

DILATION OF THE WEYL SYMBOL AND BALIAN-LOW THEOREM

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ABSTRACT. The key result of this paper describes the fact that for an important class of pseudodifferential operators the property of invertibility is preserved under minor dilations of their Weyl symbols. This observation has two implications in time-frequency analysis. First, it implies the stability of general Gabor frames under small dilations of the time-frequency set, previously known only for the lattice case. Secondly, it allows us to derive a new Balian-Low theorem (BLT) for Gabor systems with window in the standard window class and with general time-frequency families. The new BLT does not only exclude orthonormal bases and Riesz bases at critical density, but indeed it even excludes irregular Gabor frames at critical density.

1. INTRODUCTION AND MAIN RESULTS

Gabor frames are a key object in time-frequency analysis. They are families of functions obtained by applying a discrete family of time-frequency shift operators to a given function, called Gabor atom resp. window. The standard setting are so-called regular Gabor frames, where the time-frequency set is a lattice. The focus of this paper is on the general case of irregular Gabor frames, without restrictions to the (countable) set.

It is known that regular Gabor frames are robust under small dilations of the underlying set of points in the time-frequency plane [17]. This paper is motivated by the question whether also irregular Gabor frames share this property. We were led to investigate whether the invertibility of a pseudodifferential operator

$$A_\sigma f(x) = \int_{\mathbb{R}^{2n}} \sigma\left(\frac{x+y}{2}, \xi\right) e^{2\pi i \xi^T (x-y)} f(y) dy d\xi, \quad f \in L^2(\mathbb{R}^n)$$

is preserved under small dilations of the Weyl symbol σ on \mathbb{R}^{2n} . Notice that dilations of Weyl symbols are not as well-behaved as translations or modulations [21, p.91, “Warning”]. Note that standard perturbation results for linear operators do not apply in this case. In

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fact, even starting from a decent operator A_σ one does not have continuous dependence of the corresponding operators in the operator norm, even for some arbitrary small dilations.

Our first main theorem gives a positive result about dilation stability. It is formulated for operators A_σ in the Sjöstrand class, that is, we assume that the symbol σ belongs to the modulation space $M^{\infty,1}(\mathbb{R}^{2n})$. The Sjöstrand class is meanwhile recognized as a standard setting for pseudodifferential operator symbols in time-frequency analysis [27, 26]. In Section 2 we include the definition of modulation spaces, for details we refer to [12, 25].

Let $\text{GL}(n, \mathbb{R})$ be the group of invertible $n \times n$ matrices with the usual topology, induced for example by the maximum norm

$$\|\rho\|_{\max} = \max_{j,k=1,\dots,n} |\rho_{j,k}|, \quad \rho \in \text{GL}(n, \mathbb{R}).$$

The $n \times n$ identity matrix is denoted by I_n .

The following theorem is the main result of this paper. It follows from a series of technical results of general interest, details are given in Section 3.

Theorem 1.1. *Let A_σ denote the pseudodifferential operator with Weyl symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2n})$, and let $\sigma^\rho(\lambda) = \sigma(\rho\lambda)$ denote the symbol dilated by $\rho \in \text{GL}(2n, \mathbb{R})$. Let*

$$\Sigma = \{(\sigma, \rho) \in M^{\infty,1}(\mathbb{R}^{2n}) \times \text{GL}(2n, \mathbb{R}) : A_{\sigma^\rho} \text{ is invertible}\}.$$

Then Σ is an open set in $M^{\infty,1}(\mathbb{R}^{2n}) \times \text{GL}(2n, \mathbb{R})$.

Remark 1.2. Let us rephrase the key implication of Theorem 1.1 for the case that σ is fixed. Suppose the operator A_σ with Weyl symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2n})$ is invertible on $L^2(\mathbb{R}^n)$. Then there exists $\delta > 0$ such that for $\|\rho - I_{2n}\|_{\max} < \delta$ the operator A_{σ^ρ} with dilated symbol $\sigma^\rho(\lambda) = \sigma(\rho\lambda)$ is also invertible on $L^2(\mathbb{R}^n)$.

Theorem 1.1 has significant applications to Gabor frames, described next. A Gabor system is formed by the time-frequency shifts of some function $g \in L^2(\mathbb{R}^n)$ along a set Λ of time-frequency points in \mathbb{R}^{2n} ,

$$(1.1) \quad \begin{aligned} G(g, \Lambda) &= \{g_\lambda : \lambda \in \Lambda\}, \\ g_\lambda(t) &= e^{2\pi i \omega^T t} g(t - x) = M_\omega T_x g(t), \quad \lambda = (x, \omega) \in \mathbb{R}^{2n}, \end{aligned}$$

where M_ω and T_x are the modulation and translation operators respectively.

The system $G(g, \Lambda)$ is a Gabor frame if there exist $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{L^2}^2 \leq \sum_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle|^2 \leq C_2 \|f\|_{L^2}^2, \quad \text{for all } f \in L^2(\mathbb{R}^n),$$

or, equivalently, if the frame operator, given by

$$S_{g,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda,$$

is bounded and invertible on $L^2(\mathbb{R}^n)$. See [25] for the details of Gabor analysis. We assume that g belongs to the standard window class, the modulation space $M^1(\mathbb{R}^n)$, also known as

Feichtinger algebra $S_0(\mathbb{R}^n)$. For details on $M^1(\mathbb{R}^n)$, see [11, 20, 25] and Section 2. Note that the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $M^1(\mathbb{R}^n)$, so our results apply to arbitrary Schwartz atoms g in particular.

The first important consequence of Theorem 1.1 is the robustness of time-frequency sets generating irregular Gabor frames, under small dilations (including rotations, shearing and other operations close to the identity) described in the next theorem. In order to formulate it properly we write $\rho\Lambda = (\rho\lambda)_{\lambda \in \Lambda}$, for $\Lambda \subset \mathbb{R}^{2n}$ and a matrix $\rho \in \text{GL}(2n, \mathbb{R})$.

