 3 we introduce the Lax pair for the AL hierarchy and prove its equivalence with the corresponding zero-curvature formulation. These results are new. In Section 4 we discuss the Green's function of the Lax operator $\breve{L}$ and study its asymptotic expansions as the spectral parameter tends to zero and infinity. As a direct consequence of these asymptotic expansions, local conservation laws are then derived in Section 5. Our final Section 6 then introduces the basics of variational derivatives and provides a detailed derivation of the Hamiltonian formalism for the AL hierarchy.

Finally, we emphasize that our recursive and systematic approach to local conservation laws of the Ablowitz–Ladik hierarchy appears to be new. Moreover, our treatment of Poisson brackets and variational derivatives, and their connections with the diagonal Green's function of the underlying Lax operator, now puts the AL hierarchy on precisely the same level as the Toda and KdV hierarchy with respect to this particular aspect of the Hamiltonian formalism (cf. [22, Ch. 1], [23, Ch. 1]).

2. The Ablowitz–Ladik hierarchy in a nutshell

In this section we summarize the construction of the Ablowitz–Ladik hierarchy employing a Laurent polynomial recursion formalism and derive the associated sequence of Ablowitz–Ladik zero-curvature pairs. Moreover, we discuss the Burchnall–Chaundy Laurent polynomial in connection with the stationary Ablowitz–Ladik hierarchy and the underlying hyperelliptic curve. For a detailed treatment of this material we refer to [23], [25].

We denote by $\mathbb{C}^{\mathbb{Z}}$ the set of complex-valued sequences indexed by $\mathbb{Z}$.

Throughout this section we suppose the following hypothesis.

Hypothesis 2.1. *In the stationary case we assume that α, β satisfy*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0,1\},\ n \in \mathbb{Z}. \tag{2.1}$$

In the time-dependent case we assume that α, β satisfy

$$\begin{aligned} &\alpha(\,\cdot\,,t), \beta(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t \in \mathbb{R}, \quad \alpha(n,\,\cdot\,), \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \\ &\alpha(n,t)\beta(n,t) \notin \{0,1\},\ (n,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{2.2}$$

We denote by $S^{\pm}$ the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ according to

$$(S^{\pm} f)(n) = f(n \pm 1), \quad n \in \mathbb{Z}. \tag{2.3}$$

Moreover, we will frequently use the notation

$$f^{\pm} = S^{\pm} f, \quad f \in \mathbb{C}^{\mathbb{Z}}. \tag{2.4}$$

To construct the Ablowitz–Ladik hierarchy one typically introduces appropriate zero-curvature pairs of 2×2 matrices, denoted by $U(z)$ and $V_{\underline{p}}(z)$, $\underline{p} \in \mathbb{N}_0^2$ (with $z \in \mathbb{C} \setminus \{0\}$ a certain spectral parameter to be discussed later), defined recursively in the following. We take the shortest route to the construction of $V_{\underline{p}}$ and hence to that of the Ablowitz–Ladik hierarchy by starting from the recursion relation (2.5)–(2.12) below.

Define sequences $\{f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ recursively by

$$g_{0,+} = \tfrac{1}{2} c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^{+}, \quad h_{0,+} = c_{0,+}\beta, \tag{2.5}$$

$$g_{\ell+1,+} - g_{\ell+1,+}^{-} = \alpha h_{\ell,+}^{-} + \beta f_{\ell,+}, \quad \ell \in \mathbb{N}_0, \tag{2.6}$$

$$f_{\ell+1,+}^{-} = f_{\ell,+} - \alpha(g_{\ell+1,+} + g_{\ell+1,+}^{-}), \quad \ell \in \mathbb{N}_0, \tag{2.7}$$

$$h_{\ell+1,+} = h_{\ell,+}^{-} + \beta(g_{\ell+1,+} + g_{\ell+1,+}^{-}), \quad \ell \in \mathbb{N}_0, \tag{2.8}$$

and

$$g_{0,-} = \tfrac{1}{2} c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^{+}, \tag{2.9}$$

$$g_{\ell+1,-} - g_{\ell+1,-}^{-} = \alpha h_{\ell,-} + \beta f_{\ell,-}^{-}, \quad \ell \in \mathbb{N}_0, \tag{2.10}$$

$$f_{\ell+1,-} = f_{\ell,-}^{-} + \alpha(g_{\ell+1,-} + g_{\ell+1,-}^{-}), \quad \ell \in \mathbb{N}_0, \tag{2.11}$$

$$h_{\ell+1,-}^{-} = h_{\ell,-} - \beta(g_{\ell+1,-} + g_{\ell+1,-}^{-}), \quad \ell \in \mathbb{N}_0. \tag{2.12}$$

Here $c_{0,\pm} \in \mathbb{C}$ are given constants. For later use we also introduce

$$f_{-1,\pm} = h_{-1,\pm} = 0. \tag{2.13}$$

Remark 2.2. The sequences $\{f_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,+}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,+}\}_{\ell\in\mathbb{N}_0}$ can be computed recursively as follows: Assume that $f_{\ell,+}$, $g_{\ell,+}$, and $h_{\ell,+}$ are known. Equation (2.6) is a first-order difference equation in $g_{\ell+1,+}$ that can be solved directly and yields a local lattice function that is determined up to a new constant denoted by $c_{\ell+1,+}\in\mathbb{C}$. Relations (2.7) and (2.8) then determine $f_{\ell+1,+}$ and $h_{\ell+1,+}$, etc. The sequences $\{f_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, $\{g_{\ell,-}\}_{\ell\in\mathbb{N}_0}$, and $\{h_{\ell,-}\}_{\ell\in\mathbb{N}_0}$ are determined similarly.

Upon setting

$$\gamma = 1-\alpha\beta, \tag{2.14}$$

one explicitly obtains

$$\begin{aligned}
&f_{0,+}=c_{0,+}(-\alpha^+), \quad f_{1,+}=c_{0,+}\big(-\gamma^+\alpha^{++}+(\alpha^+)^2\beta\big)+c_{1,+}(-\alpha^+),\\
&g_{0,+}=\tfrac12 c_{0,+}, \quad g_{1,+}=c_{0,+}(-\alpha^+\beta)+\tfrac12 c_{1,+},\\
&h_{0,+}=c_{0,+}\beta, \quad h_{1,+}=c_{0,+}\big(\gamma\beta^- -\alpha^+\beta^2\big)+c_{1,+}\beta,\\
&f_{0,-}=c_{0,-}\alpha, \quad f_{1,-}=c_{0,-}\big(\gamma\alpha^- -\alpha^2\beta^+\big)+c_{1,-}\alpha,\\
&g_{0,-}=\tfrac12 c_{0,-}, \quad g_{1,-}=c_{0,-}(-\alpha\beta^+)+\tfrac12 c_{1,-},\\
&h_{0,-}=c_{0,-}(-\beta^+), \quad h_{1,-}=c_{0,-}\big(-\gamma^+\beta^{++}+\alpha(\beta^+)^2\big)+c_{1,-}(-\beta^+), \text{ etc.}
\end{aligned} \tag{2.15}$$

Here $\{c_{\ell,\pm}\}_{\ell\in\mathbb{N}}$ denote summation constants which naturally arise when solving the difference equations for $g_{\ell,\pm}$ in (2.6), (2.10). Subsequently, it will also be useful to work with the corresponding homogeneous coefficients $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, and $\hat h_{\ell,\pm}$, defined by the vanishing of all summation constants $c_{k,\pm}$ for $k=1,\dots,\ell$, and choosing $c_{0,\pm}=1$,

$$\hat f_{0,+}=-\alpha^+, \quad \hat f_{0,-}=\alpha, \quad \hat f_{\ell,\pm}=f_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}, \tag{2.16}$$

$$\hat g_{0,\pm}=\tfrac12, \quad \hat g_{\ell,\pm}=g_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}, \tag{2.17}$$

$$\hat h_{0,+}=\beta, \quad \hat h_{0,-}=-\beta^+, \quad \hat h_{\ell,\pm}=h_{\ell,\pm}|_{c_{0,\pm}=1,\, c_{j,\pm}=0, j=1,\dots,\ell}, \quad \ell\in\mathbb{N}. \tag{2.18}$$

By induction one infers that

$$f_{\ell,\pm}=\sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat f_{k,\pm}, \quad g_{\ell,\pm}=\sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat g_{k,\pm}, \quad h_{\ell,\pm}=\sum_{k=0}^{\ell} c_{\ell-k,\pm}\hat h_{k,\pm}. \tag{2.19}$$

In a slight abuse of notation we will occasionally stress the dependence of $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ on α,β by writing $f_{\ell,\pm}(\alpha,\beta)$, $g_{\ell,\pm}(\alpha,\beta)$, and $h_{\ell,\pm}(\alpha,\beta)$.

One can show (cf. [25]) that all homogeneous elements $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, and $\hat h_{\ell,\pm}$, $\ell\in\mathbb{N}_0$, are polynomials in α,β, and some of their shifts.

Remark 2.3. As an efficient tool to distinguish between nonhomogeneous and homogeneous quantities $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, and $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, $\hat h_{\ell,\pm}$, respectively, we now introduce the notion of degree as follows. Denote

$$f^{(r)}=S^{(r)}f, \quad f=\{f(n)\}_{n\in\mathbb{Z}}\in\mathbb{C}^{\mathbb{Z}}, \quad S^{(r)}=\begin{cases}(S^+)^r, & r\ge 0,\\ (S^-)^{-r}, & r<0,\end{cases} \quad r\in\mathbb{Z}, \tag{2.20}$$

and define

$$\deg\big(\alpha^{(r)}\big)=r, \quad \deg\big(\beta^{(r)}\big)=-r, \quad r\in\mathbb{Z}. \tag{2.21}$$

This implies

$$\begin{aligned} &\deg\big(\hat f^{(r)}_{\ell,+}\big) = \ell+1+r, \quad \deg\big(\hat f^{(r)}_{\ell,-}\big) = -\ell+r, \quad \deg\big(\hat g^{(r)}_{\ell,\pm}\big) = \pm\ell, \\ &\deg\big(\hat h^{(r)}_{\ell,+}\big) = \ell-r, \quad \deg\big(\hat h^{(r)}_{\ell,-}\big) = -\ell-1-r, \quad \ell\in\mathbb{N}_0,\ r\in\mathbb{Z}. \end{aligned} \tag{2.22}$$

Alternatively the homogeneous coefficients can be computed directly via the following nonlinear recursion relations:

Lemma 2.4. *The homogeneous quantities $\hat f_{\ell,\pm}$, $\hat g_{\ell,\pm}$, $\hat h_{\ell,\pm}$ are uniquely defined by the following recursion relations:*

$$\begin{aligned} &\hat g_{0,+} = \frac{1}{2}, \quad \hat f_{0,+} = -\alpha^+, \quad \hat h_{0,+} = \beta, \\ &\hat g_{l+1,+} = \sum_{k=0}^{l} \hat f_{l-k,+}\hat h_{k,+} - \sum_{k=1}^{l} \hat g_{l+1-k,+}\hat g_{k,+}, \\ &\hat f^-_{l+1,+} = \hat f_{l,+} - \alpha(\hat g_{l+1,+} + \hat g^-_{l+1,+}), \\ &\hat h_{l+1,+} = \hat h^-_{l,+} + \beta(\hat g_{l+1,+} + \hat g^-_{l+1,+}), \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} &\hat g_{0,-} = \frac{1}{2}, \quad \hat f_{0,-} = \alpha, \quad \hat h_{0,-} = -\beta^+, \\ &\hat g_{l+1,-} = \sum_{k=0}^{l} \hat f_{l-k,-}\hat h_{k,-} - \sum_{k=1}^{l} \hat g_{l+1-k,-}\hat g_{k,-}, \\ &\hat f_{l+1,-} = \hat f^-_{l,-} + \alpha(\hat g_{l+1,-} + \hat g^-_{l+1,-}), \\ &\hat h^-_{l+1,-} = \hat h_{l,-} - \beta(\hat g_{l+1,-} + \hat g^-_{l+1,-}). \end{aligned} \tag{2.24}$$

We also note the following useful result (cf. [25]): Assuming (2.1), we find

$$\begin{aligned} &g_{\ell,+} - g^-_{\ell,+} = \alpha h_{\ell,+} + \beta f^-_{\ell,+}, \quad \ell\in\mathbb{N}_0, \\ &g_{\ell,-} - g^-_{\ell,-} = \alpha h^-_{\ell,-} + \beta f_{\ell,-}, \quad \ell\in\mathbb{N}_0. \end{aligned} \tag{2.25}$$

Moreover, we record the following symmetries,

$$\hat f_{\ell,\pm}(c_{0,\pm},\alpha,\beta) = \hat h_{\ell,\mp}(c_{0,\mp},\beta,\alpha), \quad \hat g_{\ell,\pm}(c_{0,\pm},\alpha,\beta) = \hat g_{\ell,\mp}(c_{0,\mp},\beta,\alpha), \quad \ell\in\mathbb{N}_0. \tag{2.26}$$

Next we define the 2×2 zero-curvature matrices

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix} \tag{2.27}$$

and

$$V_{\underline{p}}(z) = i\begin{pmatrix} G^-_{\underline{p}}(z) & -F^-_{\underline{p}}(z) \\ H^-_{\underline{p}}(z) & -K^-_{\underline{p}}(z) \end{pmatrix}, \quad \underline{p}\in\mathbb{N}_0^2, \tag{2.28}$$

for appropriate Laurent polynomials $F_{\underline{p}}(z)$, $G_{\underline{p}}(z)$, $H_{\underline{p}}(z)$, and $K_{\underline{p}}(z)$ in the spectral parameter $z\in\mathbb{C}\setminus\{0\}$ to be determined shortly. By postulating the stationary zero-curvature relation,

$$0 = UV_{\underline{p}} - V^+_{\underline{p}}U, \tag{2.29}$$

one concludes that (2.29) is equivalent to the following relations

$$z(G^-_{\underline{p}} - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H^-_{\underline{p}} = 0, \tag{2.30}$$

$$z\beta F_{\underline{p}}^{-} + \alpha H_{\underline{p}} - K_{\underline{p}} + K_{\underline{p}}^{-} = 0, \tag{2.31}$$

$$-F_{\underline{p}} + zF_{\underline{p}}^{-} + \alpha(G_{\underline{p}} + K_{\underline{p}}^{-}) = 0, \tag{2.32}$$

$$z\beta(G_{\underline{p}}^{-} + K_{\underline{p}}) - zH_{\underline{p}} + H_{\underline{p}}^{-} = 0. \tag{2.33}$$

In order to make the connection between the zero-curvature formalism and the recursion relations (2.5)–(2.12), we now define Laurent polynomials $F_{\underline{p}}$, $G_{\underline{p}}$, $H_{\underline{p}}$, and $K_{\underline{p}}$, $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2$, by[1]

$$F_{\underline{p}}(z) = \sum_{\ell=1}^{p_-} f_{p_- - \ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+ - 1} f_{p_+ - 1 - \ell,+} z^{\ell}, \tag{2.34}$$

$$G_{\underline{p}}(z) = \sum_{\ell=1}^{p_-} g_{p_- - \ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+} g_{p_+ - \ell,+} z^{\ell}, \tag{2.35}$$

$$H_{\underline{p}}(z) = \sum_{\ell=0}^{p_- - 1} h_{p_- - 1 - \ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} h_{p_+ - \ell,+} z^{\ell}, \tag{2.36}$$

$$K_{\underline{p}}(z) = \sum_{\ell=0}^{p_-} g_{p_- - \ell,-} z^{-\ell} + \sum_{\ell=1}^{p_+} g_{p_+ - \ell,+} z^{\ell} = G_{\underline{p}}(z) + g_{p_-,-} - g_{p_+,+}. \tag{2.37}$$

The corresponding homogeneous quantities are defined by ($\ell \in \mathbb{N}_0$)

$$\begin{aligned}
&\widehat{F}_{0,\mp}(z) = 0, \quad \widehat{F}_{\ell,-}(z) = \sum_{k=1}^{\ell} \hat{f}_{\ell-k,-} z^{-k}, \quad \widehat{F}_{\ell,+}(z) = \sum_{k=0}^{\ell-1} \hat{f}_{\ell-1-k,+} z^{k}, \\
&\widehat{G}_{0,-}(z) = 0, \quad \widehat{G}_{\ell,-}(z) = \sum_{k=1}^{\ell} \hat{g}_{\ell-k,-} z^{-k}, \\
&\widehat{G}_{0,+}(z) = \frac{1}{2}, \quad \widehat{G}_{\ell,+}(z) = \sum_{k=0}^{\ell} \hat{g}_{\ell-k,+} z^{k}, \\
&\widehat{H}_{0,\mp}(z) = 0, \quad \widehat{H}_{\ell,-}(z) = \sum_{k=0}^{\ell-1} \hat{h}_{\ell-1-k,-} z^{-k}, \quad \widehat{H}_{\ell,+}(z) = \sum_{k=1}^{\ell} \hat{h}_{\ell-k,+} z^{k}, \\
&\widehat{K}_{0,-}(z) = \frac{1}{2}, \quad \widehat{K}_{\ell,-}(z) = \sum_{k=0}^{\ell} \hat{g}_{\ell-k,-} z^{-k} = \widehat{G}_{\ell,-}(z) + \hat{g}_{\ell,-}, \\
&\widehat{K}_{0,+}(z) = 0, \quad \widehat{K}_{\ell,+}(z) = \sum_{k=1}^{\ell} \hat{g}_{\ell-k,+} z^{k} = \widehat{G}_{\ell,+}(z) - \hat{g}_{\ell,+}.
\end{aligned} \tag{2.38}$$

The stationary zero-curvature relation (2.29), $0 = UV_{\underline{p}} - V_{\underline{p}}^{+} U$, is then equivalent to

$$-\alpha(g_{p_+,+} + g_{p_-,-}^{-}) + f_{p_+-1,+} - f_{p_--1,-}^{-} = 0, \tag{2.39}$$

$$\beta(g_{p_+,+}^{-} + g_{p_-,-}) + h_{p_+-1,+}^{-} - h_{p_--1,-} = 0. \tag{2.40}$$

[1] In this paper, a sum is interpreted as zero whenever the upper limit in the sum is strictly less than its lower limit.

Thus, varying $p_\pm \in \mathbb{N}_0$, equations (2.39) and (2.40) give rise to the stationary Ablowitz–Ladik (AL) hierarchy which we introduce as follows

$$\text{s-AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -\alpha(g_{p_+,+} + g^-_{p_-,-}) + f_{p_+-1,+} - f^-_{p_--1,-} \\ \beta(g^-_{p_+,+} + g_{p_-,-}) + h^-_{p_+-1,+} - h_{p_--1,-} \end{pmatrix} = 0, \quad \underline{p} \in \mathbb{N}_0^2. \tag{2.41}$$

Explicitly (recalling $\gamma = 1 - \alpha\beta$ and taking $p_- = p_+$ for simplicity),

$$\begin{aligned}
\text{s-AL}_{(0,0)}(\alpha,\beta) &= \begin{pmatrix} -c_{(0,0)}\alpha \\ c_{(0,0)}\beta \end{pmatrix} = 0, \\
\text{s-AL}_{(1,1)}(\alpha,\beta) &= \begin{pmatrix} -\gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0, \\
\text{s-AL}_{(2,2)}(\alpha,\beta) &= \begin{pmatrix} -\gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix} \\
&\quad + \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \text{ etc.,}
\end{aligned} \tag{2.42}$$

represent the first few equations of the stationary Ablowitz–Ladik hierarchy. Here we introduced

$$c_{\underline{p}} = (c_{p_-,-} + c_{p_+,+})/2, \quad p_\pm \in \mathbb{N}_0. \tag{2.43}$$

By definition, the set of solutions of (2.41), with $\underline{p}$ ranging in $\mathbb{N}_0^2$ and $c_{\ell,\pm} \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, represents the class of algebro-geometric Ablowitz–Ladik solutions.

Using (2.1), one can show (cf. [25]) that $g_{p_+,+} = g_{p_-,-}$ up to a lattice constant which can be set equal to zero without loss of generality. Thus, we will henceforth assume that

$$g_{p_+,+} = g_{p_-,-}, \tag{2.44}$$

which in turn implies that

$$K_{\underline{p}} = G_{\underline{p}} \tag{2.45}$$

and hence renders $V_{\underline{p}}$ in (2.28) traceless in the stationary context. (We note that equations (2.44) and (2.45) cease to be valid in the time-dependent context, though.)

Next we turn to the time-dependent Ablowitz–Ladik hierarchy. For that purpose the coefficients α and β are now considered as functions of both the lattice point and time. For each equation in the hierarchy, that is, for each $\underline{p}$, we introduce a deformation (time) parameter $t_{\underline{p}} \in \mathbb{R}$ in α, β, replacing $\alpha(n), \beta(n)$ by $\alpha(n, t_{\underline{p}}), \beta(n, t_{\underline{p}})$. Moreover, the definitions (2.27), (2.28), and (2.34)–(2.37) of $U, V_{\underline{p}}$, and $F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}, K_{\underline{p}}$, respectively, still apply. Imposing the zero-curvature relation

$$U_{t_{\underline{p}}} + UV_{\underline{p}} - V_{\underline{p}}^+ U = 0, \quad \underline{p} \in \mathbb{N}_0^2, \tag{2.46}$$

then results in the equations

$$\alpha_{t_{\underline{p}}} = i\big(zF^-_{\underline{p}} + \alpha(G_{\underline{p}} + K^-_{\underline{p}}) - F_{\underline{p}}\big), \tag{2.47}$$

$$\beta_{t_{\underline{p}}} = -i\big(\beta(G^-_{\underline{p}} + K_{\underline{p}}) - H_{\underline{p}} + z^{-1}H^-_{\underline{p}}\big), \tag{2.48}$$

$$0 = z(G^-_{\underline{p}} - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H^-_{\underline{p}}, \tag{2.49}$$

$$0 = z\beta F^-_{\underline{p}} + \alpha H_{\underline{p}} + K^-_{\underline{p}} - K_{\underline{p}}. \tag{2.50}$$

Varying $\underline{p} \in \mathbb{N}_0^2$, the collection of evolution equations

$$\mathrm{AL}_{\underline{p}}(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_{\underline{p}}} - \alpha(g_{p_+,+} + g^-_{p_-,-}) + f_{p_+-1,+} - f^-_{p_--1,-} \\ -i\beta_{t_{\underline{p}}} + \beta(g^-_{p_+,+} + g_{p_-,-}) - h_{p_--1,-} + h^-_{p_+-1,+} \end{pmatrix} = 0, \quad t_{\underline{p}} \in \mathbb{R}, \ \underline{p} \in \mathbb{N}_0^2, \tag{2.51}$$

then defines the time-dependent Ablowitz–Ladik hierarchy. Explicitly, taking $p_- = p_+$ for simplicity,

$$\begin{aligned}
&\mathrm{AL}_{(0,0)}(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_{(0,0)}} - c_{(0,0)}\alpha \\ -i\beta_{t_{(0,0)}} + c_{(0,0)}\beta \end{pmatrix} = 0, \\
&\mathrm{AL}_{(1,1)}(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_{(1,1)}} - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ -i\beta_{t_{(1,1)}} + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0, \\
&\mathrm{AL}_{(2,2)}(\alpha, \beta) \\
&\quad = \begin{pmatrix} -i\alpha_{t_{(2,2)}} - \gamma\big(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2)\big) \\ -i\beta_{t_{(2,2)}} + \gamma\big(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+) \\ -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2)\big) \end{pmatrix} \\
&\qquad + \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \ \text{etc.},
\end{aligned} \tag{2.52}$$

represent the first few equations of the time-dependent Ablowitz–Ladik hierarchy. Here we recall the definition of $c_{\underline{p}}$ in (2.43).

By (2.51), (2.6), and (2.10), the time derivative of $\gamma = 1 - \alpha\beta$ is given by

$$\gamma_{t_{\underline{p}}} = i\gamma\big((g_{p_+,+} - g^-_{p_+,+}) - (g_{p_-,-} - g^-_{p_-,-})\big), \tag{2.53}$$

or alternatively, by

$$\gamma_{t_{\underline{p}}} = i\gamma\big(\alpha z^{-1} H^-_{\underline{p}} - \alpha H_{\underline{p}} + \beta F_{\underline{p}} - z\beta F^-_{\underline{p}}\big), \tag{2.54}$$

using (2.47)–(2.50).

Remark 2.5. (*i*) The special choices $\beta = \pm\overline{\alpha}$, $c_{0,\pm} = 1$ lead to the discrete nonlinear Schrödinger hierarchy. In particular, choosing $c_{(1,1)} = -2$ yields the discrete nonlinear Schrödinger equation in its usual form (see, e.g., [7, Ch. 3] and the references cited therein), with

$$-i\alpha_t - (1 \mp |\alpha|^2)(\alpha^- + \alpha^+) + 2\alpha = 0, \tag{2.55}$$

the first nonlinear element of the hierarchy. The choice $\beta = \overline{\alpha}$ is called the *defocusing* case, $\beta = -\overline{\alpha}$ represents the *focusing* case of the discrete nonlinear Schrödinger hierarchy.
(*ii*) The alternative choice $\beta = \overline{\alpha}$, $c_{0,\pm} = \mp i$, leads to the hierarchy of Schur flows. In particular, choosing $c_{(1,1)} = 0$ yields

$$\alpha_t - (1 - |\alpha|^2)(\alpha^+ - \alpha^-) = 0 \tag{2.56}$$

as the first nonlinear element of this hierarchy (cf. [9], [16], [17], [29], [35], [44]).

3. Lax pairs for the AL hierarchy

In this section we introduce Lax pairs for the AL hierarchy and prove the equivalence of the zero-curvature and Lax representation. This result is new.

Throughout this section we suppose Hypothesis 2.1. We start by relating the homogeneous coefficients $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$ to certain matrix elements of L, where L will later be identified as the Lax difference expression associated with the Ablowitz–Ladik hierarchy. For this purpose it is useful to introduce the standard basis $\{\delta_m\}_{m\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ by

$$\delta_m = \{\delta_{m,n}\}_{n\in\mathbb{Z}},\ m \in \mathbb{Z}, \quad \delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases} \tag{3.1}$$

The scalar product in $\ell^2(\mathbb{Z})$, denoted by $(\,\cdot\,,\,\cdot\,)$, is defined by

$$(f,g) = \sum_{n\in\mathbb{Z}} \overline{f(n)}g(n), \quad f,g \in \ell^2(\mathbb{Z}). \tag{3.2}$$

In the standard basis just defined, we introduce the difference expression L by

$$L = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & -\alpha(0)\rho(-1) & -\beta(-1)\alpha(0) & -\alpha(1)\rho(0) & \rho(0)\rho(1) & & & \text{\Large 0} \\ & \rho(-1)\rho(0) & \beta(-1)\rho(0) & -\beta(0)\alpha(1) & \beta(0)\rho(1) & 0 & & \\ & & 0 & -\alpha(2)\rho(1) & -\beta(1)\alpha(2) & -\alpha(3)\rho(2) & \rho(2)\rho(3) & \\ & & & \rho(1)\rho(2) & \beta(1)\rho(2) & -\beta(2)\alpha(3) & \beta(2)\rho(3) & 0 \\ \text{\Large 0} & & & & \ddots & \ddots & \ddots & \ddots \quad \ddots \end{pmatrix} \tag{3.3}$$

$$\begin{aligned} &= \Big(- \beta(n)\alpha(n+1)\delta_{m,n} \\ &\quad + \big(\beta(n-1)\rho(n)\delta_{\mathrm{odd}}(n) - \alpha(n+1)\rho(n)\delta_{\mathrm{even}}(n)\big)\delta_{m,n-1} \\ &\quad + \big(\beta(n)\rho(n+1)\delta_{\mathrm{odd}}(n) - \alpha(n+2)\rho(n+1)\delta_{\mathrm{even}}(n)\big)\delta_{m,n+1} \\ &\quad + \rho(n+1)\rho(n+2)\delta_{\mathrm{even}}(n)\delta_{m,n+2} + \rho(n-1)\rho(n)\delta_{\mathrm{odd}}(n)\delta_{m,n-2}\Big)_{m,n\in\mathbb{Z}} \end{aligned} \tag{3.4}$$

$$\begin{aligned} &= \rho^-\rho\,\delta_{\mathrm{even}}\,S^{--} + (\beta^-\rho\,\delta_{\mathrm{even}} - \alpha^+\rho\,\delta_{\mathrm{odd}})S^- - \beta\alpha^+ \\ &\quad + (\beta\rho^+\,\delta_{\mathrm{even}} - \alpha^{++}\rho^+\,\delta_{\mathrm{odd}})S^+ + \rho^+\rho^{++}\,\delta_{\mathrm{odd}}\,S^{++}, \end{aligned} \tag{3.5}$$

where δ_{even} and δ_{odd} denote the characteristic functions of the even and odd integers,

$$\delta_{\mathrm{even}} = \chi_{2\mathbb{Z}}, \quad \delta_{\mathrm{odd}} = 1 - \delta_{\mathrm{even}} = \chi_{2\mathbb{Z}+1}. \tag{3.6}$$

In particular, terms of the form $-\beta(n)\alpha(n+1)$ represent the diagonal (n,n)-entries, $n \in \mathbb{Z}$, in the infinite matrix (3.3). In addition, we used the abbreviation

$$\rho = \gamma^{1/2} = (1 - \alpha\beta)^{1/2}. \tag{3.7}$$

Next, we introduce the unitary operator $U_{\tilde{\varepsilon}}$ in $\ell^2(\mathbb{Z})$ by

$$U_{\tilde{\varepsilon}} = \big(\tilde{\varepsilon}(n)\delta_{m,n}\big)_{(m,n)\in\mathbb{Z}^2}, \quad \tilde{\varepsilon}(n) \in \{1,-1\},\ n \in \mathbb{Z}, \tag{3.8}$$

and the sequence $\varepsilon = \{\varepsilon(n)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ by

$$\varepsilon(n) = \tilde{\varepsilon}(n-1)\tilde{\varepsilon}(n),\ n \in \mathbb{Z}. \tag{3.9}$$

Assuming $\alpha, \beta \in \ell^\infty(\mathbb{Z})$, a straightforward computation then shows that

$$\breve{L}_\varepsilon = U_{\tilde{\varepsilon}} \breve{L} U_{\tilde{\varepsilon}}^{-1}, \tag{3.10}$$

where L_ε is associated with the sequences $\alpha_\varepsilon = \alpha$, $\beta_\varepsilon = \beta$, and $\rho_\varepsilon = \varepsilon\rho$, and $\breve{L}$ and $\breve{L}_\varepsilon$ are the bounded operator realizations of L and L_ε in $\ell^2(\mathbb{Z})$, respectively. Moreover, the recursion formalism in (2.5)–(2.12) yields coefficients which are polynomials in α, β and some of their shifts and hence depends only quadratically on ρ. As a result, the choice of square root of $\rho(n)$, $n \in \mathbb{Z}$, in (3.7) is immaterial when introducing the AL hierarchy via the Lax equations (3.38).

The half-lattice (i.e., semi-infinite) version of L was recently rediscovered by Cantero, Moral, and Velázquez [11] in the special case where $\beta = \overline{\alpha}$ (see also Simon [42], [43] who coined the term CMV matrix in this context). The matrix representation of L^{-1} is then obtained from that of L in (3.3) by taking the formal adjoint of L and subsequently exchanging α and β

$$\begin{aligned} L^{-1} = \Big(& -\alpha(n)\beta(n+1)\delta_{m,n} + \big(\alpha(n-1)\rho(n)\delta_{\mathrm{even}}(n) \\ & - \beta(n+1)\rho(n)\delta_{\mathrm{odd}}(n)\big)\delta_{m,n-1} \\ & + \big(\alpha(n)\rho(n+1)\delta_{\mathrm{even}}(n) - \beta(n+2)\rho(n+1)\delta_{\mathrm{odd}}(n)\big)\delta_{m,n+1} \\ & + \rho(n+1)\rho(n+2)\delta_{\mathrm{odd}}(n)\delta_{m,n+2} + \rho(n-1)\rho(n)\delta_{\mathrm{even}}(n)\delta_{m,n-2}\Big)_{m,n\in\mathbb{Z}} \end{aligned} \tag{3.11}$$

$$\begin{aligned} &= \rho^-\rho\,\delta_{\mathrm{odd}}\,S^{--} + (\alpha^-\rho\,\delta_{\mathrm{odd}} - \beta^+\rho\,\delta_{\mathrm{even}})S^- - \alpha\beta^+ \\ &\quad + (\alpha\rho^+\,\delta_{\mathrm{odd}} - \beta^{++}\rho^+\,\delta_{\mathrm{even}})S^+ + \rho^+\rho^{++}\,\delta_{\mathrm{even}}\,S^{++}. \end{aligned} \tag{3.12}$$

L and L^{-1} define bounded operators in $\ell^2(\mathbb{Z})$ if α and β are bounded sequences. However, this is of no importance in the context of Lemma 3.1 below as we only apply the five-diagonal matrices L and L^{-1} to basis vectors of the type δ_m.

Next, we discuss a useful factorization of L. For this purpose we introduce the sequence of 2×2 matrices $\theta(n)$, $n \in \mathbb{Z}$, by

$$\theta(n) = \begin{pmatrix} -\alpha(n) & \rho(n) \\ \rho(n) & \beta(n) \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{3.13}$$

and two difference expressions D and E by their matrix representations in the standard basis (3.1) of $\ell^2(\mathbb{Z})$

$$D = \begin{pmatrix} \ddots & & 0 \\ & \theta(2n-2) & \\ & & \theta(2n) \\ 0 & & & \ddots \end{pmatrix}, \quad E = \begin{pmatrix} \ddots & & 0 \\ & \theta(2n-1) & \\ & & \theta(2n+1) \\ 0 & & & \ddots \end{pmatrix}, \tag{3.14}$$

where

$$\begin{aligned} \begin{pmatrix} D(2n-1,2n-1) & D(2n-1,2n) \\ D(2n,2n-1) & D(2n,2n) \end{pmatrix} &= \theta(2n), \\ \begin{pmatrix} E(2n,2n) & E(2n,2n+1) \\ E(2n+1,2n) & E(2n+1,2n+1) \end{pmatrix} &= \theta(2n+1), \quad n \in \mathbb{Z}. \end{aligned} \tag{3.15}$$

Then L can be factorized into

$$L = DE. \tag{3.16}$$

Explicitly, D and E are given by

$$D = \rho\,\delta_{\mathrm{even}}\,S^- - \alpha^+\,\delta_{\mathrm{odd}} + \beta\,\delta_{\mathrm{even}} + \rho^+\,\delta_{\mathrm{odd}}\,S^+, \tag{3.17}$$

$$E = \rho\,\delta_{\mathrm{odd}}\,S^- + \beta\,\delta_{\mathrm{odd}} - \alpha^+\,\delta_{\mathrm{even}} + \rho^+\,\delta_{\mathrm{even}}\,S^+, \tag{3.18}$$

and their inverses are of the form

$$D^{-1} = \rho\,\delta_{\mathrm{even}}\,S^- - \beta^+\,\delta_{\mathrm{odd}} + \alpha\,\delta_{\mathrm{even}} + \rho^+\,\delta_{\mathrm{odd}}\,S^+, \tag{3.19}$$

$$E^{-1} = \rho\,\delta_{\mathrm{odd}}\,S^- + \alpha\,\delta_{\mathrm{odd}} - \beta^+\,\delta_{\mathrm{even}} + \rho^+\,\delta_{\mathrm{even}}\,S^+. \tag{3.20}$$

The next result details the connections between L and the recursion coefficients $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$.

Lemma 3.1. *Let $n\in\mathbb{Z}$. Then the homogeneous coefficients $\{\hat f_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, $\{\hat g_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$, and $\{\hat h_{\ell,\pm}\}_{\ell\in\mathbb{N}_0}$ satisfy the following relations:*

$$\begin{aligned}
\hat f_{\ell,+}(n) &= (\delta_n, EL^\ell\delta_n)\delta_{\mathrm{even}}(n) + (\delta_n, L^\ell D\delta_n)\delta_{\mathrm{odd}}(n), \quad \ell\in\mathbb{N}_0,\\
\hat f_{\ell,-}(n) &= (\delta_n, D^{-1}L^{-\ell}\delta_n)\delta_{\mathrm{even}}(n) + (\delta_n, L^{-\ell}E^{-1}\delta_n)\delta_{\mathrm{odd}}(n), \quad \ell\in\mathbb{N}_0,\\
\hat g_{0,\pm} &= 1/2, \quad \hat g_{\ell,\pm}(n) = (\delta_n, L^{\pm\ell}\delta_n), \quad \ell\in\mathbb{N},\\
\hat h_{\ell,+}(n) &= (\delta_n, L^\ell D\delta_n)\delta_{\mathrm{even}}(n) + (\delta_n, EL^\ell\delta_n)\delta_{\mathrm{odd}}(n), \quad \ell\in\mathbb{N}_0,\\
\hat h_{\ell,-}(n) &= (\delta_n, L^{-\ell}E^{-1}\delta_n)\delta_{\mathrm{even}}(n) + (\delta_n, D^{-1}L^{-\ell}\delta_n)\delta_{\mathrm{odd}}(n) \;\; \ell\in\mathbb{N}_0.
\end{aligned} \tag{3.21}$$

Proof. Using (3.16)–(3.20) we show that the sequences defined in (3.21) satisfy the recursion relations of Lemma 2.4 respectively relation (2.6). For n even,

$$\begin{aligned}
\hat g_{\ell,+}(n) - \hat g_{\ell,+}(n-1) &= (\delta_n, DEL^{\ell-1}\delta_n) - (\delta_{n-1}, DEL^{\ell-1}\delta_{n-1})\\
&= (D^*\delta_n, EL^{\ell-1}\delta_n) - (D^*\delta_{n-1}, EL^{\ell-1}\delta_{n-1})\\
&= \beta(n)(\delta_n, EL^{\ell-1}\delta_n) + \rho(n)(\delta_{n-1}, EL^{\ell-1}\delta_n)\\
&\quad + \alpha(n)(\delta_{n-1}, EL^{\ell-1}\delta_{n-1}) - \rho(n)(\delta_n, EL^{\ell-1}\delta_{n-1})\\
&= \beta(n)\hat f_{\ell-1,+}(n) + \alpha(n)\hat h_{\ell-1,+}(n-1),
\end{aligned} \tag{3.22}$$

since $(EL^\ell)^\top = EL^\ell$ by (3.13), (3.16). Moreover,

$$\begin{aligned}
\hat f_{\ell,+}(n) &= (\delta_n, EL^\ell\delta_n) = (E^*\delta_n, L^\ell\delta_n)\\
&= -\alpha(n+1)(\delta_n, L^\ell\delta_n) + \rho(n+1)(\delta_{n+1}, L^\ell\delta_n)\\
&\quad + \alpha(n+1)(\delta_{n+1}, L^\ell\delta_{n+1}) - \alpha(n+1)(\delta_{n+1}, L^\ell\delta_{n+1})\\
&= \hat f_{\ell-1,+}(n+1) - \alpha(n+1)\big(\hat g_{\ell,+}(n+1) + \hat g_{\ell,+}(n)\big),\\
\hat h_{\ell,+}(n) &= (\delta_n, L^\ell D\delta_n) = \beta(n)(\delta_n, L^\ell\delta_n) + \rho(n)(\delta_n, L^\ell\delta_{n-1})\\
&\quad + \beta(n)(\delta_{n-1}, L^\ell\delta_{n-1}) - \beta(n)(\delta_{n-1}, L^\ell\delta_{n-1})\\
&= \hat h_{\ell-1,+}(n-1) + \beta(n)\big(\hat g_{\ell,+}(n) + \hat g_{\ell,+}(n-1)\big),
\end{aligned} \tag{3.23}$$

that is, the coefficients satisfy (2.23). The remaining cases follow analogously. □

Finally, we derive an explicit expression for the Lax pair for the Ablowitz–Ladik hierarchy, but first we need some notation. Let T be a bounded operator in $\ell^2(\mathbb{Z})$. Given the standard basis (3.1) in $\ell^2(\mathbb{Z})$, we represent T by

$$T = \big(T(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T(m,n) = (\delta_m, T\,\delta_n), \quad (m,n)\in\mathbb{Z}^2. \tag{3.24}$$

Actually, for our purpose below, it is sufficient that T is an N-diagonal matrix for some $N\in\mathbb{N}$. Moreover, we introduce the upper and lower triangular parts $T_\pm$ of

T by

$$T_\pm = \big(T_\pm(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad T_\pm(m,n) = \begin{cases} T(m,n), & \pm(n-m) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.25}$$

Next, consider the finite difference expression $P_{\underline{p}}$ defined by

$$P_{\underline{p}} = \frac{i}{2}\sum_{\ell=1}^{p_+} c_{p_+-\ell,+}\big((L^\ell)_+ - (L^\ell)_-\big) - \frac{i}{2}\sum_{\ell=1}^{p_-} c_{p_--\ell,-}\big((L^{-\ell})_+ - (L^{-\ell})_-\big) - \frac{i}{2}c_{\underline{p}}\,Q_d, \quad \underline{p}\in\mathbb{N}_0^2, \tag{3.26}$$

with L given by (3.3) and Q_d denoting the doubly infinite diagonal matrix

$$Q_d = \big((-1)^k\delta_{k,\ell}\big)_{k,\ell\in\mathbb{Z}}. \tag{3.27}$$

Before we prove that $(L, P_{\underline{p}})$ is indeed the Lax pair for the Ablowitz–Ladik hierarchy, we derive one more representation of $P_{\underline{p}}$ in terms of L.

We denote by $\ell_0(\mathbb{Z})$ the set of complex-valued sequences of compact support. If R denotes a finite difference expression, then ψ is called a weak solution of $R\psi = z\psi$, for some $z\in\mathbb{C}$, if the relation holds pointwise for each lattice point, that is, if $((R-z)\psi)(n) = 0$ for all $n\in\mathbb{Z}$.

Lemma 3.2. *Let $\psi\in\ell_0^\infty(\mathbb{Z})$. Then the difference expression $P_{\underline{p}}$ defined in* (3.26) *acts on ψ by*

$$\begin{aligned}
(P_{\underline{p}}\psi)(n) = i\bigg(&-\sum_{\ell=1}^{p_-} f_{p_--\ell,-}(n)(EL^{-\ell}\psi)(n) - \sum_{\ell=0}^{p_+-1} f_{p_+-1-\ell,+}(n)(EL^{\ell}\psi)(n) \\
&+ \sum_{\ell=1}^{p_-} g_{p_--\ell,-}(n)(L^{-\ell}\psi)(n) + \sum_{\ell=1}^{p_+} g_{p_+-\ell,+}(n)(L^{\ell}\psi)(n) \\
&+ \frac{1}{2}\big(g_{p_-,-}(n) + g_{p_+,+}(n)\big)\psi(n)\bigg)\delta_{\mathrm{odd}}(n) \\
+ i\bigg(&\sum_{\ell=0}^{p_--1} h_{p_--1-\ell,-}(n)(D^{-1}L^{-\ell}\psi)(n) + \sum_{\ell=1}^{p_+} h_{p_+-\ell,+}(n)(D^{-1}L^{\ell}\psi)(n) \\
&- \sum_{\ell=1}^{p_-} g_{p_--\ell,-}(n)(L^{-\ell}\psi)(n) - \sum_{\ell=1}^{p_+} g_{p_+-\ell,+}(n)(L^{\ell}\psi)(n) \\
&- \frac{1}{2}\big(g_{p_-,-}(n) + g_{p_+,+}(n)\big)\psi(n)\bigg)\delta_{\mathrm{even}}(n), \quad n\in\mathbb{Z}.
\end{aligned} \tag{3.28}$$

In addition, if u is a weak solution of $Lu(z) = zu(z)$, then

$$\begin{aligned}
&\big(P_{\underline{p}}u(z)\big)(n) \\
&\quad = \Big(-iF_{\underline{p}}(z,n)(Eu(z))(n) + \frac{i}{2}\big(G_{\underline{p}}(z,n) + K_{\underline{p}}(z,n)\big)u(z,n)\Big)\delta_{\mathrm{odd}}(n) \\
&\qquad + \Big(iH_{\underline{p}}(z,n)(D^{-1}u(z))(n) - \frac{i}{2}\big(G_{\underline{p}}(z,n) + K_{\underline{p}}(z,n)\big)u(z,n)\Big)\delta_{\mathrm{even}}(n), \quad n\in\mathbb{Z},
\end{aligned} \tag{3.29}$$

in the weak sense.

Proof. We consider the case where n is even and use induction on $\underline{p} = (p_-, p_+)$. The case n odd is analogous. For $\underline{p} = (0, 0)$, the formulas (3.28) and (3.26) match. Denoting by $\widehat{P}_{\underline{p}}$ the corresponding homogeneous operator where all summation constants $c_{k,\pm}$, $k = 1, \dots, p_\pm$, vanish, we have to show that

$$\begin{aligned} i\widehat{P}_{\underline{p}} &= i\widehat{P}^+_{p_+-1}L - \hat{h}_{p_+-1,+}D^{-1}L + \frac{1}{2}\big(\hat{g}_{p_+-1,+}L + \hat{g}_{p_+,+}\big) \\ &\quad + i\widehat{P}^-_{p_--1}L^{-1} - \hat{h}_{p_--1,-}D^{-1} + \frac{1}{2}\big(\hat{g}_{p_--1,-}L^{-1} + \hat{g}_{p_-,-}\big), \end{aligned} \tag{3.30}$$

where $\widehat{P}^\pm_j$ correspond to the powers of L in (3.26), $\widehat{P}^\pm_j = \frac{i}{2}\big((L^{\pm j})_\pm - (L^{\pm j})_\mp\big)$. This can be done upon considering $(\delta_m, \widehat{P}_{\underline{p}}\delta_n)$ and making appropriate case distinctions $m = n$, $m > n$, and $m < n$.

Using (3.4), (3.11), (3.16)–(3.20), (3.25), and Lemma 3.1, one verifies for instance in the case $m = n$,

$$\begin{aligned} &(\delta_n, i\widehat{P}^+_{p_+}\delta_n) \\ &= (\delta_n, i\widehat{P}^+_{p_+-1}L\delta_n) + \alpha(n+1)\hat{h}_{p_+-1,+}(n) \\ &\quad + \frac{1}{2}\big(\hat{g}_{p_+,+}(n) - \alpha(n+1)\beta(n)\hat{g}_{p_+-1,+}(n)\big) \\ &= \bigg(\delta_n, \frac{1}{2}\big((L^{p_+-1})_+ - (L^{p_+-1})_-\big)\big(\alpha^{++}\rho^+\delta_{n-1} + \alpha^+\beta\delta_n + \alpha^+\rho\delta_{n+1} - \rho^-\rho\delta_{n+2}\big)\bigg) \\ &\quad + \alpha(n+1)\hat{h}_{p_+-1,+}(n) + \frac{1}{2}\big(\hat{g}_{p_+,+}(n) - \alpha(n+1)\beta(n)\hat{g}_{p_+-1,+}(n)\big) \\ &= -\frac{1}{2}\alpha(n+1)\rho(n)(\delta_n, L^{p_+-1}\delta_{n-1}) + \frac{1}{2}\alpha(n+2)\rho(n+1)(\delta_n, L^{p_+-1}\delta_{n+1}) \\ &\quad - \frac{1}{2}\rho(n+1)\rho(n+2)(\delta_n, L^{p_+-1}\delta_{n+2}) + \alpha(n+1)\hat{h}_{p_+-1,+}(n) \\ &\quad + \frac{1}{2}\big(\hat{g}_{p_+,+}(n) - \alpha(n+1)\beta(n)\hat{g}_{p_+-1,+}(n)\big) \\ &= -\alpha(n+1)\rho(n)(\delta_n, L^{p_+-1}\delta_{n-1}) \\ &\quad - \alpha(n+1)\beta(n)\hat{g}_{p_+-1,+}(n) + \alpha(n+1)\hat{h}_{p_+-1,+}(n) \\ &= 0, \end{aligned} \tag{3.31}$$

since by Lemma 3.1,

$$\begin{aligned} \hat{g}_{p_+,+}(n) &= (\delta_n, L^{p_+-1}L\delta_n), \\ \hat{h}_{p_+-1,+}(n) &= (\delta_n, L^{p_+-1}D\delta_n) = \beta(n)(\delta_n, L^{p_+-1}\delta_n) + \rho(n)(\delta_n, L^{p_+-1}\delta_{n-1}). \end{aligned}$$

Similarly,

$$\begin{aligned} &(\delta_n, i\widehat{P}^-_{p_-}\delta_n) \\ &= (\delta_n, i\widehat{P}^-_{p_--1}L^{-1}\delta_n) - \alpha(n)\hat{h}_{p_--1,-}(n) + \frac{1}{2}\big(\hat{g}_{p_-,-}(n) - \alpha(n)\beta(n+1)\hat{g}_{p_--1,-}(n)\big) \\ &= \bigg(\delta_n, \frac{1}{2}\big((L^{1-p_-})_+ - (L^{1-p_-})_-\big)\big(\rho^+\rho^{++}\delta_{n-2} + \alpha\rho^+\delta_{n-1} - \alpha\beta^+\delta_n + \alpha^-\rho\delta_{n+1}\big)\bigg) \\ &\quad - \alpha(n)\hat{h}_{p_--1,-}(n) + \frac{1}{2}\big(\hat{g}_{p_-,-}(n) - \alpha(n)\beta(n+1)\hat{g}_{p_--1,-}(n)\big) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\rho(n-1)\rho(n)(\delta_n, L^{1-p_-}\delta_{n-2}) - \frac{1}{2}\alpha(n-1)\rho(n)(\delta_n, L^{1-p_-}\delta_{n-1}) \\
&\quad + \frac{1}{2}\alpha(n)\rho(n+1)(\delta_n, L^{1-p_-}\delta_{n+1}) - \alpha(n)\hat{h}_{p_--1,-}(n) \\
&\quad + \frac{1}{2}\big(\hat{g}_{p_-,-}(n) - \alpha(n)\beta(n+1)\hat{g}_{p_--1,-}(n)\big) \\
&= \alpha(n)\rho(n+1)(\delta_n, L^{1-p_-}\delta_{n+1}) - \alpha(n)\beta(n+1)\hat{g}_{p_--1,-}(n) - \alpha(n)\hat{h}_{p_--1,-}(n) \\
&= 0, \qquad (3.32)
\end{aligned}
$$

where we used Lemma 3.1 and (2.12) at $\ell = p_- - 2$ for the last equality. This proves the case $m = n$. The remaining cases $m > n$ and $m < n$ are settled in a similar fashion.

Equality (3.29) then follows from $Lu(z) = zu(z)$ and (2.34)–(2.37). $\square$

Next, we introduce the difference expression $P_{\underline{p}}^\top$ by

$$
\begin{aligned}
P_{\underline{p}}^\top = &-\frac{i}{2}\sum_{\ell=1}^{p_+} c_{p_+-\ell,+}\big(((L^\top)^\ell)_+ - ((L^\top)^\ell)_-\big) \\
&+ \frac{i}{2}\sum_{\ell=1}^{p_-} c_{p_--\ell,-}\big(((L^\top)^{-\ell})_+ - ((L^\top)^{-\ell})_-\big) - \frac{i}{2}c_{\underline{p}}\, Q_d, \quad \underline{p}\in\mathbb{N}_0^2,
\end{aligned} \tag{3.33}
$$

with $L^\top = ED$ the difference expression associated with the transpose of the infinite matrix (3.3) in the standard basis of $\ell^2(\mathbb{Z})$ and Q_d denoting the doubly infinite diagonal matrix in (3.27). Here we used

$$
(M_+)^\top = (M^\top)_-, \quad (M_-)^\top = (M^\top)_+ \tag{3.34}
$$

for a finite difference expression M in the standard basis of $\ell^2(\mathbb{Z})$.

For later purpose in Section 5 we now mention the analog of Lemma 3.2 for the difference expression $P_{\underline{p}}^\top$ without proof:

Lemma 3.3. *Let $\chi\in\ell_0^\infty(\mathbb{Z})$. Then the difference expression $P_{\underline{p}}^\top$ defined in* (3.33) *acts on χ by*

$$
\begin{aligned}
(P_{\underline{p}}^\top\chi)(n) = i\bigg(&-\sum_{\ell=0}^{p_--1} h_{p_--1-\ell,-}(n)(E^{-1}(L^\top)^{-\ell}\chi)(n) \\
&- \sum_{\ell=1}^{p_+} h_{p_+-\ell,+}(n)(E^{-1}(L^\top)^{\ell}\chi)(n) \\
&+ \sum_{\ell=1}^{p_-} g_{p_--\ell,-}(n)((L^\top)^{-\ell}\chi)(n) + \sum_{\ell=1}^{p_+} g_{p_+-\ell,+}(n)((L^\top)^{\ell}\chi)(n) \\
&+ \frac{1}{2}\big(g_{p_-,-}(n) + g_{p_+,+}(n)\big)\chi(n)\bigg)\delta_{\mathrm{odd}}(n) \qquad (3.35) \\
+ i\bigg(&\sum_{\ell=1}^{p_-} f_{p_--\ell,-}(n)(D(L^\top)^{-\ell}\chi)(n) \\
&+ \sum_{\ell=0}^{p_+-1} f_{p_+-1-\ell,+}(n)(D(L^\top)^{\ell}\chi)(n)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\ell=1}^{p_-} g_{p_- -\ell,-}(n)((L^\top)^{-\ell}\chi)(n) - \sum_{\ell=1}^{p_+} g_{p_+-\ell,+}(n)((L^\top)^{\ell}\chi)(n) \\
&- \frac{1}{2}\big(g_{p_-,-}(n) + g_{p_+,+}(n)\big)\chi(n)\bigg)\delta_{\mathrm{even}}(n), \quad n \in \mathbb{Z}.
\end{aligned}
$$

In addition, if χ is a weak solution of $L^\top v(z) = zv(z)$, then

$$
\begin{aligned}
&\big(P_{\underline{p}}^\top v(z)\big)(n) \\
&\quad = -i\bigg(H_{\underline{p}}(z,n)(E^{-1}v(z))(n) - \frac{1}{2}\big(G_{\underline{p}}(z,n) + K_{\underline{p}}(z,n)\big)v(z,n)\bigg)\delta_{\mathrm{odd}}(n) \\
&\qquad + i\bigg(F_{\underline{p}}(z,n)(Dv(z))(n) - \frac{1}{2}\big(G_{\underline{p}}(z,n) + K_{\underline{p}}(z,n)\big)v(z,n)\bigg)\delta_{\mathrm{even}}(n), \\
&\hspace{25em} n \in \mathbb{Z}, \quad (3.36)
\end{aligned}
$$

in the weak sense.

Given these preliminaries, one can now prove the following result, the proof of which is based on fairly tedious computations. We present them here in some detail as these results have not appeared in print before.

