

# Operators Commuting with a Discrete Subgroup of Translations

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*ABSTRACT.* We study the structure of operators from the Schwartz space  $S(\mathbb{R}^n)$  into the tempered distributions  $S'(\mathbb{R}^n)$  that commute with a discrete subgroup of translations. The formalism leads to simple derivations of recent results about the frame operator of shift-invariant systems, Gabor, and wavelet frames.

## 1. Introduction and main result

We investigate shift-invariant operators, i.e., operators that commute with a discrete group of translation operators. Special classes of such operators have received considerable attention in the context of approximation theory and shift-invariant systems, Gabor analysis, and wavelet frames; see Section 4. In the terminology of operator theory, we look at the commutant of a discrete group of translation operators. We work in the general context of continuous linear operators from the Schwartz class  $S(\mathbb{R}^n)$  into the tempered distributions  $S'(\mathbb{R}^n)$ . Since these operators are described by distributional kernels, shift-invariant operators can be treated as a special case of shift-invariant distributions. This strategy leads to a convenient formalism for shift-invariant operators and avoids many technicalities encountered in the literature.

Our study is motivated by the fact that in the theory of shift-invariant systems one encounters operators that are shift-invariant and admit a so-called Walnut series. The original examples are the Walnut representation of Gabor frame operators and a corresponding result for wavelet frames, see Section 4.2. Our main result, Theorem 1.2, shows that indeed an arbitrary shift-invariant operator can be expressed by a Walnut series.

For  $x \in \mathbb{R}^n$ , let  $T_x\varphi(t) = \varphi(t - x)$  denote the translation of a function  $\varphi$  on  $\mathbb{R}^n$ ; the notion extends to distributions. We use the following normalization for the Fourier transform  $\mathcal{F}$ ,

$$\widehat{f}(s) = \int_{\mathbb{R}^n} f(t)e^{-2\pi i s t} dt, \quad s \in \mathbb{R}^n.$$

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Recall that the space of tempered distributions  $S'(\mathbb{R}^n)$  is endowed with the weak\*-topology, that is,  $u_k \rightarrow u$  in  $S'(\mathbb{R}^n)$  if  $\langle u_k, \varphi \rangle \rightarrow \langle u, \varphi \rangle$  in  $\mathbb{C}$ , for all  $\varphi \in S(\mathbb{R}^n)$ . By  $\mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$  we denote the space of continuous operators from  $S(\mathbb{R}^n)$  into  $S'(\mathbb{R}^n)$ . It is equipped with the usual bounded convergence topology (compact-open topology), that is,  $A_k \rightarrow A$  in  $\mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$  if  $A_k \varphi \rightarrow A \varphi$  in  $S'(\mathbb{R}^n)$ , for all  $\varphi \in S(\mathbb{R}^n)$ .

The following well-known result is a structure theorem for translation-invariant operators, i.e., operators that commute with all translations  $T_x$ , for  $x \in \mathbb{R}^n$ .

**Theorem 1.1.** ([15, 22]). *Let  $A \in \mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$  satisfy  $AT_x = T_x A$ , for all  $x \in \mathbb{R}^n$ .*

- (i) *Then  $A$  is a Fourier multiplier, i.e.,  $\widehat{A\varphi} = w \cdot \widehat{\varphi}$ , for  $\varphi \in S(\mathbb{R}^n)$ , where  $w \in S'(\mathbb{R}^n)$ .*
- (ii) *The mapping  $A \mapsto w$  is continuous from  $\mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$  into  $S'(\mathbb{R}^n)$ .*

Our main result is a structure theorem for shift-invariant operators, i.e., operators which commute only with a discrete subgroup of translations.

**Theorem 1.2.** *Given  $a > 0$ , suppose that the operator  $A \in \mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$  satisfies  $AT_{ak} = T_{ak}A$ , for all  $k \in \mathbb{Z}^n$ .*

- (i) *Then there exist uniquely defined  $w_k \in S'(\mathbb{R}^n)$ , for  $k \in \mathbb{Z}^n$ , such that  $A$  can be written as*

$$\widehat{A\varphi} = \sum_{k \in \mathbb{Z}^n} w_k \cdot (T_{k/a} \widehat{\varphi}), \quad \varphi \in S(\mathbb{R}^n), \quad (1.1)$$

*with unconditional convergence of the series in  $S'(\mathbb{R}^n)$ .*

- (ii) *For each  $k \in \mathbb{Z}^n$ , the mapping  $A \mapsto w_k$  is continuous from  $\mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$  into  $S'(\mathbb{R}^n)$ .*

The proof is given in Section 3. We mention that Theorem 1.1 can be obtained by using Theorem 1.2 with, say,  $a = 1$  and  $a = \sqrt{2}$  at the same time.

**Remark 1.3.**

- (i) The distributions  $w_k \in S'(\mathbb{R}^n)$ ,  $k \in \mathbb{Z}^n$ , are obtained by inspecting the Fourier transform of the distributional kernel of  $A$ , see the proof of the theorem below.
- (ii) For example, the identity operator  $A = \text{id}_{S \rightarrow S'}$  is characterized by the “multipliers”

$$w_k = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k \in \mathbb{Z}^n \setminus \{0\}, \end{cases} \quad \text{in } S'(\mathbb{R}^n). \quad (1.2)$$

- (iii) In operator notation we have  $\mathcal{F}A\mathcal{F}^{-1} = \sum_{k \in \mathbb{Z}^n} w_k T_{k/a}$ .