For the regular case, i.e. for the case that Λ is a discrete lattice in \mathbb{R}^{2n} , this result has been given in [17], and reads as follows: Assume that $G(g, \Lambda)$ is a Gabor frame, with $g \in M^1(\mathbb{R}^n)$ and a lattice Λ , then there exists $\delta > 0$ such that for any $\rho \in \text{GL}(2n, \mathbb{R})$ with $\|\rho - I_{2n}\|_{\max} < \delta$ and any $g' \in M^1(\mathbb{R}^n)$ with $\|g' - g\|_{M^1} < \delta$, the system $G(g', \rho\Lambda)$ with dilated time-frequency lattice $\rho\Lambda$ is also a Gabor frame. The proof of this result in [17] heavily relies on the assumption that the time-frequency set Λ is a lattice and thus there is no chance to transfer the arguments of that approach to the general case. Our goal is an extension to general time-frequency sets, by providing the necessary tools applicable in this more general case. For recent developments on irregular Gabor frames see the summaries in [1, Section 1.1], [22, Section 1], [32, Section 5], or [18, 19, 34].

Theorem 1.3. *For any discrete set Λ in \mathbb{R}^{2n} the set*

$$F_\Lambda = \{(g, \rho) \in M^1(\mathbb{R}^n) \times \text{GL}(2n, \mathbb{R}) : G(g, \rho\Lambda) \text{ is a Gabor frame}\}$$

is open in $M^1(\mathbb{R}^n) \times \text{GL}(2n, \mathbb{R})$.

The proof of Theorem 1.3 will be given in Section 4.

Remark 1.4. We rephrase the key implications of Theorem 1.3 when g is fixed. Suppose that $G(g, \Lambda)$ is a Gabor frame, with $g \in M^1(\mathbb{R}^n)$. Then Theorem 1.3 implies that for some neighborhood U of I_{2n} in $\text{GL}(2n, \mathbb{R})$, every family $G(g, \rho\Lambda)$ with $\rho \in U$ is also a Gabor frame.

Our next theorem is concerned with the density of the time-frequency set Λ . We will show that a fundamental phenomenon in time-frequency analysis, the Balian-Low Theorem for Gabor orthonormal systems and Riesz bases, indeed holds for general Gabor frames, including the case of irregular time-frequency sets.

For a set X of points in \mathbb{R}^n , define its lower Beurling density $\mathcal{D}^-(X)$,

$$(1.2) \quad \mathcal{D}^-(X) = \liminf_{r \rightarrow \infty} \frac{1}{r^n} \inf_{u \in \mathbb{R}^n} \#(X \cap (u + [0, r]^n)).$$

We define the upper Beurling density $\mathcal{D}^+(X)$ in an equivalent way using \limsup and \sup instead of \liminf and \inf .

It is known if $G(g, \Lambda)$ is a frame, then Λ cannot be sparse, the lower Beurling density of Λ in \mathbb{R}^{2n} must satisfy $\mathcal{D}^-(\Lambda) \geq 1$, see [32]. Balian-Low theorems (BLTs) are results that draw the stronger conclusion $\mathcal{D}^-(\Lambda) > 1$, assuming that g is smooth and localized. Various variants of BLTs have been given in the literature, for example see [3, 4, 5, 6, 16, 23, 24, 33], we refer to [10] and [32, Section 3.7]. The original forms of BLTs were found for the case

that Λ is a lattice. The BLT for Gabor systems with more general time-frequency sets Λ in [29, Theorem 11] works for the case of orthonormal bases. Our goal is a BLT for Gabor frames, with general $g \in M^1$ and without restriction on the time-frequency set Λ .

Theorem 1.5. *If $G(g, \Lambda)$ is a Gabor frame for some $g \in M^1(\mathbb{R}^n)$, then one has $\mathcal{D}^-(\Lambda) > 1$.*

Proof. We prove the theorem by contradiction. Assume that $\mathcal{D}^-(\Lambda) = 1$. Then by Theorem 1.3 there exists $\epsilon > 0$ such that the system $G(g, \Lambda')$ with expanded time-frequency set $\Lambda' = (1 + \epsilon)\Lambda$ is also a Gabor frame. It implies

$$\mathcal{D}^-(\Lambda') = \mathcal{D}^-((1 + \epsilon)\Lambda) = \frac{1}{(1 + \epsilon)^{2n}} \mathcal{D}^-(\Lambda) < 1.$$

By [7], an extension of [38], this is impossible and we obtain a contradiction. \square

Remark 1.6. (i) The implication $\mathcal{D}^-(\Lambda) > 1$ of Theorem 1.5 is sharp, indeed the inequality cannot be improved even if some specially selected non-zero $g \in M^1(\mathbb{R}^n)$ is fixed. In fact, for any non-zero $g \in M^1(\mathbb{R}^n)$ and $\epsilon > 0$, by [2, Corollary 5.6] there exists a Gabor frame $G(g, \Lambda)$ whose time-frequency set Λ satisfies $\mathcal{D}^-(\Lambda) \leq 1 + \epsilon$.

(ii) We denote the Gaussian function in n variables by

$$(1.3) \quad \varphi_n(t) = 2^{\frac{n}{4}} e^{-\pi \|t\|_2^2}, \quad t \in \mathbb{R}^n.$$

It is known in dimension one [39] that the density condition $\mathcal{D}^-(\Lambda) > 1$ on the time-frequency set $\Lambda \subset \mathbb{R}^2$ is both necessary and sufficient for $G(\varphi_1, \Lambda)$ to be a frame. For higher dimensions this description is not longer true in the general case. A complete description in the case of rectangular lattices has been found recently [36, Theorem 4.3] and for more general lattices the problem is studied in [28]. There are plenty of examples that show that it is not possible to extend the sufficiency part to general $g \in M^1(\mathbb{R}^n)$. By Theorem 1.5 we extend the necessity part to general $g \in M^1(\mathbb{R}^n)$.

Theorem 1.5 implies a new version for Gabor systems with window in the standard class M^1 , of a recent non-existence result for time-frequency localized Riesz bases, found in [30]. In that paper, Gröchenig and Malinnikova prove the non-existence of Riesz bases when the functions used are well localized in time and frequency, we refer to [30] for more details. Applying this to the case of irregular Gabor frames we can deduce the non-existence of irregular Gabor Riesz bases with that localization. Belonging to M^1 can be interpreted as a good time-frequency localization, but it differs slightly from the type of localization used in [30], and thus our next corollary is a new variant to this theme.