Theorem 3.4. *Assume Hypothesis* 2.1. *Then, for each $\underline{p} \in \mathbb{N}_0^2$, the $\underline{p}$th stationary Ablowitz–Ladik equation* s-$\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$ *in* (2.41) *is equivalent to the vanishing of the commutator of $P_{\underline{p}}$ and L,*

$$
[P_{\underline{p}}, L] = 0. \tag{3.37}
$$

In addition, the $\underline{p}$th time-dependent Ablowitz–Ladik equation $\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$ *in* (2.51) *is equivalent to the Lax commutator equations*

$$
L_{t_{\underline{p}}}(t_{\underline{p}}) - [P_{\underline{p}}(t_{\underline{p}}), L(t_{\underline{p}})] = 0, \quad t_{\underline{p}} \in \mathbb{R}. \tag{3.38}
$$

In particular, the pair of difference expressions $(L, P_{\underline{p}})$ represents the Lax pair for the Ablowitz–Ladik hierarchy of nonlinear differential-difference evolution equations.

Proof. Let $f \in \ell_0(\mathbb{Z})$. To curb the length of this proof we will only consider the case n even. We apply formulas (3.28) to compute the commutator $([P_{\underline{p}}, L]f)(n)$ by rewriting $D^{-1}L^\ell = EL^{\ell-1}$ and using (3.5), (3.17), and (3.18). This yields

$$
\begin{aligned}
&i([P_{\underline{p}}, L]f)(n) \\
&= \bigg(\sum_{\ell=1}^{p_+} \rho^-\rho(g^-_{p_+-\ell,+} - g^{--}_{p_+-\ell,+}) \\
&\qquad - \sum_{\ell=0}^{p_+-1} \rho^-\rho(\alpha^- h^{--}_{p_+-1-\ell,+} + \beta^- f^-_{p_+-1-\ell,+})\bigg)(L^\ell f)(n-2) \\
&\quad + \bigg(\sum_{\ell=1}^{p_+} \rho\big(2\beta^- g^-_{p_+-\ell,+} - h^-_{p_+-\ell,+}\big) \\
&\qquad + \sum_{\ell=0}^{p_+-1} \rho\big((\rho^-)^2 h^{--}_{p_+-1-\ell,+} - (\beta^-)^2 f^-_{p_+-1-\ell,+}\big)\bigg)(L^\ell f)(n-1)
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg(\sum_{\ell=1}^{p_+}\rho^+\big(\beta(g^+_{p_+-\ell,+}+g^-_{p_+-\ell,+})-h_{p_+-\ell,+}\big)\\
&\quad+\sum_{\ell=0}^{p_+-1}\rho^+\big(h^-_{p_+-1-\ell,+}-\beta\beta^+f^+_{p_+-1-\ell,+}-\beta\alpha^+h_{p_+-1-\ell,+}\big)\bigg)(L^\ell f)(n+1)\\
&+\bigg(\sum_{\ell=2}^{p_++1}\big(g_{p_++1-\ell,+}-g^-_{p_++1-\ell,+}\big)\\
&\quad+\sum_{\ell=1}^{p_+}\Big(\alpha^+\big(\beta(g_{p_+-\ell,+}-g^-_{p_+-\ell,+})+h_{p_+-\ell,+}\big)-\alpha h^-_{p_+-\ell,+}\Big)\\
&\quad+\sum_{\ell=0}^{p_+-1}\Big(\beta(\alpha^+)^2h_{p_+-1-\ell,+}-\beta(\rho^+)^2f^+_{p_+-1-\ell,+}-\alpha^+h^-_{p_+-1-\ell,+}\Big)\bigg)(L^\ell f)(n)\\
&+\sum_{\ell=1}^{p_-}\rho^-\rho\big(g^-_{p_--\ell,-}-g^{--}_{p_--\ell,-}-\beta^-f^-_{p_--\ell,-}-\alpha^-h^{--}_{p_--\ell,-}\big)(L^{-\ell}f)(n-2)\\
&+\bigg(\sum_{\ell=1}^{p_-}\rho\Big(\beta^-\big(2g^-_{p_--\ell,-}-\beta^-f^-_{p_--\ell,-}\big)+\rho(\rho^-)^2h^{--}_{p_--\ell,-}\Big)\\
&\quad-\sum_{\ell=0}^{p_--1}\rho h^-_{p_--1-\ell,-}\bigg)(L^{-\ell}f)(n-1)\\
&+\bigg(\sum_{\ell=1}^{p_-}\Big(\beta\rho^+\big(g^+_{p_--\ell,-}+g^-_{p_--\ell,-}-\alpha^+h_{p_--\ell,-}-\beta^+f^+_{p_--\ell,-}\big)+\rho^+h^-_{p_--\ell,-}\Big)\\
&\quad-\sum_{\ell=0}^{p_--1}\rho^+h_{p_--1-\ell,-}\bigg)(L^{-\ell}f)(n+1)\\
&+\bigg(\sum_{\ell=1}^{p_-}\Big(\beta\alpha^+(g_{p_--\ell,-}-g^-_{p_--\ell,-}+\alpha^+h_{p_--\ell,-}+\beta^+f^+_{p_--\ell,-})-\beta f^+_{p_--\ell,-}\\
&\qquad\quad-\alpha^+h^-_{p_--\ell,-}\Big)\\
&\quad+\sum_{\ell=0}^{p_--1}\big(g_{p_--1-\ell,-}-g^-_{p_--1-\ell,-}-\alpha h^-_{p_--1-\ell,-}+\alpha^+h_{p_--1-\ell,-}\big)\bigg)(L^{-\ell}f)(n)\\
&+\frac{1}{2}\Big(\beta\rho^+\big(g^+_{p_-,-}+g^+_{p_+,+}+g_{p_-,-}+g_{p_+,+}\big)f(n+1)\\
&\quad+\beta^-\rho\big(g^-_{p_-,-}+g^-_{p_+,+}+g_{p_-,-}+g_{p_+,+}\big)f(n-1)\\
&\quad-\rho^-\rho\big(g^{--}_{p_-,-}+g^{--}_{p_+,+}-g_{p_-,-}-g_{p_+,+}\big)f(n-2)\Big),
\end{aligned}\tag{3.39}
$$

where we added the terms

$$
\begin{aligned}
0&=-\sum_{\ell=1}^{p_+}g^-_{p_+-\ell,+}(L^{\ell+1}f)(n)+\sum_{\ell=1}^{p_+}g^-_{p_+-\ell,+}(L^{\ell+1}f)(n)\\
&=-\sum_{\ell=2}^{p_++1}g^-_{p_++1-\ell,+}(L^\ell f)(n)+\sum_{\ell=1}^{p_+}g^-_{p_+-\ell,+}L(L^\ell f)(n),
\end{aligned}
$$

$$\begin{aligned}
0 &= -\sum_{\ell=1}^{p_+} h^-_{p_+-\ell,+}(D^{-1}L^\ell f)(n) + \sum_{\ell=1}^{p_+} h^-_{p_+-\ell,+}(EL^{\ell-1}f)(n)\\
&= -\sum_{\ell=1}^{p_+} h^-_{p_+-\ell,+}\Big(\alpha(L^\ell f)(n) + \rho(L^\ell f)(n-1)\Big)\\
&\quad + \sum_{\ell=0}^{p_+-1} h^-_{p_+-1-\ell,+}\Big(-\alpha^+(L^\ell f)(n) + \rho^+(L^\ell f)(n+1)\Big),
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
0 &= -\sum_{\ell=1}^{p_-} g^-_{p_--\ell,-}(L^{-\ell+1}f)(n) + \sum_{\ell=1}^{p_-} g^-_{p_--\ell,-}(L^{-\ell+1}f)(n)\\
&= -\sum_{\ell=0}^{p_--1} g^-_{p_--1-\ell,-}(L^{-\ell}f)(n) + \sum_{\ell=1}^{p_-} g^-_{p_--\ell,-}L(L^{-\ell}f)(n),
\end{aligned}$$

$$\begin{aligned}
0 &= -\sum_{\ell=1}^{p_-} h^-_{p_--\ell,-}(D^{-1}L^{-\ell+1}f)(n) + \sum_{\ell=1}^{p_-} h^-_{p_--\ell,-}(EL^{-\ell}f)(n)\\
&= -\sum_{\ell=0}^{p_--1} h^-_{p_--1-\ell,-}\Big(\alpha(L^{-\ell}f)(n) + \rho(L^{-\ell}f)(n-1)\Big)\\
&\quad + \sum_{\ell=1}^{p_-} h^-_{p_--\ell,-}\Big(-\alpha^+(L^{-\ell}f)(n) + \rho^+(L^{-\ell}f)(n+1)\Big).
\end{aligned}$$

Next we apply the recursion relations (2.5)–(2.12). In addition, we also use

$$\begin{aligned}
\alpha^+ h_{p_--\ell,-} + \beta^+ f^+_{p_--\ell,-} &= \alpha^+\big(h^+_{p_--1-\ell,-} - \beta^+(g^+_{p_--\ell,-} + g_{p_--\ell,-})\big)\\
&\quad + \beta^+\big(f_{p_--1-\ell,-} + \alpha^+(g^+_{p_--\ell,-} + g_{p_--\ell,-})\big)\\
&= g^+_{p_--\ell,-} - g_{p_--\ell,-}.
\end{aligned} \tag{3.41}$$

This implies,

$$\begin{aligned}
&i([P_{\underline{p}}, L]f)(n)\\
&= \sum_{\ell=1}^{p_+-1} \rho^-\rho\Big(g^-_{p_+-\ell,+} - g^{--}_{p_+-\ell,+} - \alpha^- h^{--}_{p_+-1-\ell,+} - \beta^- f^-_{p_+-1-\ell,+}\Big)(L^\ell f)(n-2)\\
&\quad + \sum_{\ell=1}^{p_+-1} \Big(\beta^-\rho\big(g^-_{p_+-\ell,+} - g^{--}_{p_+-\ell,+} - \alpha^- h^{--}_{p_+-1-\ell,+} - \beta^- f^-_{p_+-1-\ell,+}\big)\\
&\qquad + \rho\big(\beta^-(g^-_{p_+-\ell,+} + g^{--}_{p_+-\ell,+}) + h^{--}_{p_+-1-\ell,+} - h^-_{p_+-\ell,+}\big)\Big)(L^\ell f)(n-1)\\
&\quad + \sum_{\ell=1}^{p_+-1} \Big(\beta\rho^+\big(g^+_{p_+-\ell,+} - g_{p_+-\ell,+} - \alpha^+ h_{p_+-1-\ell,+} - \beta^+ f^+_{p_+-1-\ell,+}\big)\\
&\qquad + \rho^+\big(\beta(g_{p_+-\ell,+} + g^-_{p_+-\ell,+}) + h^-_{p_+-1-\ell,+} - h_{p_+-\ell,+}\big)\Big)(L^\ell f)(n+1)\\
&\quad + \Bigg(\sum_{\ell=1}^{p_+-1} \big(g_{p_++1-\ell,+} - g^-_{p_++1-\ell,+} - \alpha h^-_{p_+-\ell,+} + \beta\alpha^+(g^+_{p_+-\ell,+} + g_{p_+-\ell,+})
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \beta f^{+}_{p_+-1-\ell,+}\big) \\
&\quad + \sum_{\ell=1}^{p_+-1} \alpha^{+}\big(\beta(-g_{p_+-\ell,+} - g^{-}_{p_+-\ell,+}) + h_{p_+-\ell,+} - h^{-}_{p_+-1-\ell,+}\big) \\
&\quad + \sum_{\ell=0}^{p_+-1} \beta\alpha^{+}\big(g_{p_+-\ell,+} - g^{+}_{p_+-\ell,+} + \alpha^{+} h_{p_+-1-\ell,+} + \beta^{+} f^{+}_{p_+-1-\ell,+}\big)\Big)(L^{\ell} f)(n) \\
&+ \sum_{\ell=1}^{p_-} \rho^{-}\rho\big(g^{-}_{p_--\ell,-} - g^{--}_{p_--\ell,-} - \beta^{-} f^{-}_{p_--\ell,-} - \alpha^{-} h^{--}_{p_--\ell,-}\big)(L^{-\ell} f)(n-2) \\
&+ \sum_{\ell=1}^{p_--1} \Big(\beta^{-}\rho\big(g^{-}_{p_--\ell,-} - g^{--}_{p_--\ell,-} - \beta^{-} f^{-}_{p_--\ell,-} - \alpha^{-} h^{--}_{p_--\ell,-}\big) \\
&\qquad + \rho\big(\beta^{-}(g^{-}_{p_--\ell,-} + g^{--}_{p_--\ell,-}) + h^{--}_{p_--\ell,-} - h^{-}_{p_--1-\ell,-}\big)\Big)(L^{-\ell} f)(n-1) \\
&+ \sum_{\ell=1}^{p_--1} \Big(\beta\rho^{+}(g^{+}_{p_--\ell,-} - g_{p_--\ell,-} - \alpha^{+} h_{p_--\ell,-} - \beta^{+} f^{+}_{p_--\ell,-}) \\
&\qquad + \rho^{+}\big(\beta(g_{p_--\ell,-} + g^{-}_{p_--\ell,-}) + h^{-}_{p_--\ell,-} - h_{p_--1-\ell,-}\big)\Big)(L^{-\ell} f)(n+1) \\
&+ \Big(\sum_{\ell=1}^{p_--1} \Big(g_{p_--1-\ell,-} - g^{-}_{p_--1-\ell,-} - \alpha h^{-}_{p_--1-\ell,-}\big) \\
&\quad + \sum_{\ell=1}^{p_-} \beta\big(\alpha^{+}(g^{+}_{p_--\ell,-} + g_{p_--\ell,-}) - f^{+}_{p_--\ell,-}\big) \\
&\quad + \sum_{\ell=1}^{p_-} \beta\alpha^{+}\big(g_{p_--\ell,-} - g^{+}_{p_--\ell,-} + \alpha^{+} h_{p_--\ell,-} + \beta^{+} f^{+}_{p_--\ell,-}\big) \\
&\quad + \sum_{\ell=1}^{p_--1} \alpha^{+}\big(\beta(-g_{p_--\ell,-} - g^{-}_{p_--\ell,-}) - h^{-}_{p_--\ell,-} + h_{p_--1-\ell,-}\big)\Big)(L^{-\ell} f)(n) \\
&+ \rho^{-}\rho\big(g^{-}_{0,+} - g^{--}_{0,+}\big)(L^{p_+} f)(n-2) - \rho^{-}\rho\big(\alpha^{-} h^{--}_{p_+-1,+} + \beta^{-} f^{-}_{p_+-1,+}\big) f(n-2) \\
&+ \rho\big(2\beta^{-} g^{-}_{0,+} - h^{-}_{0,+}\big)(L^{p_+} f)(n-1) + \rho\big((\rho^{-})^2 h^{--}_{p_+-1,+} - (\beta^{-})^2 f^{-}_{p_+-1,+}\big) f(n-1) \\
&+ \rho^{+}\big(\beta(g^{+}_{0,+} + g^{-}_{0,+}) - h_{0,+}\big)(L^{p_+} f)(n+1) \\
&- \rho^{+}\big(\beta(\alpha^{+} h_{p_+-1,+} + \beta^{+} f^{+}_{p_+-1,+}) - h^{-}_{p_+-1,+}\big) f(n+1) \\
&+ (g_{0,+} - g^{-}_{0,+})(L^{p_++1} f)(n) \\
&+ \big(g_{1,+} - g^{-}_{1,+} - \alpha h^{-}_{0,+} + \beta\alpha^{+}(g_{0,+} - g^{-}_{0,+}) + \alpha^{+} h_{0,+}\big)(L^{p_+} f)(n) \\
&+ \big(g_{p_--1,-} - g^{-}_{p_--1,-} - \alpha h^{-}_{p_--1,-} - \beta f_{p_--1,-}\big) f(n) \\
&+ \beta\big(\alpha^{+}(g^{+}_{p_+,+} - g_{p_+,+}) + f_{p_--1,-} - f^{+}_{p_+-1,+}\big) f(n) \\
&+ \alpha^{+}\big(\beta(\beta f_{p_+-1,+} + \alpha h^{-}_{p_+-1,+}) + h_{p_--1,-} - h^{-}_{p_+-1,+}\big) f(n) \\
&- \big(\beta\alpha^{+}(g_{0,-} + g^{-}_{0,-}) - \beta f_{-1,-} + \alpha^{+} h^{-}_{0,-}\big)(L^{-p_-} f)(n) \\
&- (\alpha h^{-}_{p_+-1,+} + \beta f_{p_+-1,+})\big(\beta\rho^{+} f(n+1) + \beta^{-}\rho f(n-1) + \rho^{-}\rho f(n-2)\big) \\
&+ \Big(\beta^{-}\rho\big(g^{-}_{0,-} + g^{-}_{0,-} - \beta^{-} f^{-}_{0,-}\big) + (\rho^{-})^2 \rho h^{--}_{0,-}\Big)(L^{-p_-} f)(n-1)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\beta\rho^+\big(g_{0,-}^+ + g_{0,-}^- - \beta^+ f_{0,-}^+ - \alpha^+ h_{0,-}\big) + \rho^+ h_{0,-}^-\Big)(L^{-p_-}f)(n+1)\\
&- \rho h_{p_--1,-}^- f(n-1) - \rho^+ h_{p_--1,-} f(n+1)\\
&+ \frac{1}{2}\Big(\beta\rho^+\big(g_{p_-,-}^+ + g_{p_+,+}^+ + g_{p_-,-} + g_{p_+,+}\big) f(n+1)\\
&\quad + \beta^-\rho\big(g_{p_-,-}^- + g_{p_+,+}^- + g_{p_-,-} + g_{p_+,+}\big) f(n-1)\\
&\quad - \rho^-\rho\big(g_{p_-,-}^{--} + g_{p_+,+}^{--} - g_{p_-,-} - g_{p_+,+}\big) f(n-2)\Big)\\
=& \frac{\rho^-\rho}{2}\big(g_{p_+,+}^{--} - g_{p_-,-}^{--} + g_{p_-,-} - g_{p_+,+}\big) f(n-2)\\
&+ \Big(\rho\big(\beta^-(g_{p_+,+}^{--} + g_{p_-,-}^-) - h_{p_--1,-}^- + h_{p_+-1,+}^{--}\big)\\
&\quad + \frac{\beta^-\rho}{2}\big(g_{p_+,+}^- + g_{p_-,-} - g_{p_+,+} - g_{p_-,-}^-\big)\Big) f(n-1)\\
&+ \Big(\beta\big(\alpha^+(g_{p_+,+}^+ + g_{p_-,-}) + f_{p_--1,-} - f_{p_+-1,+}^+\big)\\
&\quad - \alpha^+\big(\beta(g_{p_+,+}^- + g_{p_-,-}) - h_{p_--1,-} + h_{p_+-1,+}^-\big)\Big) f(n)\\
&+ \Big(\rho^+\big(\beta(g_{p_+,+}^- + g_{p_-,-}) - h_{p_--1,-} + h_{p_+-1,+}^-\big)\\
&\quad + \frac{\beta\rho^+}{2}\big(g_{p_+,+} + g_{p_-,-}^+ - g_{p_+,+}^+ - g_{p_-,-}\big)\Big) f(n+1),
\end{aligned} \tag{3.42}
$$

where we also used (2.25).

Comparing coefficients finally shows that (3.38) is equivalent to

$$(\rho^-\rho)_{t_{\underline{p}}} = \rho^-\rho(C^- + C), \tag{3.43}$$

$$(\alpha\rho^-)_{t_{\underline{p}}} = \rho^- A + \alpha\rho^- C^-, \tag{3.44}$$

$$(\beta\rho^+)_{t_{\underline{p}}} = \rho^+ B + \beta\rho^+ C^+, \tag{3.45}$$

$$(\alpha^+\beta)_{t_{\underline{p}}} = \beta A^+ + \alpha^+ B, \tag{3.46}$$

where

$$A = i\big(\alpha(g_{p_+,+} + g_{p_-,-}^-) - f_{p_+-1,+} + f_{p_--1,-}^-\big), \tag{3.47}$$

$$B = i\big(-\beta(g_{p_+,+}^- + g_{p_-,-}) + h_{p_--1,-} - h_{p_+-1,+}^-\big), \tag{3.48}$$

$$C = \frac{i}{2}\big(g_{p_+,+} + g_{p_-,-}^- - g_{p_+,+}^- - g_{p_-,-}\big). \tag{3.49}$$

In particular, (2.51) implies (3.38) since, by (2.53),

$$\rho_{t_{\underline{p}}} = \frac{i}{2}\rho\big(g_{p_+,+} + g_{p_-,-}^- - g_{p_+,+}^- - g_{p_-,-}\big). \tag{3.50}$$

To prove the converse assertion (i.e., that (3.38) implies (2.51)), we argue as follows: Rewriting (3.44) and (3.45) using $\rho = \gamma^{1/2} = (1-\alpha\beta)^{1/2}$ and (3.43) yields

$$
\begin{aligned}
&(1 + \frac{\alpha\beta}{2\gamma})\alpha_{t_{\underline{p}}} + \frac{\alpha^2}{2\gamma}\beta_{t_{\underline{p}}} = A - \alpha C,\\
&\frac{\beta^2}{2\gamma}\alpha_{t_{\underline{p}}} + (1 + \frac{\alpha\beta}{2\gamma})\beta_{t_{\underline{p}}} = B - \beta C.
\end{aligned} \tag{3.51}
$$

This linear system is uniquely solvable since its determinant equals γ^{-1} and the solution reads

$$\begin{aligned} \alpha_{t_{\underline{p}}} &= A - \frac{\alpha}{2}\big(\beta A + \alpha B + 2\gamma C\big), \\ \beta_{t_{\underline{p}}} &= B - \frac{\beta}{2}\big(\beta A + \alpha B + 2\gamma C\big). \end{aligned} \tag{3.52}$$

Using (2.6) and (2.10) it is straightforward to check that $\beta A + \alpha B + 2\gamma C = 0$ which shows that the converse assertion also holds. $\square$

The Ablowitz–Ladik Lax pair in the special defocusing case, where $\beta = \overline{\alpha}$, in the finite-dimensional context, was recently discussed by Nenciu [38].

4. Green's functions and high- and low-energy expansions

In this section we discuss the Green's function of an $\ell^2(\mathbb{Z})$-realization of the difference expression L and systematically derive high- and low-energy expansions of solutions of an associated Riccati-type equation.

Throughout this section we make the following strengthened assumptions on the coefficients α and β.

Hypothesis 4.1. *Suppose that α, β satisfy*

$$\alpha, \beta \in \ell^\infty(\mathbb{Z}), \quad \alpha(n)\beta(n) \notin \{0,1\},\ n \in \mathbb{Z}. \tag{4.1}$$

Given Hypothesis 4.1 we introduce the $\ell^2(\mathbb{Z})$-realization $\breve{L}$ of the difference expression L in (3.5) by

$$\breve{L} f = Lf, \quad f \in \operatorname{dom}\big(\breve{L}\big) = \ell^2(\mathbb{Z}), \tag{4.2}$$

and similarly introduce the $\ell^2(\mathbb{Z})$-realizations of the difference expression D, E, D^{-1}, and E^{-1} in (3.17)–(3.20) by

$$\breve{D} f = Df, \quad f \in \operatorname{dom}\big(\breve{D}\big) = \ell^2(\mathbb{Z}), \tag{4.3}$$

$$\breve{E} f = Ef, \quad f \in \operatorname{dom}\big(\breve{E}\big) = \ell^2(\mathbb{Z}), \tag{4.4}$$

$$\breve{D}^{-1} f = D^{-1}f, \quad f \in \operatorname{dom}\big(\breve{D}^{-1}\big) = \ell^2(\mathbb{Z}), \tag{4.5}$$

$$\breve{E}^{-1} f = E^{-1}f, \quad f \in \operatorname{dom}\big(\breve{E}^{-1}\big) = \ell^2(\mathbb{Z}). \tag{4.6}$$

The following elementary result shows that these $\ell^2(\mathbb{Z})$-realizations are meaningful; it will be used in the proof of Lemma 4.3 below.

Lemma 4.2. *Assume Hypothesis* 4.1. *Then the operators $\breve{D}, \breve{D}^{-1}, \breve{E}, \breve{E}^{-1}, \breve{L}$, and $\breve{L}^{-1}$ are bounded on $\ell^2(\mathbb{Z})$. In addition, $\big(\breve{L} - z\big)^{-1}$ is norm analytic with respect to z in an open neighborhood of $z = 0$, and $\big(\breve{L} - z\big)^{-1} = -z^{-1}\big(I - z^{-1}\breve{L}\big)^{-1}$ is analytic with respect to $1/z$ in an open neighborhood of $1/z = 0$.*

Proof. By Hypothesis 4.1, $\rho^2 = 1 - \alpha\beta$, and (3.17)–(3.20), one infers that $\breve{D}$, $\breve{E}$, $\breve{D}^{-1}$, $\breve{E}^{-1}$ are bounded operators on $\ell^2(\mathbb{Z})$ whose norms are bounded by

$$\begin{aligned} \big\|\breve{D}\big\|, \big\|\breve{E}\big\|, \big\|\breve{D}^{-1}\big\|, \big\|\breve{E}^{-1}\big\| &\le 2\|\rho\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty \\ &\le 2(1 + \|\alpha\|_\infty + \|\beta\|_\infty). \end{aligned} \tag{4.7}$$

Since by (3.16),

$$\breve{L} = \breve{D}\breve{E}, \quad \breve{L}^{-1} = \breve{E}^{-1}\breve{D}^{-1}, \tag{4.8}$$

the assertions of Lemma 4.2 are evident (alternatively, one can of course invoke (3.5) and (3.12)). □

To introduce the Green's function of $\breve{L}$, we need to digress a bit. Introducing the transfer matrix $T(z,\,\cdot\,)$ associated with L by

$$T(z,n) = \begin{cases} \rho(n)^{-1}\begin{pmatrix} \alpha(n) & z \\ z^{-1} & \beta(n) \end{pmatrix}, & n \text{ odd}, \\ \rho(n)^{-1}\begin{pmatrix} \beta(n) & 1 \\ 1 & \alpha(n) \end{pmatrix}, & n \text{ even}, \end{cases} \quad z \in \mathbb{C}\setminus\{0\},\ n \in \mathbb{Z}, \tag{4.9}$$

recalling that $\rho = \gamma^{1/2} = (1-\alpha\beta)^{1/2}$, one then verifies that (cf. (2.27))

$$T(z,n) = A(z,n)z^{-1/2}\rho(n)^{-1}U(z,n)A(z,n-1)^{-1}, \quad z \in \mathbb{C}\setminus\{0\},\ n \in \mathbb{Z}. \tag{4.10}$$

Here we introduced

$$A(z,n) = \begin{cases} \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}, & n \text{ odd}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & n \text{ even}, \end{cases} \quad z \in \mathbb{C}\setminus\{0\},\ n \in \mathbb{Z}. \tag{4.11}$$

Next, we consider a fundamental system of solutions

$$\Psi_\pm(z,\,\cdot\,) = \begin{pmatrix} \psi_{1,\pm}(z,\,\cdot\,) \\ \psi_{2,\pm}(z,\,\cdot\,) \end{pmatrix} \tag{4.12}$$

of

$$U(z)\Psi_\pm^-(z) = \Psi_\pm(z), \quad z \in \mathbb{C}\setminus\big(\mathrm{spec}\big(\breve{L}\big)\cup\{0\}\big), \tag{4.13}$$

with $\mathrm{spec}\big(\breve{L}\big)$ denoting the spectrum of $\breve{L}$ and U given by (2.27), such that

$$\det(\Psi_-(z),\Psi_+(z)) \neq 0. \tag{4.14}$$

The precise form of $\Psi_\pm$ will be chosen as a consequence of (4.20) below. Introducing in addition,

$$\begin{pmatrix} u_\pm(z,n) \\ v_\pm(z,n) \end{pmatrix} = C_\pm z^{-n/2}\bigg(\prod_{n'=1}^{n}\rho(n')^{-1}\bigg)A(z,n)\begin{pmatrix} \psi_{1,\pm}(z,n) \\ \psi_{2,\pm}(z,n) \end{pmatrix}, \quad z \in \mathbb{C}\setminus\big(\mathrm{spec}\big(\breve{L}\big)\cup\{0\}\big),\ n \in \mathbb{Z}, \tag{4.15}$$

for some constants $C_\pm \in \mathbb{C}\setminus\{0\}$, (4.10) and (4.15) yield

$$T(z)\begin{pmatrix} u_\pm^-(z) \\ v_\pm^-(z) \end{pmatrix} = \begin{pmatrix} u_\pm(z) \\ v_\pm(z) \end{pmatrix}. \tag{4.16}$$

Moreover, one can show (cf. [28]) that

$$Lu_\pm(z) = zu_\pm(z), \quad L^\top v_\pm(z) = zv_\pm(z), \tag{4.17}$$

$$Dv_\pm(z) = u_\pm(z), \quad Eu_\pm(z) = zv_\pm(z), \tag{4.18}$$

where

$$L = DE, \quad L^\top = ED, \tag{4.19}$$

and hence $L^\top$ represents the difference expression associated with the transpose of the infinite matrix L (cf. (3.3)) in the standard basis of $\ell^2(\mathbb{Z})$. Next, we choose $\Psi_\pm(z)$ in such a manner (cf. again [28]), such that for all $n_0 \in \mathbb{Z}$,

$$\begin{pmatrix} u_\pm(z,\,\cdot\,) \\ v_\pm(z,\,\cdot\,) \end{pmatrix} \in \ell^2([n_0, \pm\infty) \cap \mathbb{Z})^2, \quad z \in \mathbb{C} \setminus \big(\mathrm{spec}\big(\breve{L}\big) \cup \{0\}\big). \tag{4.20}$$

Since by hypothesis $z \in \mathbb{C} \setminus \mathrm{spec}\big(\breve{L}\big)$, $(u_+(z,\,\cdot\,), v_+(z,\,\cdot\,))^\top$ and $(u_-(z,\,\cdot\,), v_-(z,\,\cdot\,))^\top$ are linearly independent since otherwise z would be an eigenvalue of $\breve{L}$. This is of course consistent with (4.14) and (4.15).

The Green's function of $\breve{L}$, the $\ell^2(\mathbb{Z})$-realization of the Lax difference expression L, is then of the form

$$\begin{aligned} G(z,n,n') &= \big(\delta_n, \big(\breve{L} - z\big)^{-1} \delta_{n'}\big) \qquad\qquad (4.21) \\ &= \frac{-1}{z \det \begin{pmatrix} u_+(z,0) & u_-(z,0) \\ v_+(z,0) & v_-(z,0) \end{pmatrix}} \\ &\quad \times \begin{cases} v_-(z,n')u_+(z,n), & n' < n \text{ or } n = n' \text{ even}, \\ v_+(z,n')u_-(z,n), & n' > n \text{ or } n = n' \text{ odd}, \end{cases} \quad n, n' \in \mathbb{Z}, \\ &= -\frac{1}{4z} \frac{(1-\phi_+(z,0))(1-\phi_-(z,0))}{\phi_+(z,0) - \phi_-(z,0)} \\ &\quad \times \begin{cases} v_-(z,n')u_+(z,n), & n' < n \text{ or } n = n' \text{ even}, \\ v_+(z,n')u_-(z,n), & n' > n \text{ or } n = n' \text{ odd}, \end{cases} \quad n, n' \in \mathbb{Z}, \qquad (4.22) \\ &\qquad\qquad\qquad\qquad z \in \mathbb{C} \setminus \big(\mathrm{spec}\big(\breve{L}\big) \cup \{0\}\big). \end{aligned}$$

Introducing

$$\phi_\pm(z,n) = \frac{\psi_{2,\pm}(z,n)}{\psi_{1,\pm}(z,n)}, \quad z \in \mathbb{C} \setminus \big(\mathrm{spec}\big(\breve{L}\big) \cup \{0\}\big),\ n \in \mathbb{N}, \tag{4.23}$$

then (4.13) implies that $\phi_\pm$ satisfy the Riccati-type equation

$$\alpha \phi_\pm \phi_\pm^- - \phi_\pm^- + z\phi_\pm = z\beta, \tag{4.24}$$

and one introduces in addition,

$$\mathfrak{f} = \frac{2}{\phi_+ - \phi_-}, \tag{4.25}$$

$$\mathfrak{g} = \frac{\phi_+ + \phi_-}{\phi_+ - \phi_-}, \tag{4.26}$$

$$\mathfrak{h} = \frac{2\phi_+\phi_-}{\phi_+ - \phi_-}. \tag{4.27}$$

Using the Riccati-type equation (4.24) and its consequences,

$$\alpha(\phi_+\phi_+^- - \phi_-\phi_-^-) - (\phi_+^- - \phi_-^-) + z(\phi_+ - \phi_-) = 0, \tag{4.28}$$

$$\alpha(\phi_+\phi_+^- + \phi_-\phi_-^-) - (\phi_+^- + \phi_-^-) + z(\phi_+ + \phi_-) = 2z\beta, \tag{4.29}$$

one then derives the identities

$$z(\mathfrak{g}^- - \mathfrak{g}) + z\beta\mathfrak{f} + \alpha\mathfrak{h}^- = 0, \tag{4.30}$$

$$z\beta\mathfrak{f}^- + \alpha\mathfrak{h} - \mathfrak{g} + \mathfrak{g}^- = 0, \tag{4.31}$$

$$-\mathfrak{f} + z\mathfrak{f}^- + \alpha(\mathfrak{g} + \mathfrak{g}^-) = 0, \tag{4.32}$$

$$z\beta(\mathfrak{g}^- + \mathfrak{g}) - z\mathfrak{h} + \mathfrak{h}^- = 0, \tag{4.33}$$

$$\mathfrak{g}^2 - \mathfrak{f}\mathfrak{h} = 1. \tag{4.34}$$

For the connection between $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{h}$ and the Green's function of L one finally obtains

$$\begin{aligned}
\mathfrak{f}(z,n) &= -2\alpha(n)(zG(z,n,n)+1) - 2\rho(n)z \begin{cases} G(z,n-1,n), & n \text{ even}, \\ G(z,n,n-1), & n \text{ odd}, \end{cases} \\
\mathfrak{g}(z,n) &= -2zG(z,n,n) - 1, \\
\mathfrak{h}(z,n) &= -2\beta(n)zG(z,n,n) - 2\rho(n)z \begin{cases} G(z,n,n-1), & n \text{ even}, \\ G(z,n-1,n), & n \text{ odd}, \end{cases}
\end{aligned} \tag{4.35}$$

illustrating the spectral theoretic content of $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{h}$.

We are particularly interested in the asymptotic expansion of $\phi_\pm$ in a neighborhood of the points $z = 0$ and $1/z = 0$ and turn to this topic next.

Lemma 4.3. *Assume that α, β satisfy Hypothesis 4.1. Then $\phi_\pm$ have the following convergent expansions with respect to $1/z$ around $1/z = 0$ and with respect to z around $z = 0$,*

$$\phi_\pm(z) \underset{z\to\infty}{=} \begin{cases} \sum_{j=0}^\infty \phi_{j,+}^\infty z^{-j}, \\ \sum_{j=-1}^\infty \phi_{j,-}^\infty z^{-j}, \end{cases} \tag{4.36}$$

$$\phi_\pm(z) \underset{z\to 0}{=} \begin{cases} \sum_{j=0}^\infty \phi_{j,+}^0 z^{j}, \\ \sum_{j=1}^\infty \phi_{j,-}^0 z^{j}, \end{cases} \tag{4.37}$$

where

$$\begin{aligned}
\phi_{0,+}^\infty &= \beta, \quad \phi_{1,+}^\infty = \beta^-\gamma, \\
\phi_{j+1,+}^\infty &= (\phi_{j,+}^\infty)^- - \alpha\sum_{\ell=0}^{j} (\phi_{j-\ell,+}^\infty)^- \phi_{\ell,+}^\infty, \quad j \in \mathbb{N},
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
\phi_{-1,-}^\infty &= -\frac{1}{\alpha^+}, \quad \phi_{0,-}^\infty = \frac{\alpha^{++}}{(\alpha^+)^2}\gamma^+, \\
\phi_{j+1,-}^\infty &= -\frac{\alpha^{++}}{\alpha^+}\phi_{j,-}^\infty + \alpha^{++}\sum_{\ell=0}^{j} \phi_{j-\ell,-}^\infty (\phi_{\ell,-}^\infty)^+, \quad j \in \mathbb{N}_0,
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
\phi_{0,+}^0 &= \frac{1}{\alpha}, \quad \phi_{1,+}^0 = -\frac{\alpha^-}{\alpha^2}\gamma, \\
\phi_{j+1,+}^0 &= (\phi_{j,+}^0)^+ + \alpha^+\sum_{\ell=0}^{j+1} (\phi_{j+1-\ell,+}^0)^+ \phi_{\ell,+}^0, \quad j \in \mathbb{N},
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
\phi_{1,-}^0 &= -\beta^+, \\
\phi_{j+1,-}^0 &= (\phi_{j,-}^0)^+ + \alpha^+\sum_{\ell=1}^{j} \phi_{j+1-\ell,-}^0 (\phi_{\ell,-}^0)^+, \quad j \in \mathbb{N}.
\end{aligned} \tag{4.41}$$

Proof. Since

$$\phi_\pm = \frac{\mathfrak{g} \pm 1}{\mathfrak{f}}, \tag{4.42}$$

combining Lemma 4.2, (4.21) and (4.35) proves that $\phi_\pm$ has a convergent expansion with respect to z and $1/z$ in a neighborhood of $z = 0$ and $1/z = 0$, respectively.

The explicit expansion coefficients $\phi^\infty_{j,\pm}$ are then readily derived by making the ansatz

$$\phi_\pm \underset{z\to\infty}{=} \sum_{j=-1}^{\infty} \phi^\infty_{j,\pm} z^{-j}, \quad \phi^\infty_{-1,+} = 0. \tag{4.43}$$

Inserting (4.43) into the Riccati-type equation (4.24) one finds

$$0 = \alpha\phi_\pm\phi_\pm^- - \phi_\pm^- + z(\phi_\pm - \beta) = \big(\alpha\phi^\infty_{-1,\pm}(\phi^\infty_{-1,\pm})^- + \phi^\infty_{-1,\pm}\big)z^2 + O(z), \tag{4.44}$$

which yields the case distinction above and the formulas for $\phi^\infty_{j,\pm}$. The corresponding expansion coefficients $\phi^0_{j,\pm}$ are obtained analogously by making the ansatz $\phi_\pm \underset{z\to 0}{=} \sum_{j=0}^\infty \phi^0_{j,\pm} z^j$. □

For the record we list a few explicit expressions:

$$\begin{aligned}
\phi^\infty_{0,+} &= \beta, \\
\phi^\infty_{1,+} &= \beta^-\gamma, \\
\phi^\infty_{2,+} &= \gamma\big(-\alpha(\beta^-)^2 + \beta^{--}\gamma^-\big), \\
\phi^\infty_{-1,-} &= -\frac{1}{\alpha^+}, \\
\phi^\infty_{0,-} &= \frac{\alpha^{++}}{(\alpha^+)^2}\gamma^+, \\
\phi^\infty_{1,-} &= \frac{\gamma^+}{(\alpha^+)^3}\big(\alpha^+\alpha^{+++}\gamma^{++} - (\alpha^{++})^2\big), \\
\phi^0_{0,+} &= \frac{1}{\alpha}, \\
\phi^0_{1,+} &= -\frac{\alpha^-}{\alpha^2}\gamma, \\
\phi^0_{2,+} &= \frac{\gamma}{\alpha^3}\big((\alpha^-)^2 - \alpha^{--}\alpha\gamma^-\big), \\
\phi^0_{1,-} &= -\beta^+, \\
\phi^0_{2,-} &= -\gamma^+\beta^{++}, \\
\phi^0_{3,-} &= \gamma^+\big(\alpha^+(\beta^{++})^2 - \gamma^{++}\beta^{+++}\big), \\
\phi^0_{4,-} &= \gamma^+\Big(-(\alpha^+)^2(\beta^{++})^3 \\
&\quad + \gamma^{++}\big(2\alpha^+\beta^{++}\beta^{+++} + \alpha^{++}(\beta^{+++})^2 - \gamma^{+++}\beta^{++++}\big)\Big), \text{ etc.}
\end{aligned}$$

Later on we will also need the convergent expansions of $\ln(z + \alpha^+\phi_\pm(z))$ with respect to z and $1/z$. We will separately provide all four expansions of $\ln(z + \alpha^+\phi_\pm(z))$ around $1/z = 0$ and $z = 0$ and repeatedly use the general formula

$$\ln\left(1 + \sum_{j=1}^\infty \omega_j z^{\pm j}\right) = \sum_{j=1}^\infty \sigma_j z^{\pm j}, \tag{4.45}$$

where

$$\sigma_1 = \omega_1, \qquad \sigma_j = \omega_j - \sum_{\ell=1}^{j-1} \frac{\ell}{j} \omega_{j-\ell} \sigma_\ell, \quad j \geq 2, \tag{4.46}$$

and $|z|$ as $|z| \to 0$, respectively, $1/|z|$ as $|z| \to \infty$, are assumed to be sufficiently small in (4.45). We start by expanding ϕ_+ around $1/z = 0$

$$\begin{aligned}
\ln(z + \alpha^+ \phi_+(z)) &= \ln\left(z + \alpha^+ \sum_{j=0}^{\infty} \phi_{j,+}^{\infty} z^{-j}\right) \\
&= \ln\left(1 + \alpha^+ \sum_{j=0}^{\infty} \phi_{j,+}^{\infty} z^{-j-1}\right) \\
&= \ln\left(1 + \alpha^+ \sum_{j=1}^{\infty} \phi_{j-1,+}^{\infty} z^{-j}\right) \\
&= \sum_{j=1}^{\infty} \rho_{j,+}^{\infty} z^{-j},
\end{aligned} \tag{4.47}$$

where

$$\rho_{1,+}^{\infty} = \alpha^+ \phi_{0,+}^{\infty}, \quad \rho_{j,+}^{\infty} = \alpha^+ \left(\phi_{j-1,+}^{\infty} - \sum_{\ell=1}^{j-1} \frac{\ell}{j} \phi_{j-1-\ell,+}^{\infty} \rho_{\ell,+}^{\infty}\right), \quad j \geq 2. \tag{4.48}$$

An expansion of ϕ_- around $1/z = 0$ yields

$$\begin{aligned}
\ln(z + \alpha^+ \phi_-(z)) &= \ln\left(z + \alpha^+ \sum_{j=-1}^{\infty} \phi_{j,-}^{\infty} z^{-j}\right) \\
&= \ln\left(\frac{\alpha^{++}\gamma^+}{\alpha^+}\right) + \ln\left(1 + \frac{(\alpha^+)^2}{\alpha^{++}\gamma^+} \sum_{j=1}^{\infty} \phi_{j,-}^{\infty} z^{-j}\right) \\
&= \ln\left(\frac{\alpha^{++}}{\alpha^+}\right) + \ln(\gamma^+) + \ln\left(1 + \frac{(\alpha^+)^2}{\alpha^{++}\gamma^+} \sum_{j=1}^{\infty} \phi_{j,-}^{\infty} z^{-j}\right) \\
&= \ln\left(\frac{\alpha^{++}}{\alpha^+}\right) + \ln(\gamma^+) + \sum_{j=1}^{\infty} \rho_{j,-}^{\infty} z^{-j},
\end{aligned} \tag{4.49}$$

where

$$\rho_{1,-}^{\infty} = \frac{(\alpha^+)^2}{\alpha^{++}\gamma^+} \phi_{1,-}^{\infty}, \quad \rho_{j,-}^{\infty} = \frac{(\alpha^+)^2}{\alpha^{++}\gamma^+} \left(\phi_{j,-}^{\infty} - \sum_{\ell=1}^{j-1} \frac{\ell}{j} \phi_{j-\ell,-}^{\infty} \rho_{\ell,-}^{\infty}\right), \quad j \geq 2. \tag{4.50}$$

For the expansion of ϕ_+ around $z = 0$ one gets

$$\begin{aligned}
\ln(z + \alpha^+ \phi_+(z)) &= \ln\left(z + \alpha^+ \sum_{j=0}^{\infty} \phi_{j,+}^{0} z^{j}\right) \\
&= \ln\left(\frac{\alpha^+}{\alpha}\right) + \ln\left(1 + \frac{\alpha}{\alpha^+}(1 + \alpha^+ \phi_{1,+}^{0}) z + \alpha \sum_{j=2}^{\infty} \phi_{j,+}^{0} z^{j}\right) \\
&= \ln\left(\frac{\alpha^+}{\alpha}\right) + \sum_{j=1}^{\infty} \rho_{j,+}^{0} z^{j},
\end{aligned} \tag{4.51}$$

where

$$\rho_{1,+}^0 = \frac{\alpha}{\alpha^+}\big(1+\alpha^+\phi_{1,+}^0\big), \quad \rho_{2,+}^0 = \alpha\phi_{2,+}^0 - \frac{1}{2}(\rho_{1,+}^0)^2,$$
$$\rho_{j,+}^0 = \alpha\bigg(\phi_{j,+}^0 - \sum_{j=1}^{j-2}\frac{\ell}{j}\phi_{j-\ell,+}^0\rho_{\ell,+}^0\bigg) - \frac{j-1}{j}\rho_{1,+}^0\rho_{j-1,+}^0, \quad j\geq 3. \tag{4.52}$$

Finally, the expansion of ϕ_- around $z=0$ is given by

$$\begin{aligned}\ln(z+\alpha^+\phi_-(z)) &= \ln\bigg(z+\alpha^+\sum_{j=1}^\infty \phi_{j,-}^0 z^j\bigg)\\ &= \ln\bigg(\gamma^+ z+\alpha^+\sum_{j=2}^\infty \phi_{j,-}^0 z^j\bigg)\\ &= \ln(z)+\ln(\gamma^+)+\ln\bigg(1+\frac{\alpha^+}{\gamma^+}\sum_{j=1}^\infty \phi_{j+1,-}^0 z^j\bigg)\\ &= \ln(z)+\ln(\gamma^+)+\sum_{j=1}^\infty \rho_{j,-}^0 z^j, \end{aligned} \tag{4.53}$$

where

$$\rho_{1,-}^0 = \frac{\alpha^+}{\gamma^+}\phi_{2,-}^0, \quad \rho_{j,-}^0 = \frac{\alpha^+}{\gamma^+}\bigg(\phi_{j+1,-}^0 - \sum_{\ell=1}^{j-1}\frac{\ell}{j}\phi_{j+1-\ell,-}^0\rho_{\ell,-}^0\bigg), \quad j\geq 2. \tag{4.54}$$

Explicitly, the first expansion coefficients are given by

$$\begin{aligned}
\rho_{1,+}^\infty &= \alpha^+\beta,\\
\rho_{2,+}^\infty &= -\tfrac{1}{2}(\alpha^+\beta)^2 + \gamma\alpha^+\beta^-,\\
\rho_{3,+}^\infty &= \tfrac{1}{3}(\alpha^+\beta)^3 - \gamma\big(\gamma^-\alpha^+\beta^{--} - (\alpha^+)^2\beta^-\beta - \alpha\alpha^+(\beta^-)^2\big),\\
\rho_{1,-}^\infty &= -\alpha^{+++}\beta^{++} + (S^+-I)\frac{\alpha^{++}}{\alpha^+},\\
\rho_{1,+}^0 &= \alpha^-\beta + (S^+-I)\frac{\alpha^-}{\alpha},\\
\rho_{1,-}^0 &= -\alpha^+\beta^{++},\\
\rho_{2,-}^0 &= \tfrac{1}{2}(\alpha^+\beta^{++})^2 - \gamma^{++}\alpha^+\beta^{+++},\\
\rho_{3,-}^0 &= -\tfrac{1}{3}(\alpha^+\beta^{++})^3\\
&\quad + \gamma^{++}\big(-\gamma^{+++}\alpha^+\beta^{++++} + (\alpha^+)^2\beta^{++}\beta^{+++} + \alpha^+\alpha^{++}(\beta^{+++})^2\big), \text{ etc.}
\end{aligned} \tag{4.55}$$

The next result shows that $\hat{g}_{j,+}$ and $\pm j\rho_{j,\pm}^\infty$, respectively $\hat{g}_{j,-}$ and $\pm j\rho_{j,\pm}^0$, are equal up to terms that are total differences, that is, are of the form $(S^+-I)d_{j,\pm}$ for some sequence $d_{j,\pm}$. The exact form of $d_{j,\pm}$ will not be needed later. In the proof we will heavily use the equations (4.30)–(4.34).

Lemma 4.4. *Suppose Hypothesis* 4.1 *holds. Then*

$$\hat{g}_{j,+} = -j\rho_{j,+}^\infty + (S^+-I)d_{j,+} = j\rho_{j,-}^\infty + (S^+-I)e_{j,+}, \quad j\in\mathbb{N}, \tag{4.56}$$
$$\hat{g}_{j,-} = -j\rho_{j,+}^0 + (S^+-I)d_{j,-} = j\rho_{j,-}^0 + (S^+-I)e_{j,-}, \quad j\in\mathbb{N}, \tag{4.57}$$

for some polynomials $d_{j,\pm}$, $e_{j,\pm}$, $j \in \mathbb{N}$, *in* α *and* β *and certain shifts thereof.*

Proof. We consider the case for $\hat{g}_{j,+}$ first. Our aim is to show that

$$\frac{d}{dz}\ln(z+\alpha^+\phi_+) = -\frac{1}{2z}\mathfrak{g} + \frac{1}{2z} + (S^+-I)K + (S^+-I)M, \tag{4.58}$$

where

$$K = \frac{1}{2}\bigg(\mathfrak{g}\frac{\dot{\mathfrak{f}}}{\mathfrak{f}} - \dot{\mathfrak{g}}\bigg), \quad M = \frac{1}{2}\frac{\dot{\mathfrak{f}}}{\mathfrak{f}}, \tag{4.59}$$

which implies (4.56) by (4.47). Here $\cdot$ denotes d/dz.