(iv) Theorem 1.2 says that if  $A$  commutes with all shifts contained in the lattice  $\Lambda = a\mathbb{Z}^n$ , then  $A$  can be expressed as a series of weighted shifts with respect to the dual lattice  $\Lambda^\perp = \frac{1}{a}\mathbb{Z}^n$ . By an obvious modification the result can be formulated for general lattices  $\Lambda = L\mathbb{Z}^n$ , given by  $L \in \text{GL}(n, \mathbb{R})$ ; note that  $\Lambda^\perp = (L^{-1})^T \mathbb{Z}^n$  in this case.

(v) A similar expansion as in Theorem 1.2 (i) was found in [1] in the context of linear systems theory. Our proof is different and simpler by using Poisson’s summation formula at a crucial point.

By using Theorem 1.2 we obtain a conceptually simple approach to important known results in the theory of shift-invariant systems, Gabor and wavelet frames.

The article is arranged as follows. Section 2 contains preliminary results on periodic distributions. Section 3 elaborates on these preliminary results and includes the proof of Theorem 1.2.

Finally, in Section 4 we illustrate the main result by examples, that is, the result is applied to shift-invariant systems, Gabor frames, and quasi-affine systems.

## 2. Preliminary results on invariant distributions

In this section, we describe the structure of distributions that are invariant under certain classes of translations.

In our work, the bracket  $\langle u, \varphi \rangle$  denotes the duality between  $S'(\mathbb{R}^n)$  and  $S(\mathbb{R}^n)$ , e.g., if  $u$  is a bounded function,

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^n} u(x)\varphi(x) dx . \quad (2.1)$$

While this duality is not strictly preserved by the Fourier transform, we will frequently use the fact that

$$\langle u, \bar{\varphi} \rangle = \langle \widehat{u}, \widehat{\bar{\varphi}} \rangle , \quad (2.2)$$

where the bar denotes complex conjugation. For functions  $f, g$ , we use the tensor product notation  $(f \otimes g)(x, y) = f(x)g(y)$ . The same notation is also used for distributions, see [23, Section 5.1].

Given an invertible real  $n \times n$  matrix  $M$ , define the dilation of a function  $f$  on  $\mathbb{R}^n$  by

$$D_M f(x) = |\det(M)|^{-1/2} f(M^{-1}x), \quad x \in \mathbb{R}^n .$$

The notion extends to distributions as usual by duality, i.e.,  $\langle D_M u, \varphi \rangle := \langle u, D_{M^{-1}} \varphi \rangle$ , for  $u \in S'(\mathbb{R}^n)$ ,  $\varphi \in S(\mathbb{R}^n)$ .

A usual form of the Poisson summation formula (PSF) states that for  $a > 0$  and  $f \in S(\mathbb{R}^n)$ , we have

$$\sum_{k \in \mathbb{Z}^n} f(ak) = a^{-n} \sum_{k \in \mathbb{Z}^n} \widehat{f}(k/a) , \quad (2.3)$$

with absolute convergence of both series. A more general version of the PSF is formulated for lattices  $\Lambda = L\mathbb{Z}^n \subset \mathbb{R}^n$ , where  $L$  is an invertible real  $n \times n$ -matrix. If  $\Lambda^\perp = (L^{-1})^T \mathbb{Z}^n$  denotes the dual lattice and  $|\Lambda| = |\det L|$  the lattice volume, then the PSF for  $\Lambda$  is

$$\sum_{\lambda \in \Lambda} f(\lambda) = |\Lambda|^{-1} \sum_{\lambda \in \Lambda^\perp} \widehat{f}(\lambda) .$$

It is less well known that the PSF can, in fact, be formulated for more general subgroups. For an arbitrary closed subgroup  $H \subseteq \mathbb{R}^n$  with Haar measure  $dh$  the PSF works as follows: Let

$$H^\perp = \{h' \in \mathbb{R}^n : \langle h, h' \rangle \in \mathbb{Z}, \text{ for all } h \in H\}$$

be the orthogonal subgroup with Haar measure  $dh'$ , then

$$\int_H f(h) dh = c_H \cdot \int_{H^\perp} \widehat{f}(h') dh' , \quad (2.4)$$

where the constant  $c_H$  is obtained from the Haar modulus of  $H$  [21, (31.46)]. In the following we are concerned with the subgroups

$$\begin{aligned} H_1 &= \{(ak, 0) : k \in \mathbb{Z}^m\} \subset \mathbb{R}^n, & m \leq n, & \text{ and} \\ H_2 &= \{(ak, ak) : k \in \mathbb{Z}^m\} \subset \mathbb{R}^n, & n = 2m. & \end{aligned}$$

The corresponding orthogonal subgroups are

$$\begin{aligned} H_1^\perp &= \{(k/a, t) : k \in \mathbb{Z}^m, t \in \mathbb{R}^{n-m}\} \subset \mathbb{R}^n, \quad \text{and} \\ H_2^\perp &= \{(t, k/a - t) : k \in \mathbb{Z}^m, t \in \mathbb{R}^m\} \subset \mathbb{R}^n. \end{aligned}$$

These are discrete subgroups, but not full-rank lattices, so the orthogonal subgroups are not lattices. The PSF (2.4) for  $H_1$  and  $H_2$  reads as follows.