Corollary 1.7. *There is no Gabor Riesz basis for $L^2(\mathbb{R}^n)$, with atom $g \in M^1(\mathbb{R}^n)$.*

Proof. The corollary follows from Theorem 1.5, since it is known that Gabor Riesz bases have density exactly equal to $\mathcal{D}^-(\Lambda) = 1$ (see [38]). \square

Theorem 1.5 also has an immediate consequence to sampling in the Bargmann-Fock space. The Bargmann-Fock space \mathcal{F} consists of entire functions in \mathbb{C}^n with the norm

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^n} |F(z)|^2 e^{-\pi|z|^2} dz < \infty.$$

Recall that a discrete set $\Gamma \subset \mathbb{C}^n$ is a set of sampling for \mathcal{F} , if the following expression defines an equivalent norm on \mathcal{F} :

$$\sum_{z_i \in \Gamma} |F(z_i)|^2 e^{-\pi|z_i|^2}.$$

The Beurling density of a set of points in the complex plane is defined by the obvious modification of (1.2). It has been known that the sets of sampling of \mathcal{F} have lower density bigger than or equal to 1. We improve this known result by showing that a strict inequality takes place.

Corollary 1.8. *Any set of sampling $\Gamma \subset \mathbb{C}^n$ for the Bargmann-Fock space satisfies $\mathcal{D}^-(\Gamma) > 1$.*

Proof. It is known (see [25, Section 3.4]) that Γ is a set of sampling for \mathcal{F} if and only if the Gabor system $G(g, \Lambda)$ is a frame, where $g = \varphi_n$ is the Gaussian function in n variables from (1.3) and

$$\Lambda = \{(x, \omega) \in \mathbb{R}^{2n} : x - i\omega \in \Gamma\};$$

The corollary is obtained by applying Theorem 1.5 with $g = \varphi_n \in M^1(\mathbb{R}^n)$. \square

Remark 1.9. (i) As we say in Remark 1.6(ii), a complete description of the sampling sets for the Fock space is only known in one dimension (see [39]). For several dimensions we refer as before to the recent papers [28, 36]. Corollary 1.8 is a necessary condition for general sampling sets in several dimensions.

(ii) Corollary 1.8 also provides a new approach to a result in [30] on sampling and interpolation in the Bargmann-Fock space \mathcal{F} . Interpolation sets for the Fock space are those sets Γ for which the interpolation problem

$$F(z_i) = a_i e^{\pi|z_i|^2/2} \quad \forall z_i \in \Gamma$$

has a solution F for every sequence $(a_i) \in l^2$. Interpolation sets of the Fock space corresponds to Riesz sets of time-frequency translates of the Gaussian function. Consequently, sets that are both sampling and interpolation correspond to Riesz bases for the Fock space. It was known that an interpolation set Γ satisfies $\mathcal{D}^+(\Gamma) \leq 1$. Thus Corollary 1.8 implies: *There does not exist a set $\Gamma \subset \mathbb{C}^n$ that is both a sampling and an interpolation set for \mathcal{F} at the same time.*

Section 2 consists of preliminary results; Section 3 contains the proof of Theorem 1.1; and Section 4 contains the proof of Theorem 1.3.

2. PRELIMINARY RESULTS

Let $\varphi = \varphi_n$ denote the Gaussian function on \mathbb{R}^n , defined in (1.3). Define the short-time Fourier transform $V_g f(\lambda) = \langle f, g_\lambda \rangle$, for $\lambda = (x, \omega) \in \mathbb{R}^{2n}$, where g_λ denotes the time-frequency shift of g as defined in (1.1). The usual description of modulation spaces used nowadays makes use of a mixed $L^{p,q}$ norm applied to the short-time Fourier transform $V_\varphi f$ of a function f ,

$$(2.1) \quad \|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\varphi f(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q}, \quad 1 \leq p, q \leq \infty,$$

with the usual modification if $p = \infty$ or $q = \infty$. We write $M^p = M^{p,p}$.

For $\rho \in \text{GL}(n, \mathbb{R})$, we denote the dilation operator applied to a function f on \mathbb{R}^n by

$$D_\rho f(z) = f(\rho^{-1}z), \quad z \in \mathbb{R}^n.$$

The next two lemmas are based on results of Sugimoto and Tomita [42], Cordero and Nicola [9]. The first lemma states that the family of dilations D_ρ , for ρ in any compact subset of $\text{GL}(n, \mathbb{R})$, is uniformly bounded on modulation spaces.

Lemma 2.1. *For any compact subset $U \subset \text{GL}(n, \mathbb{R})$ there exists $C_U > 0$ such that*

$$\|D_\rho f\|_{M^{p,q}} \leq C_U \|f\|_{M^{p,q}},$$

for all $\rho \in U$ and $f \in M^{p,q}$, for any pair (p, q) with $1 \leq p, q \leq \infty$.

Proof. Apply the explicit bounds for the norm of D_ρ in [9, Proposition 3.1]. □

The next lemma is concerned with varying the dilation parameter.

Lemma 2.2. (i) *Let $1 \leq p, q < \infty$. Then the following mapping is continuous:*

$$(h, \rho) \mapsto D_\rho h, \\ M^{p,q}(\mathbb{R}^n) \times \text{GL}(n, \mathbb{R}) \rightarrow M^{p,q}(\mathbb{R}^n)$$

(ii) *Although the mapping in (i) is not continuous into $M^\infty(\mathbb{R}^n)$ for general elements $h_0 \in M^\infty(\mathbb{R}^n)$, it is continuous at any constant function or, more generally, at any function $h_0 \in M^\infty(\mathbb{R}^n)$ which is convergent at infinity.*

Proof. (i) Since D_ρ is a topological isomorphism of $M^1(\mathbb{R}^n)$, see [20], and $D_{\rho_1} D_{\rho_2} = D_{\rho_1 \rho_2}$, it is sufficient to prove the continuity at $\rho = I_n$. If $h = \varphi_n$ is the Gaussian function, then the desired convergence

$$(2.2) \quad \|D_\rho h - h\|_{M^1} \rightarrow 0, \quad \text{as } \rho \rightarrow I_n,$$

is a consequence of [9, Lemma 3.1]. By basic properties of modulation spaces, the validity of (2.2) also extends to finite linear combinations of time-frequency shifted versions of φ_n . The set of these finite linear combinations is dense in $M^{p,q}(\mathbb{R}^n)$, since we assume that $p, q \neq \infty$. Hence, the validity of (2.2) for all $h \in M^{p,q}(\mathbb{R}^n)$ follows by a standard continuity argument and the uniform boundedness stated in Lemma 2.1.