Since $\phi_+ = (\mathfrak{g}+1)/\mathfrak{f}$,

$$\frac{d}{dz}\ln(z+\alpha^+\phi_+) = \frac{1+\alpha^+\dot{\phi}_+}{z+\alpha^+\phi_+} = \frac{\mathfrak{f}^2+\alpha^+(\mathfrak{f}\dot{\mathfrak{g}}-\dot{\mathfrak{f}}\mathfrak{g}-\dot{\mathfrak{f}})}{\mathfrak{f}(z\mathfrak{f}+\alpha^+\mathfrak{g}+\alpha^+)}\,\frac{z\mathfrak{f}+\alpha^+\mathfrak{g}-\alpha^+}{z\mathfrak{f}+\alpha^+\mathfrak{g}-\alpha^+}. \tag{4.60}$$

Next we treat the denominator of (4.60) using (4.30), (4.32),

$$\begin{aligned}
\mathfrak{f}\big((z\mathfrak{f}+\alpha^+\mathfrak{g})^2-(\alpha^+)^2\big) &= \mathfrak{f}\big((z\mathfrak{f}+\alpha^+\mathfrak{g})^2-(\alpha^+)^2(\mathfrak{g}^2-\mathfrak{f}\mathfrak{h})\big)\\
&= z\mathfrak{f}^2\Big(z\mathfrak{f}+\alpha^+\mathfrak{g}+\alpha^+\mathfrak{g}+(\alpha^+)^2\frac{1}{z}\mathfrak{h}\Big)\\
&= z\mathfrak{f}^2\big(\mathfrak{f}^+-\alpha^+\mathfrak{g}^++\alpha^+\mathfrak{g}^+-\alpha^+\beta^+\mathfrak{f}^+\big)\\
&= z\gamma^+\mathfrak{f}^2\mathfrak{f}^+.
\end{aligned} \tag{4.61}$$

Expanding the numerator in (4.60) and applying (4.30), (4.32), and their derivatives with respect to z as well as $2\mathfrak{g}\dot{\mathfrak{g}} = \dot{\mathfrak{f}}\mathfrak{h}+\mathfrak{f}\dot{\mathfrak{h}}$ yields

$$\begin{aligned}
&\big(z\mathfrak{f}+\alpha^+\mathfrak{g}-\alpha^+\big)\big(\mathfrak{f}^2+\alpha^+(\mathfrak{f}\dot{\mathfrak{g}}-\dot{\mathfrak{f}}\mathfrak{g}-\dot{\mathfrak{f}})\big)\\
&= \mathfrak{f}\big(z\mathfrak{f}^2+z\alpha^+(\mathfrak{f}\dot{\mathfrak{g}}-\dot{\mathfrak{f}}\mathfrak{g})+\alpha^+\mathfrak{f}\mathfrak{g}+\tfrac{1}{2}(\alpha^+)^2(\mathfrak{f}\dot{\mathfrak{h}}-\dot{\mathfrak{f}}\mathfrak{h})-\alpha^+(\mathfrak{f}+z\dot{\mathfrak{f}}+\alpha^+\dot{\mathfrak{g}})\big)\\
&= \frac{\mathfrak{f}}{2}\Big(2z\mathfrak{f}^2+z\alpha^+\mathfrak{f}\dot{\mathfrak{g}}+z\mathfrak{f}(-\alpha^+\dot{\mathfrak{g}}^+-z\dot{\mathfrak{f}}-\mathfrak{f}+\dot{\mathfrak{f}}^+)-z\alpha^+\dot{\mathfrak{f}}\mathfrak{g}+z\dot{\mathfrak{f}}(\alpha^+\mathfrak{g}^++z\mathfrak{f}-\mathfrak{f}^+)\\
&\quad +\alpha^+\mathfrak{f}\mathfrak{g}+\mathfrak{f}(-\alpha^+\mathfrak{g}^+-z\mathfrak{f}+\mathfrak{f}^+)+\alpha^+\mathfrak{f}(-\beta^+\mathfrak{f}^+-z\beta^+\dot{\mathfrak{f}}^+-\mathfrak{g}+\mathfrak{g}^+-z\dot{\mathfrak{g}}+z\dot{\mathfrak{g}}^+)\\
&\quad +\alpha^+\dot{\mathfrak{f}}(z\beta^+\mathfrak{f}^++z\mathfrak{g}-z\mathfrak{g}^+)-2\alpha^+(\mathfrak{f}+z\dot{\mathfrak{f}}+\alpha^+\dot{\mathfrak{g}})\Big)\\
&= \frac{\mathfrak{f}}{2}\Big(\gamma^+\mathfrak{f}\mathfrak{f}^++z\gamma^+\mathfrak{f}\dot{\mathfrak{f}}^+-z\gamma^+\dot{\mathfrak{f}}\mathfrak{f}^+-2\alpha^+(\mathfrak{f}+z\dot{\mathfrak{f}}+\alpha^+\dot{\mathfrak{g}})\Big).
\end{aligned} \tag{4.62}$$

In summary,

$$\frac{d}{dz}\ln(z+\alpha^+\phi_+) = \frac{1}{2z}+(S^+-I)M-\frac{\alpha^+}{z\gamma^+\mathfrak{f}\mathfrak{f}^+}\big(\mathfrak{f}+z\dot{\mathfrak{f}}+\alpha^+\dot{\mathfrak{g}}\big). \tag{4.63}$$

We multiply the numerator on the right-hand side by $-2 = -2(\mathfrak{g}^2-\mathfrak{f}\mathfrak{h})$ and use again (4.30), (4.32), and their derivatives:

$$\begin{aligned}
&2\alpha^+\big(\mathfrak{f}\mathfrak{h}-\mathfrak{g}^2\big)\big(\mathfrak{f}+z\dot{\mathfrak{f}}+\alpha^+\dot{\mathfrak{g}}\big)\\
&= 2\alpha^+\mathfrak{f}\mathfrak{h}(\mathfrak{f}+z\dot{\mathfrak{f}}+\alpha^+\dot{\mathfrak{g}})-2\alpha^+\mathfrak{f}\mathfrak{g}^2-2z\alpha^+\dot{\mathfrak{f}}\mathfrak{g}^2-(\alpha^+)^2\mathfrak{g}(\dot{\mathfrak{f}}\mathfrak{h}+\mathfrak{f}\dot{\mathfrak{h}})\\
&= \alpha^+\mathfrak{f}\mathfrak{h}(\mathfrak{f}+z\dot{\mathfrak{f}})+\alpha^+\mathfrak{f}\dot{\mathfrak{g}}(-z\beta^+\mathfrak{f}^+-z\mathfrak{g}+z\mathfrak{g}^+)\\
&\quad +\mathfrak{f}(-z\beta^+\mathfrak{f}^+-z\mathfrak{g}+z\mathfrak{g}^+)(\dot{\mathfrak{f}}^+-\alpha^+\dot{\mathfrak{g}}^+)\\
&\quad -\alpha^+\mathfrak{f}\mathfrak{g}^2+\mathfrak{f}\mathfrak{g}(\alpha^+\mathfrak{g}^++z\mathfrak{f}-\mathfrak{f}^+)-z\alpha^+\dot{\mathfrak{f}}\mathfrak{g}^2+z\dot{\mathfrak{f}}\mathfrak{g}(\alpha^+\mathfrak{g}^++z\mathfrak{f}-\mathfrak{f}^+)\\
&\quad +\alpha^+\dot{\mathfrak{f}}\mathfrak{g}(z\beta^+\mathfrak{f}^++z\mathfrak{g}-z\mathfrak{g}^+)+\alpha^+\mathfrak{f}\mathfrak{g}(\beta^+\mathfrak{f}^++z\beta^+\dot{\mathfrak{f}}^++\mathfrak{g}-\mathfrak{g}^++z\dot{\mathfrak{g}}-z\dot{\mathfrak{g}}^+)\\
&= \alpha^+\mathfrak{f}\mathfrak{h}(\mathfrak{f}+z\dot{\mathfrak{f}})+\alpha^+\mathfrak{f}\dot{\mathfrak{g}}(-z\beta^+\mathfrak{f}^+-z\mathfrak{g}+z\mathfrak{g}^+)+\mathfrak{f}\dot{\mathfrak{f}}^+(-z\beta^+\mathfrak{f}^+-z\mathfrak{g}+z\mathfrak{g}^+)
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha^+ \mathfrak{f}\dot{\mathfrak{g}}^+(z\beta^+\mathfrak{f}^+ + z\mathfrak{g} - z\mathfrak{g}^+) + z\mathfrak{f}^2\mathfrak{g} - \gamma^+\mathfrak{f}\mathfrak{f}^+\mathfrak{g} + z^2\dot{\mathfrak{f}}\mathfrak{f}\mathfrak{g} \\
&- z\gamma^+\dot{\mathfrak{f}}\mathfrak{f}^+\mathfrak{g} + z\beta^+\mathfrak{f}\dot{\mathfrak{f}}^+(-\alpha^+\mathfrak{g}^+ - z\mathfrak{f} + \mathfrak{f}^+) + z\alpha^+\mathfrak{f}\mathfrak{g}(\dot{\mathfrak{g}} - \dot{\mathfrak{g}}^+) \\
={}& \alpha^+\mathfrak{f}\mathfrak{h}(\mathfrak{f} + z\dot{\mathfrak{f}}) - z\alpha^+\beta^+\mathfrak{f}\mathfrak{f}^+(\dot{\mathfrak{g}} - \dot{\mathfrak{g}}^+) - z\mathfrak{f}(\alpha^+\mathfrak{g} + z\mathfrak{f} - \mathfrak{f}^+)(\dot{\mathfrak{g}} - \dot{\mathfrak{g}}^+) \\
&- z\mathfrak{f}\dot{\mathfrak{f}}^+\mathfrak{g} + z\dot{\mathfrak{f}}^2\mathfrak{g} - \gamma^+\mathfrak{f}\mathfrak{f}^+\mathfrak{g} + z^2\dot{\mathfrak{f}}\mathfrak{f}\mathfrak{g} - z\gamma^+\dot{\mathfrak{f}}\mathfrak{f}^+\mathfrak{g} + z\gamma^+\mathfrak{f}\dot{\mathfrak{f}}^+\mathfrak{g}^+ - z^2\beta^+\mathfrak{f}^2\dot{\mathfrak{f}}^+ \\
={}& z\alpha^+\mathfrak{f}\dot{\mathfrak{f}}\mathfrak{h} - z\mathfrak{f}^2(z\beta^+\dot{\mathfrak{f}}^+ + z\dot{\mathfrak{g}} - z\dot{\mathfrak{g}}^+ - \tfrac{1}{z}\alpha^+\mathfrak{h}) + z\mathfrak{f}\mathfrak{g}(\mathfrak{f} + z\dot{\mathfrak{f}} - \alpha^+\dot{\mathfrak{g}} - \dot{\mathfrak{f}}^+ + \alpha^+\dot{\mathfrak{g}}^+) \\
&+ z\gamma^+\mathfrak{f}\mathfrak{f}^+(\dot{\mathfrak{g}} - \dot{\mathfrak{g}}^+) - \gamma^+\mathfrak{f}\mathfrak{f}^+\mathfrak{g} - z\gamma^+\dot{\mathfrak{f}}\mathfrak{f}^+\mathfrak{g} + z\gamma^+\mathfrak{f}\dot{\mathfrak{f}}^+\mathfrak{g}^+ \\
={}& z\alpha^+\mathfrak{f}(\dot{\mathfrak{f}}\mathfrak{h} + \mathfrak{f}\dot{\mathfrak{h}} - 2\mathfrak{g}\dot{\mathfrak{g}}) + z\gamma^+\mathfrak{f}\mathfrak{f}^+(\dot{\mathfrak{g}} - \dot{\mathfrak{g}}^+) - \gamma^+\mathfrak{f}\mathfrak{f}^+\mathfrak{g} - z\gamma^+\dot{\mathfrak{f}}\mathfrak{f}^+\mathfrak{g} + z\gamma^+\mathfrak{f}\dot{\mathfrak{f}}^+\mathfrak{g}^+ \\
={}& z\gamma^+\mathfrak{f}\mathfrak{f}^+(\dot{\mathfrak{g}} - \dot{\mathfrak{g}}^+) - \gamma^+\mathfrak{f}\mathfrak{f}^+\mathfrak{g} - z\gamma^+\dot{\mathfrak{f}}\mathfrak{f}^+\mathfrak{g} + z\gamma^+\mathfrak{f}\dot{\mathfrak{f}}^+\mathfrak{g}^+.
\end{aligned} \tag{4.64}
$$

Inserting this in (4.63) finally yields (4.58). The result for $\hat{g}_{j,-}$ is derived similarly starting from $\phi_- = (\mathfrak{g} - 1)/\mathfrak{f}$. □

5. Local conservation laws

Throughout this section (and with the only exception of Theorem 5.5) we make the following assumption:

Hypothesis 5.1. *Suppose that* $\alpha, \beta \colon \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ *satisfy*

$$
\begin{aligned}
&\sup_{(n,t_{\underline{p}})\in\mathbb{Z}\times\mathbb{R}} \big(|\alpha(n,t_{\underline{p}})| + |\beta(n,t_{\underline{p}})|\big) < \infty, \\
&a(n,\,\cdot\,),\, b(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \quad \alpha(n,t_{\underline{p}})\beta(n,t_{\underline{p}}) \notin \{0,1\},\ (n,t_{\underline{p}}) \in \mathbb{Z}\times\mathbb{R}.
\end{aligned} \tag{5.1}
$$

In accordance with the notation introduced in (4.2)–(4.6) we denote the bounded difference operator defined on $\ell^2(\mathbb{Z})$, generated by the finite difference expression $P_{\underline{p}}$ in (3.26), by the symbol $\breve{P}_{\underline{p}}$. Similarly, the bounded finite difference operator in $\ell^2(\mathbb{Z})$ generated by $P_{\underline{p}}^{\top}$ in (3.33) is then denoted by $\breve{P}_{\underline{p}}^{\top}$.

We start with the following existence result.

Theorem 5.2. *Assume Hypothesis* 5.1 *and suppose* α, β *satisfy* $\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$ *for some* $\underline{p} \in \mathbb{N}_0^2$. *In addition, let* $t_{\underline{p}} \in \mathbb{R}$ *and* $z \in \mathbb{C} \setminus \big(\mathrm{spec}\big(\breve{L}(t_{\underline{p}})\big) \cup \{0\}\big)$. *Then there exist Weyl–Titchmarsh-type solutions* $u_\pm = u_\pm(z,n,t_{\underline{p}})$ *and* $v_\pm = v_\pm(z,n,t_{\underline{p}})$ *such that for all* $n_0 \in \mathbb{Z}$,

$$
\begin{pmatrix} u_\pm(z,\,\cdot\,,t_{\underline{p}}) \\ v_\pm(z,\,\cdot\,,t_{\underline{p}}) \end{pmatrix} \in \ell^2([n_0,\pm\infty)\cap\mathbb{Z})^2, \quad u_\pm(z,n,\,\cdot\,), v_\pm(z,n,\,\cdot\,) \in C^1(\mathbb{R}), \tag{5.2}
$$

and $u_\pm$ *and* $v_\pm$ *simultaneously satisfy the following equations in the weak sense*

$$
\breve{L}(t_{\underline{p}})u_\pm(z,\,\cdot\,,t_{\underline{p}}) = zu_\pm(z,\,\cdot\,,t_{\underline{p}}), \tag{5.3}
$$

$$
u_{\pm,t_{\underline{p}}}(z,\,\cdot\,,t_{\underline{p}}) = \breve{P}_{\underline{p}}(t_{\underline{p}})u_\pm(z,\,\cdot\,,t_{\underline{p}}), \tag{5.4}
$$

and

$$
\breve{L}^{\top}(t_{\underline{p}})v_\pm(z,\,\cdot\,,t_{\underline{p}}) = zv_\pm(z,\,\cdot\,,t_{\underline{p}}), \tag{5.5}
$$

$$
v_{\pm,t_{\underline{p}}}(z,\,\cdot\,,t_{\underline{p}}) = -\breve{P}_{\underline{p}}^{\top}(t_{\underline{p}})v_\pm(z,\,\cdot\,,t_{\underline{p}}), \tag{5.6}
$$

respectively.

Proof. Applying $\big(\breve{L}(t_{\underline{p}}) - zI\big)^{-1}$ to δ_0 (cf. (4.21)) yields the existence of Weyl–Titchmarsh-type solutions $\tilde{u}_\pm$ of $Lu = zu$ satisfying (5.2). Next, using the Lax commutator equation (3.38) one computes

$$\begin{aligned} z\tilde{u}_{\pm,t_{\underline{p}}} &= (L\tilde{u}_\pm)_{t_{\underline{p}}} = L_{t_{\underline{p}}}\tilde{u}_\pm + L\tilde{u}_{\pm,t_{\underline{p}}} = [P_{\underline{p}}, L]\tilde{u}_\pm + L\tilde{u}_{\pm,t_{\underline{p}}} \\ &= zP_{\underline{p}}\tilde{u}_\pm - LP_{\underline{p}}\tilde{u}_\pm + L\tilde{u}_{\pm,t_{\underline{p}}} \end{aligned} \tag{5.7}$$

and hence

$$(L - zI)(\tilde{u}_{\pm,t_{\underline{p}}} - P_{\underline{p}}\tilde{u}_\pm) = 0. \tag{5.8}$$

Thus, $\tilde{u}_\pm$ satisfy

$$\tilde{u}_{\pm,t_{\underline{p}}} - P_{\underline{p}}\tilde{u}_\pm = C_\pm\tilde{u}_\pm + D_\pm\tilde{u}_\mp. \tag{5.9}$$

Introducing $\tilde{u}_\pm = c_\pm u_\pm$, and choosing $c_\pm$ such that $c_{\pm,t_{\underline{p}}} = C_\pm c_\pm$, one obtains

$$u_{\pm,t_{\underline{p}}} - P_{\underline{p}}u_\pm = D_\pm u_\mp. \tag{5.10}$$

Since $u_\pm \in \ell^2([n_0, \pm\infty) \cap \mathbb{Z})$, $n_0 \in \mathbb{Z}$, and α, β satisfy Hypothesis 5.1, (3.29) shows that $P_{\underline{p}}u_\pm \in \ell^2([n_0, \pm\infty) \cap \mathbb{Z})$. Moreover, since

$$u_\pm(z, n, t_{\underline{p}}) = d_\pm(t_{\underline{p}})(\breve{L}(t_{\underline{p}}) - zI)^{-1}\delta_0)(n) \tag{5.11}$$

for $n \in [\pm 1, \infty) \cap \mathbb{Z}$ and some $d_\pm \in C^1(\mathbb{R})$, the calculation

$$u_{\pm,t_{\underline{p}}} = d_{\pm,t_{\underline{p}}}\big(\breve{L} - zI\big)^{-1}\delta_0 - d_\pm\big(\breve{L} - zI\big)^{-1}\breve{L}_{t_{\underline{p}}}\big(\breve{L} - zI\big)^{-1}\delta_0 \tag{5.12}$$

also yields $u_{\pm,t_{\underline{p}}} \in \ell^2([n_0, \pm\infty) \cap \mathbb{Z})$. But then $D_\pm = 0$ in (5.10) since $u_\mp \notin \ell^2([n_0, \pm\infty) \cap \mathbb{Z})$. This proves (5.4).

Equations (5.2), (5.5), and (5.6) for $v_\pm$ are proved similarly replacing $L, P_{\underline{p}}$ by $L^\top, P_{\underline{p}}^\top$ and observing that (3.38) implies

$$L^\top_{t_{\underline{p}}}(t_{\underline{p}}) + \big[P_{\underline{p}}^\top(t_{\underline{p}}), L^\top(t_{\underline{p}})\big] = 0, \quad t_{\underline{p}} \in \mathbb{R}. \tag{5.13}$$

□

For the remainder of this section we will always refer to the Weyl–Titchmarsh solutions $u_\pm$, $v_\pm$ introduced in Theorem 5.2. Given $u_\pm$, $v_\pm$, we now introduce

$$\Psi_\pm(z, \cdot\,, t_{\underline{p}}) = \begin{pmatrix} \psi_{1,\pm}(z, \cdot\,, t_{\underline{p}}) \\ \psi_{2,\pm}(z, \cdot\,, t_{\underline{p}}) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \big(\mathrm{spec}\big(\breve{L}(t_{\underline{p}})\big) \cup \{0\}\big),\ t_{\underline{p}} \in \mathbb{R}, \tag{5.14}$$

by (cf. (4.15))

$$\begin{aligned} \begin{pmatrix} \psi_{1,\pm}(z, n, t_{\underline{p}}) \\ \psi_{2,\pm}(z, n, t_{\underline{p}}) \end{pmatrix} &= D(t_{\underline{p}})z^{n/2}\bigg(\prod_{n'=1}^{n} \rho(n', t_{\underline{p}})\bigg)A(z, n)^{-1}\begin{pmatrix} u_\pm(z, n, t_{\underline{p}}) \\ v_\pm(z, n, t_{\underline{p}}) \end{pmatrix}, \\ &\qquad z \in \mathbb{C} \setminus \big(\mathrm{spec}\big(\breve{L}(t_{\underline{p}})\big) \cup \{0\}\big),\ (n, t_{\underline{p}}) \in \mathbb{Z} \times \mathbb{R}, \end{aligned} \tag{5.15}$$

with the choice of normalization

$$D(t_{\underline{p}}) = \exp\bigg(\frac{i}{2}\int_0^{t_{\underline{p}}} ds\,\big(g_{p_+,+}(0, s) - g_{p_-,-}(0, s)\big)\bigg)D(0), \quad t_{\underline{p}} \in \mathbb{R}, \tag{5.16}$$

for some constant $D(0) \in \mathbb{C} \setminus \{0\}$.

Lemma 5.3. *Assume Hypothesis* 5.1 *and suppose* α, β *satisfy* $\mathrm{AL}_{\underline{p}}(\alpha,\beta)=0$ *for some* $\underline{p}\in\mathbb{N}_0^2$. *In addition, let* $t_{\underline{p}}\in\mathbb{R}$ *and* $z\in\mathbb{C}\setminus\big(\mathrm{spec}\big(\breve{L}(t_{\underline{p}})\big)\cup\{0\}\big)$. *Then* $\Psi_\pm(z,\,\cdot\,,t_{\underline{p}})$ *defined in* (5.15) *satisfy*

$$U(z,\,\cdot\,,t_{\underline{p}})\Psi_\pm^-(z,\,\cdot\,,t_{\underline{p}})=\Psi_\pm(z,\,\cdot\,,t_{\underline{p}}), \tag{5.17}$$

$$\Psi_{\pm,t_{\underline{p}}}(z,\,\cdot\,,t_{\underline{p}})=V_{\underline{p}}^+(z,\,\cdot\,,t_{\underline{p}})\Psi_\pm(z,\,\cdot\,,t_{\underline{p}}). \tag{5.18}$$

In addition, $\Psi_-(z,\,\cdot\,,t_{\underline{p}})$ *and* $\Psi_+(z,\,\cdot\,,t_{\underline{p}})$ *are linearly independent.*

Proof. Equation (5.17) is equivalent to

$$\begin{pmatrix}\psi_{1,\pm}\\ \psi_{2,\pm}\end{pmatrix}=\begin{pmatrix}z\psi_{1,\pm}^-+\alpha\psi_{2,\pm}^-\\ z\beta\psi_{1,\pm}^-+\psi_{2,\pm}^-\end{pmatrix}. \tag{5.19}$$

Using (4.11) and (5.15) one obtains

$$\begin{pmatrix}\psi_{1,\pm}\\ \psi_{2,\pm}\end{pmatrix}=Dz^{n/2}\bigg(\prod_{n'=1}^{n}\rho(n')\bigg)\begin{cases}\begin{pmatrix}z^{-1/2}u_\pm\\ z^{1/2}v_\pm\end{pmatrix}, & n \text{ odd},\\[2mm] \begin{pmatrix}v_\pm\\ u_\pm\end{pmatrix}, & n\text{ even}.\end{cases} \tag{5.20}$$

Inserting (5.20) into (5.19), one finds that (5.19) is equivalent to (4.16), thereby proving (5.17).

Equation (5.18) is equivalent to

$$\begin{pmatrix}\psi_{1,\pm,t_{\underline{p}}}\\ \psi_{2,\pm,t_{\underline{p}}}\end{pmatrix}=i\begin{pmatrix}G_{\underline{p}}\psi_{1,\pm}-F_{\underline{p}}\psi_{2,\pm}\\ H_{\underline{p}}\psi_{1,\pm}-K_{\underline{p}}\psi_{2,\pm}\end{pmatrix}. \tag{5.21}$$

We first consider the case when n is odd. Using (5.20), the right-hand side of (5.21) reads

$$i\begin{pmatrix}G_{\underline{p}}\psi_{1,\pm}-F_{\underline{p}}\psi_{2,\pm}\\ H_{\underline{p}}\psi_{1,\pm}-K_{\underline{p}}\psi_{2,\pm}\end{pmatrix}=iDz^{(n-1)/2}\bigg(\prod_{n'=1}^{n}\rho(n')\bigg)\begin{pmatrix}G_{\underline{p}}u_\pm-zF_{\underline{p}}v_\pm\\ H_{\underline{p}}u_\pm-zK_{\underline{p}}v_\pm\end{pmatrix}. \tag{5.22}$$

Equation (5.20) then implies

$$\begin{aligned}\begin{pmatrix}\psi_{1,\pm,t_{\underline{p}}}\\ \psi_{2,\pm,t_{\underline{p}}}\end{pmatrix}&=D_{t_{\underline{p}}}z^{n/2}\bigg(\prod_{n'=1}^{n}\rho(n')\bigg)\begin{pmatrix}z^{-1/2}u_\pm\\ z^{1/2}v_\pm\end{pmatrix}\\ &\quad+Dz^{n/2}\bigg(\prod_{n'=1}^{n}\rho(n')\bigg)\begin{pmatrix}z^{-1/2}u_{\pm,t_{\underline{p}}}\\ z^{1/2}v_{\pm,t_{\underline{p}}}\end{pmatrix}+Dz^{n/2}\bigg(\partial_{t_{\underline{p}}}\prod_{n'=1}^{n}\rho(n')\bigg)\begin{pmatrix}z^{-1/2}u_\pm\\ z^{1/2}v_\pm\end{pmatrix}.\end{aligned} \tag{5.23}$$

Next, one observes that

$$\begin{aligned}&\bigg(\partial_{t_{\underline{p}}}\prod_{n'=1}^{n}\rho(n')\bigg)\bigg(\prod_{n'=1}^{n}\rho(n')\bigg)^{-1}=\partial_{t_{\underline{p}}}\ln\bigg(\prod_{n'=1}^{n}\rho(n')\bigg)=\frac{1}{2}\partial_{t_{\underline{p}}}\ln\bigg(\prod_{n'=1}^{n}\rho(n')^2\bigg)\\ &\quad=\frac{1}{2}\partial_{t_{\underline{p}}}\ln\bigg(\prod_{n'=1}^{n}\gamma(n')\bigg)=\frac{1}{2}\sum_{n'=1}^{n}\frac{\gamma_{t_{\underline{p}}}(n')}{\gamma(n')}.\end{aligned} \tag{5.24}$$

Thus, (5.23) reads

$$D^{-1}z^{-(n-1)/2}\bigg(\prod_{n'=1}^{n}\rho(n')\bigg)^{-1}\begin{pmatrix}\psi_{1,\pm,t_{\underline{p}}}\\ \psi_{2,\pm,t_{\underline{p}}}\end{pmatrix} \tag{5.25}$$

$$= \begin{pmatrix} u_{\pm,t_{\underline{p}}} \\ zv_{\pm,t_{\underline{p}}} \end{pmatrix} + \frac{1}{2}\bigg(\sum_{n'=1}^{n} \frac{\gamma_{t_{\underline{p}}}(n')}{\gamma(n')}\bigg) \begin{pmatrix} u_{\pm} \\ zv_{\pm} \end{pmatrix} + \frac{D_{t_{\underline{p}}}}{D} \begin{pmatrix} u_{\pm} \\ zv_{\pm} \end{pmatrix}. \tag{5.26}$$

Combining (5.22) and (5.26) one finds that (5.21) is equivalent to

$$\begin{pmatrix} u_{\pm,t_{\underline{p}}} \\ v_{\pm,t_{\underline{p}}} \end{pmatrix} + \frac{1}{2}\bigg(\sum_{n'=1}^{n} \frac{\gamma_{t_{\underline{p}}}(n')}{\gamma(n')}\bigg) \begin{pmatrix} u_{\pm} \\ v_{\pm} \end{pmatrix} + \frac{D_{t_{\underline{p}}}}{D} \begin{pmatrix} u_{\pm} \\ v_{\pm} \end{pmatrix} = i \begin{pmatrix} G_{\underline{p}} u_{\pm} - zF_{\underline{p}} v_{\pm} \\ z^{-1} H_{\underline{p}} u_{\pm} - K_{\underline{p}} v_{\pm} \end{pmatrix}. \tag{5.27}$$

Using (2.53), (2.37), and (5.16) we find

$$\begin{aligned} \sum_{n'=1}^{n} \frac{\gamma_{t_{\underline{p}}}(n')}{\gamma(n')} &= i \sum_{n'=1}^{n} (I - S^{-})\big(g_{p_+,+} - g_{p_-,-}\big) \\ &= i\big((g_{p_+,+}(n) - g_{p_-,-}(n)) - (g_{p_+,+}(0) - g_{p_-,-}(0))\big) \\ &= i\big(G_{\underline{p}} - K_{\underline{p}}\big) - 2\frac{D_{t_{\underline{p}}}}{D}. \end{aligned} \tag{5.28}$$

From (3.29), (3.36), (5.4), and (5.6) one obtains (we recall that n is assumed to be odd)

$$\begin{pmatrix} u_{\pm,t_{\underline{p}}} \\ v_{\pm,t_{\underline{p}}} \end{pmatrix} = i \begin{pmatrix} -zF_{\underline{p}} v_{\pm} + \frac{1}{2}(G_{\underline{p}} + K_{\underline{p}}) u_{\pm} \\ z^{-1} H_{\underline{p}} u_{\pm} - \frac{1}{2}(G_{\underline{p}} + K_{\underline{p}}) v_{\pm} \end{pmatrix}, \tag{5.29}$$

using (4.18).

Inserting (5.28) into (5.27), we see that it reduces to (5.29), thereby proving (5.21) in the case when n is odd. The case with n even follows from analogous computations.

Linear independence of $\Psi_-(z,\cdot,t_{\underline{p}})$ and $\Psi_+(z,\cdot,t_{\underline{p}})$ follows from

$$\begin{aligned} \begin{pmatrix} \psi_{1,-}(z,n,t_{\underline{p}}) & \psi_{1,+}(z,n,t_{\underline{p}}) \\ \psi_{2,-}(z,n,t_{\underline{p}}) & \psi_{2,+}(z,n,t_{\underline{p}}) \end{pmatrix} &= D(t_{\underline{p}}) z^{n/2} \bigg(\prod_{n'=1}^{n} \rho(n',t_{\underline{p}})\bigg) A(z,n)^{-1} \\ &\quad \times \begin{pmatrix} u_-(z,n,t_{\underline{p}}) & u_+(z,n,t_{\underline{p}}) \\ v_-(z,n,t_{\underline{p}}) & v_+(z,n,t_{\underline{p}}) \end{pmatrix}, \end{aligned} \tag{5.30}$$

the fact that $\rho(n,t_{\underline{p}}) \neq 0$, $\det(A(z,n)) = (-1)^{n+1}$, and from

$$\det\left(\begin{pmatrix} u_-(z,n,t_{\underline{p}}) & u_+(z,n,t_{\underline{p}}) \\ v_-(z,n,t_{\underline{p}}) & v_+(z,n,t_{\underline{p}}) \end{pmatrix} \right) \neq 0, \quad (n,t_{\underline{p}}) \in \mathbb{Z} \times \mathbb{R}, \tag{5.31}$$

since by hypothesis $z \in \mathbb{C} \setminus \mathrm{spec}\big(\breve{L}(t_{\underline{p}})\big)$. ☐

In the following we will always refer to the solutions $\Psi_{\pm}$ introduced in (5.14)–(5.16).

The next result recalls the existence of a propagator $W_{\underline{p}}$ associated with $P_{\underline{p}}$. (Below we denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators defined on the Hilbert space $\mathcal{H}$.)

Theorem 5.4. *Assume Hypothesis* 5.1 *and suppose* α, β *satisfy* $\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$ *for some* $\underline{p} \in \mathbb{N}_0^2$. *Then there is a propagator* $W_{\underline{p}}(s,t) \in \mathcal{B}(\ell^2(\mathbb{Z}))$, $(s,t) \in \mathbb{R}^2$, *satisfying*

$$(i)\ \ W_{\underline{p}}(t,t) = I, \quad t \in \mathbb{R}, \tag{5.32}$$

$$(ii)\ \ W_{\underline{p}}(r,s)W_{\underline{p}}(s,t) = W_{\underline{p}}(r,t), \quad (r,s,t) \in \mathbb{R}^3, \tag{5.33}$$

$$(iii)\ W_{\underline{p}}(s,t) \text{ is jointly strongly continuous in } (s,t) \in \mathbb{R}^2, \tag{5.34}$$

such that for fixed $t_0 \in \mathbb{R}$, $f_0 \in \ell^2(\mathbb{Z})$,

$$f(t) = W_{\underline{p}}(t, t_0) f_0, \quad t \in \mathbb{R}, \tag{5.35}$$

satisfies

$$\frac{d}{dt} f(t) = \breve{P}_{\underline{p}}(t) f(t), \quad f(t_0) = f_0. \tag{5.36}$$

Moreover, $\breve{L}(t)$ *is similar to* $\breve{L}(s)$ *for all* $(s,t) \in \mathbb{R}^2$,

$$\breve{L}(s) = W_{\underline{p}}(s,t) \breve{L}(t) W_{\underline{p}}(s,t)^{-1}, \quad (s,t) \in \mathbb{R}^2. \tag{5.37}$$

This extends to appropriate functions of $\breve{L}(t)$ *and so, in particular, to its resolvent* $\big(\breve{L}(t) - zI\big)^{-1}$, $z \in \mathbb{C} \setminus \sigma\big(\breve{L}(t)\big)$, *and hence also yields*

$$\sigma\big(\breve{L}(s)\big) = \sigma\big(\breve{L}(t)\big), \quad (s,t) \in \mathbb{R}^2. \tag{5.38}$$

Consequently, the spectrum of $\breve{L}(t)$ *is independent of* $t \in \mathbb{R}$.

Proof. (5.32)–(5.36) are standard results which follow, for instance, from Theorem X.69 of [40] under even weaker hypotheses on α, β. In particular, the propagator $W_{\underline{p}}$ admits the norm convergent Dyson series

$$W_{\underline{p}}(s,t) = I + \sum_{k \in \mathbb{N}} \int_s^t dt_1 \int_s^{t_1} dt_2 \cdots \int_s^{t_{k-1}} dt_k \, \breve{P}_{\underline{p}}(t_1) \breve{P}_{\underline{p}}(t_2) \cdots \breve{P}_{\underline{p}}(t_k), \quad (s,t) \in \mathbb{R}^2. \tag{5.39}$$

Fixing $s \in \mathbb{R}$ and introducing the operator-valued function

$$\breve{K}(t) = W_{\underline{p}}(s,t) \breve{L}(t) W_{\underline{p}}(s,t)^{-1}, \quad t \in \mathbb{R}, \tag{5.40}$$

one computes

$$\breve{K}'(t) f = W_{\underline{p}}(s,t)\big(\breve{L}'(t) - \big[\breve{P}_{\underline{p}}(t), \breve{L}(t)\big]\big) W_{\underline{p}}(s,t)^{-1} f = 0, \quad t \in \mathbb{R}, \ f \in \ell^2(\mathbb{Z}), \tag{5.41}$$

using the Lax commutator equation (3.38). Thus, $\breve{K}$ is independent of $t \in \mathbb{R}$ and hence taking $t = s$ in (5.40) then yields $\breve{K} = \breve{L}(s)$ and thus proves (5.37). □

Next we briefly recall the Ablowitz–Ladik initial value problem in a setting convenient for our purpose.

Theorem 5.5. *Let* $t_{0,\underline{p}} \in \mathbb{R}$ *and suppose* $\alpha^{(0)}, \beta^{(0)} \in \ell^q(\mathbb{Z})$ *for some* $q \in [1, \infty) \cup \{\infty\}$. *Then the* $\underline{p}$*th Ablowitz–Ladik initial value problem*

$$\mathrm{AL}_{\underline{p}}(\alpha, \beta) = 0, \quad (\alpha, \beta)\big|_{t_{\underline{p}} = t_{0,\underline{p}}} = \big(\alpha^{(0)}, \beta^{(0)}\big) \tag{5.42}$$

for some $\underline{p} \in \mathbb{N}_0^2$, *has a unique, local, and smooth solution in time, that is, there exists a* $T_0 > 0$ *such that*

$$\alpha(\,\cdot\,), \, \beta(\,\cdot\,) \in C^\infty((t_{0,\underline{p}} - T_0, t_{0,\underline{p}} + T_0), \ell^q(\mathbb{Z})). \tag{5.43}$$

Proof. This follows from standard results in [8, Sect. 4.1]. More precisely, local existence and uniqueness as well as smoothness of the solution of the initial value problem (5.42) (cf. (2.51)) follows from [8, Theorem 4.1.5] since $f_{p_\pm - 1, \pm}$, $g_{p_\pm, \pm}$, and $h_{p_\pm - 1, \pm}$ depend polynomially on α, β and certain of their shifts, and the fact that the Ablowitz–Ladik flows are autonomous. □

For an analogous result in connection with the Toda hierarchy we refer to [24] and [46, Sect. 12.2].

Remark 5.6. In the special defocusing case, where $\beta = \overline{\alpha}$ and hence $\breve{L}(t)$, $t \in \mathbb{R}$, is unitary, one obtains

$$\sup_{(n,t_{\underline{p}})\in\mathbb{N}\times(t_{0,\underline{p}}-T_0,t_{0,\underline{p}}+T_0)} |\alpha(n,t_{\underline{p}})| \leq 1 \tag{5.44}$$

using $\gamma = 1-|\alpha|^2$ and $\gamma_{t_{\underline{p}}} = i\gamma\big((g_{p_+,+}-g^-_{p_+,+})-(g_{p_-,-}-g^-_{p_-,-})\big)$ in (2.53). A further application of [8, Proposition 4.1.22] then yields a unique, global, and smooth solution of the $\underline{p}$th AL initial value problem (5.42). Moreover, the same argument shows that if α satisfies Hypothesis 5.1 and the $\underline{p}$th AL equation $\mathrm{AL}_{\underline{p}}(\alpha,\overline{\alpha}) = 0$, then α is actually smooth with respect to $t_{\underline{p}} \in \mathbb{R}$, that is,

$$\alpha(n,\,\cdot\,) \in C^\infty(\mathbb{R}), \quad n \in \mathbb{Z}. \tag{5.45}$$

Equation (5.18), that is, $\Psi_{\pm,t_{\underline{p}}} = V^+_{\underline{p}}\Psi_\pm$, implies that

$$\begin{aligned}
\partial_{t_{\underline{p}}} \ln\left(\frac{\psi^+_{1,\pm}}{\psi_{1,\pm}}\right) &= (S^+ - I)\partial_{t_{\underline{p}}} \ln(\psi_{1,\pm}) \\
&= (S^+ - I)\frac{\partial_{t_{\underline{p}}}\psi_{1,\pm}}{\psi_{1,\pm}} \\
&= i(S^+ - I)(G_{\underline{p}} - F_{\underline{p}}\phi_\pm).
\end{aligned} \tag{5.46}$$

On the other hand, equation (5.17), that is, $U\Psi^-_\pm = \Psi_\pm$, yields

$$\partial_{t_{\underline{p}}} \ln\left(\frac{\psi^+_{1,\pm}}{\psi_{1,\pm}}\right) = \partial_{t_{\underline{p}}} \ln(z + \alpha^+\phi_\pm), \tag{5.47}$$

and thus one concludes that

$$\partial_{t_{\underline{p}}} \ln(z + \alpha^+\phi_\pm) = i(S^+ - I)(G_{\underline{p}} - F_{\underline{p}}\phi_\pm). \tag{5.48}$$

Below we will refer to (5.48$\pm$) according to the upper or lower sign in (5.48). Expanding (5.48$\pm$) in powers of z and $1/z$ then yields the following conserved densities:

Theorem 5.7. *Assume Hypothesis* 5.1 *and suppose α, β satisfy $\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$ for some $\underline{p} \in \mathbb{N}_0^2$. Then the following infinite sequences of local conservation laws hold:*
Expansion of (5.48+) *at* $1/z = 0$:

$$\partial_{t_{\underline{p}}}\rho^\infty_{j,+} = i(S^+ - I)\bigg(g_{p_- - j,-} - \sum_{\ell=0}^{j-1} f_{p_- - j+\ell,-}\phi^\infty_{\ell,+} - \sum_{\ell=0}^{p_+-1} f_{p_+-1-\ell,+}\phi^\infty_{j+\ell,+}\bigg), \quad j = 1,\dots,p_-, \tag{5.49}$$

$$\partial_{t_{\underline{p}}}\rho^\infty_{j,+} = -i(S^+ - I)\bigg(\sum_{\ell=1}^{p_-} f_{p_- - \ell,-}\phi^\infty_{j-\ell,+} + \sum_{\ell=0}^{p_+-1} f_{p_+-1-\ell,+}\phi^\infty_{j+\ell,+}\bigg), \quad j \geq p_- + 1, \tag{5.50}$$

where $\rho^\infty_{j,+}$ and $\phi^\infty_{j,+}$ are given by (4.48) *and* (4.38).

Expansion of (5.48−) *at* $1/z = 0$:

$$\partial_{t_{\underline{p}}} \rho^{\infty}_{j,-} = i(S^+ - I)\bigg(g_{p_- - j,-} - \sum_{\ell=-1}^{j-1} f_{p_- + \ell - j,-} \phi^{\infty}_{\ell,-} - \sum_{\ell=0}^{p_+ - 1} f_{p_+ - 1 - \ell,+} \phi^{\infty}_{j+\ell,-} \bigg), \quad j = 1, \dots, p_-, \tag{5.51}$$

$$\partial_{t_{\underline{p}}} \rho^{\infty}_{j,-} = -i(S^+ - I)\bigg(\sum_{\ell=1}^{p_-} f_{p_- - \ell,-} \phi^{\infty}_{j-\ell,-} + \sum_{\ell=0}^{p_+ - 1} f_{p_+ - 1 - \ell,+} \phi^{\infty}_{j+\ell,-} \bigg), \quad j \geq p_- + 1, \tag{5.52}$$

where $\rho^{\infty}_{j,-}$ *and* $\phi^{\infty}_{j,-}$ *are given by* (4.50) *and* (4.39).

Expansion of (5.48+) *at* $z = 0$:

$$\partial_{t_{\underline{p}}} \rho^{0}_{j,+} = i(S^+ - I)\bigg(g_{p_+ - j,+} - \sum_{\ell=1}^{p_-} \phi^{0}_{j+\ell,+} f_{p_- - \ell,-} - \sum_{\ell=0}^{j} \phi^{0}_{\ell,+} f_{p_+ - 1 - j + \ell,+} \bigg), \quad j = 1, \dots, p_+ - 1, \tag{5.53}$$

$$\partial_{t_{\underline{p}}} \rho^{0}_{p_+,+} = i(S^+ - I)\bigg(g_{0,+} - \sum_{\ell=1}^{p_-} \phi^{0}_{j+\ell,+} f_{p_- - \ell,-} - \sum_{\ell=0}^{p_+ - 1} \phi^{0}_{j+\ell-p_+ + 1,+} f_{\ell,+} \bigg), \tag{5.54}$$

$$\partial_{t_{\underline{p}}} \rho^{0}_{j,+} = -i(S^+ - I)\bigg(\sum_{\ell=1}^{p_-} \phi^{0}_{j+\ell,+} f_{p_- - \ell,-} + \sum_{\ell=0}^{p_+ - 1} \phi^{0}_{j+\ell-p_+ + 1,+} f_{\ell,+} \bigg), \quad j \geq p_+ + 1, \tag{5.55}$$

where $\rho^{0}_{j,+}$ *and* $\phi^{0}_{j,+}$ *are given by* (4.52) *and* (4.40).

Expansion of (5.48−) *at* $z = 0$:

$$\partial_{t_{\underline{p}}} \rho^{0}_{j,-} = i(S^+ - I)\bigg(g_{p_+ - j,+} - \sum_{\ell=1}^{j} \phi^{0}_{\ell,-} f_{p_+ - j + \ell - 1,+} - \sum_{\ell=1}^{p_-} \phi^{0}_{j+\ell,-} f_{p_- - \ell,-} \bigg), \quad j = 1, \dots, p_+, \tag{5.56}$$

$$\partial_{t_{\underline{p}}} \rho^{0}_{j,-} = -i(S^+ - I)\bigg(\sum_{\ell=j+1-p_+}^{j} \phi^{0}_{\ell,-} f_{p_+ - j + \ell - 1,+} + \sum_{\ell=1}^{p_-} \phi^{0}_{j+\ell,-} f_{p_- - \ell,-} \bigg), \quad j \geq p_+ + 1, \tag{5.57}$$

where $\rho^{0}_{j,-}$ *and* $\phi^{0}_{j,-}$ *are given by* (4.54) *and* (4.41).

Proof. The proof consists of expanding (5.48±) in powers of z and $1/z$ and applying (4.47)–(4.54).

Expansion of (5.48+) at $1/z = 0$: For the right-hand side of (5.48+) one finds

$$\begin{aligned} G_{\underline{p}} - F_{\underline{p}} \phi_+ &= \sum_{\ell=1}^{p_-} g_{p_- - \ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+} g_{p_+ - \ell,+} z^{\ell} \\ &\quad - \bigg(\sum_{\ell=1}^{p_-} f_{p_- - \ell,-} z^{-\ell} + \sum_{\ell=0}^{p_+ - 1} f_{p_+ - 1 - \ell,+} z^{\ell} \bigg) \sum_{j=0}^{\infty} \phi^{\infty}_{j,+} z^{-j} \\ &= g_{0,+} z^{p_+} + \sum_{j=0}^{p_+ - 1} \bigg(g_{p_+ - j,+} - \sum_{\ell=0}^{p_+ - j - 1} f_{p_+ - j - 1 - \ell,+} \phi^{\infty}_{\ell,+} \bigg) z^{j} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{p_-}\bigg(g_{p_--j,-}-\sum_{\ell=0}^{j-1}f_{p_--j+\ell,-}\phi^\infty_{\ell,+}-\sum_{\ell=0}^{p_+-1}f_{p_+-1-\ell,+}\phi^\infty_{j+\ell,+}\bigg)z^{-j}\\
&-\sum_{j=p_-+1}^{\infty}\bigg(\sum_{\ell=1}^{p_-}f_{p_--\ell,-}\phi^\infty_{j-\ell,+}+\sum_{\ell=0}^{p_+-1}f_{p_+-1-\ell,+}\phi^\infty_{j+\ell,+}\bigg)z^{-j}. \quad (5.58)
\end{aligned}
$$

Here we used that all positive powers vanish because of (5.48). This yields the following additional formulas:

Conservation laws derived from ϕ_+ at $1/z=0$:

$$
\begin{aligned}
&(S^+-1)\bigg(g_{p_+-j,+}-\sum_{\ell=0}^{p_+-j-1}f_{p_+-j-1-\ell,+}\phi^\infty_{\ell,+}\bigg)=0, \quad j=0,\dots,p_+-1,\\
&(S^+-1)g_{0,+}=0. \quad (5.59)
\end{aligned}
$$

Expansion of (5.48−) at $1/z=0$: The right-hand side of (5.48−) yields

$$
\begin{aligned}
G_{\underline{p}}-F_{\underline{p}}\phi_- &= \sum_{\ell=1}^{p_-}g_{p_--\ell,-}z^{-\ell}+\sum_{\ell=0}^{p_+}g_{p_+-\ell,+}z^{\ell}\\
&\quad-\bigg(\sum_{\ell=1}^{p_-}f_{p_--\ell,-}z^{-\ell}+\sum_{\ell=0}^{p_+-1}f_{p_+-1-\ell,+}z^{\ell}\bigg)\sum_{j=-1}^{\infty}\phi^\infty_{j,-}z^{-j}\\
&=\sum_{j=1}^{p_+}\bigg(g_{p_+-j,+}-\sum_{\ell=-1}^{p_+-j-1}f_{p_+-j-1-\ell,+}\phi^\infty_{\ell,-}\bigg)z^{j}\\
&\quad+\bigg(g_{p_+,+}-\sum_{\ell=0}^{p_+-1}f_{p_+-1-\ell,+}\phi^\infty_{\ell,-}-f_{p_--1,-}\phi^\infty_{-1,-}\bigg)\\
&\quad+\sum_{j=1}^{p_-}\bigg(g_{p_--j,-}-\sum_{\ell=-1}^{j-1}f_{p_-+\ell-j,-}\phi^\infty_{\ell,-}-\sum_{\ell=0}^{p_+-1}f_{p_+-1-\ell,+}\phi^\infty_{j+\ell,-}\bigg)z^{-j}\\
&\quad-\sum_{j=p_-+1}^{\infty}\bigg(\sum_{\ell=1}^{p_-}f_{p_--\ell,-}\phi^\infty_{j-\ell,-}+\sum_{\ell=0}^{p_+-1}f_{p_+-1-\ell,+}\phi^\infty_{j+\ell,-}\bigg)z^{-j}. \quad (5.60)
\end{aligned}
$$

Conservation laws derived from ϕ_+ at $1/z=0$:

$$
(S^+-I)\bigg(g_{p_+-j,+}-\sum_{\ell=-1}^{p_+-j-1}f_{p_+-j-1-\ell,+}\phi^\infty_{\ell,-}\bigg)=0, \quad j=1,\dots,p_+, \quad (5.61)
$$

$$
\begin{aligned}
&i(S^+-I)\bigg(g_{p_+,+}-\sum_{\ell=0}^{p_+-1}f_{p_+-1-\ell,+}\phi^\infty_{\ell,-}-f_{p_--1,-}\phi^\infty_{-1,-}\bigg)\\
&\quad=\partial_{t_{\underline{p}}}\ln\bigg(\frac{\alpha^{++}}{\alpha^+}\bigg)+\partial_{t_{\underline{p}}}\ln(\gamma^+). \quad (5.62)
\end{aligned}
$$

Expansion of (5.48+) at $z=0$: For the right-hand side of (5.48+) one finds

$$
G_{\underline{p}}-F_{\underline{p}}\phi_+=\sum_{j=1}^{p_-}\bigg(g_{p_--j,-}-\sum_{\ell=0}^{p_--j}\phi_{\ell,+}f_{p_--j-\ell,-}\bigg)z^{-j} \quad (5.63)
$$

$$+\sum_{j=0}^{p_+-1}\bigg(g_{p_+-j,+}-\sum_{\ell=1}^{p_-}\phi^0_{j+\ell,+}f_{p_--\ell,-}-\sum_{\ell=0}^{j}\phi^0_{\ell,+}f_{p_+-1-j+\ell,+}\bigg)z^j$$
$$+\sum_{j=p_+}^{\infty}\bigg(g_{0,+}\chi_{jp_+}-\sum_{\ell=1}^{p_-}\phi^0_{j+\ell,+}f_{p_--\ell,-}-\sum_{\ell=0}^{p_+-1}\phi^0_{j+\ell-p_++1,+}f_{\ell,+}\bigg)z^j.$$

Conservation laws derived from ϕ_+ at $z=0$:

$$(S^+-I)\bigg(g_{j,-}-\sum_{\ell=0}^{j}\phi^0_{\ell,+}f_{j-\ell,-}\bigg)=0,\quad j=1,\dots,p_-, \tag{5.64}$$

$$(S^+-I)\bigg(g_{p_+,+}-\phi^0_{0,+}f_{p_+-1,+}-\sum_{\ell=1}^{p_-}\phi^0_{\ell,+}f_{p_--\ell,-}\bigg)=\partial_{t_{\underline{p}}}\ln\bigg(\frac{\alpha}{\alpha^+}\bigg). \tag{5.65}$$

Expansion of (5.48−) at $z=0$: For the right-hand side of (5.48−) one finds

$$\begin{aligned}G_{\underline{p}}-F_{\underline{p}}\phi_- &= g_{0,-}z^{-p_-}+\sum_{j=1}^{p_--1}\bigg(g_{p_--j,-}-\sum_{\ell=1}^{p_--j}\phi^0_{\ell,-}f_{p_--j-\ell,-}\bigg)z^{-j} \\ &\quad+g_{p_+,+}-\sum_{\ell=1}^{p_-}\phi^0_{\ell,-}f_{p_--\ell,-} \\ &\quad+\sum_{j=1}^{p_+}\bigg(g_{p_+-j,+}-\sum_{\ell=1}^{j}\phi^0_{\ell,-}f_{p_+-j+\ell-1,+}-\sum_{\ell=1}^{p_-}\phi^0_{j+\ell,-}f_{p_--\ell,-}\bigg)z^j \\ &\quad-\sum_{j=p_++1}^{\infty}\bigg(\sum_{\ell=j+1-p_+}^{j}\phi^0_{\ell,-}f_{p_+-j+\ell-1,+}+\sum_{\ell=1}^{p_-}\phi^0_{j+\ell,-}f_{p_--\ell,-}\bigg)z^j.\end{aligned} \tag{5.66}$$

Conservation laws derived from ϕ_- at $z=0$:

$$(S^+-I)g_{0,-}=0, \tag{5.67}$$

$$(S^+-I)\bigg(g_{j,-}-\sum_{\ell=1}^{j}\phi^0_{\ell,-}f_{j-\ell,-}\bigg)=0,\quad j=1,\dots,p_--1, \tag{5.68}$$

$$(S^+-I)\bigg(g_{p_+,+}-\sum_{\ell=1}^{p_-}\phi^0_{\ell,-}f_{p_--\ell,-}\bigg)=\partial_{t_{\underline{p}}}\ln(\gamma^+). \tag{5.69}$$

Combining these expansions with (4.47)–(4.53) finishes the proof. □

Remark 5.8. (i) There is a certain redundancy in the conservation laws (5.49)–(5.57) as can be observed from Lemma 4.4. Equations (4.56)–(4.57) imply

$$\rho^\infty_{j,+}=-\rho^\infty_{j,-}+\frac{1}{j}(S^+-I)(d_{j,+}-e_{j,+}),\quad j\in\mathbb{N}, \tag{5.70}$$

$$\rho^0_{j,+}=-\rho^0_{j,-}+\frac{1}{j}(S^+-I)(d_{j,-}-e_{j,-}),\quad j\in\mathbb{N}. \tag{5.71}$$

Thus one can, for instance, transfer (5.49)–(5.50) into (5.51)–(5.52).
(ii) In addition to the conservation laws listed in Theorem 5.7, we recover the familiar conservation law (cf. (2.53))

$$\partial_{t_{\underline{p}}}\ln(\gamma)=i(I-S^-)(g_{p_+,+}-g_{p_-,-}),\quad \underline{p}\in\mathbb{N}_0^2. \tag{5.72}$$

(iii) Another consequence of Theorem 5.7 and Lemma 4.4 is that for α, β satisfying Hypothesis 5.1 and $\alpha, \beta \in C^1(\mathbb{R}, \ell^2(\mathbb{Z}))$, one has

$$\frac{d}{dt_{\underline{p}}} \sum_{n\in\mathbb{Z}} \ln(\gamma(n,t_{\underline{p}})) = 0, \quad \frac{d}{dt_{\underline{p}}} \sum_{n\in\mathbb{Z}} \hat{g}_{j,\pm}(n,t_{\underline{p}}) = 0, \quad j\in\mathbb{N},\ \underline{p}\in\mathbb{N}_0^2. \tag{5.73}$$

Remark 5.9. The infinite sequence of conservation laws has been studied in the literature; we refer to [4], [7, Ch. 3], [14], [50], and [52]. In particular, Zhang and Chen [50] study local conservation laws for the full 4×4 Ablowitz–Ladik system in a similar way to the one employed here. However, they only expand their equation around a point that corresponds to $1/z = 0$. The systematic derivation of infinite sequences of conserved densities and currents (cf. the corresponding discussion in the introduction) as presented in Theorem 5.7 appears to be new.

The two local conservation laws coming from expansions around $z = 0$ are essentially the same since the two conserved densities, $\rho_{j,+}^0$ and $\rho_{j,+}^0$, differ by a first-order difference expression (cf. Remark 5.8). A similar argument applies to the expansions around $1/z = 0$. That there are two independent sequences of conservation laws is also clear from (5.73) which yields that $\sum_{n\in\mathbb{Z}} \hat{g}_{j,\pm}(n,t_{\underline{p}})$ are time-independent. One observes that the quantities $\hat{g}_{j,+}$, $j\in\mathbb{N}$, are related to the expansions around $1/z=0$, that is, to $\rho_{j,\pm}^\infty$, while $\hat{g}_{j,-}$, $j\in\mathbb{N}$, are related to $\rho_{j,\pm}^0$ (cf. Lemma 4.4). In addition to the two infinite sequences of polynomial conservation laws, there is a logarithmic conservation law (cf. (5.72) and (5.73)).

The first conservation laws explicitly read as follows:
$p_+ = p_- = 1$:

$$\partial_{t_{(1,1)}} \rho_{j,\pm}^\infty = -i(S^+ - I)(f_{0,-}\phi_{j-1,\pm}^\infty + f_{0,+}\phi_{j,\pm}^\infty), \quad j\geq 1, \tag{5.74}$$

$$\partial_{t_{(1,1)}} \rho_{j,\pm}^0 = -i(S^+ - I)(f_{0,-}\phi_{j+1,\pm}^0 + f_{0,+}\phi_{j,\pm}^0), \quad j\geq 1. \tag{5.75}$$

For $j=1$ this yields using (4.55)

$$\begin{aligned}
\partial_{t_{(1,1)}} \rho_{1,+}^\infty &= \partial_{t_{(1,1)}} \alpha^+\beta = i(S^+ - I)(-c_{0,-}\alpha\beta + c_{0,+}\alpha^+\beta^-\gamma),\\
\partial_{t_{(1,1)}} \rho_{1,-}^\infty &= \partial_{t_{(1,1)}} \Big(-\alpha^{+++}\beta^{++} + (S^+ - I)\frac{\alpha^{++}}{\alpha^+}\Big)\\
&= i(S^+ - I)\Big(c_{0,+}\frac{\alpha^{+++}}{\alpha^+}\gamma^+\gamma^{++} - c_{0,-}\frac{\alpha\alpha^{++}}{(\alpha^+)^2}\gamma^+ - c_{0,+}\Big(\frac{\alpha\alpha^{++}}{\alpha^+}\Big)^2\gamma^+\Big),\\
\partial_{t_{(1,1)}} \rho_{1,+}^0 &= \partial_{t_{(1,1)}} \Big(\alpha^-\beta + (S^+ - I)\frac{\alpha^-}{\alpha}\Big)\\
&= i(S^+ - I)\Big(c_{0,-}\frac{\alpha^{--}}{\alpha}\gamma^-\gamma - c_{0,+}\frac{\alpha^-\alpha^+}{\alpha^2}\gamma - c_{0,-}\Big(\frac{\alpha^-}{\alpha}\Big)^2\gamma\Big),\\
\partial_{t_{(1,1)}} \rho_{1,-}^0 &= \partial_{t_{(1,1)}} \alpha^+\beta^{++} = i(S^+ - I)(c_{0,+}\alpha^+\beta^+ - c_{0,-}\alpha\beta^{++}\gamma^+).
\end{aligned} \tag{5.76}$$

This shows in particular that we obtain two sets of conservation laws (one from expanding near ∞ and the other from expanding near 0), where the first few equations of each set explicitly read ($p_+ = p_- = 1$):

$$\begin{aligned}
j=1: \quad & \partial_{t_{(1,1)}} \alpha^+\beta = i(S^+ - I)(-c_{0,-}\alpha\beta + c_{0,+}\alpha^+\beta^-\gamma),\\
& \partial_{t_{(1,1)}} \alpha\beta^+ = i(S^+ - I)(c_{0,+}\alpha\beta - c_{0,-}\alpha^-\beta^+\gamma),
\end{aligned} \tag{5.77}$$

$$\begin{aligned}
j=2: \quad & \partial_{t_{(1,1)}} \big(-\tfrac{1}{2}(\alpha^+\beta)^2 + \gamma\alpha^+\beta^-\big)\\
& \quad = i(S^+ - I)\gamma\big(- c_{0,-}\alpha\beta^- - c_{0,+}\alpha\alpha^+(\beta^-)^2 + c_{0,+}\gamma^-\alpha^+\beta^{--}\big),
\end{aligned}$$

$$\begin{aligned}&\partial_{t_{(1,1)}}\big(\tfrac{1}{2}(\alpha\beta^+)^2-\gamma^+\alpha\beta^{++}\big)\\&\quad=i(S^+-I)\gamma\big(-c_{0,+}\alpha\beta^+-c_{0,-}\alpha^-\alpha(\beta^+)^2+c_{0,-}\gamma^+\alpha^-\beta^{++}\big).\end{aligned}\tag{5.78}$$

Using Lemma 4.4, one observes that one can replace $\rho^{\infty,0}_{j,\pm}$ in Theorem 5.7 by $\hat g_{j,\pm}$ by suitably adjusting the right-hand sides in (5.49)–(5.57).