**Lemma 2.1.**

(i) Let  $a > 0$  and  $m \leq n$ . Then for  $f \in \mathcal{S}(\mathbb{R}^n)$ , we have

$$\sum_{k \in \mathbb{Z}^m} f(ak, 0) = a^{-m} \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^{n-m}} \widehat{f}(k/a, t) dt. \quad (2.5)$$

(ii) Let  $a > 0$  and  $n = 2m$ . Then for  $f \in \mathcal{S}(\mathbb{R}^n)$ , we have

$$\sum_{k \in \mathbb{Z}^m} f(ak, ak) = a^{-m} \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^m} \widehat{f}(t, k/a - t) dt. \quad (2.6)$$

We note that the statements for  $H_1$  and for  $H_2$  are equivalent by a suitable coordinate transform applied to  $f$ . Indeed, let  $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$  and note that  $H_2 = MH_1$ . Then for  $f_1 := D_{M^{-1}}f$ , we have  $f_1(ak, 0) = f(ak, ak)$  and

$$\begin{aligned} \widehat{f}_1(k/a, t) &= (D_{M^{-1}}f)^\wedge(k/a, t) \\ &= (D_M \widehat{f})(k/a, t) = \widehat{f}(t, k/a - t), \quad k \in \mathbb{Z}^m, t \in \mathbb{R}^m, \end{aligned} \quad (2.7)$$

so (i) and (ii) are equivalent.

Generally, the PSF can be seen as an identity of distributions. Given  $H$  as above, let  $\mu_H$  denote the (suitably normalized) Haar measure of  $H$ , identified as a tempered distribution on  $\mathbb{R}^n$ , i.e.,  $\langle \mu_H, f \rangle = \int_H f(h) dh$ . Then

$$\widehat{\mu_H} = c_H \cdot \mu_{H^\perp} \quad (2.8)$$

and the original form (2.4) of the PSF is equivalent by (2.2). Now based on (2.8) and standard relations of the Fourier transform we have

$$\widehat{\mu_H * f} = \widehat{\mu_H} \cdot \widehat{f} = c_H \cdot \mu_{H^\perp} \cdot \widehat{f}, \quad f \in \mathcal{S}(\mathbb{R}^n)$$

Thus, given  $f \in \mathcal{S}(\mathbb{R}^n)$ , the periodization  $u := \mu_H * f$  along  $H$  can be expressed by the formula

$$\begin{aligned} \langle u, \overline{\varphi} \rangle &= \langle \widehat{\mu_H * f}, \overline{\varphi} \rangle = c_H \langle \mu_{H^\perp} \cdot \widehat{f}, \overline{\varphi} \rangle \\ &= c_H \langle \mu_{H^\perp}, \widehat{f} \cdot \overline{\varphi} \rangle = c_H \int_{H^\perp} \widehat{f}(h') \overline{\varphi}(h') dh'. \end{aligned} \quad (2.9)$$

With  $H = H_1, H_2$  as above, (2.9) yields the following identities.

**Lemma 2.2.**

(i) Let  $a > 0$  and  $m \leq n$ . For  $f \in \mathcal{S}(\mathbb{R}^n)$ , we define

$$u(x, y) = \sum_{k \in \mathbb{Z}^m} f(x - ak, y), \quad x \in \mathbb{R}^m, y \in \mathbb{R}^{n-m}.$$

Then we have

$$\langle u, \bar{\varphi} \rangle = \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^m} w_k(t) \overline{\widehat{\varphi}(k/a, t)} dt, \quad \varphi \in S(\mathbb{R}^n), \quad (2.10)$$

where  $w_k(t) = a^{-m} \widehat{f}(k/a, t)$ .

(ii) Let  $a > 0$  and  $n = 2m$ . For  $f \in S(\mathbb{R}^n)$ , we define

$$u(x, y) = \sum_{k \in \mathbb{Z}^m} f(x - ak, y - ak), \quad x, y \in \mathbb{R}^m.$$

Then we have

$$\langle u, \bar{\varphi} \rangle = \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^m} w_k(t) \overline{\widehat{\varphi}(t, k/a - t)} dt, \quad \varphi \in S(\mathbb{R}^n), \quad (2.11)$$

where  $w_k(t) = a^{-m} \widehat{f}(t, k/a - t)$ .

While this is a result for distributions given explicitly in the form of a periodization along  $H$ , our next result holds for arbitrary  $H$ -periodic distributions.

### Proposition 2.3.

(i) Let  $u \in S'(\mathbb{R}^n)$  satisfy  $u = T_{(ak, 0)}u$ , for all  $k \in \mathbb{Z}^m$ , where  $a > 0$  and  $m \leq n$ . Then there exist uniquely defined  $w_k \in S'(\mathbb{R}^{n-m})$ , for  $k \in \mathbb{Z}^m$ , such that

$$\langle u, \bar{\varphi} \rangle = \sum_{k \in \mathbb{Z}^m} \langle w_k, \overline{R_k \widehat{\varphi}} \rangle, \quad \varphi \in S(\mathbb{R}^n), \quad (2.12)$$

where  $R_k \widehat{\varphi}(t) = \widehat{\varphi}(k/a, t)$ , for  $t \in \mathbb{R}^{n-m}$ , with unconditional convergence of the series. Moreover, the mapping  $u \mapsto w_k$  is continuous from  $S'(\mathbb{R}^n)$  into  $S'(\mathbb{R}^{n-m})$ , for each  $k \in \mathbb{Z}^m$ .