(ii) First, the discontinuity claim is verified by simple examples, such as noticing the fact that $\|1 - \exp(2\pi i \rho t)\|_{M^\infty} \not\rightarrow 0$, as $\rho \rightarrow I_n$ in $\text{GL}(n, \mathbb{R})$.

Next, suppose $h_0 \in M^\infty(\mathbb{R}^n)$ is convergent at infinity. Then $h_0(x) = \text{const} + \tilde{h}_0(x)$, where \tilde{h}_0 lies in the closure of the test functions. From approximating \tilde{h}_0 by finite linear combinations of time-frequency shifts of the Gaussian function we see that $\|D_\rho \tilde{h}_0 - D_{\rho_0} \tilde{h}_0\|_{M^\infty} \rightarrow 0$, as $\rho \rightarrow \rho_0$. Hence,

$$(2.3) \quad \begin{aligned} & \|D_\rho h_0 - D_{\rho_0} h_0\|_{M^\infty} \\ & \leq \underbrace{\|D_\rho \text{const} - D_{\rho_0} \text{const}\|_{M^\infty}}_{=0} + \underbrace{\|D_\rho \tilde{h}_0 - D_{\rho_0} \tilde{h}_0\|_{M^\infty}}_{\rightarrow 0} \rightarrow 0, \quad \rho \rightarrow \rho_0. \end{aligned}$$

From (2.3) and the uniform boundedness property in Lemma 2.1 we obtain

$$\begin{aligned} \|D_\rho h - D_{\rho_0} h_0\|_{M^\infty} & \leq \|D_\rho h - D_\rho h_0\|_{M^\infty} + \|D_\rho h_0 - D_{\rho_0} h_0\|_{M^\infty} \\ & \leq C_U \underbrace{\|h - h_0\|_{M^\infty}}_{\rightarrow 0} + \underbrace{\|D_\rho h_0 - D_{\rho_0} h_0\|_{M^\infty}}_{\rightarrow 0} \rightarrow 0, \quad \rho \rightarrow \rho_0. \end{aligned}$$

□

The next lemma is a continuity result that we use later, combining dilation and convolution in certain modulation spaces. The proof makes use of the Wiener amalgam spaces $W^{p,q}$, defined like modulation spaces but with the role of time and frequency interchanged,

$$\|f\|_{W^{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\varphi f(x, \omega)|^p d\omega \right)^{q/p} dx \right)^{1/q}, \quad 1 \leq p, q \leq \infty.$$

The Wiener amalgam spaces are the isomorphic image of the modulation spaces under the Fourier transform. In particular, we have $\|f\|_{M^{p,q}} = \|\hat{f}\|_{W^{p,q}}$, see [31]. We also notice (see [14]) that $M^p = M^{p,p} = W^{p,p}$ is invariant under the Fourier transform, $1 \leq p \leq \infty$. Note that $W^{p_1, q_1} \subseteq W^{p_2, q_2}$ if and only if $p_1 \geq p_2$ and $q_1 \leq q_2$.

Results for modulation spaces mostly have a counterpart for Wiener amalgam spaces, obtained applying by the Fourier transform. For example, the claims formulated in Lemma 2.1 and Lemma 2.2 thus imply analogous results for Wiener amalgam spaces $W^{p,q}$. Also in the proof of the next lemma we make use of this correspondence.

Lemma 2.3. *The mapping*

$$(f, g, \rho) \mapsto f * D_\rho g, \\ M^{\infty,1}(\mathbb{R}^n) \times M^{1,\infty}(\mathbb{R}^n) \times \text{GL}(n, \mathbb{R}) \rightarrow M^{\infty,1}(\mathbb{R}^n),$$

is continuous.

Proof. We prove the equivalent claim obtained by applying the Fourier transform, that is, we show that the mapping

$$(2.4) \quad (f, g, \rho) \mapsto f \cdot D_\rho g, \\ W^{\infty,1}(\mathbb{R}^n) \times W^{1,\infty}(\mathbb{R}^n) \times \text{GL}(n, \mathbb{R}) \rightarrow W^{\infty,1}(\mathbb{R}^n),$$

is continuous. There are various simple multiplication relations of Wiener amalgam spaces, see [15, Theorem 2.11], we will use the fact that for any $1 \leq p \leq \infty$, the mapping

$$(2.5) \quad \begin{aligned} & (f, g) \mapsto f \cdot g, \\ & W^{p,1}(\mathbb{R}^n) \times W^{1,q}(\mathbb{R}^n) \rightarrow W^{p,1}(\mathbb{R}^n), \end{aligned}$$

is continuous. We also use the equivalent of Lemma 2.1 by applying the Fourier transform, that is, the uniform boundedness on compact sets of the mapping

$$(2.6) \quad \begin{aligned} & (g, \rho) \mapsto D_\rho g, \\ & W^{1,\infty}(\mathbb{R}^n) \times \text{GL}(n, \mathbb{R}) \rightarrow W^{1,\infty}(\mathbb{R}^n). \end{aligned}$$