6. Hamiltonian formalism, variational derivatives

We start this section by a short review of variational derivatives for discrete systems. Consider the functional

$$\begin{aligned}&\mathcal{G}\colon \ell^1(\mathbb{Z})^\kappa\to\mathbb{C},\\&\mathcal{G}(u)=\sum_{n\in\mathbb{Z}}G\big(u(n),u^{(+1)}(n),u^{(-1)}(n),\dots,u^{(k)}(n),u^{(-k)}(n)\big)\end{aligned}\tag{6.1}$$

for some $\kappa\in\mathbb{N}$ and $k\in\mathbb{N}_0$, where $G\colon\mathbb{Z}\times\mathbb{C}^{2r\kappa}\to\mathbb{C}$ is C^1 with respect to the $2r\kappa$ complex-valued entries and where

$$u^{(s)}=S^{(s)}u,\quad S^{(s)}=\begin{cases}(S^+)^s u & \text{if } s\geq 0,\\(S^-)^{-s}u & \text{if } s<0,\end{cases}\quad u\in\ell^\infty(\mathbb{Z})^\kappa.\tag{6.2}$$

For brevity we write

$$G(u(n))=G\big(u(n),u^{(+1)}(n),u^{(-1)}(n),\dots,u^{(k)}(n),u^{(-k)}(n)\big).\tag{6.3}$$

The functional $\mathcal{G}$ is Frechet-differentiable and one computes for any $v\in\ell^1(\mathbb{Z})^\kappa$ for the differential $d\mathcal{G}$

$$\begin{aligned}(d\mathcal{G})_u(v)&=\frac{d}{d\epsilon}\mathcal{G}(u+\epsilon v)\big|_{\epsilon=0}\\&=\sum_{n\in\mathbb{Z}}\bigg(\frac{\partial G(u(n))}{\partial u}v(n)+\frac{\partial G(u(n))}{\partial u^{(+1)}}v^{(+1)}(n)+\frac{\partial G(u(n))}{\partial u^{(-1)}}v^{(-1)}(n)\\&\qquad+\dots+\frac{\partial G(u(n))}{\partial u^{(k)}}v^{(k)}(n)+\frac{\partial G(u(n))}{\partial u^{(-k)}}v^{(-k)}(n)\bigg)\\&=\sum_{n\in\mathbb{Z}}\bigg(\frac{\partial G(u(n))}{\partial u}+S^{(-1)}\frac{\partial G(u(n))}{\partial u^{(+1)}}+S^{(+1)}\frac{\partial G(u(n))}{\partial u^{(-1)}}\\&\qquad+\dots+S^{(-k)}\frac{\partial G(u(n))}{\partial u^{(k)}}+S^{(k)}\frac{\partial G(u(n))}{\partial u^{(-k)}}\bigg)v(n)\\&=\sum_{n\in\mathbb{Z}}\frac{\delta G}{\delta u}(n)v(n),\end{aligned}\tag{6.4}$$

where we introduce the gradient and the variational derivative of $\mathcal{G}$ by

$$\begin{aligned}(\nabla\mathcal{G})_u&=\frac{\delta G}{\delta u}\\&=\frac{\partial G}{\partial u}+S^{(-1)}\frac{\partial G}{\partial u^{(+1)}}+S^{(+1)}\frac{\partial G}{\partial u^{(-1)}}+\dots+S^{(-k)}\frac{\partial G}{\partial u^{(k)}}+S^{(k)}\frac{\partial G}{\partial u^{(-k)}},\end{aligned}\tag{6.5}$$

assuming

$$\{G(u(n))\}_{n\in\mathbb{Z}},\ \bigg\{\frac{\partial G(u(n))}{\partial u^{(\pm j)}}\bigg\}_{n\in\mathbb{Z}}\in\ell^1(\mathbb{Z}),\quad j=1,\dots,k.\tag{6.6}$$

To establish the connection with the Ablowitz–Ladik hierarchy we make the following assumption for the remainder of this section.

Hypothesis 6.1. *Suppose*

$$\alpha, \beta \in \ell^1(\mathbb{Z}), \quad \alpha(n)\beta(n) \notin \{0,1\}, \; n \in \mathbb{Z}. \tag{6.7}$$

Next, let $\mathcal{G}$ be a functional of the type

$$\begin{aligned}
&\mathcal{G}\colon \ell^\infty(\mathbb{Z})^2 \to \mathbb{C}, \\
&\mathcal{G}(\alpha,\beta) = \sum_{n\in\mathbb{Z}} G(\alpha(n), \beta(n), \dots, \alpha(n+k), \beta(n+k), \alpha(n-k), \beta(n-k)) \\
&\qquad\quad = \sum_{n\in\mathbb{Z}} G(\alpha(n), \beta(n)),
\end{aligned} \tag{6.8}$$

where $G(\alpha, \beta)$ is polynomial in α, β and some of their shifts. The gradient $\nabla\mathcal{G}$ and symplectic gradient $\nabla_s\mathcal{G}$ of $\mathcal{G}$ are then defined by

$$(\nabla\mathcal{G})_{\alpha,\beta} = \begin{pmatrix} (\nabla\mathcal{G})_\alpha \\ (\nabla\mathcal{G})_\beta \end{pmatrix} = \begin{pmatrix} \frac{\delta\mathcal{G}}{\delta\alpha} \\ \frac{\delta\mathcal{G}}{\delta\beta} \end{pmatrix} \tag{6.9}$$

and

$$(\nabla_s\mathcal{G})_{\alpha,\beta} = \mathcal{D}(\nabla\mathcal{G})_{\alpha,\beta} = \mathcal{D}\begin{pmatrix} (\nabla\mathcal{G})_\alpha \\ (\nabla\mathcal{G})_\beta \end{pmatrix}, \tag{6.10}$$

respectively. Here $\mathcal{D}$ is defined by

$$\mathcal{D} = \gamma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma = 1 - \alpha\beta. \tag{6.11}$$

In addition, we introduce the bilinear form

$$\begin{aligned}
&\Omega\colon \ell^1(\mathbb{Z})^2 \times \ell^1(\mathbb{Z})^2 \to \mathbb{C}, \\
&\Omega(u,v) = \sum_{n\in\mathbb{Z}} (\mathcal{D}^{-1}u)(n) \cdot v(n).
\end{aligned} \tag{6.12}$$

One then concludes that

$$\begin{aligned}
\Omega(\mathcal{D}u, v) &= \sum_{n\in\mathbb{Z}} u(n)\cdot v(n) = \sum_{n\in\mathbb{Z}} \big(u_1(n)v_1(n) + u_2(n)v_2(n)\big) \\
&= \langle u, v\rangle_{\ell^2(\mathbb{Z})^2}, \quad u, v \in \ell^1(\mathbb{Z})^2,
\end{aligned} \tag{6.13}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\ell^2(\mathbb{Z})^2}$ denotes the "real" inner product in $\ell^2(\mathbb{Z})^2$, that is,

$$\begin{aligned}
&\langle\,\cdot\,,\,\cdot\,\rangle_{\ell^2(\mathbb{Z})^2}\colon \ell^2(\mathbb{Z})^2 \times \ell^2(\mathbb{Z})^2 \to \mathbb{C}, \\
&\langle u, v\rangle_{\ell^2(\mathbb{Z})^2} = \sum_{n\in\mathbb{Z}} u(n)\cdot v(n) = \sum_{n\in\mathbb{Z}} \big(u_1(n)v_1(n) + u_2(n)v_2(n)\big).
\end{aligned} \tag{6.14}$$

In addition, one obtains

$$(d\mathcal{G})_{\alpha,\beta}(v) = \langle(\nabla\mathcal{G})_{\alpha,\beta}, v\rangle_{\ell^2(\mathbb{Z})^2} = \Omega(\mathcal{D}(\nabla\mathcal{G})_{\alpha,\beta}, v) = \Omega((\nabla_s\mathcal{G})_{\alpha,\beta}, v). \tag{6.15}$$

Given two functionals $\mathcal{G}_1, \mathcal{G}_2$ we define their Poisson bracket by

$$\begin{aligned}
\{\mathcal{G}_1, \mathcal{G}_2\} &= d\mathcal{G}_1(\nabla_s\mathcal{G}_2) = \Omega(\nabla_s\mathcal{G}_1, \nabla_s\mathcal{G}_2) \\
&= \Omega(\mathcal{D}\nabla\mathcal{G}_1, \mathcal{D}\nabla\mathcal{G}_2) = \langle\nabla\mathcal{G}_1, \mathcal{D}\nabla\mathcal{G}_2\rangle_{\ell^2(\mathbb{Z})^2} \\
&= \sum_{n\in\mathbb{Z}} \begin{pmatrix} \frac{\delta G_1}{\delta\alpha}(n) \\ \frac{\delta G_1}{\delta\beta}(n) \end{pmatrix} \cdot \mathcal{D} \begin{pmatrix} \frac{\delta G_2}{\delta\alpha}(n) \\ \frac{\delta G_2}{\delta\beta}(n) \end{pmatrix}.
\end{aligned} \tag{6.16}$$

Since $\Omega(\,\cdot\,,\,\cdot\,)$ is a weakly non-degenerate closed 2-form, both the Jacobi identity

$$\{\{\mathcal{G}_1,\mathcal{G}_2\},\mathcal{G}_3\}+\{\{\mathcal{G}_2,\mathcal{G}_3\},\mathcal{G}_1\}+\{\{\mathcal{G}_3,\mathcal{G}_1\},\mathcal{G}_2\}=0, \tag{6.17}$$

as well as the Leibniz rule

$$\{\mathcal{G}_1,\mathcal{G}_2\mathcal{G}_3\}=\{\mathcal{G}_1,\mathcal{G}_2\}\mathcal{G}_3+\mathcal{G}_2\{\mathcal{G}_1,\mathcal{G}_3\}, \tag{6.18}$$

hold as discussed in [32, Theorem 48.8].

If $\mathcal{G}$ is a smooth functional and (α,β) develops according to a Hamiltonian flow with Hamiltonian $\mathcal{H}$, that is,

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}_t=(\nabla_s\mathcal{H})_{\alpha,\beta}=\mathcal{D}(\nabla\mathcal{H})_{\alpha,\beta}=\mathcal{D}\begin{pmatrix}\frac{\delta H}{\delta\alpha}\\ \frac{\delta H}{\delta\beta}\end{pmatrix}, \tag{6.19}$$

then

$$\begin{aligned}\frac{d\mathcal{G}}{dt}&=\frac{d}{dt}\sum_{n\in\mathbb{Z}}G(\alpha(n),\beta(n))\\ &=\sum_{n\in\mathbb{Z}}\begin{pmatrix}\frac{\delta G}{\delta\alpha}(n)\\ \frac{\delta G}{\delta\beta}(n)\end{pmatrix}\cdot\begin{pmatrix}\alpha(n)\\ \beta(n)\end{pmatrix}_t=\sum_{n\in\mathbb{Z}}\begin{pmatrix}\frac{\delta G}{\delta\alpha}(n)\\ \frac{\delta G}{\delta\beta}(n)\end{pmatrix}\cdot\mathcal{D}\begin{pmatrix}\frac{\delta H}{\delta\alpha}(n)\\ \frac{\delta H}{\delta\beta}(n)\end{pmatrix}\\ &=\{\mathcal{G},\mathcal{H}\}.\end{aligned} \tag{6.20}$$

Here, and in the remainder of this section, time-dependent equations such as (6.20) are viewed locally in time, that is, assumed to hold on some open t-interval $\mathbb{I}\subseteq\mathbb{R}$.

If a functional $\mathcal{G}$ is in involution with the Hamiltonian $\mathcal{H}$, that is,

$$\{\mathcal{G},\mathcal{H}\}=0, \tag{6.21}$$

then it is conserved in the sense that

$$\frac{d\mathcal{G}}{dt}=0. \tag{6.22}$$

Next, we turn to the specifics of the AL hierarchy. We define

$$\widehat{\mathcal{G}}_{\ell,\pm}=\sum_{n\in\mathbb{Z}}\hat{g}_{\ell,\pm}(n). \tag{6.23}$$

Lemma 6.2. *Assume Hypothesis* 6.1 *and* $v\in\ell^1(\mathbb{Z})$. *Then,*

$$(d\widehat{\mathcal{G}}_{\ell,\pm})_\beta(v)=\sum_{n\in\mathbb{Z}}\frac{\delta\hat{g}_{\ell,\pm}(n)}{\delta\beta}v(n)=\pm\ell\sum_{n\in\mathbb{Z}}(\delta_n,L^{\pm\ell-1}M_\beta(v)\delta_n),\quad \ell\in\mathbb{N}, \tag{6.24}$$

$$(d\widehat{\mathcal{G}}_{\ell,\pm})_\alpha(v)=\sum_{n\in\mathbb{Z}}\frac{\delta\hat{g}_{\ell,\pm}(n)}{\delta\alpha}v(n)=\pm\ell\sum_{n\in\mathbb{Z}}(\delta_n,L^{\pm\ell-1}M_\alpha(v)\delta_n),\quad \ell\in\mathbb{N}, \tag{6.25}$$

where

$$\begin{aligned}M_\beta(v)&=-v\alpha^+ +\Big(\big(v^-\rho-\beta^-\frac{v\alpha}{2\rho}\big)\delta_{\rm even}+\alpha^+\frac{v\alpha}{2\rho}\delta_{\rm odd}\Big)S^-\\ &\quad+\Big(\big(v\rho^+-\beta\frac{v^+\alpha^+}{2\rho^+}\big)\delta_{\rm even}+\alpha^{++}\frac{v^+\alpha^+}{2\rho^+}\delta_{\rm odd}\Big)S^+\\ &\quad-\Big(\rho\frac{v^-\alpha^-}{2\rho^-}+\rho^-\frac{v\alpha}{2\rho}\Big)\delta_{\rm even}S^{--}-\Big(\rho^+\frac{v^{++}\alpha^{++}}{2\rho^{++}}+\rho^{++}\frac{v^+\alpha^+}{2\rho^+}\Big)\delta_{\rm odd}S^{++},\\ M_\alpha(v)&=-v^+\beta-\Big(\big(v^+\rho-\alpha^+\frac{v\beta}{2\rho}\big)\delta_{\rm odd}+\beta^-\frac{v\beta}{2\rho}\delta_{\rm even}\Big)S^-\end{aligned} \tag{6.26}$$

$$
\begin{aligned}
&- \Big(\big(v^{++}\rho^{+} - \alpha^{++} \frac{v^{+}\beta^{+}}{2\rho^{+}} \big) \delta_{\mathrm{odd}} - \beta \frac{v^{+}\beta^{+}}{2\rho^{+}} \delta_{\mathrm{even}} \Big) S^{+} \qquad (6.27)\\
&- \Big(\rho \frac{v^{-}\beta^{-}}{2\rho^{-}} + \rho^{-} \frac{v\beta}{2\rho} \Big) \delta_{\mathrm{even}} S^{--} - \Big(\rho^{+} \frac{v^{++}\beta^{++}}{2\rho^{++}} + \rho^{++} \frac{v^{+}\beta^{+}}{2\rho^{+}} \Big) \delta_{\mathrm{odd}} S^{++}.
\end{aligned}
$$

Proof. We first consider the derivative with respect to β. By a slight abuse of notation we write $L = L(\beta)$. Using (6.23) and (3.21) one finds

$$
\begin{aligned}
(d\widehat{\mathcal{G}}_{\ell,\pm})_\beta v &= \frac{d}{d\epsilon} \mathcal{G}(\beta + \epsilon v)\big|_{\epsilon=0} = \sum_{n\in\mathbb{Z}} \frac{d}{d\epsilon} \hat{g}_{\ell,\pm}(\beta + \epsilon v)(n)\big|_{\epsilon=0}\\
&= \sum_{n\in\mathbb{Z}} \bigg(\delta_n, \frac{d}{d\epsilon} L(\beta+\epsilon v)^{\pm\ell} \delta_n \bigg)\bigg|_{\epsilon=0}. \qquad (6.28)
\end{aligned}
$$

Next, one considers

$$
\begin{aligned}
\frac{d}{d\epsilon} L(\beta+\epsilon v)^{\ell}|_{\epsilon=0} &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \big(L(\beta+\epsilon v)^{\ell} - L(\beta)^{\ell} \big)\\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \Big((L(\beta+\epsilon v) - L(\beta)) L(\beta)^{\ell-1}\\
&\qquad + L(\beta)(L(\beta+\epsilon v) - L(\beta)) L(\beta)^{\ell-2}\\
&\qquad + \cdots + L(\beta)^{\ell-1} (L(\beta+\epsilon v) - L(\beta)) \Big)\\
&= M_\beta L(\beta)^{\ell-1} + L(\beta) M_\beta L(\beta)^{\ell-2} + \cdots + L(\beta)^{\ell-1} M_\beta, \qquad (6.29)
\end{aligned}
$$

where

$$
\begin{aligned}
M_\beta &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \big(L(\beta+\epsilon v) - L(\beta) \big)\\
&= \bigg(- v(n)\alpha(n+1)\delta_{m,n} + \Big(\big(v(n-1)\rho(n) - \beta(n-1) \frac{v(n)\alpha(n)}{2\rho(n)} \big) \delta_{\mathrm{odd}}(n)\\
&\quad + \alpha(n+1) \frac{v(n)\alpha(n)}{2\rho(n)} \delta_{\mathrm{even}}(n) \Big) \delta_{m,n-1}\\
&\quad + \Big(\big(v(n)\rho(n+1) - \beta(n) \frac{v(n+1)\alpha(n+1)}{2\rho(n+1)} \big) \delta_{\mathrm{odd}}(n) \qquad (6.30)\\
&\quad + \alpha(n+2) \frac{v(n+1)\alpha(n+1)}{2\rho(n+1)} \delta_{\mathrm{even}}(n) \Big) \delta_{m,n+1}\\
&\quad - \Big(\rho(n+1) \frac{v(n+2)\alpha(n+2)}{2\rho(n+2)} + \rho(n+2) \frac{v(n+1)\alpha(n+1)}{2\rho(n+1)} \Big) \delta_{\mathrm{even}}(n) \delta_{m,n+2}\\
&\quad - \Big(\rho(n) \frac{v(n-1)\alpha(n-1)}{2\rho(n-1)} + \rho(n-1) \frac{v(n)\alpha(n)}{2\rho(n)} \Big) \delta_{\mathrm{odd}}(n) \delta_{m,n-2} \bigg)_{m,n\in\mathbb{Z}}.
\end{aligned}
$$

Similarly one obtains

$$
\begin{aligned}
&\frac{d}{d\epsilon} L(\beta+\epsilon v)^{-\ell}|_{\epsilon=0} \qquad (6.31)\\
&\quad = - \Big(L(\beta)^{-1} M_\beta L(\beta)^{-\ell} + L(\beta)^{-2} M_\beta L(\beta)^{-\ell+1} + \cdots + L(\beta)^{-\ell} M_\beta L(\beta)^{-1} \Big).
\end{aligned}
$$

Inserting the expression (6.29) into (6.28) one finds

$$
(d\widehat{\mathcal{G}}_{\ell,+})_\beta v = \sum_{n\in\mathbb{Z}} \bigg(\delta_n, \frac{d}{d\epsilon} L(\beta+\epsilon v)^{\ell} \delta_n \bigg)\bigg|_{\epsilon=0}
$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \bigg(\delta_n, \sum_{k=0}^{\ell-1} L^k M_\beta L^{\ell-1-k}\delta_n\bigg) \\
&= \sum_{k=0}^{\ell-1}\sum_{n\in\mathbb{Z}} (\delta_n, L^k M_\beta L^{\ell-1-k}\delta_n) \\
&= \sum_{k=0}^{\ell-1}\sum_{n\in\mathbb{Z}} (\delta_n, (L^{\ell-1}M_\beta + [L^k M_\beta, L^{\ell-1-k}])\delta_n) \\
&= \sum_{k=0}^{\ell-1}\sum_{n\in\mathbb{Z}} \Big((\delta_n, L^{\ell-1}M_\beta\delta_n) + (\delta_n, [L^k M_\beta, L^{\ell-1-k}]\delta_n)\Big) \\
&= \ell\sum_{n\in\mathbb{Z}} (\delta_n, L^{\ell-1}M_\beta\delta_n) + \sum_{k=0}^{\ell-1} \operatorname{tr}\big([L^k M_\beta, L^{\ell-1-k}]\big) \\
&= \ell\sum_{n\in\mathbb{Z}} (\delta_n, L^{\ell-1}M_\beta\delta_n).
\end{aligned} \tag{6.32}
$$

Similarly, using (6.28) and (6.31), one concludes that

$$
(d\widehat{\mathcal{G}}_{\ell,-})_\beta v = -\ell\sum_{n\in\mathbb{Z}} (\delta_n, L^{-\ell-1}M_\beta\delta_n). \tag{6.33}
$$

For the derivative with respect to α we set $L = L(\alpha)$ and replace M_β by

$$
\begin{aligned}
M_\alpha &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(L(\alpha+\epsilon v) - L(\alpha)\big) \\
&= \bigg(- v(n+1)\beta(n)\delta_{m,n} + \Big(- \beta(n-1)\frac{v(n)\beta(n)}{2\rho(n)}\delta_{\mathrm{odd}}(n) \\
&\quad - \big(v(n+1)\rho(n) - \alpha(n+1)\frac{v(n)\beta(n)}{2\rho(n)}\big)\delta_{\mathrm{even}}(n)\Big)\delta_{m,n-1} \\
&\quad - \Big(\beta(n)\frac{v(n+1)\beta(n+1)}{2\rho(n+1)}\delta_{\mathrm{odd}}(n) + \big(v(n+2)\rho(n+1) \\
&\quad + \alpha(n+2)\frac{v(n+1)\beta(n+1)}{2\rho(n+1)}\big)\delta_{\mathrm{even}}(n)\Big)\delta_{m,n+1} \\
&\quad - \Big(\rho(n+1)\frac{v(n+2)\beta(n+2)}{2\rho(n+2)} + \rho(n+2)\frac{v(n+1)\beta(n+1)}{2\rho(n+1)}\Big)\delta_{\mathrm{even}}(n)\delta_{m,n+2} \\
&\quad - \Big(\rho(n)\frac{v(n-1)\beta(n-1)}{2\rho(n-1)} + \rho(n-1)\frac{v(n)\beta(n)}{2\rho(n)}\Big)\delta_{\mathrm{odd}}(n)\delta_{m,n-2}\bigg)_{m,n\in\mathbb{Z}}.
\end{aligned} \tag{6.34}
$$

□

Lemma 6.3. *Assume Hypothesis* 6.1. *Then the following relations hold:*

$$
\frac{\delta\hat g_{\ell,+}}{\delta\beta} = \frac{\ell}{\gamma}\big(\hat f_{\ell-1,+} - \alpha\hat g_{\ell,+}\big), \quad \ell\in\mathbb{N}, \tag{6.35}
$$

$$
\frac{\delta\hat g_{\ell,-}}{\delta\beta} = -\frac{\ell}{\gamma}\big(\hat f^-_{\ell-1,-} + \alpha\hat g^-_{\ell,-}\big), \quad \ell\in\mathbb{N}. \tag{6.36}
$$

Proof. We consider (6.35) first. By (3.21) one concludes that

$$\begin{aligned}\hat f_{\ell-1,+}(n) - \alpha(n)\hat g_{\ell,+}(n)\\ = (\delta_n, EL^{\ell-1}\delta_n)\delta_{\text{even}}(n) + (\delta_n, L^{\ell-1}D\delta_n)\delta_{\text{odd}}(n) - \alpha(n)(\delta_n, L^{\ell}\delta_n).\end{aligned} \tag{6.37}$$

Thus one has to show that

$$\sum_{n\in\mathbb{Z}}(\delta_n, L^{\ell-1}M_\beta\delta_n) = \sum_{n\in\mathbb{Z}} \frac{v(n)}{\rho(n)^2}\big(\hat f_{\ell-1,+}(n) - \alpha(n)\hat g_{\ell,+}(n)\big), \tag{6.38}$$

since this implies (6.35), using (6.24). By (6.30), (3.17), and (3.18), and assuming $v \in \ell^1(\mathbb{Z})$, one obtains

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}(\delta_n, L^{\ell-1}M_\beta\delta_n)\\
&= \sum_{n\in\mathbb{Z}}\Big(- v\alpha^+(\delta_n, L^{\ell-1}\delta_n) + v^-\rho(\delta_n, L^{\ell-1}\delta_{n-1})\delta_{\text{odd}} + v\rho^+(\delta_n, L^{\ell-1}\delta_{n+1})\delta_{\text{odd}}\\
&\quad - \frac{v\alpha}{2\rho}\big(- \alpha^+(\delta_n, L^{\ell-1}\delta_{n-1})\delta_{\text{even}} + \rho^+(\delta_{n+1}, L^{\ell-1}\delta_{n-1})\delta_{\text{even}}\big)\\
&\quad - \frac{v\alpha}{2\rho}\big(\beta^-(\delta_n, L^{\ell-1}\delta_{n-1})\delta_{\text{odd}} + \rho^-(\delta_n, L^{\ell-1}\delta_{n-2})\delta_{\text{odd}}\big)\\
&\quad - \frac{v^+\alpha^+}{2\rho^+}\big(- \alpha^{++}(\delta_n, L^{\ell-1}\delta_{n+1})\delta_{\text{even}} + \rho^{++}(\delta_n, L^{\ell-1}\delta_{n+1})\delta_{\text{even}}\big)\\
&\quad - \frac{v^+\alpha^+}{2\rho^+}\big(\beta(\delta_n, L^{\ell-1}\delta_{n+1})\delta_{\text{odd}} + \rho(\delta_{n-1}, L^{\ell-1}\delta_{n+1})\delta_{\text{odd}}\big)\Big)\\
&= \sum_{n\in\mathbb{Z}}\Big(- v\alpha^+(\delta_n, L^{\ell-1}\delta_n) + v\rho^+(\delta_{n+1}, L^{\ell-1}\delta_n)\delta_{\text{even}} + v\rho^+(\delta_n, L^{\ell-1}\delta_{n+1})\delta_{\text{odd}}\\
&\quad - \frac{v\alpha}{2\rho}\big((\delta_n, EL^{\ell-1}\delta_{n-1})\delta_{\text{even}} + (\delta_n, L^{\ell-1}D\delta_{n-1})\delta_{\text{odd}}\big)\\
&\quad - \frac{v^+\alpha^+}{2\rho^+}\big((\delta_n, L^{\ell-1}D\delta_{n+1})\delta_{\text{even}} + (\delta_n, EL^{\ell-1}\delta_{n+1})\delta_{\text{odd}}\big)\Big)\\
&= \sum_{n\in\mathbb{Z}}\Big(v(\delta_n, EL^{\ell-1}\delta_n)\delta_{\text{even}} + v(\delta_n, L^{\ell-1}D\delta_n)\delta_{\text{odd}}\\
&\quad - \frac{v\alpha}{2\rho}\big((\delta_n, EL^{\ell-1}\delta_{n-1})\delta_{\text{even}} + (\delta_n, L^{\ell-1}D\delta_{n-1})\delta_{\text{odd}}\\
&\quad + (\delta_{n-1}, L^{\ell-1}D\delta_n)\delta_{\text{odd}} + (\delta_{n-1}, EL^{\ell-1}\delta_n)\delta_{\text{even}}\big)\Big)\\
&= \sum_{n\in\mathbb{Z}}\Big(\frac{v}{\rho^2}\big(\hat f_{\ell-1,+} - \alpha\hat g_{\ell,+}\big) + \frac{v\alpha}{2\rho}\big((\delta_{n-1}, EL^{\ell-1}\delta_n)\delta_{\text{even}}\\
&\quad + (\delta_n, L^{\ell-1}D\delta_{n-1})\delta_{\text{odd}} - (\delta_n, EL^{\ell-1}\delta_{n-1})\delta_{\text{even}} - (\delta_{n-1}, L^{\ell-1}D\delta_n)\delta_{\text{odd}}\big)\Big),
\end{aligned} \tag{6.39}$$

where we used (3.21) and

$$\begin{aligned}
\hat g_{\ell,+} &= (\delta_n, L^{\ell-1}DE\delta_n)\\
&= \beta(\delta_n, L^{\ell-1}D\delta_n)\delta_{\text{odd}} + \rho(\delta_n, L^{\ell-1}D\delta_{n-1})\delta_{\text{odd}}\\
&\quad + \beta(\delta_n, EL^{\ell-1}\delta_n)\delta_{\text{even}} + \rho(\delta_{n-1}, EL^{\ell-1}\delta_n)\delta_{\text{even}}\\
&= \beta\hat f_{\ell-1,+} + \rho(\delta_n, L^{\ell-1}D\delta_{n-1})\delta_{\text{odd}} + \rho(\delta_{n-1}, EL^{\ell-1}\delta_n)\delta_{\text{even}}.
\end{aligned} \tag{6.40}$$

Hence it remains to show that

$$\begin{aligned}&(\delta_{n-1}, EL^{\ell-1}\delta_n)\delta_{\mathrm{even}} + (\delta_n, L^{\ell-1}D\delta_{n-1})\delta_{\mathrm{odd}} \\ &\quad - (\delta_n, EL^{\ell-1}\delta_{n-1})\delta_{\mathrm{even}} - (\delta_{n-1}, L^{\ell-1}D\delta_n)\delta_{\mathrm{odd}} = 0,\end{aligned} \tag{6.41}$$

but this follows from $(EL^\ell)^\top = EL^\ell$ respectively $(L^\ell D)^\top = L^\ell D$ by (3.13), (3.16).

In the case (6.36) one similarly shows that

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(\delta_n, L^{-\ell-1}M_\beta\delta_n) = -\sum_{n\in\mathbb{Z}}\frac{v(n+1)}{\rho(n)^2}\Big((\delta_n, D^{-1}L^{-\ell+1}\delta_n)\delta_{\mathrm{even}}(n) \\ + (\delta_n, L^{-\ell+1}E^{-1}\delta_n)\delta_{\mathrm{odd}}(n) + \alpha(n+1)(\delta_n, L^{-\ell}\delta_n)\Big).\end{aligned} \tag{6.42}$$

□

Lemma 6.4. *Assume Hypothesis* 6.1. *Then the following relations hold:*

$$\frac{\delta\hat{g}_{\ell,+}}{\delta\alpha} = -\frac{\ell}{\gamma}\big(\hat{h}^-_{\ell-1,+} + \beta\hat{g}^-_{\ell,+}\big), \quad \ell\in\mathbb{N}, \tag{6.43}$$

$$\frac{\delta\hat{g}_{\ell,-}}{\delta\alpha} = \frac{\ell}{\gamma}\big(\hat{h}_{\ell-1,-} - \beta\hat{g}_{\ell,-}\big), \quad \ell\in\mathbb{N}. \tag{6.44}$$

Proof. We consider (6.43) first. Using (6.25), (6.34), (3.17), and (3.18), and assuming $v\in\ell^1(\mathbb{Z})$ one obtains

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}(\delta_n, L^{\ell-1}M_\alpha\delta_n) \\ &= \sum_{n\in\mathbb{Z}}\Big(- v^+\beta(\delta_n, L^{\ell-1}\delta_n) - v^+\rho(\delta_n, L^{\ell-1}\delta_{n-1})\delta_{\mathrm{even}} - v^{++}\rho^+(\delta_n, L^{\ell-1}\delta_{n+1})\delta_{\mathrm{even}} \\ &\quad - \frac{v\beta}{2\rho}\big(\beta^-(\delta_n, L^{\ell-1}\delta_{n-1})\delta_{\mathrm{odd}} + \rho^-(\delta_n, L^{\ell-1}\delta_{n-2})\delta_{\mathrm{odd}}\big) \\ &\quad - \frac{v\beta}{2\rho}\big(- \alpha^+(\delta_n, L^{\ell-1}\delta_{n-1})\delta_{\mathrm{even}} + \rho^+(\delta_{n+1}, L^{\ell-1}\delta_{n-1})\delta_{\mathrm{even}}\big) \\ &\quad - \frac{v^+\beta^+}{2\rho^+}\big(- \alpha^{++}(\delta_n, L^{\ell-1}\delta_{n+1})\delta_{\mathrm{even}} + \rho^{++}(\delta_n, L^{\ell-1}\delta_{n+2})\delta_{\mathrm{even}}\big) \\ &\quad - \frac{v^+\beta^+}{2\rho^+}\big(\beta(\delta_n, L^{\ell-1}\delta_{n+1})\delta_{\mathrm{odd}} + \rho(\delta_{n-1}, L^{\ell-1}\delta_{n+1})\delta_{\mathrm{odd}}\big)\Big) \\ &= \sum_{n\in\mathbb{Z}}\Big(- v^+\big((\delta_n, L^{\ell-1}D\delta_n)\delta_{\mathrm{even}} + (\delta_n, EL^{\ell-1}\delta_n)\delta_{\mathrm{odd}}\big) \\ &\quad - \frac{v\beta}{2\rho}\big((\delta_n, EL^{\ell-1}\delta_{n-1})\delta_{\mathrm{even}} + (\delta_n, L^{\ell-1}D\delta_{n-1})\delta_{\mathrm{odd}}\big) \\ &\quad - \frac{v^+\beta^+}{2\rho^+}\big((\delta_n, L^{\ell-1}D\delta_{n+1})\delta_{\mathrm{even}} + (\delta_n, EL^{\ell-1}\delta_{n+1})\delta_{\mathrm{odd}}\big)\Big) \\ &= -\sum_{n\in\mathbb{Z}}\frac{v}{\rho^2}\big(\hat{h}^-_{\ell-1,+} + \beta\hat{g}^-_{\ell,+}\big),\end{aligned} \tag{6.45}$$

since by (3.21),

$$\begin{aligned}2\alpha\hat{h}^-_{\ell-1,+} + 2\hat{g}^-_{\ell,+} = \rho\big((\delta_n, L^{\ell-1}D\delta_{n-1})\delta_{\mathrm{odd}} + (\delta_n, EL^{\ell-1}\delta_{n-1})\delta_{\mathrm{even}} \\ + (\delta_{n-1}, L^{\ell-1}D\delta_n)\delta_{\mathrm{odd}} + (\delta_{n-1}, EL^{\ell-1}\delta_n)\delta_{\mathrm{even}}\big).\end{aligned} \tag{6.46}$$

The result (6.44) follows similarly. □

Next, we introduce the Hamiltonians

$$\widehat{\mathcal{H}}_0 = \sum_{n\in\mathbb{Z}} \ln(\gamma(n)), \qquad \widehat{\mathcal{H}}_{p_\pm,\pm} = \frac{1}{p_\pm} \sum_{n\in\mathbb{Z}} \hat{g}_{p_\pm,\pm}(n), \quad p_\pm \in \mathbb{N}, \tag{6.47}$$

$$\mathcal{H}_{\underline{p}} = \sum_{\ell=1}^{p_+} c_{p_+-\ell,+} \widehat{\mathcal{H}}_{\ell,+} + \sum_{\ell=1}^{p_-} c_{p_--\ell,-} \widehat{\mathcal{H}}_{\ell,-} + c_{\underline{p}} \widehat{\mathcal{H}}_0, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2. \tag{6.48}$$

(We recall that $c_{\underline{p}} = (c_{p,-} + c_{p,+})/2$.)

Theorem 6.5. *Assume Hypothesis* 6.1. *Then the following relations hold:*

$$\mathrm{AL}_{\underline{p}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_{t_{\underline{p}}} \\ -i\beta_{t_{\underline{p}}} \end{pmatrix} + \mathcal{D}\nabla\mathcal{H}_{\underline{p}} = 0, \quad \underline{p} \in \mathbb{N}_0^2. \tag{6.49}$$

Proof. This follows directly from Lemmas 6.3 and 6.4,

$$\begin{aligned} (\nabla\widehat{\mathcal{H}}_{\ell,+})_\alpha &= \frac{1}{\gamma}\big(- \beta\hat{g}_{\ell,+}^- - \hat{h}_{\ell-1,+}^- \big), \quad (\nabla\widehat{\mathcal{H}}_{\ell,+})_\beta = \frac{1}{\gamma}\big(- \alpha\hat{g}_{\ell,+} + \hat{f}_{\ell-1,+} \big), \\ (\nabla\widehat{\mathcal{H}}_{\ell,-})_\alpha &= \frac{1}{\gamma}\big(- \beta\hat{g}_{\ell,-} + \hat{h}_{\ell-1,-} \big), \quad (\nabla\widehat{\mathcal{H}}_{\ell,-})_\beta = \frac{1}{\gamma}\big(- \alpha\hat{g}_{\ell,-}^- - \hat{f}_{\ell-1,-}^- \big), \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ell \in \mathbb{N}, \end{aligned} \tag{6.50}$$

together with (2.19). ☐

Theorem 6.6. *Assume Hypothesis* 5.1 *and suppose that* α, β *satisfy* $\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0$ *for some* $\underline{p} \in \mathbb{N}_0^2$. *Then,*

$$\frac{d\mathcal{H}_{\underline{r}}}{dt_{\underline{p}}} = 0, \quad \underline{r} \in \mathbb{N}_0^2. \tag{6.51}$$

Proof. From Lemma 4.4 and Theorem 5.7 one obtains

$$\frac{d\hat{g}_{r_\pm,\pm}}{dt_{\underline{p}}} = (S^+ - I) J_{r_\pm,\pm}, \quad r_\pm \in \mathbb{N}_0, \tag{6.52}$$

for some $J_{r_\pm,\pm}$, $r_\pm \in \mathbb{N}_0$, which are polynomials in α and β and certain shifts thereof. Using definition (6.48) of $\mathcal{H}_{\underline{r}}$, the result (6.51) follows in the homogeneous case and then by linearity in the general case. ☐

Theorem 6.7. *Assume Hypothesis* 6.1 *and let* $\underline{p}, \underline{r} \in \mathbb{N}_0^2$. *Then,*

$$\{\mathcal{H}_{\underline{p}}, \mathcal{H}_{\underline{r}}\} = 0, \tag{6.53}$$

that is, $\mathcal{H}_{\underline{p}}$ *and* $\mathcal{H}_{\underline{r}}$ *are in involution for all* $\underline{p}, \underline{r} \in \mathbb{N}_0^2$.

Proof. By Theorem 5.5, there exists $T > 0$ such that the initial value problem

$$\mathrm{AL}_{\underline{p}}(\alpha,\beta) = 0, \quad (\alpha,\beta)\big|_{t_{\underline{p}}=0} = \big(\alpha^{(0)}, \beta^{(0)}\big), \tag{6.54}$$

where $\alpha^{(0)}, \beta^{(0)}$ satisfy Hypothesis 6.1, has unique, local, and smooth solutions $\alpha(t), \beta(t)$ satisfying Hypothesis 6.1 for each $t \in [0, T)$. For this solution we know that

$$\frac{d}{dt_{\underline{p}}} \mathcal{H}_{\underline{p}}(t) = \{\mathcal{H}_{\underline{r}}(t), \mathcal{H}_{\underline{p}}(t)\} = 0. \tag{6.55}$$

Next, let $t \downarrow 0$. Then

$$0 = \{\mathcal{H}_{\underline{r}}(t), \mathcal{H}_{\underline{p}}(t)\} \underset{t\downarrow 0}{\to} \{\mathcal{H}_{\underline{r}}(0), \mathcal{H}_{\underline{p}}(0)\} = \{\mathcal{H}_{\underline{r}}, \mathcal{H}_{\underline{p}}\}\big|_{(\alpha,\beta)=(\alpha^{(0)},\beta^{(0)})}. \tag{6.56}$$

Since $\alpha^{(0)}, \beta^{(0)}$ are arbitrary coefficients satisfying Hypothesis 6.1 one concludes (6.53). $\square$

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References

[1] M. J. Ablowitz. Nonlinear evolution equations – continuous and discrete. *SIAM Rev.* 19:663–684, 1977.

[2] M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, Cambridge, 1991.

[3] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations. *J. Math. Phys.* 16:598–603, 1975.

[4] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.* 17:1011–1018, 1976.

[5] M. J. Ablowitz and J. F. Ladik. A nonlinear difference scheme and inverse scattering. *Studies Appl. Math.* 55:213–229, 1976.

[6] M. J. Ablowitz and J. F. Ladik. On the solution of a class of nonlinear partial difference equations. *Studies Appl. Math.* 57:1–12, 1977.

[7] M. J. Ablowitz, B. Prinari, and A. D. Trubatch. *Discrete and Continuous Nonlinear Schrödinger Systems*. London Mathematical Society Lecture Note Series, Vol. 302, Cambridge University Press, Cambridge, 2004.

[8] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications.* 2nd ed., Springer, New York, 1988.

[9] G. S. Ammar and W. B. Gragg. Schur flows for orthogonal Hessenberg matrices. *Hamiltonian and Gradient Flows, Algorithms and Control*, A. Bloch (ed.), Fields Inst. Commun., Vol. 3, Amer. Math. Soc., Providence, RI, 1994, pp. 27–34.

[10] M. J. Cantero and B. Simon. Poisson brackets of orthogonal polynomials. *J. Approx. Th.*, to appear.

[11] M. J. Cantero, L. Moral, and L. Velázquez. Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. *Lin. Algebra Appl.* 362:29–56, 2003.

[12] S.-C. Chiu and J. F. Ladik. Generating exactly soluble nonlinear discrete evolution equations by a generalized Wronskian technique. *J. Math. Phys.* 18:690–700, 1977.

[13] A. K. Common. A solution of the initial value problem for half-infinite integrable lattice systems. *Inverse Problems* 8:393–408, 1992.

[14] H.-Y. Ding, Y.-P. Sun, and X.-X. Xu. A hierarchy of nonlinear lattice soliton equations, its integrable coupling systems and infinitely many conservation laws. *Chaos, Solitons and Fractals* 30:227–234, 2006.

[15] N. M. Ercolani and G. I. Lozano. A bi-Hamiltonian structure for the integrable, discrete non-linear Schrödinger system. *Physica D* 218:105–121, 2006.

[16] L. Faybusovich and M. Gekhtman. On Schur flows. *J. Phys. A* 32:4671–4680, 1999.

[17] L. Faybusovich and M. Gekhtman. Elementary Toda orbits and integrable lattices. *J. Math. Phys.* 41:2905–2921, 2000.

[18] M. Gekhtman and I. Nenciu. Multi-hamiltonian structure for the finite defocusing Ablowitz–Ladik equation. *Commun. Pure Appl. Math.*, to appear.

[19] X. Geng and H. H. Dai. Nonlinearization of the Lax pairs for discrete Ablowitz–Ladik hierarchy. *J. Math. Anal. Appl.* 327:829–853, 2007.

[20] X. Geng, H. H. Dai, and J. Zhu. Decomposition of the discrete Ablowitz–Ladik hierarchy. *Stud. Appl. Math.* 118:281–312, 2007.

[21] J. S. Geronimo, F. Gesztesy, and H. Holden. Algebro-geometric solutions of the Baxter–Szegő difference equation. *Commun. Math. Phys.* 258:149–177, 2005.

[22] F. Gesztesy and H. Holden. *Soliton Equations and Their Algebro-Geometric Solutions. Volume I:* $(1+1)$*-Dimensional Continuous Models.* Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge University Press, Cambridge, 2003.

[23] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. *Soliton Equations and Their Algebro-Geometric Solutions. Volume II:* $(1+1)$*-Dimensional Discrete Models.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, to appear.

[24] F. Gesztesy, H. Holden, and G. Teschl. The algebro-geometric Toda hierarchy initial value problem for complex-valued initial data. *Rev. Mat. Iberoamericana*, to appear.

[25] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. The Ablowitz–Ladik hierarchy revisited. To appear in Operator Theory, Advances and Applications, Birkhäuser, Basel; arXiv:nlin/0702058.

[26] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. Algebro-geometric finite-band solutions of the Ablowitz–Ladik hierarchy. *Int. Math. Res. Notices*, 2007:1–55, rnm082.

[27] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. The algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy, arXiv:nlin/07063370.

[28] F. Gesztesy and M. Zinchenko. Weyl–Titchmarsh theory for CMV operators associated with orthogonal polynomials on the unit circle. *J. Approx. Th.* 139:172–213, 2006.

[29] L. Golinskii. Schur flows and orthogonal polynomials on the unit circle. *Sbornik Math.* 197:1145–1165, 2006.

[30] P. E. Hydon. Multisymplectic conservation laws for differential and differential-difference equations. *Proc. Roy. Soc. London Ser. A* 461:1627–1637, 2005.

[31] R. Killip and I. Nenciu. CMV: The unitary analogue of Jacobi matrices. *Commun. Pure Appl. Math.* 59:1–41, 2006.

[32] A. Kriegl and P. W. Michor. *The Convenient Setting of Global Analysis.* Mathematical Surveys and Monographs, Vol. 53, Amer. Math. Soc., Providence, R.I., 1997.

[33] G. I. Lozano. Poisson Geometry of the Ablowitz–Ladik Equations. Ph.D. Thesis, University of Arizona, 2004.

[34] P. D. Miller, N. M. Ercolani, I. M. Krichever, and C. D. Levermore. Finite genus solutions to the Ablowitz–Ladik equations. *Comm. Pure Appl. Math.* 48:1369–1440, 1995.

[35] A. Mukaihira and Y. Nakamura. Schur flow for orthogonal polynomials on the unit circle and its integrable discretization. *J. Comput. Appl. Math.* 139:75–94, 2002.

[36] I. Nenciu. Lax Pairs for the Ablowitz–Ladik System via Orthogonal Polynomials on the Unit Circle. Ph.D. Thesis, California Institute of Technology, Pasadena, CA, 2005.

[37] I. Nenciu. Lax pairs for the Ablowitz–Ladik system via orthogonal polynomials on the unit circle. *Int. Math. Res. Notices* 11:647–686, 2005.

[38] I. Nenciu. CMV matrices in random matrix theory and integrable systems: a survey. *J. Phys. A* 39:8811–8822, 2006.

[39] I. Nenciu. Poisson brackets for orthogonal polynomials on the unit circle. Preprint, 2007.

[40] M. Reed and B. Simon. *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness.* Academic Press, New York, 1975.

[41] R. J. Schilling. A systematic approach to the soliton equations of a discrete eigenvalue problem. *J. Math. Phys.* 30:1487–1501, 1989.

[42] B. Simon. *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Part 2: Spectral Theory*, AMS Colloquium Publication Series, Vol. 54, Amer. Math. Soc., Providence, R.I., 2005.

[43] B. Simon. CMV matrices: Five years after. *J. Comp. Appl. Math.* 208:120–154, 2007.

[44] B. Simon. Zeros of OPUC and long time asymptotics of Schur and related flows. *Inverse Probl. Imaging* 1:189–215, 2007.

[45] Yu. B. Suris. *The Problem of Integrable Discretization: Hamiltonian Approach.* Birkhäuser, Basel, 2003.

[46] G. Teschl. *Jacobi Operators and Completely Integrable Nonlinear Lattices.* Math. Surveys and Monographs, Vol. 72, Amer. Math. Soc., Providence, R.I., 2000.

[47] V. E. Vekslerchik. Functional representation of the Ablowitz–Ladik hierarchy. *J. Phys. A* 31:1087–1099, 1998.

[48] V. E. Vekslerchik. Functional representation of the Ablowitz–Ladik hierarchy. II. *J. Nonlin. Math. Phys.* 9:157–180, 2002.

[49] Y. Zeng and S. Rauch-Wojciechowski. Restricted flows of the Ablowitz–Ladik hierarchy and their continuous limits. *J. Phys. A* 28:113–134, 1995.

[50] D. J. Zhang and D. Y. Chen. The conservation laws of some discrete soliton systems. *Chaos, Solitons and Fractals* 14:573–579, 2002.

[51] D. J. Zhang and D. Y. Chen. Hamiltonian structure of discrete soliton systems. *J. Phys. A* 35:7225–7241, 2002.

[52] D. J. Zhang, T. K. Ning, J. B. Bi, and D. Y. Chen. New symmetries for the Ablowitz–Ladik hierarchies. *Phys. Lett. A* 359:458–466, 2006.

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ON THE SPATIAL ASYMPTOTICS OF SOLUTIONS OF THE ABLOWITZ–LADIK HIERARCHY

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Dedicated with great pleasure to Peter W. Michor on the occasion of his 60th birthday.

Abstract. We show that for decaying solutions of the Ablowitz–Ladik system, the leading asymptotic term is time independent. In addition, two arbitrary bounded solutions of the Ablowitz–Ladik system which are asymptotically close at the initial time stay close. All results are also derived for the associated hierarchy.

1. Introduction

When solving completely integrable wave equations via the inverse scattering transform, a method developed by Gardner et al. [12] in 1967 for the Korteweg–de Vries (KdV) equation, one intends to prove existence of solutions within the respective class. In particular, short-range perturbations of the background solution should remain short-range during the time evolution. So to what extend are spatial asymptotical properties time independent?

For the KdV equation, this question was answered by Bondareva and Shubin [9], [10], who considered the Cauchy problem for initial conditions which have a prescribed asymptotic expansion in terms of powers of the spatial variable and showed that the leading term of the expansion is time independent. Teschl [18] considered the initial value problem for the Toda lattice in the class of decaying solutions and obtained time independence of the leading term.

In this note we want to address the same question for the Ablowitz–Ladik (AL) system, an integrable discretization of the AKNS-ZS system derived by Ablowitz and Ladik ([3]–[6]) in the mid seventies. The AL system is given by

$$\begin{aligned} &-i\alpha_t - (1-\alpha\beta)(\alpha^- + \alpha^+) + 2\alpha = 0,\\ &-i\beta_t + (1-\alpha\beta)(\beta^- + \beta^+) - 2\beta = 0, \end{aligned} \tag{1.1}$$

where $\alpha = \alpha(n,t)$, $\beta = \beta(n,t)$, $(n,t) \in \mathbb{Z} \times \mathbb{R}$, are complex valued sequences and $f^\pm(n,t) = f(n \pm 1, t)$. In the defocusing ($\beta = \overline{\alpha}$) and focusing case ($\beta = -\overline{\alpha}$), (1.1) is a discrete analog of the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}q_t + q_{xx} \pm 2q|q|^2 = 0.$$

We refer to the monographs [2], [7], or [13] for further information.

Our main result in Theorem 2.4 yields that the dominant term of suitably decaying solutions $\alpha(n,t)$, $\beta(n,t)$ of (1.1), for instance weighted ℓ^{2p} sequences whose

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spatial difference is in ℓ^p, $1 \le p < \infty$, is time independent. For example,

$$\alpha(n,t) = \frac{a}{n^\delta} + O\Big(\frac{1}{n^{\min(2\delta,\delta+1)}}\Big), \quad \beta(n,t) = \frac{b}{n^\delta} + O\Big(\frac{1}{n^{\min(2\delta,\delta+1)}}\Big), \quad n \to \infty, \tag{1.2}$$

holds for fixed t, provided it holds at the initial time $t = t_0$. Here $a, b \in \mathbb{C}$ and $\delta \ge 0$. A similar expression is valid for $n \to -\infty$.

The inverse scattering transform for the AL system with vanishing boundary conditions was studied in [4]. Ablowitz, Biondini, and Prinari [1] (compare also Vekslerchik and Konotop [19]) considered nonvanishing steplike boundary conditions $\alpha(n) \to \alpha_0 e^{\mathrm{i}\theta_\pm}$ as $|n| \to \infty$, $\alpha_0 > 0$, in the class

$$\sum_{j=n}^{\pm\infty} (\alpha(j) - \alpha_0 e^{\mathrm{i}\theta_\pm}) < \infty \tag{1.3}$$

for the defocusing discrete NLS equation. Quasi-periodic boundary conditions for the AL hierarchy will be considered in Michor [17]. As mentioned, a crucial step is to show that short-range perturbations like (1.3) of solutions stay short-range. Here we show in general that arbitrary bounded solutions of the AL system which are asymptotically close at the initial time stay close.

In Section 2 we derive our results for the AL system and extend them in Section 3 to the AL hierarchy, a completely integrable hierarchy of nonlinear evolution equations whose first nonlinear member is (1.1).