(ii) Let  $n = 2m$ . Let  $u \in S'(\mathbb{R}^n)$  satisfy  $u = T_{(ak, ak)}u$ , for all  $k \in \mathbb{Z}^m$ , where  $a > 0$ . Then there exist uniquely defined  $w_k \in S'(\mathbb{R}^m)$ , for  $k \in \mathbb{Z}^m$ , such that

$$\langle u, \bar{\varphi} \rangle = \sum_{k \in \mathbb{Z}^m} \langle w_k, \overline{Q_k \widehat{\varphi}} \rangle, \quad \varphi \in S(\mathbb{R}^n), \quad (2.13)$$

where  $Q_k \widehat{\varphi}(t) = \widehat{\varphi}(t, k/a - t)$ , for  $t \in \mathbb{R}^m$ , with unconditional convergence of the series. Moreover, the mapping  $u \mapsto w_k$  is continuous from  $S'(\mathbb{R}^n)$  into  $S'(\mathbb{R}^m)$ , for each  $k \in \mathbb{Z}^m$ .

The proof will provide an explicit construction of the distributions  $w_k$ .

### Proof.

(i) Let  $\psi \in S(\mathbb{R}^m)$  be such that  $\sum_{k \in \mathbb{Z}^m} T_{ak} \psi = 1$  or, equivalently,

$$\widehat{\psi}(k/a) = \begin{cases} 1, & k = 0, \\ 0, & k \in \mathbb{Z}^m \setminus \{0\}. \end{cases}$$

First, for  $k \in \mathbb{Z}^m$ , define a distribution  $w_k \in S'(\mathbb{R}^{n-m})$  by

$$\langle w_k, \varphi \rangle = a^{-m} \langle \widehat{u}, (T_{k/a} \widehat{\psi}) \otimes \varphi \rangle, \quad \varphi \in S(\mathbb{R}^{n-m}). \quad (2.14)$$

Using the definition of  $w_k$ , we will verify the series expansion (2.12) for  $\widehat{u}$ . Given  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , let

$$\Phi(x, y) = \psi(x) \sum_{k \in \mathbb{Z}^m} \varphi(x - ak, y), \quad x \in \mathbb{R}^m, y \in \mathbb{R}^{n-m}.$$

Then the series defining  $\Phi$  converges unconditionally in  $\mathcal{S}(\mathbb{R}^n)$ , cf. [23, p.178]. Using the assumption on  $\psi$  and the fact that  $u = T_{(ak,0)}u$  we have that

$$\begin{aligned} \langle u, \varphi \rangle &= \left\langle u, \left( \sum_{k \in \mathbb{Z}^m} T_{(ak,0)} \psi \otimes 1 \right) \cdot \varphi \right\rangle \\ &= \sum_{k \in \mathbb{Z}^m} \langle u, (T_{(ak,0)} \psi \otimes 1) \cdot \varphi \rangle \\ &= \sum_{k \in \mathbb{Z}^m} \langle u, T_{(ak,0)} [(\psi \otimes 1) \cdot T_{(-ak,0)} \varphi] \rangle \\ &= \sum_{k \in \mathbb{Z}^m} \langle u, (\psi \otimes 1) \cdot T_{(-ak,0)} \varphi \rangle = \langle u, \Phi \rangle. \end{aligned} \tag{2.15}$$

Next, let  $\varphi_y(x) := \varphi(x, y)$  and

$$\Phi_y(x) := \psi(x) \sum_{k \in \mathbb{Z}^m} \varphi_y(x - ak), \quad x \in \mathbb{R}^m, y \in \mathbb{R}^{n-m}.$$

The standard PSF (2.3) implies

$$\sum_{k \in \mathbb{Z}^m} T_{ak} \varphi_y(x) = a^{-m} \sum_{k \in \mathbb{Z}^m} \widehat{\varphi}_y(k/a) e^{2\pi i k x / a}, \quad x \in \mathbb{R}^m.$$

Hence,

$$\begin{aligned} \widehat{\Phi}_y(s) &= \left( \psi \cdot \sum_{k \in \mathbb{Z}^m} T_{ak} \varphi_y \right)^\wedge(s) \\ &= a^{-m} \left( \psi \cdot \sum_{k \in \mathbb{Z}^m} \widehat{\varphi}_y(k/a) e^{2\pi i k \cdot / a} \right)^\wedge(s) \\ &= a^{-m} \sum_{k \in \mathbb{Z}^m} \widehat{\psi}(s - k/a) \widehat{\varphi}_y(k/a), \quad s \in \mathbb{R}^m, y \in \mathbb{R}^{n-m}, \end{aligned}$$

and thus we obtain

$$\begin{aligned} \widehat{\Phi}(s, t) &= a^{-m} \sum_{k \in \mathbb{Z}^m} \widehat{\psi}(s - k/a) \widehat{\varphi}(k/a, t) \\ &= a^{-m} \sum_{k \in \mathbb{Z}^m} (T_{k/a} \widehat{\psi} \otimes R_k \widehat{\varphi})(s, t), \quad s \in \mathbb{R}^m, t \in \mathbb{R}^{n-m}. \end{aligned} \tag{2.16}$$

Since for fixed  $t \in \mathbb{R}^{n-m}$ , the function  $\widehat{\varphi}(k/a, t)$  decays rapidly in  $k$  and the Schwartz seminorms of  $T_{k/a} \widehat{\psi}$  grow only polynomially in  $k$ , it is easily verified that the series in (2.16) converges unconditionally in  $\mathcal{S}(\mathbb{R}^n)$ ; see [17, Lemma 11.2.1 or Proposition 11.2.4] for a similar argument.