By combining (2.6) and continuity of (2.5) with $p = \infty$ we observe that the mapping in (2.4) is uniformly bounded on compact sets. Therefore we only need to prove the continuity on dense subsets. To this end we assume that f is compactly supported, with $\text{supp } f = \Omega$. Let $U \subset \text{GL}(n, \mathbb{R})$ be an arbitrary compact set. Let $\Omega' = \bigcup_{\rho \in U} \rho^{-1}\Omega$, and let $h \in M^1 = W^{1,1}$ such that $h|_{\Omega'} = 1$. Then $(D_\rho h)|_\Omega = 1$, for all $\rho \in U$, and we obtain

$$(2.7) \quad f = f \cdot D_\rho h, \quad \text{for all } \rho \in U.$$

By the continuity of (2.5) with $p = 1$ and $q = \infty$, since $h \in M^1$ is fixed, the mapping

$$\begin{aligned} & g \mapsto h \cdot g, \\ & W^{1,\infty}(\mathbb{R}^n) \rightarrow M^1(\mathbb{R}^n), \end{aligned}$$

is continuous. Hence, by using Lemma 2.2(i), the mapping

$$\begin{aligned} & (g, \rho) \mapsto D_\rho(h \cdot g), \\ & W^{1,\infty}(\mathbb{R}^n) \times \text{GL}(n, \mathbb{R}) \rightarrow M^1(\mathbb{R}^n), \end{aligned}$$

is continuous. Consequently, by the continuity of (2.5) with $p = \infty$ and $q = 1$, the mapping

$$(2.8) \quad \begin{aligned} & (f, g, \rho) \mapsto f \cdot D_\rho(h \cdot g), \\ & W^{\infty,1} \times W^{1,\infty}(\mathbb{R}^n) \times \text{GL}(n, \mathbb{R}) \rightarrow W^{\infty,1}(\mathbb{R}^n), \end{aligned}$$

is continuous. Since (2.7) implies

$$f \cdot D_\rho g = f \cdot D_\rho(h \cdot g), \quad \text{for all } \rho \in U,$$

we conclude from (2.8) that the continuity on dense subsets is verified. \square

In order to verify functorial properties for the modulation spaces $M^{\infty,1}$ defined over different domains the following technical lemma will be useful. The lemma describes the Fourier domain analysis and synthesis of functions or distributions in $M^{\infty,1}(\mathbb{R}^n)$. We denote by $B(r, x) \subset \mathbb{R}^n$ the ball of radius $r > 0$ and center $x \in \mathbb{R}^n$.

Lemma 2.4. (i) For $r > 0$, there exists $C_r > 0$ such that for any sequence of functions $f_1, f_2, \dots \in L^\infty(\mathbb{R}^n)$ with $\text{diam}(\text{supp}(\widehat{f}_k)) \leq r$ and $\sum_{k=1}^\infty \|f_k\|_{L^\infty} < \infty$, we have that the series $f = \sum_{k=1}^\infty f_k$ converges in $M^{\infty,1}(\mathbb{R}^n)$ and

$$\|f\|_{M^{\infty,1}} \leq C_r \cdot \sum_{k=1}^\infty \|f_k\|_{L^\infty}.$$

(ii) For $r > 0$, there exists $C'_r > 0$ such that for any $f \in M^{\infty,1}(\mathbb{R}^n)$, there exist $f_1, f_2, \dots \in L^\infty(\mathbb{R}^n)$ with $\text{diam}(\text{supp}(\widehat{f}_k)) \leq r$ such that we have $f = \sum_{k=1}^\infty f_k$, with (absolute) convergence in $M^{\infty,1}(\mathbb{R}^n)$, and

$$\sum_{k=1}^\infty \|f_k\|_{L^\infty(\mathbb{R}^n)} \leq C'_r \cdot \|f\|_{M^{\infty,1}}.$$

Proof. (i) Let $h \in \mathcal{S}(\mathbb{R}^n)$, $h \neq 0$, with $\text{supp}(\widehat{h}) \subseteq B(1,0)$. Note if the window φ in the definition of the norm for $M^{\infty,1}(\mathbb{R}^n)$ in (2.1) is replaced by any non-zero Schwartz function, we obtain an equivalent norm (see [25]). Hence there exists $C > 0$ such that for $g \in M^{\infty,1}$,

$$\|g\|_{M^{\infty,1}} \leq C \int_{\mathbb{R}^n} \sup_x |V_h g(x, \omega)| d\omega.$$

Suppose g satisfies $\text{supp}(\widehat{g}) \subseteq B(r, \omega_g)$. Then $\langle T_\omega \widehat{h}, \widehat{g} \rangle = 0$, for $\omega \notin B(r+1, \omega_g)$, and thus we have $V_h g(x, \omega) = 0$, for $|\omega - \omega_g| > r+1$. Therefore,

$$\begin{aligned} \|g\|_{M^{\infty,1}} &\leq C \int_{\mathbb{R}^n} \sup_x |V_h g(x, \omega)| d\omega \\ (2.9) \quad &= C \int_{B(r+1, \omega_g)} \sup_x |V_h g(x, \omega)| d\omega \\ &\leq CC'(r+1)^n \|V_h g\|_{L^\infty} \leq C_r \|g\|_{L^\infty}. \end{aligned}$$

We apply (2.9) to $g = f_k$, for $k = 1, 2, \dots$, and summation over k gives us (i).

(ii) Recall that the Fourier transform of $M^{\infty,1}$ equals the Wiener amalgam space $W^{\infty,1}$, resp. $W(\mathcal{F}L^\infty, l^1)$ in the notation of [13], consisting of functions locally with Fourier transform in L^∞ and globally in L^1 . For given $r > 0$ one can choose a bounded uniformly partition of unity $(\psi_m)_{m \in \mathbb{Z}^n} = \{T_k \psi\}_{m \in \mathbb{Z}^n}$, with $\widehat{\psi} \in L^1$ and $\text{supp} \psi \subseteq B(r, 0)$, e.g. a tensor product of triangular functions. Hence $\|\psi_m\|_{\mathcal{F}L^1} = \|\widehat{\psi}\|_{L^1}$, for all m .