2. The initial value problem for the Ablowitz–Ladik system

Let us begin by recalling some basic facts on the system (1.1). We will only consider bounded solutions and hence require

Hypothesis H. 2.1. *Suppose that* $\alpha, \beta \colon \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ *satisfy*

$$\begin{aligned} &\sup_{(n,t)\in\mathbb{Z}\times\mathbb{R}} \big(|\alpha(n,t)| + |\beta(n,t)|\big) < \infty, \\ &\alpha(n,\,\cdot\,),\ \beta(n,\,\cdot\,) \in C^1(\mathbb{R}),\ n \in \mathbb{Z}, \quad \alpha(n,t)\beta(n,t) \notin \{0,1\},\ (n,t) \in \mathbb{Z}\times\mathbb{R}. \end{aligned} \tag{2.1}$$

The AL system (1.1) is equivalent to the zero-curvature equation

$$U_t + UV - V^+ U = 0, \tag{2.2}$$

where

$$U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix}, \quad V(z) = i \begin{pmatrix} z - 1 - \alpha\beta^- & \alpha - \alpha^- z^{-1} \\ \beta^- z - \beta & 1 + \alpha^-\beta - z^{-1} \end{pmatrix} \tag{2.3}$$

for the spectral parameter $z \in \mathbb{C} \setminus \{0\}$. The AL system can also be formulated in terms of Lax pairs, see [15]. Then (1.1) is equivalent to the Lax equation

$$\frac{d}{dt} L(t) - [P(t), L(t)] = 0, \qquad t \in \mathbb{R}, \tag{2.4}$$

where L reads in the standard basis of $\ell^2(\mathbb{Z})$ (abbreviate $\rho = (1 - \alpha\beta)^{1/2}$)

$$L = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & 0 & -\alpha(0)\rho(-1) & -\beta(-1)\alpha(0) & -\alpha(1)\rho(0) & \rho(0)\rho(1) & & & \mathbf{0} \\ & & \rho(-1)\rho(0) & \beta(-1)\rho(0) & -\beta(0)\alpha(1) & \beta(0)\rho(1) & 0 & & \\ & & & 0 & -\alpha(2)\rho(1) & -\beta(1)\alpha(2) & -\alpha(3)\rho(2) & \rho(2)\rho(3) & \\ & & & & \rho(1)\rho(2) & \beta(1)\rho(2) & -\beta(2)\alpha(3) & \beta(2)\rho(3) & 0 \\ \mathbf{0} & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.5}$$

and P is given by

$$P = \tfrac{i}{2}\big(L_+ - L_- + (L^{-1})_- - (L^{-1})_+ + 2Q_d\big).$$

Here Q_d is the doubly infinite diagonal matrix $Q_d = \big((-1)^k \delta_{k,\ell}\big)_{k,\ell\in\mathbb{Z}}$ and $L_\pm$ denote the upper and lower triangular parts of L,

$$L_\pm = \big(L_\pm(m,n)\big)_{(m,n)\in\mathbb{Z}^2}, \quad L_\pm(m,n) = \begin{cases} L(m,n), & \pm(n-m) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

The Lax equation (2.4) implies existence of a propagator $W(s,t)$ such that the family of operators $L(t)$, $t \in \mathbb{R}$, is similar,

$$L(s) = W(s,t)L(t)W(s,t)^{-1}, \quad s,t \in \mathbb{R}.$$

By [13, Sec. 3.8] or [15], existence, uniqueness, and smoothness of local solutions of the AL initial value problem follow from [8, Thm 4.1.5], since the AL flows are autonomous.

Theorem 2.2. *Let $t_0 \in \mathbb{R}$ and suppose $(\alpha_0, \beta_0) \in M = \ell^p(\mathbb{Z}) \oplus \ell^p(\mathbb{Z})$ for some $p \in [1,\infty) \cup \{\infty\}$. Then there exists $T > 0$ and a unique local integral curve $t \mapsto (\alpha(t), \beta(t))$ in $C^\infty((t_0 - T, t_0 + T), M)$ of the Ablowitz–Ladik system* (1.1) *such that $(\alpha,\beta)\big|_{t=t_0} = (\alpha_0, \beta_0)$.*

Our first lemma shows that the leading asymptotics as $n \to \pm\infty$ are preserved by the AL flow. We only state the result for the AL system, whose proof follows as the one of Lemma 3.2. Define

$$\|(\alpha,\beta)\|_{w,p} = \begin{cases} \Big(\sum\limits_{n\in\mathbb{Z}} w(n)\big(|\alpha(n)|^p + |\beta(n)|^p\big)\Big)^{1/p}, & 1 \le p < \infty, \\ \sup\limits_{n\in\mathbb{Z}} w(n)\big(|\alpha(n)| + |\beta(n)|\big), & p = \infty. \end{cases} \tag{2.7}$$

Lemma 2.3. *Let $w(n) \ge 1$ be some weight with* $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$. *Fix $1 \le p \le \infty$ and suppose $(\alpha(n,t), \beta(n,t))$ and $(\tilde\alpha(n,t), \tilde\beta(n,t))$ are arbitrary bounded solutions of the* AL *system* (1.1). *If*

$$\|(\alpha(t) - \tilde\alpha(t), \beta(t) - \tilde\beta(t))\|_{w,p} < \infty \tag{2.8}$$

holds for one $t = t_0 \in \mathbb{R}$, then it holds for all $t \in (t_0 - T, t_0 + T)$.

But even the leading term is preserved by the time evolution.

Theorem 2.4. *Let $w(n) \ge 1$ be some weight with* $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$. *Fix $1 \le p \le \infty$ and suppose α_0, β_0 and $\tilde\alpha_0$, $\tilde\beta_0$ are bounded sequences such that*

$$\begin{matrix} \|(\alpha_0,\beta_0)\|_{w,2p} < \infty, \quad \|(\alpha_0 - \alpha_0^+, \beta_0 - \beta_0^+)\|_{w,p} < \infty, \\ \|(\tilde\alpha_0,\tilde\beta_0)\|_{w,p} < \infty, \end{matrix} \qquad \textit{if } 1 \le p < \infty,$$

$$\begin{matrix} \|(\alpha_0,\beta_0)\|_{w,\infty} < \infty, \quad \|(\alpha_0 - \alpha_0^+, \beta_0 - \beta_0^+)\|_{w^2,\infty} < \infty, \\ \|(\tilde\alpha_0,\tilde\beta_0)\|_{w^2,\infty} < \infty, \end{matrix} \qquad \textit{if } p = \infty.$$

Let $(\alpha(t), \beta(t))$, $t \in (-T, T)$, be the unique solution of the Ablowitz–Ladik system (1.1) *corresponding to the initial conditions*

$$\alpha(0) = \alpha_0 + \tilde\alpha_0, \qquad \beta(0) = \beta_0 + \tilde\beta_0. \tag{2.9}$$

Then this solution is of the form

$$\alpha(t) = \alpha_0 + \tilde{\alpha}(t), \qquad \beta(t) = \beta_0 + \tilde{\beta}(t), \tag{2.10}$$

where $\|(\tilde{\alpha}(t), \tilde{\beta}(t))\|_{w,p} < \infty$, *respectively,* $\|(\tilde{\alpha}(t), \tilde{\beta}(t))\|_{w^2,\infty} < \infty$.

Proof. The proof relies on the idea to consider our differential equation in two nested spaces of sequences, the Banach space of all $(\alpha(n), \beta(n))$ with sup norm, and the Banach space with norm $\|.\|_{w,p}$, as follows. Plugging $(\alpha_0 + \tilde{\alpha}(t), \beta_0 + \tilde{\beta}(t))$ into the AL equations (1.1) yields a differential equation for $(\tilde{\alpha}(t), \tilde{\beta}(t))$

$$\begin{aligned}
\mathrm{i}\tilde{\alpha}_t(t) &= -\big(1 - (\alpha_0 + \tilde{\alpha}(t))(\beta_0 + \tilde{\beta}(t))\big)\big(\tilde{\alpha}^+(t) + \alpha_0^+ + \tilde{\alpha}^-(t) + \alpha_0^-\big) + 2\tilde{\alpha}(t) + 2\alpha_0 \\
&= \alpha_0 - \alpha_0^- + \alpha_0 - \alpha_0^+ + \alpha_0\beta_0(\alpha_0^+ + \alpha_0^-) \\
&\quad + \tilde{\alpha}(t)\big(2 + (\beta_0 + \tilde{\beta}(t))(\tilde{\alpha}^+(t) + \alpha_0^+ + \tilde{\alpha}^-(t) + \alpha_0^-)\big) \\
&\quad + \tilde{\beta}(t)\alpha_0\big(\tilde{\alpha}^+(t) + \alpha_0^+ + \tilde{\alpha}^-(t) + \alpha_0^-\big) \\
&\quad + \tilde{\alpha}^+(t)(\alpha_0\beta_0 - 1) + \tilde{\alpha}^-(t)(\alpha_0\beta_0 - 1), \\
\mathrm{i}\tilde{\beta}_t(t) &= \big(1 - (\alpha_0 + \tilde{\alpha}(t))(\beta_0 + \tilde{\beta}(t))\big)\big(\tilde{\beta}^+(t) + \beta_0^+ + \tilde{\beta}^-(t) + \beta_0^-\big) - 2\tilde{\beta}(t) - 2\beta_0 \\
&= \beta_0^+ - \beta_0 + \beta_0^- - \beta_0 - \alpha_0\beta_0(\beta_0^+ + \beta_0^-) \\
&\quad - \tilde{\beta}(t)\big(2 + (\alpha_0 + \tilde{\alpha}(t))(\tilde{\beta}^+(t) + \beta_0^+ + \tilde{\beta}^-(t) + \beta_0^-)\big) \\
&\quad - \tilde{\alpha}(t)\beta_0\big(\tilde{\beta}^+(t) + \beta_0^+ + \tilde{\beta}^-(t) + \beta_0^-)\big) \\
&\quad - \tilde{\beta}^+(t)(\alpha_0\beta_0 - 1) - \tilde{\beta}^-(t)(\alpha_0\beta_0 - 1).
\end{aligned} \tag{2.11}$$

The requirement on $w(n)$ implies that the shift operators are continuous with respect to the norm $\|.\|_{w,p}$ and the same is true for the multiplication operator with a bounded sequence. Therefore, using the generalized Hölder inequality yields that (2.11) is a system of inhomogeneous linear differential equations in the Banach space with norm $\|.\|_{w,p}$ and has a local solution with respect to this norm (see e.g. [11] for the theory of ordinary differential equations in Banach spaces). Since $w(n) \geq 1$, this solution is bounded and the corresponding coefficients $(\tilde{\alpha}, \tilde{\beta})$ coincide with the solution (α, β) of the AL system (1.1) from Theorem 2.2.

Moreover, $(\tilde{\alpha}(t), \tilde{\beta}(t))$ are uniformly bounded for $t \in (-T, T)$, as writing (2.11) in integral form yields

$$\|(\tilde{\alpha}(t), \tilde{\beta}(t))\|_{w,p} \leq \|(\tilde{\alpha}(0), \tilde{\beta}(0))\|_{w,p} + tC\|(\alpha_0, \beta_0)\|_{w,2p} + C\int_0^t \|(\tilde{\alpha}(s), \tilde{\beta}(s))\|_{w,p} ds$$

for some constants C. $\square$

Example (1.2) in the introduction follows if we let $\tilde{\alpha}_0 = \tilde{\beta}_0 \equiv 0$ and

$$\alpha_0(n) = \frac{a}{n^\delta}, \quad \beta_0(n) = \frac{b}{n^\delta}, \quad a, b \in \mathbb{C}, \quad \delta \geq 0,$$

for $n > 0$, $\alpha_0(n) = \beta_0(n) = 0$ for $n \leq 0$. Now choose $p = \infty$ with

$$w(n) = \begin{cases} (1+n)^{\min(\delta, (\delta+1)/2)}, & n > 0, \\ 1, & n \leq 0, \end{cases}$$

and apply Theorem 2.4.

Finally, we remark that if a solution $(\alpha(n,t), \beta(n,t))$ vanishes at two consecutive points $n = n_0$, $n = n_0 + 1$ in an arbitrarily small time intervall $t \in (t_1, t_2)$, then it vanishes identically for all (n, t) in $\mathbb{Z}\times\mathbb{R}$, see [16]. In particular, a compact support of the solution is not preserved. The corresponding result for the AL hierarchy is derived in [16] as well.

3. Extension to the Ablowitz–Ladik hierarchy

In this section we show how our results extend to the AL hierarchy. The hierarchy can be constructed by generalizing the matrix $V(z)$ in the zero-curvature equation (2.2) to a 2×2 matrix $V_{\underline{r}}(z)$, $\underline{r} = (r_-, r_+) \in \mathbb{N}_0^2$, with Laurent polynomial entries, see [13, Sec. 3.2] or [14]. Suppose that $U(z)$ and $V_{\underline{r}}(z)$ satisfy the zero-curvature equation

$$U_t + UV_{\underline{r}} - V_{\underline{r}}^+ U = 0. \tag{3.1}$$

Then the coefficients $\{f_{\ell,\pm}\}_{\ell=0,\dots,r_\pm-1}$, $\{g_{\ell,\pm}\}_{\ell=0,\dots,r_\pm}$, and $\{h_{\ell,\pm}\}_{\ell=0,\dots,r_\pm-1}$ of the Laurent polynomials in the entries of $V_{\underline{r}}(z)$ are recursively defined by

$$\begin{aligned} g_{0,+} &= \tfrac{1}{2}c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^+, \quad h_{0,+} = c_{0,+}\beta, \\ g_{\ell+1,+} - g_{\ell+1,+}^- &= \alpha h_{\ell,+}^- + \beta f_{\ell,+}, \quad 0 \le \ell \le r_+ - 1, \\ f_{\ell+1,+}^- &= f_{\ell,+} - \alpha(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad 0 \le \ell \le r_+ - 2, \\ h_{\ell+1,+} &= h_{\ell,+}^- + \beta(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad 0 \le \ell \le r_+ - 2, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} g_{0,-} &= \tfrac{1}{2}c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^+, \\ g_{\ell+1,-} - g_{\ell+1,-}^- &= \alpha h_{\ell,-} + \beta f_{\ell,-}^-, \quad 0 \le \ell \le r_- - 1, \\ f_{\ell+1,-} &= f_{\ell,-}^- + \alpha(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad 0 \le \ell \le r_- - 2, \\ h_{\ell+1,-}^- &= h_{\ell,-} - \beta(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad 0 \le \ell \le r_- - 2. \end{aligned} \tag{3.3}$$

Note that $g_{\ell,\pm}$ are only defined up to summation constants $\{c_{\ell,\pm}\}_{\ell=0,\dots,r_\pm}$ by the difference equations in (3.2), (3.3). In addition, the zero-curvature equation (3.1) is equivalent to

$$0 = i \begin{pmatrix} 0 & \begin{matrix} -i\alpha_t - \alpha(g_{r_+,+} + g_{r_-,-}^-) \\ +f_{r_+-1,+} - f_{r_--1,-}^- \end{matrix} \\ \begin{matrix} z\big(-i\beta_t + \beta(g_{r_+,+}^- + g_{r_-,-}) \\ -h_{r_--1,-} + h_{r_+-1,+}^-\big) \end{matrix} & 0 \end{pmatrix}.$$

Varying $\underline{r} \in \mathbb{N}_0^2$, the collection of evolution equations

$$\begin{aligned} \mathrm{AL}_{\underline{r}}(\alpha,\beta) = \begin{pmatrix} -i\alpha_t - \alpha(g_{r_+,+} + g_{r_-,-}^-) + f_{r_+-1,+} - f_{r_--1,-}^- \\ -i\beta_t + \beta(g_{r_+,+}^- + g_{r_-,-}) - h_{r_--1,-} + h_{r_+-1,+}^- \end{pmatrix} = 0, \\ t\in\mathbb{R},\ \underline{r} = (r_-, r_+) \in \mathbb{N}_0^2, \end{aligned} \tag{3.4}$$

then defines the time-dependent Ablowitz–Ladik hierarchy. Explicitly, taking $r_- = r_+$ for simplicity, the first few equations are

$$\begin{aligned} \mathrm{AL}_{(0,0)}(\alpha,\beta) &= \begin{pmatrix} -i\alpha_t - c_{(0,0)}\alpha \\ -i\beta_t + c_{(0,0)}\beta \end{pmatrix} = 0, \\ \mathrm{AL}_{(1,1)}(\alpha,\beta) &= \begin{pmatrix} -i\alpha_t - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ -i\beta_t + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0, \end{aligned}$$

$$\mathrm{AL}_{(2,2)}(\alpha,\beta) = \begin{pmatrix} -i\alpha_t - \gamma\big(c_{0,+}\alpha^{++}\gamma^{+} + c_{0,-}\alpha^{--}\gamma^{-} \\ -\alpha(c_{0,+}\alpha^{+}\beta^{-} + c_{0,-}\alpha^{-}\beta^{+}) - \beta(c_{0,-}(\alpha^{-})^2 + c_{0,+}(\alpha^{+})^2)\big) \\ -i\beta_t + \gamma\big(c_{0,-}\beta^{++}\gamma^{+} + c_{0,+}\beta^{--}\gamma^{-} \\ -\beta(c_{0,+}\alpha^{+}\beta^{-} + c_{0,-}\alpha^{-}\beta^{+}) - \alpha(c_{0,+}(\beta^{-})^2 + c_{0,-}(\beta^{+})^2)\big) \end{pmatrix} + \begin{pmatrix} -\gamma(c_{1,-}\alpha^{-} + c_{1,+}\alpha^{+}) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^{-} + c_{1,-}\beta^{+}) + c_{(2,2)}\beta \end{pmatrix} = 0, \text{ etc.,} \tag{3.5}$$

where we abbreviated $c_{\underline{r}} = (c_{r,-} + c_{r,+})/2$ and $\gamma = 1 - \alpha\beta$. Different ratios of $c_{0,+}/c_{0,-}$ lead to different hierarchies. The AL system (1.1) corresponds to the case $\underline{r} = (1,1)$, $c_{0,\pm} = 1$, and $c_{(1,1)} = -2$. The special choices $\beta = \pm\overline{\alpha}$, $c_{0,\pm} = 1$ lead to the discrete NLS hierarchy, the choices $\beta = \overline{\alpha}$, $c_{0,\pm} = \mp\mathrm{i}$ yield the hierarchy of Schur flows. The AL hierarchy is invariant under the scaling transform

$$\{(\alpha(n),\beta(n))\}_{n\in\mathbb{Z}} \to \{(c\,\alpha(n),\beta(n)/c)\}_{n\in\mathbb{Z}}, \quad c \in \mathbb{C}\backslash\{0\}. \tag{3.6}$$

Hence choosing $c = e^{\mathrm{i}c_{\underline{r}}t}$ it is no restriction to assume $c_{\underline{r}} = 0$.

By [15], the AL hierarchy is equivalent to the Lax equation

$$\frac{d}{dt}L(t) - [P_{\underline{r}}(t), L(t)] = 0, \quad t \in \mathbb{R}, \quad \underline{r} \in \mathbb{N}_0^2, \tag{3.7}$$

where L is the doubly infinite five-diagonal matrix (2.5) and (recall (2.6))

$$P_{\underline{r}} = \frac{i}{2}\sum_{\ell=1}^{r_+} c_{r_+-\ell,+}\big((L^\ell)_+ - (L^\ell)_-\big) - \frac{i}{2}\sum_{\ell=1}^{r_-} c_{r_--\ell,-}\big((L^{-\ell})_+ - (L^{-\ell})_-\big) - \frac{i}{2}c_{\underline{r}}Q_d.$$

Since the AL flows are autonomous and $f_{r_\pm-1,\pm}$, $g_{r_\pm,\pm}$, and $h_{r_\pm-1,\pm}$ depend polynomially on α,β and their shifts, [8, Thm 4.1.5] implies local existence, uniqueness, and smoothness of the solution of the initial value problem of the hierarchy as well (see [13, Sec. 3.8], [15]).

Theorem 3.1. *Let $t_0 \in \mathbb{R}$ and suppose $\alpha_0, \beta_0 \in \ell^p(\mathbb{Z})$ for some $p \in [1,\infty) \cup \{\infty\}$. Then the $\underline{r}$th Ablowitz–Ladik initial value problem*

$$\mathrm{AL}_{\underline{r}}(\alpha,\beta) = 0, \quad (\alpha,\beta)\big|_{t=t_0} = (\alpha_0,\beta_0) \tag{3.8}$$

for some $\underline{r} \in \mathbb{N}_0^2$, has a unique, local, and smooth solution in time, that is, there exists a $T > 0$ such that $\alpha(\,\cdot\,)$, $\beta(\,\cdot\,) \in C^\infty((t_0 - T, t_0 + T), \ell^p(\mathbb{Z}))$.

Next we show that short-range perturbations of bounded solutions remain short-range. In fact, we will be more general to include perturbations of steplike background solutions as for example (1.3).

Lemma 3.2. *Let $w(n) \geq 1$ be some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$ and fix $1 \leq p \leq \infty$. Suppose $(\alpha(t),\beta(t))$ and $(\alpha_{\ell,r}(t), \beta_{\ell,r}(t))$ are arbitrary bounded solutions of some equation $\mathrm{AL}_{\underline{r}}$ in the AL hierarchy and abbreviate*

$$\tilde{\alpha}(n,t) = \begin{cases} \alpha_r(n,t), & n \geq 0, \\ \alpha_\ell(n,t), & n < 0, \end{cases} \qquad \tilde{\beta}(n,t) = \begin{cases} \beta_r(n,t), & n \geq 0, \\ \beta_\ell(n,t), & n < 0. \end{cases} \tag{3.9}$$

If

$$\|(\alpha(t) - \tilde{\alpha}(t), \beta(t) - \tilde{\beta}(t))\|_{w,p} < \infty \tag{3.10}$$

holds for one $t = t_0 \in \mathbb{R}$, then it holds for all $t \in (t_0 - T, t_0 + T)$.

Proof. Without loss we assume that $t_0 = 0$. First we derive the differential equation for the differences $\delta(n,t) = \big(\alpha(n,t) - \tilde{\alpha}(n,t), \beta(n,t) - \tilde{\beta}(n,t)\big)$ in the Banach space of pairs of bounded sequences $\delta = (\delta_1, \delta_2)$ for which the norm $\|\delta\|_{w,p}$ is finite.

Let us show by induction on $r_\pm$ that $f_{r_\pm-1,\pm}(t) - \tilde{f}_{r_\pm-1,\pm}(t)$, $g_{r_\pm,\pm}(t) - \tilde{g}_{r_\pm,\pm}(t)$, and $h_{r_\pm-1,\pm}(t) - \tilde{h}_{r_\pm-1,\pm}(t)$ can be written as a linear combination of shifts of δ with the coefficients depending only on $(\alpha(t), \beta(t))$ and $(\alpha_{\ell,r}(t), \beta_{\ell,r}(t))$. It suffices to consider the homogeneous case where $c_{j,\pm} = 0$, $1 \le j \le r_\pm$, since all involved sums are finite. In this case [14, Lemma A.3] yields that $f_{j,+}$, $g_{j,+}$, and $h_{j,+}$ can be recursively computed from $f_{0,+} = -\alpha^+$, $g_{0,+} = \frac{1}{2}$, and $h_{0,+} = \beta$ via

$$\begin{aligned} f^-_{\ell+1,+} &= f_{\ell,+} - \alpha(g_{\ell+1,+} + g^-_{\ell+1,+}), \\ h_{\ell+1,+} &= h^-_{\ell,+} + \beta(g_{\ell+1,+} + g^-_{\ell+1,+}), \\ g_{\ell+1,+} &= \sum_{k=0}^{\ell} f_{\ell-k,+} h_{k,+} - \sum_{k=1}^{\ell} g_{\ell+1-k,+} g_{k,+}, \end{aligned}$$

and similarly for the minus sign and $\tilde{f}_{j,\pm}$, $\tilde{g}_{j,\pm}$, and $\tilde{h}_{j,\pm}$. The fact that $(\tilde{\alpha}, \tilde{\beta})$ does not solve $\mathrm{AL}_{\underline{r}}$ only affects finitely many terms and gives rise to an inhomogeneous term $B_{\underline{r}}(t)$ which is nonzero only for a finite number of terms.

Hence δ satisfies an inhomogeneous linear differential equation of the form

$$\mathrm{i}\frac{d}{dt}\delta(t) = \sum_{|j| \le \max(r_-, r_+)} A_{\underline{r},j}(t)(S^+)^j \delta(t) + B_{\underline{r}}(t)$$

Here $S^\pm(\delta_1(n,t), \delta_2(n,t)) = (\delta_1(n \pm 1, t), \delta_2(n \pm 1, t))$ are the shift operators,

$$A_{\underline{r},j}(n,t) = \begin{pmatrix} A^{11}_{\underline{r},j}(n,t) & A^{12}_{\underline{r},j}(n,t) \\ A^{21}_{\underline{r},j}(n,t) & A^{22}_{\underline{r},j}(n,t) \end{pmatrix},$$

are multiplication operators with bounded 2×2 matrix-valued sequences, and $B_{\underline{r}}(n,t) = \big(B_{\underline{r},1}(n,t), B_{\underline{r},2}(n,t)\big)$ with $B_{r,i}(n,t) = 0$ for $|n| > \max(r_-, r_+)$. All entries of $A_{\underline{r},j}(t)$ and $B_{\underline{r}}(t)$ are polynomials with respect to $(\alpha(n+j,t), \beta(n+j,t))$, $(\alpha_{\ell,r}(n+j,t), \beta_{\ell,r}(n+j,t))$, $|j| \le \max(r_-, r_+)$. Thus $\|B_{\underline{r}}(t)\|_{w,p} \le D_{\underline{r}}$, where the constant depends only on the sup norms of $(\alpha(t), \beta(t))$ and $(\alpha_{\ell,r}(t), \beta_{\ell,r}(t))$. Moreover, by our assumption the shift operators are continuous,

$$\|S^\pm\| = \begin{cases} \sup_{n\in\mathbb{Z}} |\frac{w(n)}{w(n\pm1)}|^{1/p}, & p \in [1,\infty), \\ \sup_{n\in\mathbb{Z}} |\frac{w(n)}{w(n\pm1)}|, & p = \infty, \end{cases}$$

and the same is true for the multiplication operators $A_{\underline{r},j}(t)$ whose norms depend only on the supremum of the entries by Hölder's inequality, that is, again on the sup norms of $(\alpha(t), \beta(t))$ and $(\alpha_{\ell,r}(t), \beta_{\ell,r}(t))$. Consequently, for $t \in (-T, T)$ there is a constant such that $\sum_{|j|\le\max(r_-,r_+)} \|A_{\underline{r},j}(t)\| \|(S^+)^j\| \le C_{\underline{r}}$. Hence

$$\|\delta(t)\|_{w,p} \le \|\delta(0)\|_{w,p} + \int_0^t \big(C_{\underline{r}} \|\delta(s)\|_{w,p} + D_{\underline{r}}\big)$$

and Gronwall's inequality implies

$$\|\delta(t)\|_{w,p} \le \|\delta(0)\|_{w,p} \mathrm{e}^{C_{\underline{r}} t} + \frac{D_{\underline{r}}}{C_{\underline{r}}} \big(\mathrm{e}^{C_{\underline{r}} t} - 1\big).$$

Since $w(n) \ge 1$, this solution is again bounded and hence coincides with the solution of the AL equation from Theorem 3.1. □

For certain equations in the AL hierarchy, i.e. for certain configurations of summation coefficients $\{c_{j,\pm}\}$, our main result remains valid.

Theorem 3.3. *Let* $\underline{r} = (r_-, r_+) \in \mathbb{N}_0^2 \backslash (0,0)$ *and assume that* $c_{j,\pm} \in \mathbb{C}$, $j = 0, \dots, r_\pm$, *satisfy*

$$\sum_{j=0}^{r_+-1} c_{j,+} + \sum_{j=0}^{r_--1} c_{j,-} = 0. \tag{3.11}$$

Let $w(n) \geq 1$ *be some weight with* $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$. *Fix* $1 \leq p \leq \infty$ *and suppose* α_0, β_0 *and* $\tilde{\alpha}_0$, $\tilde{\beta}_0$ *are bounded sequences such that*

$$\begin{aligned} &\|(\alpha_0, \beta_0)\|_{w,2p} < \infty, \quad \|(\alpha_0 - \alpha_0^+, \beta_0 - \beta_0^+)\|_{w,p} < \infty, \\ &\|(\tilde{\alpha}_0, \tilde{\beta}_0)\|_{w,p} < \infty, \end{aligned} \qquad \text{if } 1 \leq p < \infty,$$

$$\begin{aligned} &\|(\alpha_0, \beta_0)\|_{w,\infty} < \infty, \quad \|(\alpha_0 - \alpha_0^+, \beta_0 - \beta_0^+)\|_{w^2,\infty} < \infty, \\ &\|(\tilde{\alpha}_0, \tilde{\beta}_0)\|_{w^2,\infty} < \infty, \end{aligned} \qquad \text{if } p = \infty.$$

Let $(\alpha(t), \beta(t))$, $t \in (-T, T)$, *be the unique solution of the equation* $\mathrm{AL}_{\underline{r}}(\alpha, \beta) = 0$ *with summation coefficients* $\{c_{j,\pm}\}_{j=0}^{r_\pm}$, *corresponding to the initial conditions*

$$\alpha(0) = \alpha_0 + \tilde{\alpha}_0, \qquad \beta(0) = \beta_0 + \tilde{\beta}_0. \tag{3.12}$$

Then this solution is of the form

$$\alpha(t) = \alpha_0 + \tilde{\alpha}(t), \qquad \beta(t) = \beta_0 + \tilde{\beta}(t), \tag{3.13}$$

where $\|(\tilde{\alpha}(t), \tilde{\beta}(t))\|_{w,p} < \infty$, *respectively,* $\|(\tilde{\alpha}(t), \tilde{\beta}(t))\|_{w^2,\infty} < \infty$.

Proof. The proof is similar to the one of Theorem 2.4. From $\mathrm{AL}_{\underline{r}}(\alpha, \beta) = 0$ we obtain an inhomogeneous differential equation for $(\tilde{\alpha}, \tilde{\beta})$. The homogeneous part is a finite sum over shifts of $(\tilde{\alpha}, \tilde{\beta})$. The inhomogeneous part consists of products of α_0, β_0 and their shifts, whose $\|.\|_{w,p}$ norm is finite by Hölder's inequality, and of sums of the form $c_{j,\pm}\alpha_0$, $c_{j,\pm}\beta_0$ and shifts thereof,

$$\begin{aligned} -\big(&c_{0,+}S^{+r_+} + c_{1,+}S^{+r_+-1} + \dots + c_{r_+-1,+}S^{+1} + c_{\underline{r}} \\ &+ c_{0,-}S^{-r_-} + c_{1,-}S^{-r_--1} + \dots + c_{r_--1,-}S^{-1}\big)\alpha_0, \end{aligned}$$

(and analogously for β_0) from which restriction (3.11) arises. Again $S^{\pm j}$ denote the shift operators $S^{\pm j}\alpha_0(n) = \alpha_0(n \pm j)$. The requirement $\|(\alpha_0 - \alpha_0^+, \beta_0 - \beta_0^+)\|_{w,p} < \infty$ yields the algebraic constraint (3.11) for $c_{j,\pm}$. Finally, note that it is no restriction to assume $c_{\underline{r}} = 0$ by (3.6). □

For example, we obtain such decaying solutions for $\mathrm{AL}_{(0,1)}(\alpha, \beta)$ if $c_{0,+} = 0$, for $\mathrm{AL}_{(1,1)}(\alpha, \beta)$ if $c_{0,+} = -c_{0,-}$ (or $c_{0,+} + c_{0,-} + c_{\underline{1}} = 0$ as in Theorem 2.4).

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References

[1] M. J. Ablowitz, G. Biondini, and B. Prinari, *Inverse scattering transform for the integrable discrete nonlinear Schrödinger equation with nonvanishing boundary conditions*, Inverse Problems **23**, 1711–1758 (2007).

[2] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[3] M. J. Ablowitz and J. F. Ladik, *Nonlinear differential-difference equations*, J. Math. Phys. **16**, 598–603 (1975).
[4] M. J. Ablowitz and J. F. Ladik, *Nonlinear differential-difference equations and Fourier analysis*, J. Math. Phys. **17**, 1011–1018 (1976).
[5] M. J. Ablowitz and J. F. Ladik, *A nonlinear difference scheme and inverse scattering*, Studies Appl. Math. **55**, 213–229 (1976).
[6] M. J. Ablowitz and J. F. Ladik, *On the solution of a class of nonlinear partial difference equations*, Studies Appl. Math. **57**, 1–12 (1977).
[7] M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems*, London Mathematical Society Lecture Note Series, Vol. 302, Cambridge University Press, Cambridge, 2004.
[8] R. Abraham, J.E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2^{nd} ed. Springer, New York, 1988.
[9] I. N. Bondareva, *The Korteweg-de Vries equation in classes of increasing functions with prescribed asymptotic behaviour as* $|x| \to \infty$, Math. USSR Sb. **50-1**, 125–135 (1983).
[10] I. N. Bondareva and M. Shubin, *Increasing asymptotic solutions of the Korteweg-de Vries equation and its higher analogues*, Sov. Math. Dokl. **26-3**, 716–719 (1982).
[11] K. Deimling, *Ordinary Differential Equations on Banach Spaces*, Lecture Notes in Mathematics **596**, Springer, Berlin, 1977.
[12] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *A method for solving the Korteweg-de Vries equation*, Phys. Rev. Letters **19**, 1095–1097 (1967).
[13] F. Gesztesy, H. Holden, J. Michor, and G. Teschl, *Soliton Equations and Their Algebro-Geometric Solutions. Volume II:* $(1+1)$*-Dimensional Discrete Models*, Cambridge Studies in Advanced Mathematics **114**, Cambridge University Press, Cambridge, 2008.
[14] F. Gesztesy, H. Holden, J. Michor, and G. Teschl, *The Ablowitz–Ladik hierarchy revisited*, in Methods of Spectral Analysis in Math. Physics, J. Janas et al. (ed.), 139–190, Oper. Theory Adv. Appl. **186**, Birkhäuser, Basel, 2008.
[15] F. Gesztesy, H. Holden, J. Michor, and G. Teschl, *Local conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy*, Stud. Appl. Math. **120-4**, 361–423 (2008).
[16] H. Krüger and G. Teschl, *Unique continuation for discrete nonlinear wave equations*, arXiv:0904.0011.
[17] J. Michor, *Inverse scattering transform for the Ablowitz–Ladik Hierarchy with quasi-periodic background*, in preparation.
[18] G. Teschl, *On the spatial asymptotics of solutions of the Toda lattice*, Discrete Contin. Dyn. Sys. **27-3**, 1233–1239 (2010).
[19] V. E. Vekslerchik and V. V. Konotop, *Discrete nonlinear Schrödinger equation under non-vanishing boundary conditions*, Inverse Problems **8**, 889–909 (1992).

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Part 2

Inverse scattering transform for the Toda hierarchy

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SCATTERING THEORY FOR JACOBI OPERATORS WITH GENERAL STEPLIKE QUASI-PERIODIC BACKGROUND

IRYNA EGOROVA, JOHANNA MICHOR, AND GERALD TESCHL

To Vladimir Aleksandrovich Marchenko and Leonid Andreevich Pastur, our teachers and inspiring colleagues.

ABSTRACT. We develop direct and inverse scattering theory for Jacobi operators with steplike coefficients which are asymptotically close to different finite-gap quasi-periodic coefficients on different sides. We give a complete characterization of the scattering data, which allow unique solvability of the inverse scattering problem in the class of perturbations with finite first moment.

1. INTRODUCTION

In this paper we consider direct and inverse scattering theory for Jacobi operators with steplike quasi-periodic finite-gap background, using the Marchenko [15] approach.

Scattering theory for Jacobi operators is a classical topic with a long tradition. Originally developed on an informal level by Case in [5], the first rigorous results for the case of a constant background were given by Guseinov [12] with further extensions by Teschl [19], [20]. The case of periodic backgrounds was completely solved in [24] (who in fact handle almost periodic operators with a homogenous Cantor type spectrum) respectively [8] using different approaches. Moreover, the case of a steplike situation, where the coefficients are asymptotically close to two different quasi-periodic finite-gap operators, was solved in [11] (see also [1], [7]) under the restriction that the two background operators are isospectral. It is the purpose of the present paper to remove this restriction.

We should also mention that scattering theory for Jacobi operators is directly applicable to the investigation of the Toda lattice with initial data in the above mentioned classes. See for example [3], [6], [23] for steplike constant backgrounds, and [9], [10], [13], [14], and [16] for periodic backgrounds. For further possible applications and additional references we refer to the discussion in [11].

Finally, let us give a brief overview of the remaining sections. After recalling some necessary facts on algebro-geometric quasi-periodic finite-gap operators in Section 2, we construct the transformation operators and investigate the properties of the scattering data in Section 3. In Section 4 we derive the Gel'fand-Levitan-Marchenko equation and show that it uniquely determines the operator. In addition, we formulate necessary conditions for the scattering data to uniquely determine our Jacobi operator. Our final Section 5 shows that our necessary conditions for the scattering data are also sufficient.

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2. Step-like finite-band backgrounds

First we need to recall some facts on quasi-periodic finite-band Jacobi operators which contain all periodic operators as a special case. We refer to [20, Chapter 9] and [8] for details.

Let $H_q^\pm$ be two quasi-periodic finite-band Jacobi operators,[1]

$$H_q^\pm f(n) = a_q^\pm(n) f(n+1) + a_q^\pm(n-1) f(n-1) + b_q^\pm(n) f(n), \quad f \in \ell^2(\mathbb{Z}), \tag{2.1}$$

associated with the Riemann surface of the square root

$$P_\pm(z) = -\prod_{j=0}^{2g_\pm+1} \sqrt{z - E_j^\pm}, \qquad E_0^\pm < E_1^\pm < \cdots < E_{2g_\pm+1}^\pm, \tag{2.2}$$

where $g_\pm \in \mathbb{N}$ and $\sqrt{\cdot}$ is the standard root with branch cut along $(-\infty, 0)$. In fact, $H_q^\pm$ are uniquely determined by fixing a Dirichlet divisor $\sum_{j=1}^{g^\pm}(\mu_j^\pm, \sigma_j^\pm)$, where $\mu_j^\pm \in [E_{2j-1}^\pm, E_{2j}^\pm]$ and $\sigma_j^\pm \in \{-1, 1\}$. The spectra of $H_q^\pm$ consist of $g_\pm + 1$ bands

$$\sigma_\pm := \sigma(H_q^\pm) = \bigcup_{j=0}^{g_\pm} [E_{2j}^\pm, E_{2j+1}^\pm]. \tag{2.3}$$

We will identify the set $\mathbb{C} \setminus \sigma(H_q^\pm)$ with the upper sheet of the Riemann surface. The upper and lower sides of the cuts over the spectrum are denoted by σ^{u} and σ^{l} and the symmetric points on these cuts by λ^{u} and λ^{l}, that is,

$$f(\lambda^u) = \lim_{\epsilon \downarrow 0} f(\lambda + i\epsilon), \quad f(\lambda^l) = \lim_{\epsilon \downarrow 0} f(\lambda - i\epsilon), \quad \lambda \in \sigma_\pm.$$

We will develop the scattering theory for the operator

$$H f(n) = a(n-1) f(n-1) + b(n) f(n) + a(n) f(n+1), \quad n \in \mathbb{Z}, \tag{2.4}$$

whose coefficients are asymptotically close to the coefficients of $H_q^\pm$ on the corresponding half-axes:

$$\sum_{n=0}^{\pm\infty} |n| \Big(|a(n) - a_q^\pm(n)| + |b(n) - b_q^\pm(n)| \Big) < \infty. \tag{2.5}$$

The special case $H_q^- = H_q^+$ has been exhaustively studied in [8] (see also [24]) and the case where H_q^- and H_q^+ are in the same isospectral class $\sigma_- = \sigma_+$ was treated in [11]. Several results are straightforward generalizations, in such situations we will simply refer to [8], [11] and only point out possible differences.

Let $\psi_q^\pm(z, n)$ be the Floquet solutions of the spectral equations

$$H_q^\pm \psi(n) = z\psi(n), \qquad z \in \mathbb{C}, \tag{2.6}$$

that decay for $z \in \mathbb{C} \setminus \sigma_\pm$ as $n \to \pm\infty$. They are uniquely defined by the condition $\psi_q^\pm(z, 0) = 1$, $\psi_q^\pm(z, \cdot) \in \ell^2(\mathbb{Z}_\pm)$. The solution $\psi_q^+(z, n)$ (resp. $\psi_q^-(z, n)$) coincides with the upper (resp. lower) branch of the Baker–Akhiezer functions of H_q^+ (resp. H_q^-), see [20]. The second solutions $\breve{\psi}_q^\pm(z, n)$ are given by the other branch of

[1] Everywhere in this paper the sub or super index "+" (resp. "−") refers to the background on the right (resp. left) half-axis.

the Baker–Akhiezer functions and satisfy $\breve{\psi}_q^{\pm}(z,\cdot) \in \ell^2(\mathbb{Z}_{\mp})$ as $z \in \mathbb{C} \setminus \sigma_{\pm}$. Their Wronskian is equal to

$$W_q^{\pm}(\breve{\psi}_q^{\pm}(z), \psi_q^{\pm}(z)) = \pm \frac{1}{\rho_{\pm}(z)}, \tag{2.7}$$

where

$$\rho_{\pm}(z) = \frac{\prod_{j=1}^{g_{\pm}}(z - \mu_j^{\pm})}{P_{\pm}(z)} \tag{2.8}$$

satisfy by our choice of the branch for the square root

$$\mathrm{Im}(\rho_{\pm}(\lambda^{\mathrm{u}})) > 0, \quad \mathrm{Im}(\rho_{\pm}(\lambda^{\mathrm{l}})) < 0, \quad \lambda \in \sigma_{\pm}. \tag{2.9}$$

In (2.7) the following notation is used

$$W_{q,n}^{\pm}(f,g) := a_q^{\pm}(n)\left(f(n)g(n+1) - f(n+1)g(n)\right). \tag{2.10}$$

Note that $\psi_q^{\pm}(z,n)$, $\breve{\psi}_q^{\pm}(z,n)$ have continuous limits as $z \to \lambda^{\mathrm{u,l}} \in \sigma_{\pm}^{\mathrm{u,l}} \setminus \partial\sigma_{\pm}$, where

$$\partial\sigma_{\pm} = \{E_0^{\pm}, ..., E_{2g_{\pm}+1}^{\pm}\},$$

and they satisfy the symmetry property

$$\psi_q^{\pm}(\lambda^{\mathrm{l}}, n) = \overline{\psi_q^{\pm}(\lambda^{\mathrm{u}}, n)} = \breve{\psi}_q^{\pm}(\lambda^{\mathrm{u}}, n), \quad \lambda \in \sigma_{\pm}. \tag{2.11}$$

The points $(\mu_j^{\pm}, \sigma_j^{\pm})$, $1 \le j \le g_{\pm}$, form the divisors of poles of the Baker–Akhiezer functions. Correspondingly, the sets of Dirichlet eigenvalues $\{\mu_1^{\pm}, ..., \mu_{g_{\pm}}^{\pm}\}$ can be divided in three disjoint subsets

$$\begin{aligned} M^{\pm} &= \{\mu_j^{\pm} \mid \mu_j^{\pm} \in \mathbb{R}\backslash\sigma_{\pm} \text{ is a pole of } \psi_q^{\pm}(z,1)\}, \\ \breve{M}^{\pm} &= \{\mu_j^{\pm} \mid \mu_j^{\pm} \in \mathbb{R}\backslash\sigma_{\pm} \text{ is a pole of } \breve{\psi}_q^{\pm}(z,1)\}, \\ \hat{M}^{\pm} &= \{\mu_j^{\pm} \mid \mu_j^{\pm} \in \partial\sigma_{\pm}\}. \end{aligned} \tag{2.12}$$

In order to remove the singularities of $\psi_q^{\pm}(z,n)$, $\breve{\psi}_q^{\pm}(z,n)$ we introduce

$$\begin{aligned} \delta_{\pm}(z) &:= \prod_{\mu_j^{\pm} \in M_{\pm}} (z - \mu_j^{\pm}), \\ \hat{\delta}_{\pm}(z) &:= \prod_{\mu_j^{\pm} \in M_{\pm}} (z - \mu_j^{\pm}) \prod_{\mu_j^{\pm} \in \hat{M}_{\pm}} \sqrt{z - \mu_j^{\pm}}, \\ \breve{\delta}_{\pm}(z) &:= \prod_{\mu_j^{\pm} \in \breve{M}_{\pm}} (z - \mu_j^{\pm}) \prod_{\mu_j^{\pm} \in \hat{M}_{\pm}} \sqrt{z - \mu_j^{\pm}}, \end{aligned} \tag{2.13}$$

where $\prod = 1$ if there are no multipliers, and set

$$\tilde{\psi}_q^{\pm}(z,n) = \delta_{\pm}(z)\psi_q^{\pm}(z,n), \quad \hat{\psi}_q^{\pm}(z,n) = \hat{\delta}_{\pm}(z)\psi_q^{\pm}(z,n). \tag{2.14}$$

Lemma 2.1. *The Floquet solutions $\psi_q^{\pm}$, $\breve{\psi}_q^{\pm}$ have the following properties:*

(i) *The functions $\psi_q^{\pm}(z,n)$ (resp. $\breve{\psi}_q^{\pm}(z,n)$) are holomorphic as functions of z in the domain $\mathbb{C} \setminus (\sigma_{\pm} \cup M_{\pm})$ (resp. $\mathbb{C} \setminus (\sigma_{\pm} \cup \breve{M}_{\pm})$), take real values on the set $\mathbb{R} \setminus \sigma_{\pm}$, and have simple poles at the points of the set $M_{\pm}$ (resp. $\breve{M}_{\pm}$).*

They are continuous up to the boundary $\sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}}$ *except at the points in* $\hat{M}_\pm$ *and satisfy the symmetry property* (2.11). *For* $E \in \hat{M}_\pm$, *they satisfy*

$$\psi_q^\pm(z,n) = O\left(\frac{1}{\sqrt{z-E}}\right), \quad \breve{\psi}_q^\pm(z,n) = O\left(\frac{1}{\sqrt{z-E}}\right), \quad z \to E \in \hat{M}_\pm.$$

Moreover, the estimate

$$\hat{\psi}_q^\pm(z,n) - \hat{\psi}_q^\pm(E,n) = O(\sqrt{z-E}), \qquad E \in \partial\sigma_\pm, \tag{2.15}$$

is valid.

(ii) *The following asymptotic expansions hold as* $z \to \pm\infty$

$$\psi_q^\pm(z,n) = z^{\mp n}\Big(\prod_{j=0}^{n-1}{}^{*}\; a_q^\pm(j)\Big)^{\pm 1}\Big(1 \pm \frac{1}{z}\sum_{j=0}^{n-1}{}^{*}\; b_q^\pm(j+{}^{1}_{0}) + O(\frac{1}{z^2})\Big), \tag{2.16}$$

where

$$\prod_{j=n_0}^{n-1}{}^{*}\; f(j) = \begin{cases} \prod\limits_{j=n_0}^{n-1} f(j), & n > n_0, \\ 1, & n = n_0, \\ \prod\limits_{j=n}^{n_0-1} f(j)^{-1}, & n < n_0, \end{cases} \qquad \sum_{j=n_0}^{n-1}{}^{*}\; f(j) = \begin{cases} \sum\limits_{j=n_0}^{n-1} f(j), & n > n_0, \\ 0, & n = n_0, \\ -\sum\limits_{j=n}^{n_0-1} f(j), & n < n_0. \end{cases}$$

(iii) *The functions* $\psi_q^\pm(\lambda,n)$ *form a complete orthogonal system on the spectrum with respect to the weight*

$$d\omega_\pm(\lambda) = \frac{1}{2\pi\mathrm{i}}\rho_\pm(\lambda)d\lambda, \tag{2.17}$$

namely

$$\oint_{\sigma_\pm} \overline{\psi_q^\pm(\lambda,m)}\psi_q^\pm(\lambda,n)d\omega_\pm(\lambda) = \delta(n,m), \tag{2.18}$$

where

$$\oint_{\sigma_\pm} f(\lambda)d\lambda := \int_{\sigma_\pm^{\mathrm{u}}} f(\lambda^{\mathrm{u}})d\lambda - \int_{\sigma_\pm^{\mathrm{l}}} f(\lambda^{\mathrm{l}})d\lambda. \tag{2.19}$$

Here $\delta(n,m) = 1$ *if* $n = m$ *and* $\delta(n,m) = 0$ *else is the Kronecker delta.*

3. Scattering data

Now let H be a steplike operator with coefficients $a(n)$, $b(n)$ satisfying (2.5). The two solutions $\psi_\pm(z,n)$ of the spectral equation

$$H\psi = z\psi, \quad z \in \mathbb{C}, \tag{3.1}$$

which are asymptotically close to the Floquet solutions $\psi_q^\pm(z,n)$ of the background equations (2.6) as $n \to \pm\infty$, are called Jost solutions. They can be represented as (see [8])

$$\psi_\pm(z,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_q^\pm(z,m), \tag{3.2}$$

where the functions $K_\pm(n,.)$ are real valued and satisfy the estimate

$$|K_\pm(n,m)| \le C_\pm(n) \sum_{j=[\frac{m+n}{2}]}^{\pm\infty} \Big(|a(j)-a_q^\pm(j)| + |b(j)-b_q^\pm(j)|\Big), \quad \pm m > \pm n > 0. \tag{3.3}$$

The functions $C_\pm(n) > 0$ decrease monotonically as $n \to \pm\infty$. Moreover, we have

$$\begin{aligned}
a(n) &= a_q^+(n)\frac{K_+(n+1,n+1)}{K_+(n,n)},\\
a(n) &= a_q^-(n)\frac{K_-(n,n)}{K_-(n+1,n+1)},\\
b(n) &= b_q^+(n) + a_q^+(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q^+(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)},\\
b(n) &= b_q^-(n) + a_q^-(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q^-(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)},
\end{aligned} \tag{3.4}$$

which implies (cf. [8]) the following asymptotic behavior of the Jost solutions as $z \to \pm\infty$ using (3.2), (2.16),

$$\psi_\pm(z,n) = z^{\mp n} K_\pm(n,n)\Big(\prod_{j=0}^{n-1}{}^{*}\; a_q^\pm(j)\Big)^{\pm 1}\Big(1+\Big(B_\pm(n)\pm\sum_{j=1}^{n}{}^{*}\; b_q^\pm(j-{}^{0}_{1})\Big)\frac{1}{z}+O(\frac{1}{z^2})\Big), \tag{3.5}$$

where

$$B_\pm(n) = \sum_{m=n\pm 1}^{\pm\infty} (b_q^\pm(m) - b(m)). \tag{3.6}$$

For $\lambda \in \sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}}$ a second pair of solutions of (3.1) is given by

$$\breve{\psi}_\pm(\lambda,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\breve{\psi}_q^\pm(\lambda,m), \quad \lambda \in \sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}}, \tag{3.7}$$

which cannot be continued to the complex plane. Note that $\breve{\psi}_\pm(\lambda,n) = \overline{\psi_\pm(\lambda,n)}$, $\lambda \in \sigma_\pm$, and from (2.5), (3.2) we conclude

$$W(\overline{\psi_\pm(\lambda)}, \psi_\pm(\lambda)) = W_q^\pm(\breve{\psi}_q^\pm(\lambda), \psi_q^\pm(\lambda)) = \pm\rho_\pm(\lambda)^{-1}. \tag{3.8}$$

The Jost solutions $\psi_\pm$ are holomorphic in the domains $\mathbb{C}\setminus(\sigma_\pm \cup M_\pm)$ and inherit almost all properties of their background counterparts listed in Lemma 2.1. As before, we set

$$\tilde{\psi}_\pm(z,n) = \delta_\pm(z)\psi_\pm(z,n), \quad \hat{\psi}_\pm(z,n) = \hat{\delta}_\pm(z)\psi_\pm(z,n). \tag{3.9}$$

The following Lemma is proven in [8].

Lemma 3.1. *The Jost solutions have the following properties.*

(i) *For all n, the functions $\psi_\pm(z,n)$ are holomorphic in the domain $\mathbb{C}\setminus(\sigma_\pm \cup M_\pm)$ with respect to z and continuous up to the boundary $(\sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}})\setminus\partial\sigma_\pm$, where*

$$\psi_\pm(\lambda^{\mathrm{u}},n) = \overline{\psi_\pm(\lambda^{\mathrm{l}},n)}, \quad \lambda \in (\sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}})\setminus\partial\sigma_\pm. \tag{3.10}$$

The functions $\psi_\pm(z,n)$ are real valued for $z\in\mathbb{R}\setminus\sigma_\pm$ and have simple poles at $\mu_j\in M_\pm$. Moreover, $\hat\psi_\pm$ are continuous up to the boundary $\sigma_\pm^{\mathrm{u}}\cup\sigma_\pm^{\mathrm{l}}$.

(ii) *At the band edges we have for $\lambda\in\sigma_\pm^{\mathrm{u,l}}$*

$$\begin{aligned}\psi_\pm(\lambda,n)-\overline{\psi_\pm(\lambda,n)}&=o(1), & E&\in\partial\sigma_\pm\setminus\hat M_\pm,\\ \psi_\pm(\lambda,n)+\overline{\psi_\pm(\lambda,n)}&=o\Big(\frac{1}{\sqrt{\lambda-E}}\Big), & E&\in\hat M_\pm.\end{aligned}\tag{3.11}$$

Next, we introduce the sets

$$\sigma^{(2)}:=\sigma_+\cap\sigma_-,\quad \sigma_\pm^{(1)}=\operatorname{clos}(\sigma_\pm\setminus\sigma^{(2)}),\quad \sigma:=\sigma_+\cup\sigma_-,\tag{3.12}$$

where σ is the (absolutely) continuous spectrum of H and $\sigma_+^{(1)}\cup\sigma_-^{(1)}$ resp. $\sigma^{(2)}$ are the parts which are of multiplicity one resp. two. We will denote the interior of the spectrum by $\operatorname{int}(\sigma)$, that is, $\operatorname{int}(\sigma):=\sigma\setminus\partial\sigma$.

In addition to the continuous part, H has a finite number of eigenvalues situated in the gaps, $\sigma_d=\{\lambda_1,\dots,\lambda_p\}\subset\mathbb{R}\setminus\sigma$ (see, e.g., [18]). For every eigenvalue we introduce the corresponding norming constants

$$\gamma_{\pm,k}^{-1}=\sum_{n\in\mathbb{Z}}|\tilde\psi_\pm(\lambda_k,n)|^2,\quad 1\le k\le p.\tag{3.13}$$

Moreover, $\tilde\psi_\pm(\lambda_k,n)=c_k^\pm\tilde\psi_\mp(\lambda_k,n)$ with $c_k^+c_k^-=1$.

Let

$$W(z):=W(\psi_-(z),\psi_+(z))\tag{3.14}$$

be the Wronskian of two Jost solutions. This function is meromorphic in the domain $\mathbb{C}\setminus\sigma$ with possible poles at the points $M_+\cup M_-\cup(\hat M_+\cap\hat M_-)$ and with possible square root singularities at the points $\hat M_+\cup\hat M_-\setminus(\hat M_+\cap\hat M_-)$. Set

$$\tilde W(z)=W(\tilde\psi_-(z),\tilde\psi_+(z)),\quad \hat W(z)=W(\hat\psi_-(z),\hat\psi_+(z)),\tag{3.15}$$

then $\hat W(\lambda)$ is holomorphic in the domain $\mathbb{C}\setminus\mathbb{R}$ and continuous up to the boundary. But unlike to $W(z)$ and $\tilde W(z)$, the function $\hat W(\lambda)$ may not take real values on the set $\mathbb{R}\setminus\sigma$ and complex conjugated values on the different sides of the spectrum. That is why it is more convenient to characterize the spectral properties of the operator H by means of the function $\tilde W$, which can have singularities at the points of the sets $\hat M_+\cup\hat M_-$. We will study the precise character of these singularities in Lemma 3.2 below.