Now using (2.16) and (2.14), we calculate

$$\begin{aligned}
\langle \widehat{u}, \widehat{\Phi} \rangle &= a^{-m} \left\langle \widehat{u}, \sum_{k \in \mathbb{Z}^n} \overline{T_{k/a} \widehat{\psi} \otimes R_k \widehat{\varphi}} \right\rangle \\
&= a^{-m} \sum_{k \in \mathbb{Z}^m} \langle \widehat{u}, \overline{T_{k/a} \widehat{\psi} \otimes R_k \widehat{\varphi}} \rangle \\
&= \sum_{k \in \mathbb{Z}^n} \langle w_k, \overline{R_k \widehat{\varphi}} \rangle.
\end{aligned} \tag{2.17}$$

Finally, combining (2.15) and (2.17), we obtain

$$\langle u, \overline{\varphi} \rangle = \langle u, \widehat{\Phi} \rangle = \langle \widehat{u}, \widehat{\Phi} \rangle = \sum_{k \in \mathbb{Z}^m} \langle w_k, \overline{R_k \widehat{\varphi}} \rangle.$$

To show the uniqueness of the distributions  $w_k$ , for  $k \in \mathbb{Z}^m$ , assume that  $\sum_{k \in \mathbb{Z}^n} \delta_{k/a} \otimes w_k = \sum_{k \in \mathbb{Z}^n} \delta_{k/a} \otimes w'_k$  for some other sequence of  $w'_k \in S'(\mathbb{R}^{n-m})$ . This means that for all  $\varphi_1 \in S'(\mathbb{R}^m)$  and all  $\varphi_2 \in S'(\mathbb{R}^{n-m})$  we have

$$\left\langle \sum_{k \in \mathbb{Z}^n} \delta_{k/a} \otimes w_k, \varphi_1 \otimes \varphi_2 \right\rangle = \sum_{k \in \mathbb{Z}^n} \varphi_1(k/a) \langle w_k, \varphi_2 \rangle = \sum_{k \in \mathbb{Z}^n} \varphi_1(k/a) \langle w'_k, \varphi_2 \rangle.$$

If for each  $k \in \mathbb{Z}^n$  we choose some  $\varphi_1 \in S(\mathbb{R}^m)$  such that  $\varphi_1(l/a) = \delta_{k,l}$ , then we obtain  $\langle w_k, \varphi_2 \rangle = \langle w'_k, \varphi_2 \rangle$ , for all  $\varphi_2 \in S(\mathbb{R}^{n-m})$ , and thus  $w_k = w'_k$  for  $k \in \mathbb{Z}^n$ .

The uniqueness also implies that the definition of the  $w_k$  is independent of the auxiliary function  $\psi$  in (2.14).

The continuity of the mapping  $u \mapsto w_k$  from  $S'(\mathbb{R}^n)$  into  $S'(\mathbb{R}^{n-m})$ , for each  $k \in \mathbb{Z}^m$ , follows immediately from (2.14).

(ii) Recall that  $H_2 = MH_1$  for  $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ . Define  $u_1 := D_{M^{-1}}u \in S'(\mathbb{R}^n)$ . Since  $T_x D_{M^{-1}} = D_{M^{-1}} T_{Mx}$  and  $u$  is  $H_2$ -invariant,  $u_1$  is  $H_1$ -invariant. Hence, (ii) follows from (i) by setting  $Q_k = R_k D_{M^t}$ . Then  $Q_k \varphi(t) = \varphi(t, k/a - t)$  as in (2.7) and

$$\begin{aligned}
\langle u, \overline{\varphi} \rangle &= \langle D_M u_1, \overline{\varphi} \rangle = \langle u_1, \overline{D_{M^{-1}} \varphi} \rangle \\
&= \sum_{k \in \mathbb{Z}^m} \left\langle w_k, \overline{R_k (D_{M^{-1}} \varphi)^\wedge} \right\rangle \\
&= \sum_{k \in \mathbb{Z}^m} \langle w_k, \overline{R_k D_{M^t} \widehat{\varphi}} \rangle = \sum_{k \in \mathbb{Z}^m} \langle w_k, \overline{Q_k \widehat{\varphi}} \rangle, \quad \varphi \in S(\mathbb{R}^n). \quad \square
\end{aligned}$$

Note that Lemma 2.2 (i), (ii) is a special case of Proposition 2.3 (i), (ii) where the distributions  $w_k$  are indeed Schwartz functions and they are given explicitly. In fact, the continuous dependence of the  $w_k$  on  $u$  allows us to determine the  $w_k$  explicitly also in certain other cases, such as discussed next.