Since $\mathcal{F}L^1$ is a Banach algebra acting pointwise on $\mathcal{F}L^\infty$, the equivalence theorem ([13, Theorem 2]) tells us that the splitting $\widehat{f} = \sum \widehat{f} \psi_m$ provides the required decomposition (after enumerating \mathbb{Z}^n) with $\widehat{f}_m = \widehat{f} \psi_m$ and $\sum_m \|f_m\|_{L^\infty}$ is an equivalent norm on $M^{\infty,1}$. \square

We use the tensor product notation $f \otimes g(x, y) = f(x)g(y)$. The restriction to \mathbb{R}^n of a function on $\mathbb{R}^{n+n'}$ is denoted by $Rf(x) = f(x, 0)$, $x \in \mathbb{R}^n$.

Lemma 2.5. (i) The mapping

$$(f, g) \mapsto f \otimes g, \\ M^{\infty,1}(\mathbb{R}^n) \times M^{\infty,1}(\mathbb{R}^n) \rightarrow M^{\infty,1}(\mathbb{R}^{2n}),$$

is continuous.

(ii) The mapping

$$f \mapsto Rf, \\ M^{\infty,1}(\mathbb{R}^{2n}) \rightarrow M^{\infty,1}(\mathbb{R}^n),$$

is continuous.

Proof. (i) The tensor product property is a consequence of the identity ($1 \leq p, q \leq \infty$):

$$\|f \otimes g\|_{M^{p,q}} = \|f\|_{M^{p,q}} \|g\|_{M^{p,q}}.$$

(ii) Let $f \in M^{\infty,1}(\mathbb{R}^{n+n'})$ and $r > 0$. First, using Lemma 2.4(ii) there exist $F_1, F_2, \dots \in L^\infty(\mathbb{R}^{n+n'})$ with $\text{diam}(\text{supp}(\widehat{F}_k)) \leq r$, $f = \sum F_k$, and

$$\sum_{k=1}^{\infty} \|F_k\|_{L^\infty(\mathbb{R}^{n+n'})} < \infty.$$

By Lemma 2.4(i) there exists $C_r > 0$ such that

$$\|f\|_{M^{\infty,1}(\mathbb{R}^{n+n'})} \leq C_r \cdot \sum_{k=1}^{\infty} \|F_k\|_{L^\infty(\mathbb{R}^{n+n'})}.$$

Next, let $f_k = RF_k$, for $k = 1, 2, \dots$. Then for $k = 1, 2, \dots$, we have $f_k \in L^\infty(\mathbb{R}^n)$ and $\text{diam}(\text{supp}(\widehat{f}_k)) \leq r$. In fact, since F_k is bandlimited, hence continuous, we have $\|f_k\|_{L^\infty(\mathbb{R}^n)} \leq \|F_k\|_{L^\infty(\mathbb{R}^{n+n'})}$, for $k = 1, 2, \dots$. Thus, finally, by Lemma 2.4(ii) there exists $C'_r > 0$ such that the restricted function $RF = \sum_{k=1}^{\infty} RF_k = \sum_{k=1}^{\infty} f_k$ converges in $M^{\infty,1}(\mathbb{R}^n)$ and indeed

$$\|RF\|_{M^{\infty,1}(\mathbb{R}^n)} \leq C'_r \cdot \sum_{k=1}^{\infty} \|f_k\|_{L^\infty(\mathbb{R}^n)} \\ \leq C'_r \cdot \sum_{k=1}^{\infty} \|F_k\|_{L^\infty(\mathbb{R}^{n+n'})} \leq C'_r \cdot C_r \cdot \|f\|_{M^{\infty,1}(\mathbb{R}^{n+n'})}.$$

□

Remark 2.6. Since dilation is a topological isomorphism on $M^{\infty,1}$, we note that Lemma 2.5(ii) indeed holds for the restriction to an arbitrary subspace of \mathbb{R}^n . In particular, we can apply the lemma to the diagonal restriction operator R_Δ on \mathbb{R}^{2n} , defined by

$$R_\Delta f(t) = f(t, t), \quad t \in \mathbb{R}^n.$$

3. PROOF OF THEOREM 1.1

The proof of the theorem is a consequence of the following lemma.

Lemma 3.1. *Let $M = (M, \odot, 1)$ be a unital Banach algebra and let $G = (G, \cdot, e)$ be a topological group acting on M by the family of bounded linear operators $(K_g)_{g \in G}$ on M , with $K_{gg'} = K_g K_{g'}$ and $K_g^{-1} = K_{g^{-1}}$. Suppose that*

- (i) *the mapping $(\sigma, g) \mapsto K_g \sigma$, $M \times G \rightarrow M$ is continuous at the identity $1 \in M$, and*
 - (ii) *the mapping $(\sigma, \tau, g) \mapsto K_{g^{-1}}(K_g \sigma \odot K_g \tau)$, $M \times M \times G \rightarrow M$ is continuous at $e \in G$.*
- Then the set $F = \{(\sigma, g) \in M \times G : K_g \sigma \text{ is invertible}\}$ is open in $M \times G$.*

Proof. First, since M is a Banach algebra, the set $M_0 = \{\sigma \in M : \sigma \text{ invertible}\}$ is open in M , and the inversion $\sigma \mapsto \sigma^{-1}$, $M_0 \rightarrow M_0$, is continuous. Let

$$\begin{aligned} u(\sigma, g) &= K_{g^{-1}}(K_g \sigma \odot K_g \sigma^{-1}), \\ v(\sigma, g) &= K_g \sigma \odot K_g \sigma^{-1}, \end{aligned} \quad \sigma \in M_0, \quad g \in G.$$

Assumption (ii) and continuity of inversion in M_0 imply continuity of the mapping

$$\begin{aligned} (\sigma, g) &\mapsto u(\sigma, g), \\ M_0 \times G &\rightarrow M, \end{aligned}$$