Note that outside the spectrum the function $\tilde W(z)$ vanishes precisely at the eigenvalues. However, it might also vanish inside the spectrum at points in $\partial\sigma_-\cup\partial\sigma_+$. We will call such points virtual levels of the operator H,

$$\sigma_v:=\{E\in\sigma:\ \hat W(E)=0\},\tag{3.16}$$

and we will show that $\sigma_v\subseteq\partial\sigma\cup(\partial\sigma_+^{(1)}\cap\partial\sigma_-^{(1)})$ in Lemma 3.2. All other points E of the set $\partial\sigma_+\cup\partial\sigma_-$ correspond to the generic case $\hat W(E)\neq0$.

Our next aim is to derive the properties of the scattering matrix. Introduce the scattering relations

$$T_\mp(\lambda)\psi_\pm(\lambda,n)=\overline{\psi_\mp(\lambda,n)}+R_\mp(\lambda)\psi_\mp(\lambda,n),\quad \lambda\in\sigma_\mp^{\mathrm{u,l}},\tag{3.17}$$

where the transmission and reflection coefficients are defined as usual,

$$T_\pm(\lambda):=\frac{W(\overline{\psi_\pm(\lambda)},\psi_\pm(\lambda))}{W(\psi_\mp(\lambda),\psi_\pm(\lambda))},\quad R_\pm(\lambda):=-\frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\mp(\lambda),\,\psi_\pm(\lambda))},\quad \lambda\in\sigma_\pm^{\mathrm{u,l}}.\tag{3.18}$$

The equalities in (3.18) imply the identity

$$\frac{1}{T_+(\lambda)\rho_+(\lambda)} = \frac{1}{T_-(\lambda)\rho_-(\lambda)} = W(\lambda), \quad \lambda \in \sigma^{(2)},$$

where $W(\lambda)$ is the Wronskian of two Jost solutions (3.14). This Wronskian plays an important role in the characterization of the properties of the scattering matrix. Namely, the following result is valid.

Lemma 3.2. *The entries of the scattering matrix have the following properties:*

I.

$$\begin{array}{llll} (\mathbf{a}) & T_\pm(\lambda^{\mathrm{u}}) = \overline{T_\pm(\lambda^{\mathrm{l}})}, & & \lambda \in \sigma_\pm, \\ & R_\pm(\lambda^{\mathrm{u}}) = \overline{R_\pm(\lambda^{\mathrm{l}})}, & & \lambda \in \sigma_\pm, \\ (\mathbf{b}) & \dfrac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}} = R_\pm(\lambda), & & \lambda \in \sigma_\pm^{(1)}, \\ (\mathbf{c}) & 1 - |R_\pm(\lambda)|^2 = \dfrac{\rho_\pm(\lambda)}{\rho_\mp(\lambda)} |T_\pm(\lambda)|^2, & & \lambda \in \sigma^{(2)}, \\ (\mathbf{d}) & \overline{R_\pm(\lambda)} T_\pm(\lambda) + R_\mp(\lambda) \overline{T_\pm(\lambda)} = 0, & & \lambda \in \sigma^{(2)}. \end{array}$$

II. *The functions $T_\pm(\lambda)$ can be extended analytically to the domain $\mathbb{C} \setminus (\sigma \cup M_\pm \cup \breve{M}_\pm)$ and satisfy*

$$\frac{1}{T_+(z)\rho_+(z)} = \frac{1}{T_-(z)\rho_-(z)} = W(z). \tag{3.19}$$

The function $W(z)$ has the following properties:

(a) *The function $\tilde{W}(z) = \delta_+(z)\delta_-(z)W(z)$ is holomorphic on $\mathbb{C} \setminus \sigma$ with simple zeros at the eigenvalues λ_k, where*

$$\left(\frac{d\tilde{W}}{dz}(\lambda_k)\right)^2 = \frac{1}{\gamma_{+,k}\gamma_{-,k}}. \tag{3.20}$$

Moreover,

$$\overline{\tilde{W}(\lambda^{\mathrm{u}})} = \tilde{W}(\lambda^{\mathrm{l}}), \quad \lambda \in \sigma, \qquad \tilde{W}(z) \in \mathbb{R}, \quad z \in \mathbb{R} \setminus \sigma. \tag{3.21}$$

(b) *The function $\hat{W}(z) = \hat{\delta}_+(z)\hat{\delta}_-(z)W(z)$ is continuous on the set $\mathbb{C}\setminus\sigma$ up to the boundary $\sigma^{\mathrm{u}} \cup \sigma^{\mathrm{l}}$. It can have zeros on the set $\partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$ and does not vanish at the other points of the spectrum σ. If $\hat{W}(E) = 0$ as $E \in \partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$, then*

$$\frac{1}{\hat{W}(\lambda)} = O\left(\frac{1}{\sqrt{\lambda - E}}\right), \quad \text{for } \lambda \in \sigma \text{ close to } E. \tag{3.22}$$

Moreover,

$$\frac{1}{\hat{W}(z)} = O\left((z - E)^{-1/2-\varepsilon}\right), \quad \text{for } z \text{ close to } E. \tag{3.23}$$

(c) *In addition,*

$$T_+(\infty) = T_-(\infty) > 0. \tag{3.24}$$

III. **(a)** *The reflection coefficients $R_\pm(\lambda)$ are continuous functions on* $\mathrm{int}(\sigma_\pm^{\mathrm{u,l}})$.

(b) *If $E \in \partial\sigma_+ \cap \partial\sigma_-$ and $\hat{W}(E) \neq 0$, then the functions $R_\pm(\lambda)$ are also continuous at E. Moreover,*

$$R_\pm(E) = \begin{cases} -1 & \text{for } E \notin \hat{M}_\pm, \\ 1 & \text{for } E \in \hat{M}_\pm. \end{cases} \tag{3.25}$$

Proof. **I**. The symmetry property **(a)** follows from formulas (3.18) and (3.10). For **(b)**, use (3.18) and observe that $\psi_\mp(\lambda)$ are real valued for $\lambda \in \operatorname{int}(\sigma_\pm^{(1)})$. Let $\lambda \in \operatorname{int}(\sigma^{(2)})$. By (3.17),

$$|T_\pm|^2 W(\psi_\mp, \overline{\psi_\mp}) = (|R_\pm|^2 - 1) W(\psi_\pm, \overline{\psi_\pm}),$$

and property **(c)** follows from (3.8). The consistency condition **(d)** can be derived directly from definition (3.18).

II. The identity (3.19) follows from (3.18). **(a)** The Wronskian inherits the properties of $\psi_\pm(z)$, so it remains to show (3.20). If $\hat{W}(z_0) = 0$ for $z_0 \in \mathbb{C} \setminus \sigma$, then

$$\tilde{\psi}_\pm(z_0, n) = c^\pm \tilde{\psi}_\mp(z_0, n) \tag{3.26}$$

for some constants $c^\pm$ (depending on z_0), which satisfy $c^- c^+ = 1$. In particular, each zero of $\tilde{W}$ (or $\hat{W}$) outside the continuous spectrum is a point of the discrete spectrum of H and vice versa.

Let $\gamma_{\pm,j}$ be the norming constants defined in (3.13) for some point of the discrete spectrum λ_j. By virtue of [20], Lemma 2.4,

$$\begin{aligned} \frac{d}{dz} W(\tilde{\psi}_-(z), \tilde{\psi}_+(z))\Big|_{\lambda_j} &= W_n(\tilde{\psi}_-(\lambda_j), \tfrac{d}{dz}\tilde{\psi}_+(\lambda_j)) + W_n(\tfrac{d}{dz}\tilde{\psi}_-(\lambda_j), \tilde{\psi}_+(\lambda_j)) \\ &= -\sum_{k\in\mathbb{Z}} \tilde{\psi}_-(\lambda_j, k)\tilde{\psi}_+(\lambda_j, k) = -\frac{1}{c_j^\pm \gamma_{\pm,j}}. \end{aligned} \tag{3.27}$$

Since $c_j^- c_j^+ = 1$, we obtain (3.20).

(b) Continuity of $\hat{W}$ up to the boundary follows from the corresponding property of $\hat{\psi}_\pm(z,n)$. We begin with the investigation of the possible zeros of this function on the spectrum.

First let $\lambda_0 \in \operatorname{int}(\sigma^{(2)}) := \sigma^{(2)} \setminus \partial\sigma^{(2)}$, that is, $\hat{\delta}_- \neq 0$ and $\hat{\delta}_+ \neq 0$. Suppose $W(\lambda_0) = 0$, then $\psi_+(\lambda_0, n) = c\,\psi_-(\lambda_0, n)$ and $\overline{\psi_+(\lambda_0, n)} = \bar{c}\,\overline{\psi_-(\lambda_0, n)}$, i.e. $W(\psi_+, \overline{\psi_+}) = |c|^2 W(\psi_-, \overline{\psi_-})$. But this implies opposite signs for ρ_+, ρ_- by (3.8), $\operatorname{sign} \rho_+(\lambda_0) = -\operatorname{sign} \rho_-(\lambda_0)$, which contradicts (2.9).

Let $\lambda_0 \in \operatorname{int}(\sigma_\pm^{(1)})$ and $\tilde{W}(\lambda_0) = 0$. The point λ_0 can coincide with a pole $\mu \in M_\mp$. But $\psi_\pm(\lambda_0, n)$ and $\overline{\psi_\pm(\lambda_0, n)}$ are linearly independent and bounded, and $\tilde{\psi}_\mp(\lambda_0, n) \in \mathbb{R}$. If $W(\lambda_0) = 0$, then $\tilde{\psi}_\mp = c_1^\pm \psi_\pm = c_2^\pm \overline{\psi_\pm}$ which implies $W(\psi_\pm, \overline{\psi_\pm})(\lambda_0) = 0$, a contradiction.

In the general mutual location of the background spectra the case $\lambda_0 = E \in (\partial\sigma^{(2)} \cap \operatorname{int}(\sigma_\pm)) \subset \operatorname{int}(\sigma)$ can occur. If $\hat{W}(E) = 0$, then $W(\psi_\pm, \hat{\psi}_\mp)(E) = 0$, where $\hat{\psi}_\mp$ are defined by (3.9). The values of $\hat{\psi}_\mp(E, \cdot)$ are either purely real or purely imaginary, therefore $W(\overline{\psi_\pm}, \hat{\psi}_\mp)(E) = 0$, that is, $\overline{\psi_\pm(E,n)}$ and $\psi_\pm(E,n)$ are linearly dependent, which is impossible at inner points of the set $\sigma_\pm$.

Thus, the virtual level σ_v of H defined in (3.16) can only be located on the set $\partial\sigma_- \cap \partial\sigma_+$, that is,

$$\sigma_v \subseteq \partial\sigma \cup \big(\partial\sigma_-^{(1)} \cap \partial\sigma_+^{(1)}\big). \tag{3.28}$$

To prove (3.22), take $E \in \sigma_v$ and assume for example $E \in \sigma_+$. By (3.17) and (3.19),

$$\frac{\hat{\delta}_+(\lambda)\hat{\psi}_-(\lambda, n)}{\hat{\delta}_-(\lambda)\rho_+(\lambda)W(\lambda)} = \hat{\delta}_+(\lambda)\overline{\psi_+(\lambda, n)} + R_+(\lambda)\hat{\psi}_+(\lambda, n).$$

Choose n_0 such that $\hat{\psi}_-(E,n_0)\neq 0$. By continuity we also have $\hat{\psi}_-(\lambda,n_0)\neq 0$ in a small vicinity of E. Then

$$\frac{\hat{\delta}_+(\lambda)\overline{\psi_+(\lambda,n_0)}+R_+(\lambda)\hat{\psi}_+(\lambda,n_0)}{\hat{\psi}_-(\lambda,n_0)}=O(1),\quad \lambda\to E.$$

Accordingly,

$$\frac{1}{\hat{W}(\lambda)}=O\Bigg(\frac{\prod_{j=1}^{g_+}(\lambda-\mu_j^+)}{\hat{\delta}_+^2(\lambda)\sqrt{\lambda-E}}\Bigg)=O\left(\frac{1}{\sqrt{\lambda-E}}\right),\quad \lambda\in\sigma_+,$$

which proves (3.22). To see (3.23) note that

$$g(z,n)=\frac{\psi_+(z,n)\psi_-(z,n)}{W(z)}$$

is a Herglotz function. Moreover, we can assume that $\mu_j\neq E$ and choose n such that $\psi_\pm(E,n_0)\neq 0$. Hence it remains to show the corresponding estimate for $g(z)=g(z,n_0)$. Since the continuous spectrum of H is purely absolutely continuous, we deduce from Stieltjes inversion formula that

$$g(z)=\frac{1}{\pi}\int_{E-\delta}^{E+\delta}\frac{\mathrm{Im}(g(\lambda))}{\lambda-z}d\lambda+\tilde{g}(z),\qquad \delta>0,$$

where $\tilde{g}(z)$ is holomorphic near E. By (3.22) we infer $(\lambda-E)^{1/2+\varepsilon}\mathrm{Im}(g(\lambda))$ is Hölder continuous and the result follows from [17, Eq. (29.8)].

(c) Equation (3.24) follows from (3.5).

III. **(a)** follows from the corresponding properties of $\psi_\pm(z)$ and from **II**, **(b)**. To show **III**, **(b)** we use that by (3.18) the reflection coefficients have the representation

$$R_\pm(\lambda)=-\frac{W(\overline{\psi_\pm(\lambda)},\psi_\mp(\lambda))}{W(\psi_\pm(\lambda),\psi_\mp(\lambda))}=\mp\frac{W(\overline{\psi_\pm(\lambda)},\psi_\mp(\lambda))}{W(\lambda)}\tag{3.29}$$

and are continuous on both sides of the set $\mathrm{int}(\sigma_\pm)\setminus(M_\mp\cup\hat{M}_\mp)$. Moreover,

$$|R_\pm(\lambda)|=\left|\frac{W(\overline{\hat{\psi}_\pm(\lambda)},\hat{\psi}_\mp(\lambda))}{\hat{W}(\lambda)}\right|,$$

where the denominator does not vanish on the set $\sigma_\pm\setminus\sigma_v$. Hence $R_\pm(\lambda)$ are continuous on this set since both numerator and denominator are.

Next, let $E\in\partial\sigma_\pm\setminus\sigma_v$ (in particular $\hat{W}(E)\neq 0$). Then, if $E\notin\hat{M}_\pm$, we use (3.29) in the form

$$R_\pm(\lambda)=-1\mp\frac{\hat{\delta}_\pm(\lambda)W(\psi_\pm(\lambda)-\overline{\psi_\pm(\lambda)},\hat{\psi}_\mp(\lambda))}{\hat{W}(\lambda)},\tag{3.30}$$

which shows $R_\pm(\lambda)\to -1$ since $\psi_\pm(\lambda)-\overline{\psi_\pm(\lambda)}\to 0$ by Lemma 3.1, (2). This settles the first case in (3.25). Similarly, if $E\in\hat{M}_\pm$, we use (3.29) in the form

$$R_\pm(\lambda)=1\pm\frac{\hat{\delta}_\pm(\lambda)W(\psi_\pm(\lambda)+\overline{\psi_\pm(\lambda)},\hat{\psi}_\mp(\lambda))}{\hat{W}(\lambda)},\tag{3.31}$$

which shows $R_\pm(\lambda)\to 1$ since $\hat{\delta}_\pm(\lambda)=O(\sqrt{\lambda-E})$ and $\psi_\pm(\lambda)+\overline{\psi_\pm(\lambda)}=o\big(\frac{1}{\sqrt{\lambda-E}}\big)$ by Lemma 3.1, (2). This settles the second case in (3.25) as well. □

4. The Gel'fand-Levitan-Marchenko equation

The aim of this section is to derive the inverse scattering problem equation (the Gel'fand-Levitan-Marchenko equation) and to discuss some additional properties of the scattering data which are consequences of this equation.

Theorem 4.1. *The inverse scattering problem (the GLM) equation has the form*

$$K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l) F_\pm(l,m) = \frac{\delta(n,m)}{K_\pm(n,n)}, \qquad \pm m \geq \pm n, \tag{4.1}$$

where

$$\begin{aligned} F_\pm(m,n) = &\oint_{\sigma_\pm} R_\pm(\lambda) \psi_q^\pm(\lambda,m) \psi_q^\pm(\lambda,n) d\omega_\pm \\ &+ \int_{\sigma_\mp^{(1),u}} |T_\mp(\lambda)|^2 \psi_q^\pm(\lambda,m) \psi_q^\pm(\lambda,n) d\omega_\mp + \sum_{k=1}^p \gamma_{\pm,k} \tilde\psi_q^\pm(\lambda_k,n) \tilde\psi_q^\pm(\lambda_k,m). \end{aligned} \tag{4.2}$$

Proof. Consider a closed contour Γ_ϵ consisting of a large circular arc and some contours inside this arc, which envelope the spectrum σ at a small distance ε from the spectrum. Let $\pm m \geq \pm n$. The residue theorem, (2.17), (3.5), (3.20), and equality $\tilde\psi_\mp(\lambda_k,n) = c_j^\mp \tilde\psi_\pm(\lambda_k,n)$ imply

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_\epsilon} \frac{\psi_\mp(\lambda,n)\psi_q^\pm(\lambda,m)}{W(\lambda)} d\lambda &= \frac{\delta(n,m)}{K_\pm(n,n)} + \sum_{k=1}^p \mathrm{Res}_{\lambda_k} \left(\frac{\tilde\psi_\mp(\lambda,n)\tilde\psi_q^\pm(\lambda,m)}{\tilde W(\lambda)} \right) \\ &= \frac{\delta(n,m)}{K_\pm(n,n)} - \sum_{k=1}^p \gamma_{\pm,k} \tilde\psi_\pm(\lambda_k,n) \tilde\psi_q^\pm(\lambda_k,m), \end{aligned} \tag{4.3}$$

since the integrand is meromorphic on $\mathbb{C}\backslash\sigma$ with simple poles at the eigenvalues λ_k and at ∞ if $m = n$. It is continuous till the boundary except at the points $E \in \partial\sigma_+ \cup \partial\sigma_-$ where

$$\frac{\psi_\mp(\lambda,n)\psi_q^\pm(\lambda,m)}{W(\lambda)} = O\left(\frac{1}{\sqrt{\lambda - E}}\right), \qquad E \in \partial\sigma_+ \cup \partial\sigma_-, \tag{4.4}$$

by (3.22). On the other hand, as $\epsilon \to 0$,

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \oint_{\sigma} & \frac{\psi_\mp(\lambda,n)\psi_q^\pm(\lambda,m)}{W(\lambda)} d\lambda = \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\sigma_\pm} \frac{\big(\overline{\psi_\pm(\lambda,n)} + R_\pm(\lambda)\psi_\pm(\lambda,n)\big)\psi_q^\pm(\lambda,m)}{T_\pm(\lambda)W(\lambda)} d\lambda \\ &\quad + \frac{1}{2\pi\mathrm{i}} \oint_{\sigma_\mp^{(1)}} \frac{\psi_\mp(\lambda,n)\psi_q^\pm(\lambda,m)}{W(\lambda)} d\lambda \\ &= \oint_{\sigma_\pm} \overline{\psi_\pm(\lambda,n)} \psi_q^\pm(\lambda,m) d\omega^\pm + \oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,n)\psi_q^\pm(\lambda,m) d\omega^\pm \\ &\quad + \frac{1}{2\pi\mathrm{i}} \int_{\sigma_\mp^{(1),u}} \psi_q^\pm(\lambda,m) \left(\frac{\psi_\mp(\lambda,n)}{W(\lambda)} - \frac{\overline{\psi_\mp(\lambda,n)}}{\overline{W(\lambda)}} \right) d\lambda. \end{aligned} \tag{4.5}$$

It remains to treat the last integrand. By (3.17) and Lemma 3.2, **I**,

$$\overline{\psi_\mp(\lambda,n)} = T_\mp(\lambda)\psi_\pm(\lambda,n) - R_\mp(\lambda)\psi_\mp(\lambda,n) = T_\mp(\lambda)\psi_\pm(\lambda,n) - \frac{T_\mp(\lambda)}{\overline{T_\mp(\lambda)}}\psi_\mp(\lambda,n),$$

and therefore

$$\frac{\psi_\mp(n)}{W} - \frac{\overline{\psi_\mp(n)}}{\overline{W}} = \frac{\overline{WT_\mp}+WT_\mp}{|W|^2\overline{T_\mp}}\psi_\mp(n) - \frac{T_\mp}{\overline{W}}\psi_\pm(n) = -\frac{T_\mp}{\overline{W}}\psi_\pm(n),$$

since $\overline{WT_\mp}+WT_\mp = 2\mathrm{Re}(WT_\mp) = 0$ on $\sigma_\mp$. In summary, (4.3) and (4.5) yield

$$\begin{aligned}\frac{\delta(n,m)}{K_\pm(n,n)} &= K_\pm(n,m) + \oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,n)\psi_q^\pm(\lambda,m)d\omega^\pm \\ &\quad + \int_{\sigma_\mp^{(1),u}} |T_\mp(\lambda)|^2\psi_\pm(\lambda,n)\psi_q^\pm(\lambda,m)d\omega^\mp + \sum_{j=1}^p \gamma_{\pm,j}\tilde\psi_\pm(\lambda_j,n)\tilde\psi_q^\pm(\lambda_j,m),\end{aligned}$$

and applying (3.2) finishes the proof. □

As it is shown in [8], the estimate (3.3) for $K_\pm(n,m)$ implies the following estimates for $F_\pm(n,m)$.

Lemma 4.2. *The kernel of the GLM equation satisfies the following properties.* **IV**. *There exist functions $C_\pm(n) > 0$ and $q_\pm(n) \ge 0$, $n \in \mathbb{Z}_\pm$, such that $C_\pm(n)$ decay as $n \to \pm\infty$, $|n|q(n) \in \ell^1(\mathbb{Z}_\pm)$, and*

$$\begin{aligned}&|F_\pm(n,m)| \le C_\pm(n)\sum_{j=n+m}^{\pm\infty} q(j),\\ &\sum_{n=n_0}^{\pm\infty} |n|\big|F_\pm(n,n) - F_\pm(n\pm1,n\pm1)\big| < \infty, \qquad (4.6)\\ &\sum_{n=n_0}^{\pm\infty} |n|\big|a_q^\pm(n)F_\pm(n,n+1) - a_q^\pm(n-1)F_\pm(n-1,n)\big| < \infty.\end{aligned}$$

In summary, we have obtained the following necessary conditions for the scattering data:

Theorem 4.3. *The scattering data*

$$\begin{aligned}\mathcal{S} = \Big\{&R_+(\lambda),\, T_+(\lambda),\, \lambda\in\sigma_+^{\mathrm{u,l}};\, R_-(\lambda),\, T_-(\lambda),\, \lambda\in\sigma_-^{\mathrm{u,l}};\\ &\lambda_1,\dots,\lambda_p \in \mathbb{R}\setminus(\sigma_+\cup\sigma_-),\, \gamma_{\pm,1},\dots,\gamma_{\pm,p}\in\mathbb{R}_+\Big\}\end{aligned} \qquad (4.7)$$

satisfy the properties **I-III** *listed in Lemma 3.2. The functions $F_\pm(n,m)$, defined in (4.2), satisfy property* **IV** *in Lemma 4.2.*

In fact, the conditions on the scattering data given in Theorem 4.3 are both necessary and sufficient for the solution of the scattering problem in the class (2.5). The sufficiency of these conditions as well as the algorithm for the solution of the inverse problem will be discussed in the next section.

5. The inverse scattering problem

Let $H_q^\pm$ be two arbitrary quasi-periodic Jacobi operators associated with sequences $a_q^\pm(n), b_q^\pm(n)$ as introduced in Section 2. Let $\mathcal{S}$ be given scattering data with corresponding kernels $F_\pm(n,m)$ satisfying the necessary conditions of Theorem 4.3.

First we show that the GLM equations (4.1) can be solved for $K_\pm(n,m)$ if $F_\pm(n,m)$ are given.

Lemma 5.1. *Under condition* **IV**, *the GLM equations (4.1) have unique real-valued solutions* $K_\pm(n,\cdot) \in \ell^1(n,\pm\infty)$ *satisfying the estimates*

$$|K_\pm(n,m)| \le C_\pm(n) \sum_{j=\left[\frac{n+m}{2}\right]}^{\pm\infty} q(j), \quad \pm m > \pm n. \tag{5.1}$$

Here the functions $q_\pm(n)$ *and* $C_\pm(n)$ *are of the same type as in (4.6).*

Moreover, the following estimates are valid

$$\begin{aligned} &\sum_{n=n_0}^{\pm\infty} |n|\big|K_\pm(n,n) - K_\pm(n\pm 1, n\pm 1)\big| < \infty, \\ &\sum_{n=n_0}^{\pm\infty} |n|\big|a_q^\pm(n)K_\pm(n,n+1) - a_q^\pm(n-1)K_\pm(n-1,n)\big| < \infty. \end{aligned} \tag{5.2}$$

Proof. The solvability of (4.1) under condition (4.6) and the estimates (5.1), (5.2) follow completely analogous to the corresponding result in [8, Theorem 7.5]. To prove uniqueness, first note that the GLM equations are generated by compact operators. Thus, it is sufficient to prove that the equation

$$f(m) + \sum_{\ell=n}^{\pm\infty} F_\pm(\ell,m) f(\ell) = 0 \tag{5.3}$$

has only the trivial solution in the space $\ell^1(n,\pm\infty)$. The proof is similar for the "+" and "−" cases, hence we give it only for the "+" case. Let $f(\ell)$, $\ell > n$, be a nontrivial solution of (5.3) and set $f(\ell) = 0$ for $\ell \le n$. Since $F_+(\ell,n)$ is real-valued, we can assume that $f(\ell)$ is real-valued. Abbreviate by

$$\widehat{f}(\lambda) = \sum_{m\in\mathbb{Z}} \psi_q^+(\lambda,m) f(m) \tag{5.4}$$

the generalized Fourier transform, generated by the spectral decomposition (2.18) (cf. [22]). Recall that $\widehat{f}(\lambda) \in L^1_{loc}(\sigma_+^{\mathrm{u}} \cup \sigma_+^{\mathrm{l}})$.

Multiplying (5.3) by $f(m)$, summing over $m \in \mathbb{Z}$, and applying (2.18), (4.2), (5.4), and condition **I**, **(a)**, we have

$$\begin{aligned} &2\int_{\sigma_+^u} |\widehat{f}(\lambda)|^2 d\omega_+(\lambda) + 2\mathrm{Re}\int_{\sigma_+^u} R_+(\lambda)\widehat{f}(\lambda)^2 d\omega_+(\lambda) \\ &+ \int_{\sigma_-^{(1),u}} \widehat{f}(\lambda)^2 |T_-(\lambda)|^2 d\omega_-(\lambda) + \sum_{k=1}^p \gamma_{+,k}\Big(\sum_{n\in\mathbb{Z}} \tilde\psi_q^+(\lambda_k,n) f(n)\Big)^2 = 0. \end{aligned} \tag{5.5}$$

The last two summands in (5.5) are nonnegative since $\widehat{f}(\lambda)\in\mathbb{R}$ for $\lambda\in\sigma_-^{(1)}$ and $\tilde\psi_q^+(\lambda_k)\in\mathbb{R}$. We estimate the first two integrands by

$$|\widehat{f}(\lambda)|^2+\mathrm{Re}R_+(\lambda)\widehat{f}(\lambda)^2\geq|\widehat{f}(\lambda)|^2-|R_+(\lambda)\widehat{f}(\lambda)^2|\geq\big(1-|R_+(\lambda)|\big)|\widehat{f}(\lambda)|^2$$

and drop the last summand in (5.5), thus obtaining

$$2\int_{\sigma^{(2),\mathrm{u}}}(1-|R_+(\lambda)|)|\widehat{f}(\lambda)|^2d\omega_+(\lambda)+\int_{\sigma_-^{(1),\mathrm{u}}}\widehat{f}(\lambda)^2|T_-(\lambda)|^2d\omega_-(\lambda)\leq 0. \tag{5.6}$$

Here we also used that

$$\int_{\sigma_+^{(1),\mathrm{u}}}(1-|R_+(\lambda)|)|\widehat{f}(\lambda)|^2d\omega_+(\lambda)=0,$$

which follows from condition **I**, **(b)**. Since $|R_+(\lambda)|<1$ for $\lambda\in\mathrm{int}(\sigma^{(2)})$ and $\omega_-(\lambda)>0$ for $\lambda\in\mathrm{int}(\sigma_-^{(1)})$ we conclude that

$$\widehat{f}(\lambda)=0\quad\text{for}\quad\lambda\in\sigma^{(2)}\cup\sigma_-^{(1)}=\sigma_-.$$

The function $\widehat{f}(z)$ can be defined by formula (5.4) as a meromorphic function on $\mathbb{C}\setminus\sigma_+$. By our analysis it is even meromorphic on $\mathbb{C}\setminus\sigma_+^{(1)}$ and vanishes on σ_-. Thus $\widehat{f}(z)$ and hence also $f(m)$ are equal to zero. □

Next, define the sequences $a_\pm, b_\pm$ by

$$\begin{aligned}
a_+(n)&=a_q^+(n)\frac{K_+(n+1,n+1)}{K_+(n,n)},\\
a_-(n)&=a_q^-(n)\frac{K_-(n,n)}{K_-(n+1,n+1)},\\
b_+(n)&=b_q^+(n)+a_q^+(n)\frac{K_+(n,n+1)}{K_+(n,n)}-a_q^+(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)},\\
b_-(n)&=b_q^-(n)+a_q^-(n-1)\frac{K_-(n,n-1)}{K_-(n,n)}-a_q^-(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)},
\end{aligned}\tag{5.7}$$

and note that estimate (5.2) implies

$$n\left\{|a_\pm(n)-a_q^\pm(n)|+|b_\pm-b_q^\pm(n)|\right\}\in\ell^1(\mathbb{Z}_\pm).\tag{5.8}$$

Lemma 5.2. *The functions $\psi_\pm(z,n)$, defined by*

$$\psi_\pm(z,n)=\sum_{m=n}^{\pm\infty}K_\pm(n,m)\psi_q^\pm(z,m),\tag{5.9}$$

solve the equations

$$a_\pm(n-1)\psi_\pm(z,n-1)+b_\pm(n)\psi_\pm(z,n)+a_\pm(n)\psi_\pm(z,n+1)=z\psi_\pm(z,n),\tag{5.10}$$

where $a_\pm(n), b_\pm(n)$ are defined by (5.7).

Proof. Consider the two operators[2]

$$(H_\pm y)(n)=a_\pm(n-1)y_\pm(n-1)+b_\pm(n)y_\pm(n)+a_\pm(n)y_\pm(n+1),\quad n\in\mathbb{Z}.$$

[2]We don't know that $H_\pm$ is limit point at $\mp\infty$ yet, but this will not be used.

Define two discrete integral operators

$$(\mathcal{K}_\pm f)(n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m) f(m).$$

Then (cf. [8]) the following identity is valid

$$H_\pm \mathcal{K}_\pm = \mathcal{K}_\pm H_q^\pm,$$

which implies (5.10). □

The remaining problem is to show that $a_+(n) \equiv a_-(n)$, $b_+(n) \equiv b_-(n)$ under conditions **II** and **III** on the scattering data $\mathcal{S}$.

Theorem 5.3. *Let the scattering data $\mathcal{S}$, defined as in (4.7), satisfy conditions* **I**, **(a)–(c)**, **II**, **III**, **(a)**, *and* **IV**. *Then each of the GLM equations (4.1) has unique solutions $K_\pm(n,m)$, satisfying the estimate (5.2). The functions $a_\pm(n), b_\pm(n)$, defined by (5.7), satisfy (5.8).*

Under the additional conditions **III**, **(b)** *and* **I**, **(d)**, *these functions coincide, $a_+(n) \equiv a_-(n) = a(n)$, $b_+(n) \equiv b_-(n) = b(n)$, and the data $\mathcal{S}$ are the scattering data for the Jacobi operator associated with the sequences $a(n), b(n)$.*

The proof of Theorem 5.3 takes up the remaining section and is split into several lemmas for the convenience of the reader.

To prove uniqueness of the reconstructed potential we follow the method proposed in [15]. Recall that, by Lemma 2.1 (iii), the functions $\psi_q^\pm(\lambda,n)$ form an orthonormal basis with corresponding generalized Fourier transform. Split the kernel of the GLM equation (4.2) into three summands $F_\pm(m,n) = F_{r,\pm}(m,n) + F_{h,\pm}(m,n) + F_{d,\pm}(m,n)$ and set

$$G_\pm(n,m) := \sum_{l=n}^{\pm\infty} K_\pm(n,l) F_{r,\pm}(l,n). \tag{5.11}$$

Then one obtains as in [8, Theorem 8.2] that the functions $h_\mp(\lambda,n)$, defined by

$$\begin{aligned} h_\mp(\lambda,n) = \frac{1}{T_\pm(\lambda)} \Bigg(& \frac{\breve{\psi}_q^\pm(\lambda,n)}{K_\pm(n,n)} + \sum_{m=n\mp1}^{\mp\infty} G_\pm(n,m) \breve{\psi}_q^\pm(\lambda,m) \\ & \mp \int_{\sigma_\mp^{(1),u}} |T_\mp(\xi)|^2 \psi_\pm(\xi,n) \frac{W_{q,n-1}^\pm(\psi_q^\pm(\xi), \breve{\psi}_q^\pm(\lambda))}{\xi-\lambda} d\omega_\mp(\xi) \\ & \pm \sum_{k=1}^p \gamma_{\pm,k} \tilde{\psi}_\pm(\lambda_k,n) \frac{W_{q,n-1}^\pm(\tilde{\psi}_q^\pm(\lambda_k), \breve{\psi}_q^\pm(\lambda))}{\lambda-\lambda_k} \Bigg), \end{aligned} \tag{5.12}$$

satisfy

$$T_\pm(\lambda) h_\mp(\lambda,n) = \overline{\psi_\pm(\lambda,n)} + R_\pm(\lambda)\psi_\pm(\lambda,n), \quad \lambda \in \sigma_\pm^{u,l}. \tag{5.13}$$

Despite the fact that $h_\mp(\lambda,n)$ are defined via the background solutions corresponding to the opposite half-axis $\mathbb{Z}_\pm$, they share a series of properties with $\psi_\mp(\lambda,n)$. Namely, we prove

Lemma 5.4. *Let $h_\mp(z,n)$ be defined by formula (5.12) on the set $\sigma_\pm^{\mathrm{u,l}}$.*

(i) *The functions $\tilde{h}_\mp(z,n) = \delta_\mp(z) h_\mp(z,n)$ admit analytic extensions to the domain $\mathbb{C} \setminus \sigma$.*

(ii) *The functions $\tilde h_\mp(z,n)$ are continuous up to the boundary $\sigma^{\mathrm{u,l}}$ except possibly at the points $\partial\sigma_+\cup\partial\sigma_-$. Furthermore,*

$$\begin{aligned}&\tilde h_\mp(\lambda^{\mathrm u},n)=\tilde h_\mp(\lambda^{\mathrm l},n)\in\mathbb R, &&\lambda\in\mathbb R\setminus\sigma_\mp,\\ &\tilde h_\mp(\lambda^{\mathrm u},n)=\overline{\tilde h_\mp(\lambda^{\mathrm l},n)}, &&\lambda\in\operatorname{int}(\sigma_\mp).\end{aligned}\tag{5.14}$$

(iii) *For large z the functions $h_\mp(z,n)$ have the following asymptotic behavior*

$$h_\mp(z,n)=\frac{z^{\pm n}}{K_\pm(n,n)T_\pm(\infty)}\Big(\prod_{j=0}^{n-1}{}^{*}\ a_q^\pm(j)\Big)^{\mp1}\Big(1+O(\frac{1}{z})\Big),\quad z\to\infty.\tag{5.15}$$

(iv) *We have*

$$\begin{aligned}W^\pm(h_\mp(z),\psi_\pm(z)):=&a_\pm(n)\big(h_\mp(z,n)\psi_\pm(z,n+1)-h_\mp(z,n+1)\psi_\pm(z,n)\big)\\ \equiv&\pm W(z),\end{aligned}$$

where $W(z)$ is defined by (3.19).

Remark 5.5. *Note that we did not establish the connection between the function $W(z)$ and the functions $W^\pm(\psi_+(z,n),\psi_-(z,n))$, which can depend on n, because ψ_+ and ψ_- are the solutions of Jacobi equations corresponding to possibly different operators H_+ and H_-.*

Proof. (i). To show that $\tilde h_\mp(z,n)$ have analytic extensions to $\mathbb C\setminus\sigma$, we study each term in (5.12) separately.

First of all, note that due to the representation

$$T_\pm(z)=\frac{1}{\rho_\pm(z)W(z)}=\frac{\hat\delta_\mp(z)}{\breve\delta_\pm(z)}\frac{\sqrt{\prod_{j=0}^{2g_\pm+1}(z-E_j^\pm)}}{\hat W(z)},\tag{5.16}$$

the functions $\tilde\zeta_\mp(z,n)=\delta_\mp(z)\zeta_\mp(z,n)$, where

$$\zeta_\mp(z,n):=\frac{\breve\psi_q^\pm(z,n)}{T_\pm(z)},\tag{5.17}$$

can be continued analytically to $\mathbb C\setminus\sigma$. This also holds for the second term since $G_\pm(n,\cdot)\in\ell^1(\mathbb Z)$ are real-valued.

Next we discuss the properties of the Cauchy-type integral in the representation (5.12). We represent the third summand in (5.12) multiplied by $T_\pm^{-1}(z)$ as

$$\Theta_\mp(z,n):=\mp\frac{1}{2\pi\mathrm i}\int_{\sigma_\mp^{(1),\mathrm u}}\theta_\mp(z,\xi,n)\frac{d\xi}{\xi-z},\tag{5.18}$$

where

$$\begin{aligned}\theta_\mp(z,\xi,n)&=-\frac{\delta_\mp(\xi)^2}{\rho_\mp(\xi)|\tilde W(\xi)|^2}\tilde\psi_\pm(\xi,n)W^\pm_{q,n-1}(\tilde\psi_q^\pm(\xi,\cdot),\zeta_\mp(z,\cdot))\\ &=-\frac{|\hat\delta_\mp(\xi)|^2}{\rho_\mp(\xi)|\hat W(\xi)|^2}\frac{|\hat\delta_\pm(\xi)|^2}{\hat\delta_\pm(\xi)^2}\hat\psi_\pm(\xi,n)W^\pm_{q,n-1}(\hat\psi_q^\pm(\xi,\cdot),\zeta_\mp(z,\cdot)).\end{aligned}\tag{5.19}$$

By property **II**, **(a)** the function $\hat W(\xi)$ has no zeros in the interior of $\sigma_\mp^{(1),\mathrm u}$. Thus, for $z\notin\sigma_\mp^{(1)}$, the functions $\theta_\mp(z,.,n)$ are bounded in the interior of $\sigma_\mp^{(1)}$ and the only possible singularities can arise at the boundary. We claim

$$\theta_\mp(z,\xi,n) = \begin{cases} O(\sqrt{\xi - E}) & \text{for } E \notin \sigma_v, \\ O\big(\frac{1}{\sqrt{\xi-E}}\big) & \text{for } E \in \sigma_v, \end{cases} \qquad E \in \partial\sigma_\mp^{(1)},\ z \neq E. \tag{5.20}$$

This follows from $\frac{|\hat{\delta}_\mp(\xi)|^2}{\rho_\mp(\xi)} = O(\sqrt{\xi - E})$ together with $\hat{W}(\xi) = O(1)$ if $E \notin \sigma_v$ and $1/\hat{W}(\xi) = O(1/\sqrt{\xi - E})$ by **II**, **(b)** if $E \in \sigma_v$. Therefore, $\theta_\mp$ are integrable and the third summand of (5.12) also inherits the properties of $\zeta_\mp(z,n)$.

Finally, the last summand in (5.12) again inherits the properties of $\tilde{\zeta}_\mp(z,n)$ except for possible additional poles at the eigenvalues λ_k. However, these cancel with the zeros of $\tilde{W}(z)$ at $z = \lambda_k$.

(ii). We consider the boundary values next. The only nontrivial term is of course the Cauchy-type integral (5.18) as $z \to \lambda \in \mathrm{int}(\sigma_\mp^{(1)})$. First of all observe that by (2.7) and (3.19),

$$\frac{W^\pm_{q,n-1}(\tilde{\psi}_q^\pm(\lambda), \breve{\psi}_q^\pm(z))}{T_\pm(z)} \to (\delta_\pm W)(\lambda),$$

where the functions $\delta_\pm W$ are bounded and nonzero for $\lambda \in \mathrm{int}(\sigma_\mp^{(1)})$ by **II**, **(a)**. Hence the Plemelj formula applied to (5.18) gives

$$\Theta_\mp(\lambda,n) = \pm\frac{\tilde{\psi}_\pm(\lambda,n)}{2\delta_\pm(\lambda)\rho_\mp(\lambda)\overline{W(\lambda)}} \mp \fint_{\sigma_\mp^{(1),\mathrm{u}}} \frac{\theta_\mp(\lambda,\xi,n)}{\xi-\lambda}d\xi, \quad \lambda \in \mathrm{int}(\sigma_\mp^{(1),\mathrm{u}}),$$

where both terms are finite. Here $\fint$ denotes the principle value integral. Therefore, the boundary values away from $\partial\sigma_+ \cup \partial\sigma_-$ exist and we have

$$h_\mp(\lambda^{\mathrm{u}},n) = \overline{h_\mp(\lambda^{\mathrm{l}},n)}, \quad \lambda \in \sigma_+ \cup \sigma_-. \tag{5.21}$$

By property **I**, **(b)**,

$$h_\mp = T_\pm^{-1}\left(R_\pm\psi_\pm + \overline{\psi_\pm}\right) = \frac{\psi_\pm}{\overline{T_\pm}} + \frac{\overline{\psi_\pm}}{T_\pm} \in \mathbb{R}, \quad \lambda \in \sigma_\pm^{(1)}, \tag{5.22}$$

from which

$$h_\mp(\lambda^{\mathrm{u}},n) = h_\mp(\lambda^{\mathrm{l}},n), \quad \lambda \in \sigma_\pm^{(1)}, \tag{5.23}$$

follows. Combining (5.21) and (5.23) yields (5.14).

(iii). Since the last two terms in (5.12) are $O(z^{-1})$, the asymptotic behavior follows from (3.5) and **II**, **(c)**.

(iv). From (5.13), (3.8), and (3.19) we obtain

$$W^\pm(h_\mp(\lambda), \psi_\pm(\lambda)) = \frac{W^\pm(\overline{\psi_\pm(\lambda)}, \psi_\pm(\lambda))}{T_\pm(\lambda)} = \frac{1}{T_\pm(\lambda)\rho_\pm(\lambda)} = \pm W(\lambda), \quad \lambda \in \sigma_\pm.$$

Hence equality holds for all $z \in \mathbb{C}$ by analytic continuation. □

Corollary 5.6. *The functions $\tilde{h}_\mp(z,n)$ admit analytic extensions to $\mathbb{C} \setminus \sigma_\mp$.*

Proof. Property (i) of Lemma 5.4 holds for $z \in \mathbb{C} \setminus \sigma$. Relation (5.14) implies that $\tilde{h}_\mp$ have no jumps across $z \in \mathrm{int}(\sigma_\pm^{(1)})$. To finish the proof we need to show that the possible remaining singularities at $E \in \partial\sigma_\pm^{(1)} \cap \partial\sigma$ are removable. This follows from (cf. (5.16))

$$\hat{\zeta}_\mp(z,n) = \frac{\hat{W}(z)}{\sqrt{\prod_{j=0}^{2g_\pm+1}(z - E_j^\pm)}} \breve{\delta}_\pm(z) \breve{\psi}_q^\pm(z,n) \tag{5.24}$$

which shows $\tilde{\zeta}_\mp(z,n) = O((z-E)^{-1/2})$ and hence $\tilde{h}_\mp(z,n) = O((z-E)^{-1/2})$ for $E \in \sigma_\pm^{(1)} \cap \partial\sigma$.

However, let us emphasize at this point that the behavior of $h_\pm(z,n)$ at the remaining edges is a more subtle question to be discussed later. □

Eliminating $\overline{\psi_\pm}$ from

$$\begin{cases} \overline{R_\pm(\lambda)}\,\overline{\psi_\pm(\lambda,n)} + \psi_\pm(\lambda,n) &= \overline{h_\mp(\lambda,n)}\,\overline{T_\pm(\lambda)} \\ R_\pm(\lambda)\,\psi_\pm(\lambda,n) + \overline{\psi_\pm(\lambda,n)} &= h_\mp(\lambda,n)\,T_\pm(\lambda) \end{cases}$$

yields

$$\psi_\pm(\lambda,n)\left(1 - |R_\pm(\lambda)|^2\right) = \overline{h_\mp(\lambda,n)}\,\overline{T_\pm(\lambda)} - \overline{R_\pm(\lambda)}\,h_\mp(\lambda,n)\,T_\pm(\lambda).$$

We apply **I**, **(c)**, **II**, and the consistency condition **I**, **(d)** to obtain

$$\begin{aligned} T_\mp(\lambda)\psi_\pm(\lambda,n) &= \overline{h_\mp(\lambda,n)} - \frac{\overline{R_\pm(\lambda)}T_\pm(\lambda)}{\overline{T_\pm(\lambda)}} h_\mp(\lambda,n) \\ &= \overline{h_\mp(\lambda,n)} + R_\mp(\lambda) h_\mp(\lambda,n), \quad \lambda \in \sigma^{(2)}. \end{aligned} \tag{5.25}$$

This equation together with (5.13) gives us a system from which we can eliminate the reflection coefficients $R_\pm$. We obtain

$$T_\pm(\lambda)\big(\psi_\pm(\lambda)\psi_\mp(\lambda) - h_\pm(\lambda)h_\mp(\lambda)\big) = \psi_\pm(\lambda)\overline{h_\pm(\lambda)} - \overline{\psi_\pm(\lambda)}h_\pm(\lambda), \quad \lambda \in \sigma^{(2),\mathrm{u,l}}. \tag{5.26}$$

Now introduce the function

$$G(z) := G(z,n) = \frac{\psi_+(z,n)\psi_-(z,n) - h_+(z,n)h_-(z,n)}{W(z)} \tag{5.27}$$

which is well defined in the domain $z \in \mathbb{C} \setminus (\sigma \cup \sigma_d \cup M_+ \cup M_-)$. By (5.26) and (3.19),

$$G(\lambda) = \Big(\psi_\pm(\lambda)\overline{h_\pm(\lambda)} - \overline{\psi_\pm(\lambda)}h_\pm(\lambda)\Big)\,\rho_\pm(\lambda), \quad \lambda \in \sigma^{(2),\mathrm{u,l}}, \tag{5.28}$$

so we need to study the properties of $G(z,n)$ as a function of z. Our aim is to prove that $G(z,n) = 0$, which will follow from the next lemma.

Lemma 5.7. *The function $G(z,n)$, defined by (5.27), has the following properties.*

(i) $G(\lambda^{\mathrm{u}},n) = G(\lambda^{\mathrm{l}},n) \in \mathbb{R}$ *for* $\lambda \in \mathbb{R} \setminus (\partial\sigma_- \cup \partial\sigma_+ \cup \sigma_d)$.

(ii) *It has removable singularities at the points $\partial\sigma_- \cup \partial\sigma_+ \cup \sigma_d$, where $\sigma_d := \{\lambda_1, ..., \lambda_p\}$.*

Proof. (i). We can rewrite $G(z,n)$ as

$$G(z,n) = \frac{\tilde{\psi}_+(z,n)\tilde{\psi}_-(z,n) - \tilde{h}_+(z,n)\tilde{h}_-(z,n)}{\tilde{W}(z)}, \tag{5.29}$$

where $\tilde h_\pm(z,n)=\delta_\pm(z)h_\pm(z,n)$ as usual. The numerator is bounded near the points under consideration and the denominator does not vanish there. Thus $G(z,n)$ has no singularities at the points $(M_+\cup M_-)\setminus\sigma_d$.

Furthermore, by Lemma 5.4, **II**, **(a)**, and Lemma 3.1 we know that $G(z,n)$ has continuous limiting values on the sets σ_- and σ_+, except possibly at the edges, and satisfies

$$G(\lambda^{\mathrm{u}},n)=\overline{G(\lambda^{\mathrm{l}},n)},\quad \lambda\in\sigma_+\cup\sigma_-.$$

Hence, if we can show that these limits are real, they will be equal and $G(z,n)$ will extend to a meromorphic function on $\mathbb{C}$, that is, (i) holds. To this aim we first observe that (5.14), (5.28), and Lemma 3.1 imply

$$G(\lambda^{\mathrm{u}},n)=G(\lambda^{\mathrm{l}},n)\in\mathbb{R},\quad \lambda\in\operatorname{int}(\sigma^{(2)}). \tag{5.30}$$

Thus, it remains to prove

$$G(\lambda^{\mathrm{u}},n)=G(\lambda^{\mathrm{l}},n)\in\mathbb{R}\quad\text{for}\quad \lambda\in\operatorname{int}(\sigma_-^{(1)})\cup\operatorname{int}(\sigma_+^{(1)}). \tag{5.31}$$

Let us show that $G(\lambda,n)$ has no jump on the set $\operatorname{int}(\sigma_-^{(1)})\cup\operatorname{int}(\sigma_+^{(1)})$. We abbreviate

$$[G]:=G(\lambda)-\overline{G(\lambda)}=\left[\frac{\psi_+\psi_-}{W}\right]-\left[\frac{h_+h_-}{W}\right],\quad \lambda\in\sigma_\pm^{(1),\mathrm{u}}, \tag{5.32}$$

and drop some dependencies until the end of this lemma for notational simplicity.

Let $\lambda\in\operatorname{int}(\sigma_\mp^{(1),\mathrm{u}})$, then $\psi_\pm,h_\pm\in\mathbb{R}$ and $\overline{T}_\mp=-(\overline{W}\,\rho_\mp)^{-1}$. By (3.19), **(I)**, **(b)**, and (5.13) we obtain for $\lambda\in\operatorname{int}(\sigma_\mp^{(1)})$

$$\left[\frac{\psi_+\psi_-}{W}\right]=\psi_\pm\left[\frac{\psi_\mp}{W}\right]=\rho_\mp\psi_\pm\left(\psi_\mp T_\mp+\overline{\psi}_\mp\overline{T}_\mp\right)=\rho_\mp h_\pm\psi_\pm|T_\mp|^2. \tag{5.33}$$

Since $\rho_\pm\in\mathbb{R}$ for $\lambda\in\operatorname{int}(\sigma_\mp^{(1),\mathrm{u}})$, (3.19) implies

$$\left[\frac{h_\mp}{W}\right]=\rho_\pm\left[h_\mp T_\pm\right].$$

The only non-real summand in (5.12) is the Cauchy-type integral. The Plemelj formula applied to this integral gives

$$[h_\mp T_\pm]=-\rho_\mp\psi_\pm|T_\mp|^2W(\psi_q^\pm,\breve\psi_q^\pm)=\rho_\mp\psi_\pm|T_\mp|^2\frac{1}{\rho_\pm},$$

and by (5.33) we get

$$\left[\frac{h_+h_-}{W}\right]=\left[\frac{\psi_+\psi_-}{W}\right]=\rho_\mp\psi_\pm h_\pm|T_\mp|^2,\quad \lambda\in\operatorname{int}(\sigma_\mp^{(1)}). \tag{5.34}$$

Since $\tilde W\neq 0$ for $\lambda\in\operatorname{int}(\sigma_\mp^{(1)})$, the function

$$\rho_\mp\psi_\pm h_\pm|T_\mp|^2=-\frac{\delta_\mp^2}{\rho_\mp}\frac{\tilde\psi_\pm\tilde h_\pm}{|\tilde W|^2}$$

is bounded on the set under consideration. Finally, (5.34) and (5.32) imply (5.31).

(ii). Now we prove that the function $G(z,n)$ has removable singularities at the points $\partial\sigma_-\cup\partial\sigma_+\cup\sigma_d$. We divide this set into four subsets

$$\Omega_1^\pm=\partial\sigma^{(2)}\cap\operatorname{int}(\sigma_\mp),\quad \Omega_2=\partial\sigma^{(2)}\cap\partial\sigma,\quad \Omega_3^\pm=\partial\sigma_\pm^{(1)}\cap\partial\sigma_\pm,\quad \Omega_4=\sigma_d. \tag{5.35}$$

Since all singularities of G are at most isolated poles, it is sufficient to show that

$$G(z)=o\big((z-E)^{-1}\big) \tag{5.36}$$

from some direction in the complex plane.

Ω_1: Consider $E \in \Omega_1^+$ (the case $E \in \Omega_1^-$ being completely analogous). We will study $\lim_{\lambda\to E} G(\lambda, n)$ as $\lambda \in \operatorname{int}(\sigma^{(2)})$ using (5.28) with the "$-$" sign. Note that $\psi_- = O(1)$, $\rho_- = O(1)$, and $\hat{W}(E) \neq 0$. Moreover, we obtain from Lemma 3.1 respectively **II** that

$$\psi_+(\lambda) = \begin{cases} O(1), & E \notin \hat{M}_+, \\ O\left(\frac{1}{\sqrt{\lambda - E}}\right), & E \in \hat{M}_+, \end{cases} \qquad \frac{1}{T_+(\lambda)} = \begin{cases} O\left(\frac{1}{\sqrt{\lambda - E}}\right), & E \notin \hat{M}_+, \\ O(1), & E \in \hat{M}_+, \end{cases}$$

which shows

$$h_-(\lambda) = \frac{\overline{\psi_+(\lambda)} + R_+(\lambda)\psi_+(\lambda)}{T_+(\lambda)} = O\left(\frac{1}{\sqrt{\lambda - E}}\right)$$

for $\lambda \in \sigma^{(2)}$. Inserting this into (5.28) shows $G(\lambda, n) = O\big(\frac{1}{\sqrt{\lambda - E}}\big)$ and finishes the case $E \in \Omega_1$.