First, the conclusions of Lemma 2.2 also hold under the more general assumption that  $f \in L^1(\mathbb{R}^n)$  (instead of  $f \in S(\mathbb{R}^n)$ ). Secondly, in Section 4 we will need the expansion (2.11) for the case when  $f$  is a tensor product, as follows.

**Lemma 2.4.** *Let  $n = 2m$  and  $f = g \otimes h \in L^2(\mathbb{R}^n)$ , for given  $g, h \in L^2(\mathbb{R}^m)$ . Let*

$$u = \sum_{k \in \mathbb{Z}^m} T_{(ak, ak)}(g \otimes h) = \sum_{k \in \mathbb{Z}^m} T_{ak} g \otimes T_{ak} h.$$

Then  $u \in S'(\mathbb{R}^n)$  and

$$\langle u, \overline{\varphi} \rangle = \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^m} w_k(t) \overline{\varphi}(t, k/a - t) dt, \quad \varphi \in S(\mathbb{R}^n), \quad (2.18)$$

with  $w_k(t) = a^{-m} \widehat{g}(t) \widehat{h}(k/a - t) \in L^1(\mathbb{R}^m)$ .

**Proof.** First, let  $Q = [0, a]^n$ . Using the Cauchy-Schwarz inequality we have for  $r, s \in \mathbb{Z}^m$ ,

$$\begin{aligned} \iint_{(ar, as) + Q} |u(x, y)|^2 dx dy &= \iint_{(ar, as) + Q} \left| \sum_{k \in \mathbb{Z}^m} g(x - ak) h(y - ak) \right|^2 dx dy \\ &= \iint_Q \left| \sum_{k \in \mathbb{Z}^m} g(x - ak) h(y - ak) \right|^2 dx dy \\ &\leq \int_{[0, a]^m} \sum_{k \in \mathbb{Z}^m} |g(x - ak)|^2 dx \int_{[0, a]^m} \sum_{k \in \mathbb{Z}^m} |h(y - ak)|^2 dy \\ &= \|g\|_{L^2}^2 \|h\|_{L^2}^2. \end{aligned} \quad (2.19)$$

Thus,  $u$  is square integrable on any cube  $Q_{r,s} = (ar, as) + Q$  and the local norm is bounded independently of the position  $(r, s) \in \mathbb{Z}^n$ . Therefore  $u$  is a tempered distribution and the mapping  $(g, h) \mapsto u$  is continuous from  $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$  into  $S'(\mathbb{R}^n)$ .

Next Proposition 2.3 asserts that

$$\langle u, \overline{\varphi} \rangle = \sum_{k \in \mathbb{Z}^m} \langle w_k, \overline{Q_k \widehat{\varphi}} \rangle, \quad \varphi \in S(\mathbb{R}^n),$$

for some  $w_k \in S'(\mathbb{R}^{n-m})$ . Using the density of  $S(\mathbb{R}^n)$  in  $L^2(\mathbb{R}^n)$ , we choose two sequences  $g_r, h_r \in S(\mathbb{R}^m)$  such that  $g_r \otimes h_r$  converges to  $f = g \otimes h$  in  $L^2(\mathbb{R}^n)$ . Set

$$u_r = \sum_{k \in \mathbb{Z}^m} T_{(ak, ak)}(g_r \otimes h_r).$$

Then by Lemma 2.2 (ii),

$$\langle u_r, \overline{\varphi} \rangle = \sum_{k \in \mathbb{Z}^m} \langle w_k^{(r)}, \overline{Q_k \widehat{\varphi}} \rangle, \quad \varphi \in S(\mathbb{R}^n),$$

with the explicit formula

$$w_k^{(r)}(t) = a^{-m} \widehat{g}_r(t) \widehat{h}_r(k/a - t) \in L^1(\mathbb{R}^m).$$

Since  $u_r \rightarrow u$  as a consequence of (2.19), Proposition 2.3 implies that  $w_k^{(r)} \rightarrow w_k$  in  $S'(\mathbb{R}^m)$ , for all  $k \in \mathbb{Z}^m$ . On the other hand,

$$w_k^{(r)}(t) \rightarrow a^{-m} \widehat{g}(t) \widehat{h}(k/a - t) \quad \text{in } L^1(\mathbb{R}^m),$$

and so  $w_k(t) = a^{-m} \widehat{g}(t) \widehat{h}(k/a - t)$ , as claimed.  $\square$

### 3. Proof of Theorem 1.2

The Schwartz kernel theorem states that there exists a topological isomorphism between the operators  $A \in \mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$  and the distributions  $\kappa \in S'(\mathbb{R}^{2n})$ . The distribution  $\kappa$  is called the kernel of  $A$  and the correspondence is given by  $\langle A\varphi, \psi \rangle = \langle \kappa, \psi \otimes \varphi \rangle$ , for  $\varphi, \psi \in S(\mathbb{R}^n)$ . The kernel theorem together with Proposition 2.3 (ii) now allows us to prove the main theorem.