Since $K_e = id$, we have $u(\sigma, e) = \sigma \odot \sigma^{-1} = 1$, and thus (i) implies that the mapping

$$\begin{aligned} (\tau, g) &\mapsto K_g \tau, \\ M \times G &\rightarrow M, \end{aligned}$$

is continuous at $\tau = u(\sigma, e)$, for any $\sigma \in M_0$. Since $K_g u(\sigma, g) = v(\sigma, g)$, we conclude altogether that the mapping

$$\begin{aligned} (\sigma, g) &\mapsto v(\sigma, g), \\ M_0 \times G &\rightarrow M, \end{aligned}$$

is continuous at $g = e$. Moreover $v(\sigma, e) = \sigma \odot \sigma^{-1} = 1$ implies that there exists a neighborhood U of (σ, e) in $M_0 \times G$ such that

$$\|K_g \sigma \odot K_g \sigma^{-1} - 1\| < 1, \quad \text{for } (\sigma, g) \in U.$$

Hence, $K_g \sigma \odot K_g \sigma^{-1}$ is invertible, for $(\sigma, g) \in U$, and this implies the existence of

$$(K_g \sigma)^{-1} = K_g \sigma^{-1} \odot (K_g \sigma \odot K_g \sigma^{-1})^{-1}, \quad \text{for } (\sigma, g) \in U.$$

We have thus shown that the set F is open. □

We will apply Lemma (3.1) to the algebra $M^{\infty, 1}$, endowed with the twisted product. To this aim we recall the following definitions and results. For $\Theta \in \text{GL}(n, \mathbb{R})$, define

$$\sigma \sharp_{\Theta} \tau(z) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \sigma(z') \tau(z'') e^{2\pi i (z - z'')^T \Theta (z - z')} dz' dz'', \quad z \in \mathbb{R}^n.$$

By selecting $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ for Θ (assuming dimension $2n$) we obtain the usual twisted product of Weyl symbols [21],

$$\sigma \# \tau = \sigma \#_J \tau.$$

Proposition 3.2. *The mapping*

$$\begin{aligned} (\sigma, \tau, \Theta) &\mapsto \sigma \#_{\Theta} \tau, \\ M^{\infty,1}(\mathbb{R}^n) \times M^{\infty,1}(\mathbb{R}^n) \times \mathrm{GL}(n, \mathbb{R}) &\rightarrow M^{\infty,1}(\mathbb{R}^n), \end{aligned}$$

is continuous.

Proof. Consider the bicharacter $B(x, y) = e^{2\pi i y^T x}$, for $x, y \in \mathbb{R}^n$. We point out that $B \in M^{1,\infty}(\mathbb{R}^{2n})$. Note that a dilation of B is the tensor product $B(x + y, x - y) = b_+(x)b_-(y)$ of the second degree characters $b_{\pm}(x) = e^{\pm 2\pi i x^T x}$. It is known that the Fourier transform of a second degree character is a multiple of another second degree character, and since by [9, Lemma 4.1] this class of functions belongs to $W^{1,\infty}(\mathbb{R}^n)$, we have $b_{\pm} \in M^{1,\infty}(\mathbb{R}^n)$. Hence we obtain $B \in M^{1,\infty}(\mathbb{R}^{2n})$.

Next, we use the notation for the restriction operator $R_{\Delta}f(z) = f(z, z)$ above and observe that

$$\begin{aligned} \sigma \#_{\Theta} \tau(z) &= \iint_{\mathbb{R}^{2n}} \sigma(z')\tau(z'')e^{2\pi i(z-z'')^T \Theta(z-z')} dz' dz'' \\ &= R_{\Delta}((\sigma \otimes \tau) * D_H B)(z), \quad \text{for } H = \begin{pmatrix} \Theta & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

By Lemma 2.5(i) the mapping $(\sigma, \tau) \mapsto \sigma \otimes \tau$ is continuous. Since the mapping $\Theta \mapsto H$ is continuous, we obtain by Lemma 2.3 that the mapping $(\sigma, \tau, \Theta) \mapsto (\sigma \otimes \tau) * D_H B$ is continuous. By Lemma 2.5(ii) and Remark 2.6 we conclude that the mapping $(\sigma, \tau, \Theta) \mapsto R((\sigma \otimes \tau) * D_H B)$ is continuous. \square

Corollary 3.3. *The mapping*

$$\begin{aligned} (\sigma, \tau, \rho) &\mapsto D_{\rho}(D_{\rho^{-1}}\sigma \# D_{\rho^{-1}}\tau), \\ M^{\infty,1}(\mathbb{R}^n) \times M^{\infty,1}(\mathbb{R}^n) \times \mathrm{GL}(n, \mathbb{R}) &\rightarrow M^{\infty,1}(\mathbb{R}^n), \end{aligned}$$

is continuous.

Proof. Since

$$D_{\rho}(D_{\rho^{-1}}\sigma \# D_{\rho^{-1}}\tau) = |\det \rho|^{-2} \sigma \#_{\Theta} \tau, \quad \text{for } \Theta = \rho^T J \rho,$$

we observe that the corollary is a consequence of Proposition 3.2. \square

Proof of Theorem 1.1. Sjöstrand proved that $(M^{\infty,1}, \#)$ is a Banach algebra [27, 26, 40, 41]. By Lemma 2.2(ii), and Corollary 3.3, we have that the family of dilations $(D_{\rho})_{\rho}$ on $(M^{\infty,1}, \#)$ satisfies the assumptions of Lemma 3.1, which implies the theorem. \square

4. PROOF OF THEOREM 1.3

A set X of points in \mathbb{R}^n is defined to be uniformly separated if there exists $a > 0$ such that $|x - y| \geq a$, for all $x, y \in X$ with $x \neq y$. A set X is called relatively separated if it is the finite union of uniformly separated subsets.

Lemma 4.1. *For every relatively separated set of points $X \subset \mathbb{R}^n$ the mapping*

$$h \mapsto \sum_{x \in X} T_x h,$$

$$M^1(\mathbb{R}^n) \rightarrow M^{\infty,1}(\mathbb{R}^n),$$

is continuous.