Ω_2: For $E \in \partial\sigma^{(2)} \cap \partial\sigma$, we use (5.28) and take the limit $\lambda \to E$ from $\sigma^{(2)}$. First of all, observe that

$$\breve{\delta}_-\left(R_-\psi_- + \overline{\psi}_-\right) = \begin{cases} O(1) & E \in \sigma_v, \\ o(1) & E \notin \sigma_v. \end{cases}$$

The case $E \in \sigma_v$ is evident. If $E \notin \sigma_v$ then (3.11) and (3.25) yield

$$\breve{\delta}_-\left(R_-\psi_- + \overline{\psi}_-\right) = \begin{cases} \breve{\delta}_-\big((\psi_- - \overline{\psi}_-) + (R_- + 1)\psi_-\big), & E \notin \hat{M}_- \\ \big(\breve{\delta}_-(\psi_- + \overline{\psi}_-) + (R_- - 1)\breve{\delta}_-\psi_-\big), & E \in \hat{M}_- \end{cases} = o(1).$$

Therefore, both for virtual and non-virtual levels the estimate

$$\breve{\delta}_-\left(R_-\psi_- + \overline{\psi}_-\right)\hat{W} = o(1), \quad E \in \partial\sigma_-, \tag{5.37}$$

is valid. Inserting (5.13) into the summand $\overline{\psi}_+ h_+ \rho_+$ of (5.28) (for the second summand we use an analogous approach) we obtain (recall (2.2))

$$\overline{\psi}_+ h_+ \rho_+ = \overline{\psi}_+ \rho_+ \rho_-(\overline{\psi}_- + R_-\psi_-)W = \frac{\overline{\psi}_+\breve{\delta}_+}{P_+P_-}\delta_+\hat{\delta}_-\breve{\delta}_-(\overline{\psi}_- + R_-\psi_-)W$$

$$= \frac{\overline{\psi}_+\breve{\delta}_+}{P_+P_-}\breve{\delta}_-\left(R_-\psi_- + \overline{\psi}_-\right)\hat{W}. \tag{5.38}$$

Combining the estimate

$$\frac{\overline{\psi}_+\breve{\delta}_+}{P_+P_-} = O\Big(\frac{1}{\lambda - E}\Big)$$

with (5.37) we have $G(z) = o\big((z - E)^{-1}\big)$ as desired.

Ω_3: Suppose that $E \in \partial\sigma_-^{(1)} \cap \partial\sigma_-$ (the case $E \in \partial\sigma_+^{(1)} \cap \partial\sigma_+$ is again analogous). Now we cannot use (5.28), so we proceed directly from formula (5.27) estimating the summands $\frac{\psi_+\psi_-}{W}$ and $\frac{h_+h_-}{W}$ separately. We investigate the limit as $\lambda \to E$ from the set $\operatorname{int}(\sigma_-^{(1)})$. By Lemma 3.1 and (3.22) we have

$$\frac{\psi_+\psi_-}{W} = \frac{\hat{\psi}_+\hat{\psi}_-}{\hat{W}} = O\left(\frac{1}{\sqrt{\lambda - E}}\right), \tag{5.39}$$

hence the first summand has the desired behavior. To estimate the second summand, we split the function $h_-(\lambda, n)$ according to

$$h_-(\lambda, n) = h_1(\lambda, n) + h_2(\lambda, n),$$

where

$$h_1(\lambda,n) = W^+_{q,n-1}(\zeta_-(\lambda,\cdot), d_-(\lambda,n,\cdot)), \quad h_2(\lambda,n) = h_-(\lambda,n) - h_1(\lambda,n),$$

$$d_-(\lambda,n,.) := \int_{\sigma_-^{(1),u}} \frac{|T_-(\xi)|^2 \psi_+(\xi,n)\psi_q^+(\xi,\cdot)}{\xi-\lambda} d\omega_-(\xi). \tag{5.40}$$

It follows from the proof of Lemma 5.4 that $h_2(\lambda) = O(\zeta_-(\lambda))$ for $\lambda \to E$. Recall that at the point under consideration singularities $E \in \{\mu_1^+,\dots,\mu_{g_+}^+\} \cup \hat M_-$ might occur (in the case $\partial\sigma_-^{(1)} \cap \partial\sigma$ one can have $E \in M_+ \cup \breve M_+$ and in the case $\partial\sigma_-^{(1)} \cap \partial\sigma_+^{(1)}$ one can have $E \in \hat M_+$). Introduce

$$\phi_q^+(z,n) := \breve\delta_+(z)\breve\psi_q^+(z,n) \tag{5.41}$$

and recall that (2.15) implies

$$\phi_q^+(z,n) - \phi_q^+(E,n) = O(\sqrt{z-E}). \tag{5.42}$$

Then (see (2.2) and (2.13)) we have

$$\frac{h_+\zeta_-}{W} = O\Big(\frac{h_+\breve\psi_q^+}{WT_+}\Big) = O\Big(\frac{h_+\hat\delta_+\breve\delta_+\breve\psi_q^+}{P_+}\Big) = O\Big(\frac{h_+\hat\delta_+}{P_+}\Big)\phi_q^+. \tag{5.43}$$

Now we distinguish two cases: (a) $E \in \partial\sigma_-^{(1)} \cap \partial\sigma_+^{(1)}$ and (b) $E \in \partial\sigma_-^{(1)} \cap \partial\sigma$.

Case (a). By (5.13) and (5.37) we have

$$\hat\delta_+ h_+ = \frac{(R_-\psi_- + \overline{\psi}_-)\hat\delta_+}{T_-} = \frac{\hat W\breve\delta_-(R_-\psi_- + \overline{\psi}_-)}{P_-} = o\Big(\frac{1}{\sqrt{\lambda-E}}\Big), \tag{5.44}$$

therefore

$$\frac{h_+(\lambda)\zeta_-(\lambda)}{W(\lambda)} = o\Big(\frac{1}{\sqrt{\lambda-E}}\Big)\frac{\phi_q^+(\lambda)}{P_+(\lambda)}. \tag{5.45}$$

As a consequence of $\frac{\phi_q^+}{P_+} = O\big(\frac{1}{\sqrt{\lambda-E}}\big)$ we obtain

$$\frac{h_+h_2}{W} = o\Big(\frac{1}{\lambda-E}\Big), \quad E \in \partial\sigma_-. \tag{5.46}$$

Next, we have to estimate

$$\frac{h_+h_1}{W} = W^+_{q,n-1}\Big(\frac{h_+\zeta_-}{W}, d_-\Big). \tag{5.47}$$

By (5.42) we can represent (5.45) as

$$\frac{h_+(\lambda)\zeta_-(\lambda)}{W(\lambda)} = o\Big(\frac{1}{\sqrt{\lambda-E}}\Big)\Big(\frac{\breve\psi_q^+(E)}{\sqrt{\lambda-E}} + O(1)\Big). \tag{5.48}$$

Then (5.47) implies

$$\begin{aligned}\frac{h_+(\lambda,n)h_1(\lambda,n)}{W(\lambda)} &= o\Big(\frac{1}{\sqrt{\lambda-E}}\Big)\Big(O(d_-(\lambda,n)) + O(d_-(\lambda,n-1)) \\ &\quad + \frac{O\left(W^+_{q,n-1}\left(\phi_q^+(E), d_-(\lambda)\right)\right)}{\sqrt{\lambda-E}}\Big).\end{aligned} \tag{5.49}$$

To estimate d_- in the first two summands we distinguish between the resonance case, $E \in \sigma_v$, and non-resonance, $E \notin \sigma_v$. First let $E \notin \sigma_v$, that is, $\hat W(E) \neq 0$.

From (5.19) and (5.20) we see that the integrand is bounded as $\lambda \to E \notin \sigma_v$, then $d_-(\lambda) = O(1)$ by [17].

If $E \in \sigma_v$, then (3.22) (see also (5.19)) yields

$$|T_-(\xi)|^2 \rho_-(\xi)\psi_+(\xi,\cdot)\psi_q^+(\xi,\cdot) = O\left(\frac{1}{\sqrt{\xi - E}}\right)$$

and [17, Eq. (29.8)] implies

$$d_-(\lambda) = o\left(\frac{1}{\sqrt{\lambda - E}}\right). \tag{5.50}$$

For the estimate of the last summand in (5.49) we use (5.19) and (5.40) to represent the integrand in $W_{q,n-1}^+\left(\breve{\psi}_q^+(E), d_-(\lambda)\right)$ as

$$\begin{aligned}|T_-(\xi)|^2 \rho_-(\xi)\psi_+(\xi,n) W_{q,n-1}^+\left(\psi_q^+(\xi), \phi_q^+(E)\right)\\ = O\Big(\frac{\sqrt{\xi - E}}{|\hat{W}(\xi)|^2} W_{q,n-1}^+\big(\hat{\psi}_q^+(\xi), \phi_q^+(E)\big)\Big).\end{aligned}$$

It follows from (2.15) and (5.41) that

$$W_{q,n-1}^+(\hat{\psi}_q^+(\xi), \phi_q^+(E)) = O\big(\sqrt{\xi - E}\big),$$

which implies together with (3.22) the boundedness of the integrand near E. Thus,

$$W_{q,n-1}^+\left(\phi_q^+(E), d_-(\lambda)\right) = O(1), \tag{5.51}$$

and combining (5.46), (5.49), (5.51), and (5.50) finishes case (a).

Case (b). Now we do not have estimate (5.37) (cf. **III**, **(b)**) at our disposal, but we can proceed as in (5.43), (5.44) since $P_+(E) \neq 0$ and arrive at

$$\frac{h_+\zeta_-}{W} = O(h_+\hat{\delta}_+) = O\Big(\frac{\hat{W}\breve{\delta}_-(R_-\psi_- + \overline{\psi}_-)}{P_-}\Big) = O\Big(\frac{\hat{W}}{\sqrt{\lambda - E}}\Big). \tag{5.52}$$

This estimate is sufficient to conclude that (5.46) is valid in case (b) as well. For h_1, we use the following estimate (cf. (5.50) and (5.52)) instead of (5.47):

$$\frac{h_+h_1}{W} = O\left(\frac{h_+\zeta_-}{W}\right) O\left(d_-\right) = O\left(\frac{\hat{W}}{\sqrt{\lambda - E}}\right) o\left(\frac{1}{\sqrt{\lambda - E}}\right).$$

Combining this with (5.46) finishes case (b).

Ω_4: Finally we have to show that the singularities of $G(z,n)$ at the points of the discrete spectrum are removable. Since $\tilde{W}(z)$ has simple zeros at $z = \lambda_k$, it suffices by (5.29) to show that

$$\tilde{h}_+(\lambda_k,n)\tilde{h}_-(\lambda_k,n) = \tilde{\psi}_-(\lambda_k,n)\tilde{\psi}_+(\lambda_k,n). \tag{5.53}$$

By Lemma 5.4, the functions $\tilde{h}_\mp = \delta_\mp h_\mp$ given in (5.12) are continuous at the points $\breve{M}_\pm$. Since $(\delta_\mp T_\pm^{-1})(\lambda_k) = 0$ and $(\delta_\mp T_\pm^{-1}\breve{\psi}_q^\pm)(\lambda_k) = 0$, only the last summand in (5.12) is non-zero. We compute the limit of this summand as $\lambda \to \lambda_k$ using (3.19),

$$\tilde{h}_\mp(\lambda_k) = -\gamma_{\pm,k}\tilde{\psi}_\pm(\lambda_k)\frac{d\tilde{W}(\lambda_k)}{d\lambda}, \tag{5.54}$$

and apply (3.20) to obtain (5.53). □

The identity $G(z,n) \equiv 0$ implies

$$\psi_+(z,n)\psi_-(z,n) - h_+(z,n)h_-(z,n) \equiv 0, \quad \forall n \in \mathbb{Z}. \tag{5.55}$$

For $z \to \infty$ we obtain by (2.16) and (5.9)

$$\psi_+(z,n)\psi_-(z,n) = K_+(n,n)K_-(n,n) \prod_{j=0}^{n-1}{}^{*} \frac{a_q^+(j)}{a_q^-(j)}(1+o(1)).$$

Formulas (5.15) and (3.24) imply

$$h_+(z,n)h_-(z,n) = \frac{1}{T_+(\infty)^2 K_+(n,n)K_-(n,n)} \prod_{j=0}^{n-1}{}^{*} \frac{a_q^-(j)}{a_q^+(j)}(1+o(1))$$

and by (5.55),

$$K_+(n,n)K_-(n,n) \prod_{j=0}^{n-1}{}^{*} \frac{a_q^+(j)}{a_q^-(j)} = \frac{1}{T_+(\infty)}.$$

The value on the left hand side does not depend on n, so using (5.7) we conclude

$$a_+(n) = a_-(n) \equiv a(n), \quad \forall n \in \mathbb{Z}. \tag{5.56}$$

It remains to prove $b_+(n) = b_-(n)$. If we eliminate the reflection coefficient $R_\pm$ from (5.13) at n and (5.25) at $n+1$ we obtain

$$\begin{aligned} G_1(\lambda,n) &:= \frac{\psi_+(\lambda,n)\psi_-(\lambda,n+1) - h_+(\lambda,n+1)h_-(\lambda,n)}{W(\lambda)} \\ &= \rho_+(\lambda)\big(\overline{h_\pm(\lambda,n+1)}\psi_\pm(\lambda,n) - \overline{\psi_\pm(\lambda,n)}h_\pm(\lambda,n+1)\big), \quad \lambda \in \sigma^{(2),\mathrm{u,l}}. \end{aligned} \tag{5.57}$$

Proceeding as for $G(\lambda,n)$ in Lemma 5.7 we can show that that the function $G_1(z,n)$ is holomorphic in $\mathbb{C}$. From (5.15), (5.9), (3.24), (2.16), (5.56), and the Liouville theorem we conclude that

$$\frac{\psi_+(z,n)\psi_-(z,n+1) - h_+(z,n+1)h_-(z,n)}{W(z)} = -1/a(n).$$

We compute the asymptotics of

$$\bar{W}(z,n) := a(n)\left(\psi_+(z,n)\psi_-(z,n+1) - h_+(z,n+1)h_-(z,n)\right) = -W(z)$$

as $z \to \infty$ and obtain (compare (3.5))

$$0 = \bar{W}(z,n) - \bar{W}(z,n-1) = (b_+(n) - b_-(n))K_+(0,0)K_-(0,0). \tag{5.58}$$

This implies in particular $b_+(n) = b_-(n) \equiv b(n)$, hence the proof of Theorem 5.3 is finished.

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References

[1] J. Bazargan and I. Egorova, *Jacobi operator with step-like asymptotically periodic coefficients*, Mat. Fiz. Anal. Geom. **10**, No.3, 425–442 (2003).

[2] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16**, No.4, 955–977 (2000).

[3] A. Boutet de Monvel, I. Egorova, and E. Khruslov, *Soliton asymptotics of the Cauchy problem solution for the Toda lattice*, Inverse Problems **13**, No.2, 223–237 (1997).

[4] A. Boutet de Monvel, I. Egorova, and G. Teschl, *Inverse Scattering theory for one-dimensional Schrödinger operators with steplike periodic potentials*, Preprint, arXiv:0707.4632.

[5] K. M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **14**, 2166–2175 (1973).

[6] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, No.1, 35–83 (1996).

[7] I. Egorova, *The scattering problem for step-like Jacobi operator*, Mat. Fiz. Anal. Geom. **9**, No.2, 188–205 (2002).

[8] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with quasi-periodic background*, Comm. Math. Phys. **264-3**, 811–842 (2006).

[9] I. Egorova, J. Michor, and G. Teschl, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817–1827 (2007).

[10] I. Egorova, J. Michor, and G. Teschl, *Soliton solutions of the Toda hierarchy on quasi-periodic background revisited*, Math. Nach. (to appear).

[11] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with steplike quasi-periodic background*, Inverse Problems **23**, 905–918 (2007).

[12] G. S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl. **17**, 1684–1688 (1976).

[13] S. Kamvissis and G. Teschl, *Stability of periodic soliton equations under short range perturbations*, Phys. Lett. A, **364-6**, 480–483 (2007).

[14] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice under short range perturbations*, Preprint, arXiv:0705.0346.

[15] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[16] J. Michor and G. Teschl, *Trace formulas for Jacobi operators in connection with scattering theory for quasi-periodic background*, Operator Theory, Analysis, and Mathematical Physics, J. Janas, et al. (eds.), 69–76, Oper. Theory Adv. Appl., 174, Birkhäuser, Basel, 2007.

[17] N. I. Muskhelishvili, *Singular integral equations*, P. Noordhoff Ltd., Groningen, 1953.

[18] G. Teschl, *Oscillation theory and renormalized oscillation theory for Jacobi operators*, J. Diff. Eqs. **129**, 532–558 (1996).

[19] G. Teschl, *On the initial value problem for the Toda and Kac-van Moerbeke hierarchies*, in Differential Equations and Mathematical Physics, R. Weikard and G. Weinstein (eds.), 375–384, AMS/IP Studies in Advanced Mathematics 16, Amer. Math. Soc., Providence, 2000.

[20] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

[21] G. Teschl, *Algebro-geometric constraints on solitons with respect to quasi-periodic backgrounds*, Bull. London Math. Soc. **39**, No.4, 677–684 (2007).

[22] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Vol. 2, Oxford, at the Clarendon Press, 1958.

[23] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. Pure Appl. Math. **44**, No.8-9, 1171–1242 (1991).

[24] A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi Matrices with the Spectrum on an Interval, a finite system of intervals or a Cantor set of positive length*, Comm. Math. Phys. **226**, 567–605 (2002).

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INVERSE SCATTERING TRANSFORM FOR THE TODA HIERARCHY WITH STEPLIKE FINITE-GAP BACKGROUNDS

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ABSTRACT. We provide a rigorous treatment of the inverse scattering transform for the entire Toda hierarchy for solutions which are asymptotically close to (in general) different finite-gap solutions as $n \to \pm\infty$.

1. INTRODUCTION

The Toda lattice is one of the most prominent discrete integrable wave equations. In particular, it can be solved via the inverse scattering method. For the classical case, where the solution is asymptotically equal to the (same) constant solution, this is of course well understood and covered in several monographs (e.g.) [12], [36], or [33]. The corresponding long-time asymptotics were first computed by Novokshenov and Habibullin [29] and were later made rigorous by Kamvissis [16] under the additional assumption that no solitons are present (the case of solitons was recently added in [23]; see also the review [22]).

The inverse scattering transform for the entire Toda hierarchy in the case of a finite-gap background was solved only recently by us in [8] as a continuation of [31]. Similar results were obtained by Khanmamedov [21]. Long-time asymptotics for such solutions have been given by Kamvissis and Teschl for the case without solitons [18], [19], [20] and by Krüger and Teschl for the case with solitons [24] (for related trace formulas and conserved quantities see [27]).

In this respect it is important to mention that even the important case of a one-soliton solution on a finite-gap background has different spatial asymptotics as $n \to \pm\infty$ and hence is not covered by the above results (see [11], [34]). Hence this clearly raises the need to extend the results from [8] to the case of solutions which are asymptotically equal to (in general) different finite-gap solutions as $n \to \pm\infty$.

In fact, the simplest case, where the solution is asymptotically equal to two different constant solutions, has already attracted considerable interest in the past. The first to solve the corresponding Cauchy problem (in the case of rapid decay with respect to the background) seems to be Oba [30]. Moreover, the long-time asymptotics were considered in [1], [3], [4], [14], [15], [17], [37].

Our aim here is to fill this gap and to provide a treatment of the inverse scattering transform for the entire Toda hierarchy in the case of steplike quasi-periodic finite-gap backgrounds. Note that since we treat the entire Toda hierarchy, our results also cover the Kac–van Moerbeke hierarchy as a special case [28].

Finally, we remark that the corresponding result for the Korteweg–de Vries equation is much more involved and was only recently solved by Egorova, Grunert, and

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Teschl [6] under some additional restrictions on the spectra of the background operators.

After introducing the Toda hierarchy in Section 2, we will first show that a solution will stay close to a given background solution in Section 3. This result implies that a short-range perturbation of a steplike finite-gap solution will stay short range for all time, and it shows that the time-dependent scattering data satisfy the hypothesis necessary for the Gel'fand–Levitan–Marchenko theory [26]. This result constitutes the main technical ingredient for the inverse scattering transform. In Section 4 we review some necessary facts on quasi-periodic finite-gap solutions and in Section 5 we compute the time dependence of the scattering data and discuss its dynamics.

2. The Toda hierarchy

In this section we introduce the Toda hierarchy using the standard Lax formalism ([25]). We first review some basic facts from [2] (see also [13], [33]).

We will only consider bounded solutions and hence require

Hypothesis H. 2.1. *Suppose $a(t)$, $b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z},\mathbb{R}), \qquad b(t) \in \ell^\infty(\mathbb{Z},\mathbb{R}), \qquad a(n,t) \neq 0, \qquad (n,t) \in \mathbb{Z}\times\mathbb{R},$$

and let $t \mapsto (a(t), b(t))$ be differentiable in $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

Associated with $a(t), b(t)$ is a Jacobi operator

$$H(t) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \qquad f \mapsto \tau(t)f, \tag{2.1}$$

where

$$\tau(t)f(n) = a(n,t)f(n+1) + a(n-1,t)f(n-1) + b(n,t)f(n) \tag{2.2}$$

and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. Moreover, choose constants $c_0 = 1$, c_j, $1 \le j \le r$, $c_{r+1} = 0$, set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{j} c_{j-\ell}\langle\delta_n, H(t)^\ell\delta_n\rangle, \\ h_j(n,t) &= 2a(n,t)\sum_{\ell=0}^{j} c_{j-\ell}\langle\delta_{n+1}, H(t)^\ell\delta_n\rangle + c_{j+1}, \end{aligned} \tag{2.3}$$

where $\langle\delta_m, A\delta_n\rangle$ denote the matrix elements of an operator A with respect to the standard basis, and consider the Lax operator

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r}(2a(t)g_j(t)S^+ - h_j(t))H(t)^{r-j} + g_{r+1}(t), \tag{2.4}$$

where $S^\pm f(n) = f(n\pm 1)$. Restricting to the two-dimensional nullspace

$$\mathrm{Ker}(\tau(t)-z), \quad z \in \mathbb{C},$$

of $\tau(t)-z$, we have the following representation of $P_{2r+2}(t)$:

$$P_{2r+2}(t)\Big|_{\mathrm{Ker}(\tau(t)-z)} = 2a(t)G_r(z,.,t)S^+ - H_{r+1}(z,.,t), \tag{2.5}$$

where $G_r(z,n,t)$ and $H_{r+1}(z,n,t)$ are monic polynomials in z of the type

$$G_r(z,n,t) = \sum_{j=0}^{r} z^j g_{r-j}(n,t),$$

$$H_{r+1}(z,n,t) = z^{r+1} + \sum_{j=0}^{r} z^j h_{r-j}(n,t) - g_{r+1}(n,t). \tag{2.6}$$

A straightforward computation shows that the Lax equation

$$\frac{d}{dt}H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{2.7}$$

is equivalent to

$$\mathrm{TL}_r(a(t),b(t))_1 = \dot{a}(n,t) - a(n,t)\Big(g_{r+1}(n+1,t) - g_{r+1}(n,t)\Big) = 0,$$

$$\mathrm{TL}_r(a(t),b(t))_2 = \dot{b}(n,t) - \Big(h_{r+1}(n,t) - h_{r+1}(n-1,t)\Big) = 0, \tag{2.8}$$

where the dot denotes a derivative with respect to t. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy $\mathrm{TL}_r(a,b) = (\mathrm{TL}_r(a,b)_1, \mathrm{TL}_r(a,b)_2) = 0$. We will always consider r as a fixed, but arbitrary, value.

Finally, we recall that the Lax equation (2.7) implies existence of a unitary propagator $U_r(t,s)$ such that the family of operators $H(t)$, $t \in \mathbb{R}$, are unitarily equivalent, $H(t) = U_r(t,s)H(s)U_r(s,t)$.

3. The initial value problem

First of all we recall the basic existence and uniqueness theorem for the Toda hierarchy (see, e.g., [31], [32], or [33, Section 12.2]).

Theorem 3.1. *Suppose $(a_0,b_0) \in M = \ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$. Then there exists a unique integral curve $t \mapsto (a(t),b(t))$ in $C^\infty(\mathbb{R},M)$ of the Toda hierarchy, that is, $\mathrm{TL}_r(a(t),b(t)) = 0$, such that $(a(0),b(0)) = (a_0,b_0)$.*

In [31] it was shown that solutions which are asymptotically close to the constant solution at the initial time stay close for all time. Our first aim is to extend this result to include perturbations of quasi-periodic finite-gap solutions. In fact, we will even be a bit more general. Set

$$\|(a,b)\|_{w,p} = \begin{cases} \left(\sum_{n\in\mathbb{Z}} w(n)\Big(|a(n)|^p + |b(n)|^p\Big)\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{n\in\mathbb{Z}} w(n)\Big(|a(n)| + |b(n)|\Big), & p = \infty. \end{cases} \tag{3.1}$$

Then

Lemma 3.2. *Let $w(n) \ge 1$ be some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$ and fix some $1 \le p \le \infty$. Suppose $a(n,t)$, $b(n,t)$ and $a_\pm(n,t)$, $b_\pm(n,t)$ are arbitrary bounded solutions of the Toda hierarchy and abbreviate*

$$\bar{a}(n,t) = \begin{cases} a_+(n,t), & n \ge 0, \\ a_-(n,t), & n < 0, \end{cases} \qquad \bar{b}(n,t) = \begin{cases} b_+(n,t), & n \ge 0, \\ b_-(n,t), & n < 0. \end{cases} \tag{3.2}$$

Then, if

$$\|\big(a(t) - \bar{a}(t), b(t) - \bar{b}(t)\big)\|_{w,p} < \infty \tag{3.3}$$

holds for one $t = t_0 \in \mathbb{R}$, then it holds for all $t \in \mathbb{R}$.

Proof. Without loss of generality we assume that $t_0 = 0$. Let us consider the differential equation for the differences $\delta(n,t) = \big(a(n,t) - \bar{a}(n,t), b(n,t) - \bar{b}(n,t)\big)$ in the Banach space of pairs of bounded sequences $\delta = (\delta_1, \delta_2)$ for which the norm $\|\delta\|_{w,p}$ is finite. We claim that δ satisfies an inhomogeneous linear differential equation of the form

$$\dot{\delta}(t) = \sum_{|j| \le r+1} A_{r,j}(t)(S^+)^j \delta(t) + B_r(t)$$

(see e.g. [5] for the theory of ordinary differential equations in Banach spaces). Here $S^{\pm}(\delta_1(n,t), \delta_2(n,t)) = (\delta_1(n \pm 1, t), \delta_2(n \pm 1, t))$ are the shift operators,

$$A_{r,j}(n,t) = \begin{pmatrix} A_{r,j}^{11}(n,t) & A_{r,j}^{12}(n,t) \\ A_{r,j}^{21}(n,t) & A_{r,j}^{22}(n,t) \end{pmatrix},$$

are multiplication operators with bounded two by two matrix-valued sequences, and

$$B_r(n,t) = \begin{pmatrix} B_{r,1}(n,t) \\ B_{r,2}(n,t) \end{pmatrix}$$

is a vector in our Banach space with $B_{r,i}(n,t) = 0$ for $|n| > \lfloor \frac{r}{2} \rfloor + 1$. All entries of $A_{r,j}(t)$ and $B_r(t)$ are polynomials with respect to $(a(n+j,t), b(n+j,t))$, $(a_{\pm}(n+j,t), b_{\pm}(n+j,t))$, $|j| \le \lfloor \frac{r}{2} \rfloor + 1$. Moreover, by our assumption the shift operators are continuous,

$$\|S^{\pm}\| = \begin{cases} \sup_{n \in \mathbb{Z}} |\frac{w(n)}{w(n\pm1)}|^{1/p}, & p \in [1, \infty), \\ \sup_{n \in \mathbb{Z}} |\frac{w(n)}{w(n\pm1)}|, & p = \infty, \end{cases}$$

and same is true for the multiplication operators $A_{r,j}(t)$ whose norms depend only on the supremum of the entries by Hölder's inequality, that is, on the sup norms of $(a(t), b(t))$ and $(a_{\pm}(t), b_{\pm}(t))$. Finally, recall that by unitary equivalence of the operator family $H(t)$, respectively $H_{\pm}(t)$, we have a uniform bound of the sup norm $\sup_n(|a(n,t)| + |b(n,t)|) \le 2\|H(t)\| = 2\|H(0)\|$, respectively $\sup_n(|a_{\pm}(n,t)| + |b_{\pm}(n,t)|) \le 2\|H_{\pm}(t)\| = 2\|H_{\pm}(0)\|$. Consequently, there is a constant such that $\sum_{|j| \le r+1} \|A_{r,j}(t)\| \|(S^+)^j\| \le C_r$. Moreover, we will show below that the vector $B_r(t)$ has only finitely many nonzero entries and thus $\|B_r(t)\|_{w,p} \le D_r$, where the constant again depends only on the sup norms of $(a(t), b(t))$ and $(a_{\pm}(t), b_{\pm}(t))$. Hence

$$\|\delta(t)\|_{w,p} \le \|\delta(0)\|_{w,p} + \int_0^t \big(C_r \|\delta(s)\|_{w,p} + D_r\big)$$

and Gronwall's inequality implies

$$\|\delta(t)\|_{w,p} \le \|\delta(0)\|_{w,p} \mathrm{e}^{C_r t} + \frac{D_r}{C_r}(\mathrm{e}^{C_r t} - 1).$$

It remains to show existence of the above differential equation. This will follow once we show that $g_{r+1}(t) - \bar{g}_{r+1}(t)$ and $h_{r+1}(t) - \bar{h}_{r+1}(t)$ can be written as a linear combination of shifts of δ with the coefficients depending only on $(a(t), b(t))$ and $(a_{\pm}(t), b_{\pm}(t))$. The fact that $(\bar{a}, \bar{b})$ does not solve TL_r only affects finitely many terms and gives rise to the inhomogeneous term $B_r(t)$ which is nonzero only for a finite number of terms.

To see that $g_{r+1}(t) - \bar{g}_{r+1}(t)$ and $h_{r+1}(t) - \bar{h}_{r+1}(t)$ can be written as a linear combination of shifts of δ we can use induction on r. It suffices to consider the homogenous case where $c_j = 0$, $1 \le j \le r$, since all involved sums are finite. In this case [33, Lemma 6.4] shows that $g_j(n,t)$, $h_j(n,t)$ can be recursively computed from $g_0(n,t) = 1$, $h_0(n,t) = 0$ via

$$g_{j+1}(n,t) = \frac{1}{2}\big(h_j(n,t) + h_j(n-1,t)\big) + b(n,t) g_j(n,t),$$

$$h_{j+1}(n,t) = 2a(n,t)^2 \sum_{l=0}^{j} g_{j-l}(n,t) g_l(n+1,t) - \frac{1}{2}\sum_{l=0}^{j} h_{j-l}(n,t) h_l(n,t)$$

and similarly for $\bar{g}_j(n,t)$, $\bar{h}_j(n,t)$. Hence the claim follows.

Finally, observe that since $w(n) \ge 1$ this solution is bounded and hence coincides with the solution of the Toda equation from Theorem 3.1. □

For closely related results we also refer to [35].

4. Quasi-periodic finite-gap solutions

As a preparation for our next section we first need to recall some facts on quasi-periodic finite-gap solutions (again see [2], [13], or [33]).

Let $H_q^\pm$ be two quasi-periodic finite-band Jacobi operators,[1]

$$H_q^\pm(t) f(n) = a_q^\pm(n,t) f(n+1) + a_q^\pm(n-1,t) f(n-1) + b_q^\pm(n,t) f(n) \tag{4.1}$$

in $\ell^2(\mathbb{Z})$ associated with the Riemann surface of the square root

$$P_\pm(z) = -\prod_{j=0}^{2g_\pm+1} \sqrt{z - E_j^\pm}, \qquad E_0^\pm < E_1^\pm < \cdots < E_{2g_\pm+1}^\pm, \tag{4.2}$$

where $g_\pm \in \mathbb{N}$ and $\sqrt{\cdot}$ is the standard root with branch cut along $(-\infty, 0)$. In fact, $H_q^\pm(t)$ are uniquely determined by fixing a Dirichlet divisor $\sum_{j=1}^{g^\pm}(\mu_j^\pm(t), \sigma_j^\pm(t))$, where $\mu_j^\pm(t) \in [E_{2j-1}^\pm, E_{2j}^\pm]$ and $\sigma_j^\pm(t) \in \{-1, 1\}$. The time evolution of the Dirichlet divisor is determined by the Dubrovin equations (cf. [33, (13.2)])and linearized by the Abel map (cf. [33, Sect. 13.2]). The spectra of $H_q^\pm(t)$ consist of $g_\pm + 1$ bands

$$\sigma_\pm := \sigma(H_q^\pm(t)) = \bigcup_{j=0}^{g_\pm} [E_{2j}^\pm, E_{2j+1}^\pm]. \tag{4.3}$$

We will identify the set $\mathbb{C} \setminus \sigma(H_q^\pm(t))$ with the upper sheet of the Riemann surface. Associated with $H_q^\pm(t)$ are the Weyl solutions

$$\psi_q^\pm(z,n,t) \in \ell^2(\pm\mathbb{N}) \tag{4.4}$$

normalized such that $\psi_q^\pm(z,0,t) = 1$. We will use the convention that for $\lambda \in \sigma_\pm$ we set $\psi_q^\pm(\lambda,n,t) = \lim_{\epsilon\downarrow 0} \psi_q^\pm(\lambda + \mathrm{i}\epsilon)$. Then

$$\hat{\psi}_q^\pm(z,n,t) = \exp\big(\alpha_r^\pm(z,t)\big)\psi_q^\pm(z,n,t) \tag{4.5}$$

satisfies

$$H_q^\pm(t)\hat{\psi}_q^\pm(z,n,t) = z\hat{\psi}_q^\pm(z,n,t), \tag{4.6}$$

$$\frac{d}{dt}\hat{\psi}_q^\pm(z,n,t) = P_{q,2r+2}^\pm(t)\hat{\psi}_q^\pm(z,n,t), \tag{4.7}$$

[1] Everywhere in this paper the sub or super index "+" (resp. "−") refers to the background on the right (resp. left) half-axis.

where ([33], (13.47))

$$\alpha_r^{\pm}(z,t) = \int_0^t \big(2a_q^{\pm}(0,s)G_{q,r}^{\pm}(z,0,s)\psi_q^{\pm}(z,1,s) - H_{q,r+1}^{\pm}(z,0,s)\big)ds. \tag{4.8}$$

Note that the integrand in this last expression might have poles if z lies in one of the spectral gaps $[E_{2j-1}^{\pm}, E_{2j}^{\pm}]$. Hence one has to understand $\alpha_r^{\pm}(z,t)$ as a limit from $z \in \mathbb{C}\backslash\mathbb{R}$ for such values of z. Alternatively one can use the expression in terms of Riemann theta functions. Moreover, $\exp\big(\alpha_r^{\pm}(z,t)\big)$ has simple poles at $\mu_j^{\pm}(0)$ and simple zeros at $\mu_j^{\pm}(t)$. We refer to the discussion in [8] for further details.

5. Inverse scattering transform

Fix two quasi-periodic finite-gap solutions $a_q^{\pm}(n,t)$, $b_q^{\pm}(n,t)$ as in the previous section. Let $a(n,t)$, $b(n,t)$ be a solution of the Toda hierarchy satisfying

$$\sum_{n=0}^{\pm\infty}(1+|n|)\Big(|a(n,t) - a_q^{\pm}(n,t)| + |b(n,t) - b_q^{\pm}(n,t)|\Big) < \infty \tag{5.1}$$

for one (and hence for any) $t_0 \in \mathbb{R}$. In [10] (see also [7], [9], [38]) we have developed scattering theory for the Jacobi operator $H(t)$ associated with $a(n,t)$, $b(n,t)$. Jost solutions, transmission and reflection coefficients now depend on an additional parameter $t \in \mathbb{R}$. The essential spectrum of $H(t)$ is (absolutely) continuous and

$$\sigma(H(t)) \equiv \sigma(H), \quad \sigma_{ess}(H) = \sigma_+ \cup \sigma_-, \quad \sigma_p(H) = \{\lambda_k\}_{k=1}^p \subseteq \mathbb{R}\backslash\sigma_{ess}(H), \tag{5.2}$$

where $p \in \mathbb{N}$ is finite. We introduce the sets

$$\sigma^{(2)} := \sigma_+ \cap \sigma_-, \quad \sigma_{\pm}^{(1)} = \operatorname{clos}(\sigma_{\pm} \setminus \sigma^{(2)}), \quad \sigma := \sigma_+ \cup \sigma_-, \tag{5.3}$$

where σ is the (absolutely) continuous spectrum of $H(t)$ and $\sigma_+^{(1)} \cup \sigma_-^{(1)}$, $\sigma^{(2)}$ are the parts which are of multiplicity one, two, respectively.

The Jost solutions $\psi_{\pm}(z,n,t)$ are normalized such that

$$\psi_{\pm}(z,n,t) = \psi_q^{\pm}(z,n,t)\,(1+o(1)) \quad \text{as } n \to \pm\infty. \tag{5.4}$$

Transmission $T_{\pm}(\lambda,t)$ and reflection $R_{\pm}(\lambda,t)$ coefficients are defined via the scattering relations

$$T_{\mp}(\lambda,t)\psi_{\pm}(\lambda,n,t) = \overline{\psi_{\mp}(\lambda,n,t)} + R_{\mp}(\lambda,t)\psi_{\mp}(\lambda,n,t), \qquad \lambda \in \sigma_{\mp}, \tag{5.5}$$

which implies

$$T_{\pm}(\lambda,t) := \frac{W(\overline{\psi_{\pm}(\lambda,t)}, \psi_{\pm}(\lambda,t))}{W(\psi_{\mp}(\lambda,t), \psi_{\pm}(\lambda,t))}, \quad R_{\pm}(\lambda,t) := -\frac{W(\psi_{\mp}(\lambda,t), \overline{\psi_{\pm}(\lambda,t)})}{W(\psi_{\mp}(\lambda,t),\, \psi_{\pm}(\lambda,t))}, \tag{5.6}$$

$\lambda \in \sigma_{\pm}$. Here $W_n(f,g) = a(n)(f(n)g(n+1) - f(n+1)g(n))$ denotes the usual Wronski determinant.

To define the norming constants we need to remove the poles of $\psi^{\pm}(z,n,t)$ by introducing

$$\tilde{\psi}^{\pm}(z,n,t) = \delta_{\pm}(z,t)\psi^{\pm}(z,n,t), \qquad \delta_{\pm}(z,t) := \prod_{\mu_j^{\pm}(t) \in M_{\pm}(t)} (z - \mu_j^{\pm}(t)), \tag{5.7}$$

where

$$M^{\pm}(t) = \{\mu_j^{\pm}(t)\,|\,\mu_j^{\pm}(t) \in \mathbb{R}\backslash\sigma_{\pm} \text{ is a pole of } \psi_q^{\pm}(z,1,t)\}. \tag{5.8}$$

The norming constants $\gamma_{\pm,k}(t)$ corresponding to $\lambda_k \in \sigma_p(H)$ are then given by

$$\gamma_{\pm,k}(t)^{-1} = \sum_{n\in\mathbb{Z}} |\tilde{\psi}_\pm(\lambda_k, n, t)|^2. \tag{5.9}$$

Lemma 5.1. *Let $(a(t), b(t))$ be a solution of the Toda hierarchy such that* (5.1) *holds. The functions*

$$\hat{\psi}_\pm(z, n, t) = \exp(\alpha_r^\pm(z, t))\psi_\pm(z, n, t) \tag{5.10}$$

satisfy

$$H(t)\hat{\psi}_\pm(z, n, t) = z\hat{\psi}_\pm(z, n, t), \qquad \frac{d}{dt}\hat{\psi}_\pm(z, n, t) = P_{2r+2}(t)\hat{\psi}_\pm(z, n, t). \tag{5.11}$$

Proof. We proceed as in [31, Theorem 3.2]. The Jost solutions $\psi_\pm(z, n, t)$ are continuously differentiable with respect to t by the same arguments as for z (compare [7, Theorem 4.2]), and the derivatives are equal to the derivatives of the Baker–Akhiezer functions as $n \to \pm\infty$.

For $z \in \mathbb{C}\backslash\sigma$, the solution $u_\pm(z, n, t)$ of (5.11) with initial condition $\psi_\pm(z, n, 0) \in \ell^2_\pm(\mathbb{Z})$ remains square summable near $\pm\infty$ for all $t \in \mathbb{R}$ (see [32] or [33, Lemma 12.16]), that is, $u_\pm(z, n, t) = C_\pm(z, t)\psi_\pm(z, n, t)$. Letting $n \to \pm\infty$ we see $C_\pm(z, t) = 1$. The general result for all $z \in \mathbb{C}$ now follows from continuity. □

This implies

Theorem 5.2. *Let $(a(t), b(t))$ be a solution of the Toda hierarchy such that* (5.1) *holds. The time evolution for the scattering data is given by*

$$\begin{aligned}
T_\pm(\lambda, t) &= T_\pm(\lambda, 0)\exp(\alpha_r^\mp(\lambda, t) - \overline{\alpha_r^\pm(\lambda, t)}),\\
R_\pm(\lambda, t) &= R_\pm(\lambda, 0)\exp(\alpha_r^\pm(\lambda, t) - \overline{\alpha_r^\pm(\lambda, t)}),\\
\gamma_{\pm,k}(t) &= \gamma_{\pm,k}(0)\frac{\delta_\pm^2(\lambda_k, 0)}{\delta_\pm^2(\lambda_k, t)}\exp(2\alpha_r^\pm(\lambda_k, t)), \qquad 1 \le k \le p.
\end{aligned} \tag{5.12}$$

Proof. The Wronskian of two solutions satisfying (5.11) does not depend on n or t (see [32], [33, Lemma 12.15]), hence

$$\begin{aligned}
T_\pm(\lambda, t) &= \frac{W(\overline{\psi_\pm(\lambda, t)}, \psi_\pm(\lambda, t))}{W(\psi_\mp(\lambda, t), \psi_\pm(\lambda, t))} = \frac{\exp(\alpha_r^\mp(\lambda, t))}{\exp(\overline{\alpha_r^\pm(\lambda, t)})}\frac{W(\overline{\hat{\psi}_\pm(\lambda, t)}, \hat{\psi}_\pm(\lambda, t))}{W(\hat{\psi}_\mp(\lambda, t), \hat{\psi}_\pm(\lambda, t))}\\
&= \exp(\alpha_r^\mp(\lambda, t) - \overline{\alpha_r^\pm(\lambda, t)})T_\pm(\lambda, 0),\\
R_\pm(\lambda, t) &= -\frac{W(\psi_\mp(\lambda), \overline{\psi_\pm(\lambda)})}{W(\psi_\mp(\lambda), \psi_\pm(\lambda))} = -\frac{\exp(\alpha_r^\pm(\lambda, t))}{\exp(\overline{\alpha_r^\pm(\lambda, t)})}\frac{W(\hat{\psi}_\mp(\lambda), \overline{\hat{\psi}_\pm(\lambda)})}{W(\hat{\psi}_\mp(\lambda), \hat{\psi}_\pm(\lambda))}\\
&= \exp(\alpha_r^\pm(\lambda, t) - \overline{\alpha_r^\pm(\lambda, t)})R_\pm(\lambda, 0).
\end{aligned}$$

The time dependence of $\gamma_{\pm,k}(t)$ follows from $\|U_r(t, 0)\tilde{\psi}_\pm(\lambda_k, ., 0)\| = \|\tilde{\psi}_\pm(\lambda_k, ., 0)\|$. □

Remark 5.3. *Note that we have*

$$\exp\left(\alpha_r^\pm(\lambda, t) - \overline{\alpha_r^\pm(\lambda, t)}\right) = \exp\left(P_\pm(\lambda)\int_0^t \frac{G^\pm_{q,r}(\lambda, 0, s)}{\prod_{j=1}^{g_\pm}(\lambda - \mu_j^\pm(s))}ds\right), \quad \lambda \in \sigma_\pm, \tag{5.13}$$

where $\mu_j^{\pm}(t)$ are the Dirichlet eigenvalues. Moreover, in case of the Toda lattice, where $r = 0$, we have $G_{q,0}^{\pm}(\lambda, n, t) = 1$ and $H_{q,1}^{\pm}(\lambda, n, t) = \lambda - b_q^{\pm}(n, t)$.

In summary, since Lemma 3.2 ensures that (5.1) remains valid for all t once it holds for the initial condition, we can compute $R_{\pm}(\lambda, 0)$ and $\gamma_{\pm,k}(0)$ from $(a(n, 0), b(n, 0))$ and then solve the Gel'fand–Levitan–Marchenko (GLM) equation to obtain the sequences $(a(n, t), b(n, t))$ as in [10]. More precisely, one needs to solve the GLM equation

$$K_{\pm}(n, m, t) + \sum_{l=n}^{\pm\infty} K_{\pm}(n, l, t) F_{\pm}(l, m, t) = \frac{\delta_n(m)}{K_{\pm}(n, n, t)}, \qquad \pm m \geq \pm n, \tag{5.14}$$

for $K_{\pm}(n, m, t)$, where according to Theorem 5.2 the kernel $F_{\pm}(m, n, t)$ is given by

Theorem 5.4. *The time dependence of the kernel of the Gel'fand–Levitan–Marchenko equation is given by*

$$\begin{aligned} F_{\pm}(m, n, t) = \frac{1}{\pi} \mathrm{Re} \int_{\sigma_{\pm}} & R_{\pm}(\lambda, 0) \hat{\psi}_q^{\pm}(\lambda, m, t) \hat{\psi}_q^{\pm}(\lambda, n, t) \frac{\prod_{j=1}^{g_{\pm}} (\lambda - \mu_j^{\pm}(0))}{P_{\pm}(\lambda)} d\lambda \\ & + \frac{1}{2\pi \mathrm{i}} \int_{\sigma_{\mp}^{(1)}} |T_{\mp}(\lambda, 0)|^2 \hat{\psi}_q^{\pm}(\lambda, m, t) \hat{\psi}_q^{\pm}(\lambda, n, t) \frac{\prod_{j=1}^{g_{\pm}} (\lambda - \mu_j^{\pm}(0))}{P_{\mp}(\lambda)} d\lambda \\ & + \sum_{k=1}^{p} \gamma_{\pm,k}(0) \breve{\psi}_q^{\pm}(\lambda_k, m, t) \breve{\psi}_q^{\pm}(\lambda_k, n, t), \end{aligned} \tag{5.15}$$

where $\breve{\psi}_q^{\pm}(z, m, t) = \delta_{\pm}(z, 0) \hat{\psi}_q^{\pm}(z, m, t)$.

Proof. The kernel $F_{\pm}(m, n, 0)$ is derived in [10, Theorem 4.1]. Observe that $\alpha_r^{\mp}(\lambda, t)$ are real valued on the set $\sigma_{\mp}^{(1)}$ and

$$\exp\left(\alpha_r^{\pm}(\lambda, t) + \overline{\alpha_r^{\pm}(\lambda, t)}\right) = \prod_{j=1}^{g_{\pm}} \frac{\lambda - \mu_j^{\pm}(t)}{\lambda - \mu_j^{\pm}(0)}, \quad \lambda \in \sigma_{\pm},$$

then our result follows from (5.10) and Theorem 5.2. □

By [10] this equation is uniquely solvable and the solution of the Toda hierarchy can be obtained from either $K_+(n, m, t)$ or $K_-(n, m, t)$ by virtue of

$$\begin{aligned} a(n, t) &= a_q^+(n, t) \frac{K_+(n+1, n+1, t)}{K_+(n, n, t)} = a_q^-(n, t) \frac{K_-(n, n, t)}{K_-(n+1, n+1, t)}, \\ b(n, t) &= b_q^+(n, t) + a_q^+(n, t) \frac{K_+(n, n+1, t)}{K_+(n, n, t)} - a_q^+(n-1, t) \frac{K_+(n-1, n, t)}{K_+(n-1, n-1, t)}, \\ &= b_q^-(n, t) + a_q^-(n-1, t) \frac{K_-(n, n-1, t)}{K_-(n, n, t)} - a_q^-(n, t) \frac{K_-(n+1, n, t)}{K_-(n+1, n+1, t)}. \end{aligned} \tag{5.16}$$

References

[1] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16**, No. 4, 955–977 (2000).

[2] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl, *Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies*, Mem. Amer. Math. Soc. **135-641** (1998).

[3] P. Deift, T. Kriecherbauer, and S. Venakides, *Forced lattice vibrations. I, II*, Comm. Pure Appl. Math. **48:11**, 1187–1249, 1251–1298 (1995).

[4] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, no. 1, 35–83 (1996).

[5] K. Deimling, *Ordinary Differential Equations on Banach Spaces*, Lecture Notes in Mathematics **596**, Springer, Berlin, 1977.

[6] I. Egorova, K. Grunert, and G. Teschl, *On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data I. Schwartz-type perturbations*, Nonlinearity (to appear).

[7] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with quasi-periodic background*, Comm. Math. Phys. **264-3**, 811-842 (2006).

[8] I. Egorova, J. Michor, and G. Teschl, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817–1827 (2007).

[9] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with steplike quasi-periodic background*, Inverse Problems **23**, 905–918 (2007).

[10] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with general steplike quasi-periodic background*, Zh. Mat. Fiz. Anal. Geom. **4:1**, 33–62 (2008).

[11] I. Egorova, J. Michor, and G. Teschl, *Soliton solutions of the Toda hierarchy on quasi-periodic background revisited*, Math. Nach. **282:4**, 526–539 (2009).

[12] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin, 1987.

[13] F. Gesztesy, H. Holden, J. Michor, and G. Teschl, *Soliton Equations and Their Algebro-Geometric Solutions. Volume II:* $(1+1)$*-Dimensional Discrete Models*, Cambridge Studies in Advanced Mathematics **114**, Cambridge University Press, Cambridge, 2008.

[14] I. M. Guseinov and A. Kh. Khanmamedov, *The* $t \to \infty$ *asymptotics of the Cauchy problem solution for the Toda chain with threshold-type initial data*, Th. and Math. Phys. **119:3**, 739–749 (1999).

[15] B. L. Holian, H. Flaschka, and D. W. McLaughlin, *Shock waves in the Toda lattice: Analysis*, Ph. Rev. A **24:5**, 2595–2623 (1981).

[16] S. Kamvissis, *On the long time behavior of the doubly infinite Toda lattice under initial data decaying at infinity,* Comm. Math. Phys., **153-3**, 479–519 (1993).

[17] S. Kamvissis, *On the Toda shock problem*, Physica D **65**, 242–266 (1993).

[18] S. Kamvissis and G. Teschl, *Stability of periodic soliton equations under short range perturbations*, Phys. Lett. A, **364-6**, 480–483 (2007).

[19] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice under short range perturbations*, arXiv:0705.0346.

[20] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice: Higher order asymptotics*, arXiv:0805.3847.

[21] A. Kh. Khanmamedov, *The solution of Cauchy's problem for the Toda lattice with limit periodic initial data*, Sb. Math. **199:3**, 449–458 (2008).

[22] H. Krüger and G. Teschl, *Long-time asymptotics of the Toda lattice for decaying initial data revisited*, Rev. Math. Phys. **21:1**, 61–109 (2009).

[23] H. Krüger and G. Teschl, *Long-time asymptotics for the Toda lattice in the soliton region*, Math. Z. **262**, 585–602 (2009).

[24] H. Krüger and G. Teschl, *Long-time asymptotics for the periodic Toda lattice in the soliton region*, Int. Math. Res. Not. (to appear).

[25] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure and Appl. Math. **21**, 467–490 (1968).

[26] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[27] J. Michor and G. Teschl, *Trace formulas for Jacobi operators in connection with scattering theory for quasi-periodic background*, in Operator Theory, Analysis and Mathematical Physics, J. Janas (ed.) et al., 69–76, Oper. Theory Adv. Appl. **174**, Birkhäuser, Basel, 2007

[28] J. Michor and G. Teschl, *On the equivalence of different Lax pairs for the Kac-van Moerbeke hierarchy*, in Modern Analysis and Applications, V. Adamyan (ed.) et al., 445–453, Oper. Theory Adv. Appl. **191**, Birkhäuser, Basel, 2009.

[29] V. Yu. Novokshenov and I. T. Habibullin, *Nonlinear differential-difference schemes integrable by the method of the inverse scattering problem. Asymptotics of the solution for* $t \to \infty$, Sov. Math. Doklady **23/2**, 304–307 (1981).

[30] R. Oba, *Doubly Infinite Toda Lattice with Antisymmetric Asymptotics*, Ph.D. Thesis, New York University, 1988.
[31] G. Teschl, *Inverse scattering transform for the Toda hierarchy*, Math. Nach. **202**, 163–171 (1999).
[32] G. Teschl, *On the Toda and Kac-van Moerbeke hierarchies*, Math. Z. **231**, 325–344 (1999).
[33] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.
[34] G. Teschl, *Algebro-geometric constraints on solitons with respect to quasi-periodic backgrounds*, Bull. London Math. Soc. **39:4**, 677–684 (2007).
[35] G. Teschl, *On the spatial asymptotics of solutions of the Toda lattice*, arXiv:0901.2717.
[36] M. Toda, *Theory of Nonlinear Lattices*, 2nd enl. ed., Springer, Berlin, 1989.
[37] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. in Pure and Applied Math. **44**, 1171–1242 (1991).
[38] A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi Matrices with the Spectrum on an Interval, a finite systems of intervals or a Cantor set of positive length*, Commun. Math. Phys. **226**, 567–605 (2002).

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SOLITON SOLUTIONS OF THE TODA HIERARCHY ON QUASI-PERIODIC BACKGROUNDS REVISITED

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Abstract. We investigate soliton solutions of the Toda hierarchy on a quasi-periodic finite-gap background by means of the double commutation method and the inverse scattering transform. In particular, we compute the phase shift caused by a soliton on a quasi-periodic finite-gap background. Furthermore, we consider short-range perturbations via scattering theory. We give a full description of the effect of the double commutation method on the scattering data and establish the inverse scattering transform in this setting.

1. Introduction

Solitons on a (quasi-)periodic background have a long tradition and are used to model localized excitements on a phonon, lattice, or magnetic field background (see, e.g., [5], [11], [13], [15], [16], [17], [18] and the references therein). Of course periodic solutions, as well as solitons travelling on a periodic background, are well understood. Nevertheless there are still several open questions.

One of them is the stability of (quasi-)periodic solutions. For the constant solution it is a classical result, that a small initial perturbation asymptotically splits in a number of stable solitons. For a (quasi-)periodic background this cannot be the case. In fact, associated with every soliton there is a phase shift (which will be explicitly computed in Section 4) and the phase shifts of all solitons will not add up to zero in general. Hence there must be something which makes up for this phase shift. Moreover, even if no solitons are present, the asymptotic limit is not the (quasi-)periodic background! A precise description of the asymptotic limit in terms of Abelian integrals on the underlying Riemann surface is given in [9] (see [10] for a proof). In particular, the asymptotic limit can be split into parts, one which stems from the discrete spectrum (solitons) and one which stems from the continuous spectrum.