**Proof of Theorem 1.2.** Let  $\kappa \in S'(\mathbb{R}^{2n})$  denote the Schwartz kernel of the operator  $A$ , that is,  $\langle A\varphi, \psi \rangle = \langle \kappa, \psi \otimes \varphi \rangle$ , for  $\varphi, \psi \in S(\mathbb{R}^n)$ . For  $k \in \mathbb{Z}^n$ , we note that

$$\begin{aligned} \langle T_{ak}AT_{-ak}\varphi, \psi \rangle &= \langle AT_{-ak}\varphi, T_{-ak}\psi \rangle \\ &= \langle \kappa, T_{-ak}\psi \otimes T_{-ak}\varphi \rangle \\ &= \langle \kappa, T_{(-ak, -ak)}(\psi \otimes \varphi) \rangle \\ &= \langle T_{(ak, ak)}\kappa, \psi \otimes \varphi \rangle, \quad \varphi, \psi \in S(\mathbb{R}^n). \end{aligned}$$

Therefore, since  $A = T_{ak}AT_{-ak}$  and the Schwartz kernel  $\kappa \in S'(\mathbb{R}^{2n})$  of an operator is unique, we have that  $\kappa = T_{(ak, ak)}\kappa$  for  $k \in \mathbb{Z}^n$ . We now apply Proposition 2.3 (ii) to the invariant distribution  $\kappa$ . Since  $\widehat{\overline{\varphi}}(x) = \widehat{\varphi}(-x)$ , we conclude that

$$\begin{aligned} \langle \widehat{A\varphi}, \widehat{\psi} \rangle &= \langle A\varphi, \overline{\psi} \rangle = \langle \kappa, \overline{\psi} \otimes \overline{\varphi} \rangle \\ &= \sum_{k \in \mathbb{Z}^n} \langle w_k, Q_k(\widehat{\overline{\psi}} \otimes \widehat{\overline{\varphi}}) \rangle \\ &= \sum_{k \in \mathbb{Z}^n} \langle w_k, \widehat{\overline{\psi}} \cdot T_{k/a}\widehat{\varphi} \rangle \\ &= \sum_{k \in \mathbb{Z}^n} \langle w_k \cdot T_{k/a}\widehat{\varphi}, \widehat{\overline{\psi}} \rangle, \quad \varphi, \psi \in S(\mathbb{R}^n). \end{aligned}$$

The continuity statement also follows from Proposition 2.3 (ii). □

## 4. Applications to shift-invariant systems

We next show how the structure theorem for operators that commute with a discrete group of translations can be used in the analysis of shift-invariant systems. Note that in the following arguments, we abide by the bilinear definition of the bracket  $\langle \cdot, \cdot \rangle$ , even though shift-invariant systems are commonly studied in the  $L^2$ -context.

### 4.1. Shift-invariant systems

Let  $a > 0$  and  $g_j \in L^2(\mathbb{R}^n)$ , for  $j \in I$ , where  $I$  denotes a finite or countable index set. A family of the form

$$G = \{T_{ak}g_j : k \in \mathbb{Z}^n, j \in I\},$$

is called a shift-invariant system. By construction, the set  $G$  is invariant under all translations  $T_{ak}$  for  $k \in \mathbb{Z}^n$ . The closed span of  $G$  in some  $L^p$  is a shift-invariant space. For shift-invariant systems, see [2, 4, 18, 30]. On a theoretical level, the main objectives are to understand the spanning and stability properties of  $G$ . These are encoded in the spectrum of the frame operator associated to  $G$ . More generally, given the shift-invariant systems  $G$  and  $H = \{T_{ak}h_j : k \in \mathbb{Z}^n, j \in I\}$ , we

define the frame type operator  $S = S_{G,H}$  by

$$Sf = \sum_{j \in I} \sum_{k \in \mathbb{Z}^n} \langle f, T_{ak} \bar{g}_j \rangle T_{ak} h_j, \quad f \in S(\mathbb{R}^n). \quad (4.1)$$

**Remark 4.1.** If  $\varphi \in S(\mathbb{R}^n)$ , it is easy to see that the sequence  $c_k = \langle \varphi, T_{ak} \bar{g} \rangle$  belongs to  $\ell^2(\mathbb{Z}^n)$ . If  $I$  is finite, then  $\langle S\varphi, \psi \rangle = \sum_{k \in \mathbb{Z}^n} \sum_{j \in I} \langle \varphi, T_{ak} \bar{g}_j \rangle \langle \psi, T_{ak} h_j \rangle$  converges absolutely for  $\varphi, \psi \in S(\mathbb{R}^n)$  and, as a consequence,  $S$  is continuous from  $S(\mathbb{R}^n)$  into  $S'(\mathbb{R}^n)$ . Hence, unconditional convergence of (4.1) is equivalent to the unconditional convergence of the outer sum.

The following structure theorem for  $S$  is an immediate consequence of Theorem 1.2. The result provides a simple approach to important results about so-called reproducing systems. In particular, we can easily characterize when the pair  $(G, H)$  is a reproducing system, that is, when  $S = \text{id}_{S \rightarrow S'}$ .

**Corollary 4.2.** *Suppose that the frame type operator  $S$ , given in (4.1), is continuous from  $S(\mathbb{R}^n)$  into  $S'(\mathbb{R}^n)$ , and the defining series over  $I$  converges  $w^*$ -unconditionally. Then  $S$  is of the form*

$$\widehat{S}\varphi = \sum_{k \in \mathbb{Z}^n} w_k \cdot T_{k/a} \widehat{\varphi}, \quad \varphi \in S(\mathbb{R}^n), \quad (4.2)$$

where

$$w_k = a^{-n} \sum_{j \in I} T_{k/a} \widehat{g}_j \cdot \widehat{h}_j, \quad \text{for } k \in \mathbb{Z}^n, \quad (4.3)$$

with unconditional convergence of the series (4.2) in  $S'(\mathbb{R}^n)$ .