Proof. Since X is relatively separated, the functional $\delta_X: f \mapsto \sum_{x \in X} f(x)$ is an element of $M^\infty(\mathbb{R}^n) = (M^1(\mathbb{R}^n))'$, the dual space of $M^1(\mathbb{R}^n)$. Next, the mapping

$$(f, g) \mapsto f * g,$$

$$M^1(\mathbb{R}^n) \times M^\infty(\mathbb{R}^n) \rightarrow M^{\infty,1}(\mathbb{R}^n),$$

is continuous, which follows by applying the Fourier transform from the corresponding multiplication embedding of Wiener amalgam spaces in [15, Theorem 2.11], cf. (2.5), or can be deduced from the more general convolution properties in [8, 12, 43]. Since $\sum_{x \in X} T_x f = \delta_X * f$ it implies the lemma, indeed for the norm of the mapping in the lemma we have

$$\left\| \sum_{x \in X} T_x h \right\|_{M^{\infty,1}} \leq C \|\delta_X\|_{M^\infty} \|h\|_{M^1},$$

where C is a global constant that does not depend on X . □

Let $G(g, \Lambda)$ be a Gabor system with $g \in M^1$ and a time-frequency set of translations Λ in \mathbb{R}^{2n} . The Weyl symbol of the (pre-)frame operator $S_{g, \Lambda}$ for $G(g, \Lambda)$ is of the form

$$\sigma_{g, \Lambda} = \sum_{\lambda \in \Lambda} T_\lambda W(g),$$

where $W(g)$ denotes the Wigner distribution of g , defined by

$$W(g)(x, \omega) = \int_{\mathbb{R}^n} g\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega t} dt.$$

We will notice that $\sigma_{g, \Lambda} \in M^{\infty,1}$. In fact, while the mapping

$$(g, \rho) \mapsto \sigma_{g, \rho \Lambda},$$

$$M^1(\mathbb{R}^n) \times \text{GL}(2n, \mathbb{R}) \rightarrow M^{\infty,1}(\mathbb{R}^{2n}),$$

is *not* continuous (except for $g = 0$), we observe the following.

Lemma 4.2. *Suppose that X is a relatively separated set of points in \mathbb{R}^{2n} . Then the mapping*

$$(g, \rho) \mapsto D_\rho \sigma_{g, \rho \Lambda},$$

$$M^1(\mathbb{R}^n) \times \text{GL}(2n, \mathbb{R}) \rightarrow M^{\infty,1}(\mathbb{R}^{2n}),$$

is continuous.

Proof. It is known that the mapping

$$g \mapsto W(g),$$

$$M^1(\mathbb{R}^n) \rightarrow M^1(\mathbb{R}^{2n}),$$

is continuous. Therefore Lemma 2.2 implies that the mapping

$$g \mapsto D_\rho W(g),$$

$$M^1(\mathbb{R}^n) \rightarrow M^1(\mathbb{R}^{2n}),$$

is continuous. Hence, by Lemma 4.1 we conclude that the mapping

$$g \mapsto \sum_{\lambda \in \Lambda} T_\lambda D_\rho W(g),$$

$$M^1(\mathbb{R}^n) \rightarrow M^{\infty,1}(\mathbb{R}^{2n}),$$

is continuous. We conclude our argument using the identity

$$D_\rho \sigma_{g, \rho \Lambda} = D_\rho \sum_{\lambda \in \rho \Lambda} T_\lambda W(g) = \sum_{\lambda \in \Lambda} T_\lambda D_\rho W(g). \quad \square$$

Using all these ingredients we can come to the proof of the main theorems.

Proof of Theorem 1.3. Suppose that $G(g_0, \Lambda)$ is a Gabor frame with $g_0 \in M^1$, hence a Bessel sequence. Consequently the set Λ in \mathbb{R}^{2n} must be relatively separated [7, Theorem 3.1]; in fact, for $g_0 \in M^1(\mathbb{R}^n)$ the two properties are equivalent [32, Theorem 12].

Next, we use that the frame property for $G(g_0, \Lambda)$ is equivalent to the invertibility of the Weyl symbol $\sigma_{g_0, \Lambda}$ with respect to twisted convolution. By Theorem 1.1 there is a neighborhood W of $\sigma_{g_0, \Lambda}$ in $M^{\infty,1}$ and a neighborhood V_1 of I in $\text{GL}(2n, \mathbb{R})$ such that

$$(4.1) \quad D_{\rho^{-1}} \sigma \text{ is invertible,} \quad \text{for all } \sigma \in W, \rho \in V_1.$$

Lemma 4.2 guarantees the existence of a neighborhood U of g_0 in M^1 and a neighborhood V_2 of I in $\text{GL}(2n, \mathbb{R})$ such that

$$(4.2) \quad D_\rho \sigma_{g, \rho \Lambda} \in W, \quad \text{for all } g \in U, \rho \in V_2.$$

Hence, setting $V = V_1 \cap V_2$, we have by (4.1) and (4.2) that

$$\sigma_{g, \rho \Lambda} = D_{\rho^{-1}}(D_\rho \sigma_{g, \rho \Lambda}) \text{ is invertible,} \quad \text{for all } g \in U, \rho \in V,$$

or equivalently, that $G(g, \Lambda)$ is a Gabor frame, for all (g, ρ) in the neighborhood $U \times V$ of (g_0, I) in $M^1 \times \text{GL}(2n, \mathbb{R})$. \square

5. FINAL COMMENTS

The results in this paper are formulated for Gabor frames $G(g, \Lambda)$ using a single window $g \in M^1(\mathbb{R}^n)$ and general relatively separated sets Λ . In fact they also hold for multi-window Gabor frames $G(g_1, \Lambda_1) \cup \dots \cup G(g_m, \Lambda_m)$ with $g_1, \dots, g_m \in M^1(\mathbb{R}^n)$. For example the multi-window version of Theorem 1.5 reads, if such a Gabor system is a frame, then $\mathcal{D}^-(\Lambda_1 \cup \dots \cup \Lambda_m) > 1$. Indeed the extension to the multi-window case is immediate, based on the fact that the Weyl symbol for the multi-window system is just $\sigma = \sigma_1 + \dots + \sigma_m$, where σ_k is the Weyl symbol for the Gabor system $G(g_k, \Lambda_k)$, for $k = 1, \dots, m$.

Note that single- and multi-window Gabor frames (especially with windows in Feichtinger's algebra) are closely related to projective modules over noncommutative tori, as pointed out by results of Luef and Manin [35, 36, 37], cf. Remark 1.6(ii).

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