The soliton part can be understood by adding/removing the solitons using a Darboux-type transformation, that is, commutation methods for the underlying Jacobi operators. Hence the purpose of the present paper is to complement [10] and provide a detailed description of the double commutation method when applied to a short-range perturbation of a quasi-periodic finite-gap solution of the Toda lattice. In particular, we are interested in the effect of one double commutation step on the scattering data.

After introducing the Toda hierarchy in Section 2 and recalling some necessary facts on algebro-geometric quasi-periodic finite-gap solutions in Section 3 we briefly review the single and double commutation methods in Section 4 and compute the

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phase shift (in the Jacobian variety) caused by inserting one eigenvalue for both methods. In Section 5 we review direct scattering theory for Jacobi operators with different (quasi-)periodic asymptotics in the same isospectral class. As our main result we give a complete description of the effect of the double commutation method on the scattering data. In addition, we provide some detailed asymptotic formulas for the Jost functions $\psi_\pm(z,n)$ (which are normalized as $n \to \pm\infty$) at the *other* side, that is, as $n \to \mp\infty$. Our final Section 6 establishes the inverse scattering transform for this setting. Our main results here are the time dependence of both the scattering data and the kernel of the Gelfand-Levitan-Marchenko equation.

2. The Toda hierarchy

In this section we introduce the Toda hierarchy using the standard Lax formalism ([12]). We first review some basic facts from [1] (see also [21]).

We will only consider bounded solutions and hence require

Hypothesis H. 2.1. *Suppose $a(t)$, $b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z},\mathbb{R}), \qquad b(t) \in \ell^\infty(\mathbb{Z},\mathbb{R}), \qquad a(n,t) \neq 0, \qquad (n,t) \in \mathbb{Z}\times\mathbb{R},$$

and let $t \mapsto (a(t), b(t))$ be differentiable in $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

Associated with $a(t), b(t)$ is a Jacobi operator

$$H(t) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \qquad f \mapsto \tau(t) f, \tag{2.1}$$

where

$$\tau(t) f(n) = a(n,t) f(n+1) + a(n-1,t) f(n-1) + b(n,t) f(n) \tag{2.2}$$

and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. Moreover, choose constants $c_0 = 1$, c_j, $1 \le j \le r$, $c_{r+1} = 0$, set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_n, H(t)^\ell \delta_n \rangle, \\ h_j(n,t) &= 2a(n,t) \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_{n+1}, H(t)^\ell \delta_n \rangle + c_{j+1}, \end{aligned} \tag{2.3}$$

and consider the Lax operator

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t) g_j(t) S^+ - h_j(t)) H(t)^{r-j} + g_{r+1}(t), \tag{2.4}$$

where $S^\pm f(n) = f(n \pm 1)$. Restricting to the two-dimensional nullspace $\mathrm{Ker}(\tau(t) - z)$, $z \in \mathbb{C}$, of $\tau(t) - z$, we have the following representation of $P_{2r+2}(t)$

$$P_{2r+2}(t)\Big|_{\mathrm{Ker}(\tau(t)-z)} = 2a(t) G_r(z,t) S^+ - H_{r+1}(z,t), \tag{2.5}$$

where $G_r(z,n,t)$ and $H_{r+1}(z,n,t)$ are monic polynomials in z of the type

$$\begin{aligned} G_r(z,n,t) &= \sum_{j=0}^{r} z^j g_{r-j}(n,t), \\ H_{r+1}(z,n,t) &= z^{r+1} + \sum_{j=0}^{r} z^j h_{r-j}(n,t) - g_{r+1}(n,t). \end{aligned} \tag{2.6}$$

A straightforward computation shows that the Lax equation

$$\frac{d}{dt}H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{2.7}$$

is equivalent to

$$\begin{aligned} \mathrm{TL}_r(a(t), b(t))_1 &= \dot{a}(t) - a(t)\Big(g_{r+1}^+(t) - g_{r+1}(t)\Big) = 0, \\ \mathrm{TL}_r(a(t), b(t))_2 &= \dot{b}(t) - \Big(h_{r+1}(t) - h_{r+1}^-(t)\Big) = 0, \end{aligned} \tag{2.8}$$

where the dot denotes a derivative with respect to t and $f^{\pm}(n) = f(n \pm 1)$. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy $\mathrm{TL}_r(a, b) = (\mathrm{TL}_r(a, b)_1, \mathrm{TL}_r(a, b)_2) = 0$. We will always consider r as a fixed, but arbitrary, value.

We recall that the Lax equation (2.7) implies existence of a unitary propagator $U_r(t, s)$ such that the family of operators $H(t)$, $t \in \mathbb{R}$, are unitarily equivalent, $H(t) = U_r(t, s)H(s)U_r(s, t)$. This also implies the basic existence and uniqueness theorem for the Toda hierarchy (see, e.g., [20], [19], or [21, Section 12.2]).

Theorem 2.2. *Suppose $(a_0, b_0) \in M = \ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$. Then there exists a unique integral curve $t \mapsto (a(t), b(t))$ in $C^\infty(\mathbb{R}, M)$ of the Toda hierarchy, that is, $\mathrm{TL}_r(a(t), b(t)) = 0$, such that $(a(0), b(0)) = (a_0, b_0)$.*

Finally, we recall the following result from [3] (compare also [20]), which says that solutions which are asymptotically close to a background solution at the initial time stay close for all time.

Lemma 2.3. *Suppose $a(n, t)$, $b(n, t)$ and $\bar{a}(n, t)$, $\bar{b}(n, t)$ are two arbitrary bounded solutions of the Toda hierarchy satisfying (2.9) for one $t_0 \in \mathbb{R}$, then (2.9) holds for all $t \in \mathbb{R}$, that is,*

$$\sum_{n\in\mathbb{Z}} w(n)\Big(|a(n, t) - \bar{a}(n, t)| + |b(n, t) - \bar{b}(n, t)|\Big) < \infty, \tag{2.9}$$

where $w(n) \geq 1$ is some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$.

3. Quasi-periodic finite-gap solutions

As a preparation for our next section we first need to recall some facts on quasi-periodic finite-gap solutions (again see [1] or [21]).

Let $\mathbb{M}$ be the Riemann surface associated with the following function

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j}, \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{3.1}$$

where $g \in \mathbb{N}$ and $\sqrt{.}$ is the standard root with branch cut along $(-\infty, 0)$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$, or $p = \infty_\pm$, and the projection onto $\mathbb{C} \cup \{\infty\}$ by $\pi(p) = z$. The sets $\Pi_\pm = \{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C}\backslash \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]\} \subset \mathbb{M}$ are called upper, lower sheet, respectively.

Now pick g numbers (the Dirichlet eigenvalues)

$$(\hat{\mu}_j)_{j=1}^g = (\mu_j, \sigma_j)_{j=1}^g \tag{3.2}$$

whose projections lie in the spectral gaps, that is, $\mu_j \in [E_{2j-1}, E_{2j}]$. Associated with these numbers is the divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ which is one at the points $\hat{\mu}_j$ and zero else. Using this divisor we introduce

$$\begin{aligned} &\underline{z}(p,n,t) = \underline{\hat{A}}_{p_0}(p) - \underline{\hat{\alpha}}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - n\underline{\hat{A}}_{\infty_-}(\infty_+) + t\underline{U}_s - \underline{\hat{\Xi}}_{p_0} \in \mathbb{C}^g, \\ &\underline{z}(n,t) = \underline{z}(\infty_+, n, t), \end{aligned} \tag{3.3}$$

where Ξ_{p_0} is the vector of Riemann constants, $\underline{U}_s$ the b-periods of the Abelian differential Ω_s defined below, and $\underline{A}_{p_0}$ ($\underline{\alpha}_{p_0}$) is Abel's map (for divisors). The hat indicates that we regard it as a (single-valued) map from $\hat{\mathbb{M}}$ (the fundamental polygon associated with $\mathbb{M}$) to $\mathbb{C}^g$. We recall that the function $\theta(\underline{z}(p,n,t))$ has precisely g zeros $\hat{\mu}_j(n,t)$ (with $\hat{\mu}_j(0,0) = \hat{\mu}_j$), where $\theta(\underline{z})$ is the Riemann theta function of $\mathbb{M}$.

Taking a stationary solution of TL_g with constants c_j, $1 \le j \le g$, as initial condition for another equation $\widehat{\mathrm{TL}}_s$ with constants $\hat{c}_j$, $1 \le j \le s$, in the Toda hierarchy (2.8) one obtains the quasi-periodic finite gap solutions of the Toda hierarchy given by (see [21, Sections 13.1, 13.2])

$$\begin{aligned} a_q(n,t)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1,t))\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))^2}, \\ b_q(n,t) &= \tilde{b} + \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln\Big(\frac{\theta(\underline{w}+\underline{z}(n,t))}{\theta(\underline{w}+\underline{z}(n-1,t))}\Big)\Big|_{\underline{w}=0}. \end{aligned} \tag{3.4}$$

The constants $\tilde{a}$, $\tilde{b}$, $c_j(g)$ depend only on the Riemann surface (see [21, Section 9.2]).

Introduce

$$\begin{aligned} \phi_q(p,n,t) &= C(n,t)\frac{\theta(\underline{z}(p,n+1,t))}{\theta(\underline{z}(p,n,t))}\exp\Big(\int_{p_0}^{p}\omega_{\infty_+,\infty_-}\Big), \\ \psi_q(p,n,t) &= C(n,0,t)\frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,0,0))}\exp\Big(n\int_{p_0}^{p}\omega_{\infty_+,\infty_-} + t\int_{p_0}^{p}\Omega_s\Big), \end{aligned} \tag{3.5}$$

where $C(n,t)$, $C(n,0,t)$ are real-valued,

$$C(n,t)^2 = \frac{\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n+1,t))}, \qquad C(n,0,t)^2 = \frac{\theta(\underline{z}(0,0))\theta(\underline{z}(-1,0))}{\theta(\underline{z}(n,t))\theta(\underline{z}(n-1,t))}, \tag{3.6}$$

and the sign of $C(n,t)$ is opposite to that of $a_q(n,t)$. $\omega_{\infty_+,\infty_-}$ is the Abelian differential of the third kind with poles at ∞_+ respectively ∞_- and Ω_s is an Abelian differential of the second kind with poles at ∞_+ respectively ∞_- whose Laurent expansion is given by the coefficients $(j+1)\hat{c}_{s-j}$ associated with $\widehat{\mathrm{TL}}_s$ (see [21, Sections 13.1, 13.2]). Then

$$\begin{aligned} \tau_q(t)\psi_q(p,n,t) &= \pi(p)\psi_q(p,n,t), \\ \frac{d}{dt}\psi_q(p,n,t) &= 2a_q(n,t)\hat{G}_s(p,n,t)\psi_q(p,n+1,t) - \hat{H}_{s+1}(p,n,t)\psi_q(p,n,t) \\ &= \hat{P}_{q,2s+2}(t)\psi_q(p,n,t), \end{aligned} \tag{3.7}$$

where we use the hat to distinguish the quantities associated with $\widehat{\mathrm{TL}}_s$ from those associated with TL_g.

The two branches $\psi_{q,\pm}(z,n,t) = \psi_q(p,n,t)$, $p = (z,\pm)$, of the Baker-Akhiezer function are linearly independent away from the branch points and their Wronskian is given by

$$W_q(\psi_{q,-}(z),\psi_{q,+}(z)) = \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^g (z-\mu_j)}.$$

Here $W_q(f,g) = a_q(n)(f(n)g(n+1) - f(n+1)g(n))$ is the usual modified Wronskian.

It is well known that the spectrum of $H_q(t)$ is time independent and consists of $g+1$ bands

$$\sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]. \tag{3.8}$$

For further information and proofs we refer to [21, Chapter 9].

Finally, let us renormalize the Baker-Akhiezer function

$$\tilde{\psi}_q(p,n,t) = \frac{\psi_q(p,n,t)}{\psi_q(p,0,t)} \tag{3.9}$$

such that $\tilde{\psi}_q(p,0,t)=1$ and let us define $\alpha_s(p,t)$ via

$$\exp\big(\alpha_s(p,t)\big) = \psi_q(p,0,t) = C(0,0,t)\frac{\theta(\underline{z}(p,0,t))}{\theta(\underline{z}(p,0,0))}\exp\Big(t\int_{p_0}^{p}\Omega_s\Big). \tag{3.10}$$

4. Commutation methods and N-soliton solutions

In this section we investigate commutation methods when applied to a quasi-periodic finite-gap background solution. In particular, we compute the phase shift (in the Jacobian variety) introduced by the solitons. This can be found for the case of one-dimensional Schrödinger operators in [7] (see also [6] for the elliptic case). The case of Jacobi operators seems to be missing and hence we provide the corresponding results to fill this gap. We want to be rather brief and refer to [8] or [21, Ch. 11] for further details in this connection. Since the time t does not play a role in this section, we will just omit it.

We start by inserting an eigenvalue using the single commutation method. Let H_q be a quasi-periodic finite-gap operator and let $\psi_{q,\pm}(z,n)$ be the branches of the Baker-Akhiezer function which are square summable near $\pm\infty$. Fix $\lambda_1 < \inf\sigma(H_q)$, $\sigma\in[-1,1]$, define

$$u_{q,\sigma}(\lambda_1,n) = \frac{1+\sigma}{2}\psi_{q,+}(\lambda_1,n) + \frac{1-\sigma}{2}\psi_{q,-}(\lambda_1,n), \tag{4.1}$$

and let $H_{q,\sigma}$ be the (self-adjoint) commuted operator associated with

$$(\tau_{q,\sigma}f)(n) = a_{q,\sigma}(n)f(n+1) + a_{q,\sigma}(n-1)f(n-1) + b_{q,\sigma}(n)f(n), \tag{4.2}$$

where (see [21, Sect. 11.2])

$$\begin{aligned} a_{q,\sigma}(n) &= -\frac{\sqrt{a_q(n)a_q(n+1)u_{q,\sigma}(\lambda_1,n)u_{q,\sigma}(\lambda_1,n+2)}}{u_{q,\sigma}(\lambda_1,n+1)},\\ b_{q,\sigma}(n) &= \lambda_1 - a_q(n)\Big(\frac{u_{q,\sigma}(\lambda_1,n)}{u_{q,\sigma}(\lambda_1,n+1)} + \frac{u_{q,\sigma}(\lambda_1,n+1)}{u_{q,\sigma}(\lambda_1,n)}\Big)\\ &= b_q(n) + \partial^*\frac{a_q(n)u_{q,\sigma}(\lambda_1,n)}{u_{q,\sigma}(\lambda_1,n+1)}. \end{aligned} \tag{4.3}$$

$H_q - \lambda_1$ and $H_{q,\sigma} - \lambda_1$ restricted to the orthogonal complements of their corresponding one-dimensional null-spaces are unitarily equivalent and hence

$$\sigma_{ac}(H_{q,\sigma}) = \sigma_{ac}(H_q), \qquad \sigma_{sc}(H_{q,\sigma}) = \sigma_{sc}(H_q) = \emptyset,$$
$$\sigma_{pp}(H_{q,\sigma}) = \begin{cases} \{\lambda_1\}, & \sigma \in (-1,1) \\ \emptyset, & \sigma \in \{-1,1\} \end{cases}.$$

Lemma 4.1. *Let H_q be a quasi-periodic finite-gap operator associated with the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and let $H_{q,\sigma}$, $-1 < \sigma < 1$, be the commuted operator associated with (4.2). Then we have*

$$a_{q,\sigma}(n) \sim a_{q,\pm 1}(n), \quad b_{q,\sigma}(n) \sim b_{q,\pm 1}(n) \qquad \text{as } n \to \pm\infty, \tag{4.4}$$

where $H_{q,\pm 1}$ are the quasi-periodic finite-gap operators associated with the Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}$ defined via

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - \underline{A}_{p_0}(p_1) - \underline{A}_{p_0}(\infty_+), \qquad p_1 = (\lambda_1, \pm). \tag{4.5}$$

Proof. That $H_{q,\pm 1}$ is associated with the divisor $\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}$ is shown in [21, Sect. 11.4] and the asymptotics follow since $u_{q,\sigma}(\lambda_1, n) \sim \frac{1\pm\sigma}{2} u_{q,\pm 1}(\lambda_1, n)$ as $n \to \mp\infty$. □

Similarly, one obtains the following result for the double commutation method. Let $\lambda_1 \in \mathbb{R}\backslash\sigma_{ess}(H_q)$, define (see [21, Sect. 11.6, (2.30)])

$$\begin{aligned} c_{q,\gamma}(\lambda_1, n) &= \frac{1}{\gamma} + \sum_{j=-\infty}^{n} \psi_{q,-}(\lambda_1, j)^2 = \frac{1}{\gamma} + W_{q,n}(\psi_{q,-}(\lambda_1), \dot{\psi}_{q,-}(\lambda_1)) \\ &= \frac{1}{\gamma} + a_q(n)\psi_{q,-}(\lambda_1, n)^2 \dot{\phi}_{q,-}(\lambda_1, n), \quad \gamma \neq 0, \end{aligned} \tag{4.6}$$

and let $H_{q,\gamma}$ be the doubly commuted operator associated with

$$\begin{aligned} a_{q,\gamma}(n) &= a_q(n) \frac{\sqrt{c_{q,\gamma}(\lambda_1, n-1) c_{q,\gamma}(\lambda_1, n+1)}}{c_{q,\gamma}(\lambda_1, n)}, \\ b_{q,\gamma}(n) &= b_q(n) - \partial^* \frac{a_q(n)\psi_{q,-}(\lambda_1, n)\psi_{q,-}(\lambda_1, n+1)}{c_{q,\gamma}(\lambda_1, n)}. \end{aligned} \tag{4.7}$$

Then

Lemma 4.2. *Let H_q be a quasi-periodic finite-gap operator associated with the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and let $H_{q,\gamma}$, $0 < \gamma < \infty$, be the doubly commuted operator associated with (4.7). Then we have*

$$\begin{aligned} a_{q,\gamma}(n) &= \begin{cases} a_q(n)(1 + O(w(\lambda_1)^{2n})) & \text{as } n \to -\infty \\ a_{q,\infty}(n)(1 + O(w(\lambda_1)^{-2n})) & \text{as } n \to +\infty \end{cases}, \\ b_{q,\gamma}(n) &= \begin{cases} b_q(n)(1 + O(w(\lambda_1)^{2n})) & \text{as } n \to -\infty \\ b_{q,\infty}(n)(1 + O(w(\lambda_1)^{-2n})) & \text{as } n \to +\infty \end{cases}, \end{aligned} \tag{4.8}$$

where $w(z) = \exp(\int_{p_0}^{(z,+)} \omega_{\infty_+,\infty_-})$ is the quasi-momentum map and $H_{q,\infty}$ is the quasi-periodic finite-gap operator associated with the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}_\infty}$ defined via

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\lambda_1}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + 2\underline{A}_{p_0}(\hat{\lambda}_1), \qquad \hat{\lambda}_1 = (\lambda_1, +). \tag{4.9}$$

Proof. Since the double commutation method can be obtained via two single commutation steps (see [21, Sect. 11.5]), the result is a consequence of our previous lemma. The asymptotics follow from (4.7) and (3.5). □

Clearly, if we add k eigenvalues $\lambda_1, \dots, \lambda_k$, then the asymptotics at $+\infty$ are given by the quasi periodic operator associated with the Dirichlet divisor

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\lambda_1,\dots,\lambda_k}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + 2\sum_{j=1}^{k} \underline{A}_{p_0}(\hat{\lambda}_j), \qquad \hat{\lambda}_j = (\lambda_j, +). \tag{4.10}$$

In particular, by choosing at least one eigenvalue in each gap, we can attain any prescribed asymptotics in the given isospectral class by Lemma 9.1 in [21].

Remark 4.3. *If $a_q(n,t)$, $b_q(n,t)$ is a quasi-periodic solution of the Toda hierarchy and $\psi_q(p,n,t)$ is the corresponding time dependent Baker-Akhiezer function, then $a_{q,\gamma}(n,t)$, $b_{q,\gamma}(n,t)$ is a solution of the Toda hierarchy which is centered at*

$$2\alpha(\lambda_1)(n - v(\lambda_1)t) + \ln(\gamma) = 0, \tag{4.11}$$

where

$$\alpha(\lambda_1) = \mathrm{Re} \int_{p_0}^{(\lambda_1,-)} \omega_{\infty_+,\infty_-}, \qquad v(\lambda_1) = -\frac{1}{\alpha(\lambda_1)} \mathrm{Re} \int_{p_0}^{(\lambda_1,-)} \Omega_s. \tag{4.12}$$

5. Scattering theory

In this section we review scattering theory for Jacobi operators with steplike quasi-periodic finite-gap background in the same isospectral class following [4]. Our only new result in this section will be a full description of the effect of the double commutation method on the scattering data in Lemma 5.5.

More precisely, we will take two quasi-periodic finite-gap operators $H_q^\pm$ associated with the sequences $a_q^\pm$, $b_q^\pm$ in the same isospectral class,

$$\sigma(H_q^+) = \sigma(H_q^-) \equiv \Sigma = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}], \tag{5.1}$$

but with possibly different Dirichlet data $\{\hat{\mu}_j^\pm\}_{j=1}^g$. We will add $\pm$ as a superscript to all data introduced in Section 3 to distinguish between the corresponding data of H_q^+ and H_q^-. To avoid excessive sub/superscripts we abbreviate

$$\psi_q^\pm(z,n) = \psi_{q,\pm}^\pm(z,n) \quad \text{and} \quad \breve{\psi}_q^\pm(z,n) = \psi_{q,\mp}^\pm(z,n), \tag{5.2}$$

that is, $\psi_q^\pm(z,n)$ is the solution of $H_q^\pm$ decaying near $\pm\infty$ and $\breve{\psi}_q^\pm(z,n)$ is the solution of $H_q^\pm$ decaying near $\mp\infty$. Note that for $\lambda \in \Sigma$ we have $\breve{\psi}_q^\pm(\lambda,n) = \overline{\psi_q^\pm(\lambda,n)}$.

Let $a(n)$, $b(n)$ be sequences satisfying

$$\sum_{n=0}^{\pm\infty} |n| \Big(|a(n) - a_q^\pm(n)| + |b(n) - b_q^\pm(n)| \Big) < \infty \tag{5.3}$$

and denote the corresponding operator by H.

Theorem 5.1. *Assume (5.3). Then there exist solutions $\psi_\pm(z,.)$, $z\in\mathbb{C}$, of $\tau\psi = z\psi$ satisfying*

$$\lim_{n\to\pm\infty} |w(z)^{\mp n}(\psi_\pm(z,n) - \psi_q^\pm(z,n))| = 0, \tag{5.4}$$

where $w(z) = \exp(\int_{p_0}^{(z,+)} \omega_{\infty_+,\infty_-})$ is the quasi-momentum map.

Theorem 5.2. *Assume (5.3). Then we have $\sigma_{ess}(H) = \Sigma$, the point spectrum of H is finite and confined to the spectral gaps of $H_q^\pm$, that is, $\sigma_p(H) = \{\rho_j\}_{j=1}^q \subset \mathbb{R}\backslash\Sigma$. Furthermore, the essential spectrum of H is purely absolutely continuous.*

Using the fact that $\psi_q^\pm(p,n)$ form an orthonormal basis for $L^2(\partial\Pi_+, d\omega^\pm)$, where

$$d\omega^\pm = \frac{\prod_{j=1}^g(\pi - \mu_j^\pm)}{R_{2g+2}^{1/2}} d\pi, \tag{5.5}$$

we can define

$$K_\pm(n,m) = 2\mathrm{Re}\int_\Sigma \psi_\pm(\lambda,n)\psi_q^\pm(\lambda,m) d\omega^\pm \tag{5.6}$$

implying

$$\psi_\pm(z,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_q^\pm(z,m). \tag{5.7}$$

Next we define the coefficients of the scattering matrix via the scattering relations

$$\psi_\mp(\lambda,n) = \alpha_\pm(\lambda)\overline{\psi_\pm(\lambda,n)} + \beta_\pm(\lambda)\psi_\pm(\lambda,n), \qquad \lambda\in\Sigma, \tag{5.8}$$

where

$$\begin{aligned} \alpha_\pm(\lambda) &= \frac{W(\psi_\pm(\lambda),\psi_\mp(\lambda))}{W(\psi_\pm(\lambda),\overline{\psi_\pm(\lambda)})} = \frac{\prod_{j=1}^g(\lambda - \mu_j^\pm)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_-(\lambda),\psi_+(\lambda)), \\ \beta_\pm(\lambda) &= \frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\pm(\lambda),\overline{\psi_\pm(\lambda)})} = \mp\frac{\prod_{j=1}^g(z - \mu_j^\pm)}{R_{2g+2}^{1/2}(z)} W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)}), \end{aligned} \tag{5.9}$$

and $W_n(f,g) = a(n)(f(n)g(n+1) - f(n+1)g(n))$ denotes the Wronskian. Transmission $T_\pm(\lambda)$ and reflection $R_\pm(\lambda)$ coefficients are then defined by

$$T_\pm(\lambda) = \alpha_\pm^{-1}(\lambda), \qquad R_\pm(\lambda) = \frac{\beta_\pm(\lambda)}{\alpha_\pm(\lambda)} = \frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\pm(\lambda),\psi_\mp(\lambda))}. \tag{5.10}$$

The norming constants $\gamma_{\pm,j}$ corresponding to $\rho_j\in\sigma_p(H)$ are given by

$$\gamma_{\pm,j}^{-1} = \sum_{n\in\mathbb{Z}} |\psi_\pm(\rho_j,n)|^2, \qquad 1\le j\le q. \tag{5.11}$$

Note that $\gamma_{\pm,j} = 0$ if ρ_j coincides with a pole $\hat\mu_\ell^\pm \in \Pi_\pm$ of $\psi_\pm(z,.)$. To avoid this, one could remove the poles by introducing $\hat\psi_\pm(z,.)$ as we did in [4]. Since this normalization cancels out in the Gel'fand-Levitan-Marchenko equation and unnecessarily complicates the formulas below, we will allow zero norming constants.

Moreover, $\psi_\pm(\rho_j,.) = c_j^\pm\psi_\mp(\rho_j,.)$ with $c_j^+c_j^- = 1$.

Lemma 5.3. *The coefficients $T_\pm(\lambda)$, $R_\pm(\lambda)$ are bounded for $\lambda \in \Sigma$, continuous for $\lambda \in \Sigma$ except at possibly the band edges E_j, and fulfill*

$$T_+(\lambda)\overline{T_-(\lambda)} + |R_\pm(\lambda)|^2 = 1, \qquad \lambda \in \Sigma, \tag{5.12}$$

$$T_\pm(\lambda)\overline{R_\pm(\lambda)} + \overline{T_\pm(\lambda)}R_\mp(\lambda) = 0, \qquad \lambda \in \Sigma. \tag{5.13}$$

In particular,

$$|T_\pm(\lambda)|^2 \prod_{j=1}^{g} \frac{\lambda - \mu_j^\pm}{\lambda - \mu_j^\mp} + |R_\pm(\lambda)|^2 = 1, \tag{5.14}$$

and hence $|R_\pm(\lambda)|^2 \le 1$ with equality only possibly at the band edges $\{E_j\}$. The transmission coefficients $T_\pm(\lambda)$ have a meromorphic continuation to $\mathbb{C}\backslash\Sigma$ with simple poles at $\mu_j^\pm$ if $\hat{\mu}_j^\pm \in \Pi_\mp$ and simple poles at ρ_j,

$$(\mathrm{Res}_{\rho_j} T_\pm(\lambda))^2 = \gamma_{+,j}\gamma_{-,j} \frac{R_{2g+2}(\rho_j)}{\prod_{l=1}^{g}(\rho_j - \mu_l^\pm)^2}. \tag{5.15}$$

In addition, $T_\pm(z) \in \mathbb{R}$ as $z \in \mathbb{R}\backslash\Sigma$ and

$$T_\pm(\infty) = \prod_{n=-\infty}^{-1} \frac{a(n)}{a_q^-(n)} \prod_{n=0}^{\infty} \frac{a(n)}{a_q^+(n)}. \tag{5.16}$$

The sets

$$S_\pm(H) = \{R_\pm(\lambda), \lambda \in \Sigma;\ (\rho_j, \gamma_{\pm,j}), 1 \le j \le q\} \tag{5.17}$$

are called left/right scattering data for H.

Theorem 5.4. *The kernel $K_\pm(n,m)$ of the transformation operator satisfies the Gel'fand-Levitan-Marchenko equation*

$$K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l)F^\pm(l,m) = \frac{\delta(n,m)}{K_\pm(n,n)}, \qquad \pm m \ge \pm n, \tag{5.18}$$

where

$$F^\pm(m,n) = 2\mathrm{Re} \int_\Sigma R_\pm(\lambda)\psi_q^\pm(\lambda,m)\psi_q^\pm(\lambda,n)d\omega^\pm + \sum_{j=1}^{q} \gamma_{\pm,j}\psi_q^\pm(\rho_j,n)\psi_q^\pm(\rho_j,m). \tag{5.19}$$

Note that the apparent poles $\mu_\ell^\pm$ cancel with the zeros of $d\omega^\pm$ and $\gamma_{\pm,j}$ at these points.

The operator H can be uniquely reconstructed from $S_\pm(H)$ by solving the Gel'fand-Levitan-Marchenko equation. We refer to [4] for further details.

Finally, we come to our principal new result in this section and investigate the connection with the double commutation method. The scattering data of the operators H, H_γ are related as follows.

Lemma 5.5. *Let H be a given Jacobi operator satisfying (5.3) and choose $\rho_{q+1} \in \mathbb{R}\backslash\sigma(H)$, $\gamma > 0$. Then the doubly commuted operator H_γ, $\gamma > 0$, defined via $\psi_-(\rho_{q+1})$ as in Section 4, satisfies*

$$a_\gamma(n) \sim \begin{cases} a_q^-(n) & \text{as } n \to -\infty \\ a_q^\infty(n) & \text{as } n \to +\infty \end{cases}, \qquad b_\gamma(n) \sim \begin{cases} b_q^-(n) & \text{as } n \to -\infty \\ b_q^\infty(n) & \text{as } n \to +\infty \end{cases}, \tag{5.20}$$

such that (5.3) still holds, where H_q^∞ is associated with Dirichlet data given by

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\mu}^\infty}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\mu}^+}) + 2\underline{A}_{p_0}(\hat{\rho}_{q+1}). \tag{5.21}$$

It has the scattering data

$$R_{-,\gamma}(\lambda) = R_-(\lambda), \tag{5.22}$$

$$R_{+,\gamma}(\lambda) = \frac{\theta(\underline{z}^\infty(\hat{\lambda},0))}{\theta(\underline{z}^+(\hat{\lambda},0))}\frac{\theta(\underline{z}^+(\hat{\lambda}^*,0))}{\theta(\underline{z}^\infty(\hat{\lambda}^*,0))} B(\lambda,\rho_{q+1})^2 R_+(\lambda), \tag{5.23}$$

$$T_{+,\gamma}(z) = C\,\frac{\theta(\underline{z}^+(\hat{z}^*,0))}{\theta(\underline{z}^\infty(\hat{z}^*,0))} B(z,\rho_{q+1}) T_+(z), \tag{5.24}$$

$$T_{-,\gamma}(z) = \frac{1}{C}\,\frac{\theta(\underline{z}^\infty(\hat{z},0))}{\theta(\underline{z}^+(\hat{z},0))} B(z,\rho_{q+1}) T_-(z), \tag{5.25}$$

where

$$B(z,\rho) = \exp\Big(-\int_{E(\rho)}^{(\rho,+)} \omega_{\hat{z}\hat{z}^*}\Big) \tag{5.26}$$

is the Blaschke factor and $\hat{\lambda} = (\lambda,+)$, $\hat{z} = (z,+)$. The constant C is given by

$$C = \sqrt{\frac{\theta(\underline{z}^\infty(\infty_+,0))\theta(\underline{z}^\infty(\infty_-,0))}{\theta(\underline{z}^+(\infty_+,0))\theta(\underline{z}^+(\infty_-,0))}} > 0. \tag{5.27}$$

The norming constants $\gamma_{-,j}$ corresponding to $\rho_j \in \sigma_p(H)$, $j = 1,\dots,q$, (cf. (5.11)) remain unchanged except for an additional eigenvalue ρ_{q+1} with norming constant $\gamma_{-,q+1} = \gamma$. The norming constants $\gamma_{+,j,\gamma}$, $j = 1,\dots,q$, are given by

$$\gamma_{+,j,\gamma} = \frac{1}{C^2}\left(\frac{\theta(\underline{z}^\infty(\hat{\rho}_j,0))}{\theta(\underline{z}^+(\hat{\rho}_j,0))}\right)^2 B(\rho_j,\rho_{q+1})^2 \gamma_{+,j}, \qquad \hat{\rho}_j = (\rho_j,+), \tag{5.28}$$

and the additional norming constant $\gamma_{+,q+1,\gamma}$ reads

$$\gamma_{+,q+1,\gamma} = C^2 \frac{\prod_{j=1}^g(\rho_{q+1}-\mu_j^+)^2}{\gamma R_{2g+2}(\rho_{q+1})}\left(\frac{\theta(\underline{z}^+(\hat{\rho}_{q+1}^*,0))}{\theta(\underline{z}^\infty(\hat{\rho}_{q+1}^*,0))} T_+(\rho_{q+1}) \mathrm{Res}_{z=\rho_{q+1}} B(z,\rho_{q+1})\right)^2. \tag{5.29}$$

Proof. First we show that (5.3) still holds. Note that

$$a_\gamma(n) = \begin{cases} a(n)(1+O(w(\rho_{q+1})^{2n})) & \text{as } n\to-\infty \\ a_\infty(n)(1+O(w(\rho_{q+1})^{-2n})) & \text{as } n\to+\infty \end{cases},$$

$$b_\gamma(n) = \begin{cases} b(n)(1+O(w(\rho_{q+1})^{2n})) & \text{as } n\to-\infty \\ b_\infty(n)(1+O(w(\rho_{q+1})^{-2n})) & \text{as } n\to+\infty \end{cases}.$$

Hence the asymptotics near $-\infty$ are clearly unchanged and for $+\infty$ it suffices to check $\gamma=\infty$. By Lemma 5.7 below,

$$\frac{c_\infty(\rho_{q+1},n+1)}{c_\infty(\rho_{q+1},n)} = \frac{c_\infty(\rho_{q+1},n+1)\psi_+(\rho_{q+1},n+1)}{c_\infty(\rho_{q+1},n)\psi_+(\rho_{q+1},n+1)} = \frac{c_{q,\infty}^+(\rho_{q+1},n+1)}{c_{q,\infty}^+(\rho_{q+1},n)}(1+C(n)),$$

where $\sum_{n\in\mathbb{N}} n|C(n)| < \infty$. Therefore

$$a_\infty(n) = a(n)\frac{\sqrt{c_\infty(\rho_{q+1},n-1)c_\infty(\rho_{q+1},n+1)}}{c_\infty(\rho_{q+1},n)} \to a_{q,\infty}(n) \tag{5.30}$$

such that $\sum_{n\in\mathbb{N}} n|a_\infty(n)-a_{q,\infty}(n)| < \infty$ and similarly for $b_\infty(n)$.

Now we turn to the scattering data. By [21, Lemma 11.19], the Jost solutions $\psi_{\pm,\gamma}(z,n)$ of H_γ are up to a constant given by
(5.31)
$$u_{\pm,\gamma}(z,n) = \frac{c_\gamma(\rho_{q+1},n)\psi_\pm(z,n) - \frac{1}{z-\rho_{q+1}}\psi_-(\rho_{q+1},n)W_{n-1}(\psi_-(\rho_{q+1}),\psi_\pm(z))}{\sqrt{c_\gamma(\rho_{q+1},n-1)c_\gamma(\rho_{q+1},n)}}.$$
Since this constant is equal to 1 for $\psi_{-,\gamma}(z,n)$ the fact that R_- is unchanged follows from its definition and [21, (11.107)]. The transmission coefficients are reconstructed from $R_-(\lambda)$ using [4, Theorem 3.6] and $R_{+,\gamma}(\lambda)$ follows then from
$$R_{+,\gamma}(\lambda) = -\frac{T_{-,\gamma}(\lambda)}{\overline{T_{-,\gamma}(\lambda)}}R_{-,\gamma}(\lambda).$$
That the norming constants $\gamma_{-,j}$ are unchanged follows from [21, Lemma 11.14]. For $\gamma_{+,j,\gamma}$, $j = 1,\dots,q+1$, we use (5.15)
$$\gamma_{+,j,\gamma} = \frac{\prod_{l=1}^g(\rho_j - \mu_l^\pm)^2(\mathrm{Res}_{\rho_j}T_{\pm,\gamma}(\lambda))^2}{R_{2g+2}(\rho_j)\gamma_{-,j}}.$$
□

Remark 5.6. *If we choose $\rho = \rho_j \in \sigma_p(H)$ and $\gamma = -\gamma_{-,j}$, then the eigenvalue ρ_j is removed from the spectrum and it is straightforward to see that an analogous result holds.*

The following result used in the previous proof is of independent interest.

Lemma 5.7. *Let H be a given Jacobi operator satisfying (5.3). Then for every $z \in \mathbb{C}\backslash\Sigma$ and every $k \in \mathbb{N}$ we have*
$$\psi_\mp(z,n)\psi_\pm(z,n+k) - \alpha_\pm(z)\psi_q^\pm(z,n+k)\bar{\psi}_q^\pm(z,n) = (k+1)|w(z)|^k C_\pm(z,n), \tag{5.32}$$
where $\sum_{\pm n\in\mathbb{N}} n|C_\pm(z,n)| < \infty$.
Similarly, one has
$$c_\infty(z,n)\psi_\pm(z,n+k) - \alpha_\pm(z)c_{q,\infty}^\pm(z,n)\psi_q^\pm(z,n+k) = \tilde{C}_\pm(z,n,k), \tag{5.33}$$
where $\sum_{\pm n\in\mathbb{N}} n|\tilde{C}_\pm(z,n,k)| < \infty$ and $c_\infty(z,n)$, $c_{q,\infty}^\pm(z,n)$ are defined as in (4.6).

Proof. The proof is an extension of [14, Lemma 3.4] and we will only consider the '+' case. First recall that the Green's functions of H_q^+ and H are given by
$$G_q^+(z,n,m) = \frac{\bar{\psi}_q^+(z,m)\psi_q^+(z,n)}{W_q(\bar{\psi}_q^+(z),\psi_q^+(z))}, \quad G(z,n,m) = \frac{\psi_-(z,m)\psi_+(z,n)}{W(\psi_-(z),\psi_+(z))}, \quad n \ge m,$$
respectively. Considering matrix elements in the second resolvent identity $(H - z)^{-1} - (H_q^+ - z)^{-1} = (H-z)^{-1}(H_q^+ - H)(H_q^+ - z)^{-1}$ we obtain
$$C_+(z,n) = \frac{|w(z)|^{-k}}{k+1}\sum_{m\in\mathbb{Z}} W(\psi_-(z),\psi_+(z))G(z,n,m)(H_q^+ - H)G_q^+(z,m,n+k)$$
for $z \in \mathbb{C}\backslash\sigma(H)$. Since the poles of $W(\psi_-(z),\psi_+(z))G(z,n,m)$ at $z \in \sigma_p(H)$ are removable, the formula holds for all $z \in \mathbb{C}\backslash\Sigma$. Estimating the right hand side using $|G(z,n,m)| \le \mathit{const}\,|w(z)|^{|n-m|}$ and $|G_q^+(z,n+k,m)| \le \mathit{const}\,|w(z)|^{|n+k-m|}$ we obtain
$$|C_+(z,n)| \le \frac{C|w(z)|^{-k}}{k+1}\sum_{m\in\mathbb{Z}} |w(z)|^{|n-m|+|n-m+k|}(2|a(m)-a_q^+(m)|+|b(m)-b_q^+(m)|).$$

We split the sum into three parts

$$|w(z)|^{|n-m|+|n-m+k|} = \begin{cases} |w(z)|^k |w(z)|^{2|n-m|} & m < n, \\ |w(z)|^k & n \le m \le n+k, \\ |w(z)|^k |w(z)|^{2|n-m+k|} & m > n+k. \end{cases}$$

Since (5.3) holds for $c(m) = 2|a(m) - a_q^+(m)| + |b(m) - b_q^+(m)|$ and we have $|w(z)| < 1$ for $z \in \mathbb{C}\backslash\Sigma$, we can apply Lemma A.1 to verify $\sum_{n\in\mathbb{N}} n|C_+(z,n)| < \infty$.

The claim (5.33) is a consequence of (4.6) and (5.32)

$$\begin{aligned} c_\infty(z,n)\psi_+(z,n+k)^2 &= \sum_{j=-\infty}^{n} \psi_-(z,j)^2 \psi_+(z,n+k)^2 \\ &= \alpha_+(z)^2 c_{q,\infty}^+(z,n)\psi_q^+(z,n+k)^2 + \tilde{C}_\pm(z,n,k)\psi_q^+(z,n+k), \end{aligned}$$

where $\tilde{C}_\pm(z,n,k)$ again can be estimated using Lemma A.1. □

6. Inverse scattering transform

Let $a(n,t)$, $b(n,t)$ be a solution of the Toda hierarchy satisfying

$$\sum_{n=0}^{\pm\infty} |n| \Big(|a(n,t) - a_q^\pm(n,t)| + |b(n,t) - b_q^\pm(n,t)| \Big) < \infty. \tag{6.1}$$

Note that by Lemma 2.3 it suffices to check (6.1) for one $t_0 \in \mathbb{R}$ (as background take $H_q^-(t)$ and insert g eigenvalues such that the asymptotics on the other side are given by $H_q^+(t)$).

Jost solutions, transmission and reflection coefficients depend now on an additional parameter $t \in \mathbb{R}$. The Jost solutions $\psi_\pm(z,n,t)$ are normalized such that

$$\tilde{\psi}_\pm(z,n,t) = \tilde{\psi}_q^\pm(z,n,t)\,(1 + o(1)) \quad \text{as } n \to \pm\infty,$$

where (c.f. (3.9))

$$\tilde{\psi}_q^\pm(z,n,t) = \frac{\psi_q^\pm(z,n,t)}{\psi_q^\pm(z,0,t)} =: \exp(-\alpha_s^\pm(z,t))\psi_q^\pm(z,n,t). \tag{6.2}$$

Moreover, we set

$$\exp(\bar{\alpha}_s^\pm(z,t)) = \bar{\psi}_q^\pm(z,0,t). \tag{6.3}$$

Note that we have $\overline{\exp(\alpha_s^\pm(z,t))} = \exp(\bar{\alpha}_s^\pm(z,t))$ for $\lambda \in \Sigma$.

Transmission and reflection coefficients are then defined via the normalized Jost solutions $\tilde{\psi}_\pm(z,n,t)$. Moreover,

$$\sigma(H(t)) \equiv \sigma(H), \qquad t \in \mathbb{R}. \tag{6.4}$$

To avoid the poles of the Baker-Akhiezer function, we will assume that none of the eigenvalues ρ_j coincides with a Dirichlet eigenvalue $\mu_k^\pm(0,0)$. This can be done without loss of generality by shifting the initial time $t_0 = 0$ if necessary.

Remark 6.1. *Due to this assumption there is no need to remove these poles for the definition of $\gamma_{\pm,j}$, as we did in [2], [4]. Since the Dirichlet eigenvalues rotate in their gap, the factor needed to remove the poles would only unnecessarily complicate the time evolution of the norming constants. Moreover, these factors would eventually cancel in the Gel'fand-Levitan-Marchenko equation, which is the only interesting object from the inverse spectral point of view in the first place.*

Lemma 6.2. *Let $(a(t), b(t))$ be a solution of the Toda hierarchy such that (5.3) holds. The functions*

$$\psi_\pm(z, n, t) = \exp(\alpha_s^\pm(z, t))\tilde{\psi}_\pm(z, n, t) \tag{6.5}$$

satisfy

$$H(t)\psi_\pm(z, n, t) = z\psi_\pm(z, n, t), \qquad \frac{d}{dt}\psi_\pm(z, n, t) = \hat{P}_{2s+2}(t)\psi_\pm(z, n, t). \tag{6.6}$$

Proof. We proceed as in [3], [20, Theorem 3.2]. The Jost solutions $\psi_\pm(z, n, t)$ are continuously differentiable with respect to t by the same arguments as for z (compare [2, Theorem 4.2]) and the derivatives are equal to the derivatives of the Baker-Akhiezer functions as $n \to \pm\infty$.

For $z \in \rho(H)$, the solution $u_\pm(z, n, t)$ of (6.6) with initial condition $\psi_\pm(z, n, 0) \in \ell^2_\pm(\mathbb{Z})$ remains square summable near $\pm\infty$ for all $t \in \mathbb{R}$ (see [19] or [21, Lemma 12.16]), that is, $u_\pm(z, n, t) = C_\pm(t)\psi_\pm(z, n, t)$. Letting $n \to \pm\infty$ we see $C_\pm(t) = 1$. The general result for all $z \in \mathbb{C}$ now follows from continuity. □

This implies

Theorem 6.3. *Let $(a(t), b(t))$ be a solution of the Toda hierarchy such that (5.3) holds. The time evolution for the scattering data is given by*

$$T_\pm(z, t) = T_\pm(z, 0)\exp(\alpha_s^\mp(z, t) - \bar{\alpha}_s^\pm(z, t)), \tag{6.7}$$

$$R_\pm(\lambda, t) = R_\pm(\lambda, 0)\exp(\alpha_s^\pm(\lambda, t) - \bar{\alpha}_s^\pm(\lambda, t)), \quad \lambda \in \Sigma, \tag{6.8}$$

$$\gamma_{\pm,j}(t) = \gamma_{\pm,j}(0)\exp(2\alpha_s^\pm(\rho_j, t)), \qquad 1 \le j \le q. \tag{6.9}$$

Proof. Since the Wronskian of two solutions satisfying (6.6) does not depend on n or t (see [19], [21, Lemma 12.15]), we have

$$\begin{aligned} T_\pm(z, t) &= \frac{W(\overline{\tilde{\psi}_\pm(z, t)}, \tilde{\psi}_\pm(z, t))}{W(\tilde{\psi}_\mp(z, t), \tilde{\psi}_\pm(z, t))} = \frac{\exp(\alpha_s^\mp(z, t) + \alpha_s^\pm(z, t))}{\exp(\bar{\alpha}_s^\pm(z, t) + \alpha_s^\pm(z, t))}\frac{W(\overline{\psi_\pm(z, t)}, \psi_\pm(z, t))}{W(\psi_\mp(z, t), \psi_\pm(z, t))} \\ &= \exp(\alpha_s^\mp(z, t) - \bar{\alpha}_s^\pm(z, t))T_\pm(z, 0). \end{aligned}$$

The result for $R_\pm(\lambda, t)$ follows similarly. The time dependence of $\gamma_{\pm,j}(t)$ follows from $\|\psi_\pm(\rho_j, ., t)\| = \|\hat{U}_s(t, 0)\psi_\pm(\rho_j, ., 0)\| = \|\psi_\pm(\rho_j, ., 0)\|$. □

Corollary 6.4. *The quantity $T_\pm(\lambda, t)\overline{T_\mp(\lambda, t)} = 1 - |R_\pm(\lambda, t)|^2$, $\lambda \in \Sigma$, does not depend on t.*

Another straightforward consequence is:

Theorem 6.5. *The time dependence of the kernel of the Gel'fand-Levitan-Marchenko equation is given by*

$$\begin{aligned} F^\pm(m, n, t) = 2\mathrm{Re} \int_\Sigma & R_\pm(\lambda, 0)\psi_q^\pm(\lambda, m, t)\psi_q^\pm(\lambda, n, t)d\omega^\pm(0) \\ &+ \sum_{j=1}^q \gamma_{\pm,j}(0)\psi_q^\pm(\rho_j, m, t)\psi_q^\pm(\rho_j, n, t). \end{aligned}$$

Proof. Just employ Theorem 6.3 to rewrite

$$F^{\pm}(m,n,t) = \int_{\partial\Pi_+} R_{\pm}(p,t)\tilde{\psi}_q^{\pm}(p,m,t)\tilde{\psi}_q^{\pm}(p,n,t)d\omega^{\pm}(t) + \sum_{j=1}^{q} \gamma_{\pm,j}(t)\tilde{\psi}_q^{\pm}(\rho_j,n,t)\tilde{\psi}_q^{\pm}(\rho_j,m,t),$$

where we also use that $\exp(\alpha_s^{\pm}(\lambda,t)+\bar{\alpha}_s^{\pm}(\lambda,t)) = \prod_{j=1}^{g}\frac{\lambda-\mu_j^{\pm}(0,t)}{\lambda-\mu_j^{\pm}(0,0)}$. □

Finally we note ([19], [21, Section 14.5])

Lemma 6.6. *Let $(a(t),b(t))$ be a solution of the Toda hierarchy such that (5.3) holds. Choose $\rho\in\mathbb{R}\backslash\sigma(H)$ and $\gamma>0$. Then $(a_\gamma(t),b_\gamma(t))$ defined via $\psi_-(\rho,n,t)$ using the double commutation method is again a solution of the Toda hierarchy such that (5.3), with $H_q^+(t)$ accordingly changed, holds.*

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Appendix A. Some estimates

In the proof of Lemma 5.7 we need the following elementary result. Consider a sequence $c\in\ell(\mathbb{Z})$ and abbreviate

$$\|c\|_\infty = \sup_{n\in\mathbb{Z}}|c(n)|, \quad \|c\|_1 = \sum_{n=0}^{\infty}|c(n)|, \quad \|c\|_{1,1} = \sum_{n=1}^{\infty} n|c(n)|. \tag{A.1}$$

Lemma A.1. *Suppose w is some complex number with $|w|<1$ and $c\in\ell(\mathbb{Z})$ satisfies $\|c\|_\infty, \|c\|_{1,1}<\infty$.*

Then

$$\|\sum_{m=0}^{\infty} c(n+m)w^m\|_{1,1} \le \frac{1}{1-|w|}\|c\|_{1,1}$$

and

$$\|\sum_{m=0}^{\infty} c(n-m)w^m\|_{1,1} \le \frac{1}{1-|w|}\|c\|_{1,1} + \frac{|w|}{(1-|w|)^2}\|c\|_1 + \frac{|w|^2}{(1-|w|)^3}\|c\|_\infty.$$

Proof. The first estimate follows from

$$\begin{aligned}\|\sum_{m=0}^{\infty} c(n+m)w^m\|_{1,1} &= \sum_{n=1}^{\infty} n\Big|\sum_{m=0}^{\infty} c(n+m)w^m\Big| \\ &\le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(n+m)|c(n+m)||w|^m \\ &= \sum_{m=0}^{\infty}\|c\|_{1,1}|w|^m = \frac{1}{1-|w|}\|c\|_{1,1}.\end{aligned}$$

Similarly, the second follows from

$$\begin{aligned}\| \sum_{m=0}^{\infty} c(n-m) w^m \|_{1,1} &= \sum_{n=0}^{\infty} n \Big| \sum_{m=0}^{\infty} c(n-m) w^m \Big| \\ &\le \sum_{m=0}^{\infty} \left(\frac{m(m-1)}{2} \|c\|_\infty + m\|c\|_1 + \|c\|_{1,1} \right) |w|^m.\end{aligned}$$

□

References

[1] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl, *Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies*, Mem. Amer. Math. Soc. **135-641**, 1998.

[2] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with quasi-periodic background*, Comm. Math. Phys. **264-3**, 811–842 (2006).

[3] I. Egorova, J. Michor, and G. Teschl, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817–1827 (2007).

[4] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with steplike quasi-periodic background*, Inverse Problems **23**, 905–918 (2007).

[5] M. G. Forest and D. W. McLaughlin, *Spectral theory*, J. Math. Phs. **23(7)**, 1248–1277 (1982).

[6] F. Gesztesy, W. Schweiger, and B. Simon, *Commutation methods applied to the mKdV-equation*, Trans. Amer. Math. Soc. **324**, No.2, 465–525 (1991).

[7] F. Gesztesy and R. Svirsky, *(m)KdV solitons on the background of quasi-periodic finite-gap solutions*, Mem. Amer. Math. Soc. **118-563**, (1995).

[8] F. Gesztesy and G. Teschl, *Commutation methods for Jacobi operators*, J. Diff. Eqs. **128**, 252–299 (1996).

[9] S. Kamvissis and G. Teschl, *Stability of periodic soliton equations under short range perturbations*, Phys. Lett. A, **364-6**, 480–483 (2007).

[10] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice under short-range perturbations*, arXiv:0705.0346.

[11] E. A. Kuznetsov and A. V. Mikhaĭlov, *Stability of stationary waves in nonlinear weakly dispersive media*, Soviet Phys. JETP **40**, No.5, 855–859 (1975).

[12] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure and Appl. Math. **21**, 467–490 (1968).

[13] L. Martínez Alonso and E. Olmedilla Moreno, *Soliton interaction and prime forms on Riemann surfaces*, Phys. Lett. A **188**, No.1, 32–38 (1994).

[14] J. Michor and G. Teschl, *Trace formulas for Jacobi operators in connection with scattering theory for quasi-periodic background*, Operator Theory, Analysis, and Mathematical Physics, J. Janas, et al. (eds.), 69–76, Oper. Theory Adv. Appl. 174, Birkhäuser, Basel, 2007.

[15] J. Pełka and J. Zagrodziński, *Effective velocity of soliton in the presence of a periodic background*, Act. Phys. Poln. A **100**, No.6, 871–877 (2001).

[16] J. Rubenstein, *Sine-Gordon equation*, J. Math. Phys. **11(1)**, 258–266 (1969).

[17] H. J. Shin, *Soliton on a cnoidal wave background in the coupled nonlinear Schrödinger equation*, J. Phys. A: Math. Gen. **37**, 8017–8030 (2004).

[18] H. J. Shin, *The dark soliton on a cnoidal wave background*, J. Phys. A: Math. Gen. **38**, 3307–3315 (2005).

[19] G. Teschl, *On the Toda and Kac-van Moerbeke hierarchies*, Math. Z. **231**, 325–344 (1999).

[20] G. Teschl, *Inverse scattering transform for the Toda hierarchy*, Math. Nach. **202**, 163–171 (1999).

[21] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

[22] G. Teschl, *Algebro-geometric constraints on solitons with respect to quasi-periodic backgrounds*, Bull. London Math. Soc. **39-4**, 677–684 (2007).

[23] M. Toda, *Theory of Nonlinear Lattices*, 2nd enl. ed., Springer, Berlin, 1989.

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