**Proof.** Given  $k \in \mathbb{Z}^n$  and  $j \in I$ , the rank-one operator  $S_{k,j} f = \langle f, T_{ak} g_j \rangle T_{ak} h_j$  has the kernel

$$T_{ak} h_j \otimes T_{ak} \bar{g}_j = T_{(ak,ak)}(h_j \otimes \bar{g}_j).$$

Hence,  $S$  has the kernel

$$\kappa = \sum_{j \in I} \sum_{k \in \mathbb{Z}^n} T_{(ak,ak)}(h_j \otimes \bar{g}_j).$$

Thus, the result follows from Theorem 1.2, and the explicit form of  $w_k$  is obtained in Example 2.4 (ii), observing that  $\widehat{\bar{g}_j}(x) = \widehat{g}_j(-x)$ .  $\square$

**Remark 4.3.** Recall that  $G$  is called a Bessel sequence if  $S_{G,G}$  is bounded on  $L^2(\mathbb{R}^n)$ . We note that if  $S_{G,H}$  is bounded on  $L^2(\mathbb{R}^n)$ , then  $G, H$  need not be Bessel sequences, cf. [14, p. 143 and p. 150, Remark (c)]. However, if  $G$  is a Bessel sequence, then the following equivalences hold.

- (i)  $G$  is a frame and  $H$  is a dual frame  $\Leftrightarrow S_{G,H} = \text{id}$ .
- (ii)  $G$  is a normalized tight frame  $\Leftrightarrow S_{G,G} = \text{id}$ .
- (iii)  $G$  is an orthonormal basis  $\Leftrightarrow$  the elements of  $G$  have norm one and  $G$  is a normalized tight frame.

We note that the same argument as in Remark 4.1 shows for Bessel sequences  $G, H$  that  $S_{G,H}$  is continuous from  $S(\mathbb{R}^n)$  to  $S'(\mathbb{R}^n)$ , and that the sum over  $I$  in (4.1) converges unconditionally.

The representation (4.2) was obtained in [30, 33, 27] under the additional hypothesis that  $S$  is bounded on  $L^2(\mathbb{R}^n)$ . Using the distributional version Theorem 1.2 makes the derivation of these important results conceptually simple and in addition avoids the technicalities connected with the  $L^2$ -boundedness. For a unified treatment of reproducing systems, see [19, 28, 30].

## 4.2. Gabor systems

Here we apply Corollary 4.2 to Gabor systems. Define the modulation operator

$$M_s f(t) = e^{2\pi i s t} f(t), \quad s, t \in \mathbb{R}^n .$$

Given  $a, b > 0$  and  $g, h \in L^2(\mathbb{R}^n)$ , the families

$$G = \{T_{ak} M_{bl} g : k, l \in \mathbb{Z}^n\}$$

and  $H = \{T_{ak} M_{bl} h\}$  are called Gabor systems, see [12, 13, 17, 24, 33]. The corresponding Gabor frame type operator

$$Sf = \sum_{k, l \in \mathbb{Z}^n} \langle f, T_{ak} M_{bl} \bar{g} \rangle T_{ak} M_{bl} h, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad (4.4)$$

commutes with the translations  $T_{ak}$ , for  $k \in \mathbb{Z}^n$ , and it follows from [14, Corollary 3.3.3 (iii)] that  $S$  is continuous from  $\mathcal{S}(\mathbb{R}^n)$  into  $\mathcal{S}'(\mathbb{R}^n)$ .

### Example 4.4.

(i) According to Theorem 1.2 the Gabor frame type operator  $S$  in (4.4) can be represented in the form (4.2) with

$$w_k = a^{-n} \sum_{l \in \mathbb{Z}^n} T_{bl} (T_{k/a} \bar{g} \cdot \widehat{h}), \quad \text{for } k \in \mathbb{Z}^n . \quad (4.5)$$

This is the so-called Walnut representation and was first obtained under more restrictive conditions in [34].

(ii) As a consequence of (1.2), we have  $S = \text{id}_{\mathcal{S} \rightarrow \mathcal{S}'}$  if and only if  $w_0 = 1$  and  $w_k = 0$  for  $k \neq 0$ . In the case of Gabor frames this is equivalent to the Wexler-Raz conditions

$$\langle h, T_{l/b} M_{k/a} \bar{g} \rangle = \begin{cases} a^n b^n, & (k, l) = (0, 0), \\ 0, & (k, l) \in \mathbb{Z}^2 \setminus \{(0, 0)\}. \end{cases} \quad (4.6)$$

**Proof.** For  $g, h \in L^2(\mathbb{R}^n)$  the series in (4.5) converges unconditionally to a  $b$ -periodic function  $w_k \in L^1([0, b]^n)$ . The equivalence of (1.2) and (4.6) follows by calculating the Fourier coefficients of  $w_k$ ,

$$\begin{aligned} \delta_{k,0} \delta_{m,0} &= \widehat{w}_k(m) = b^{-n} \int_{[0,b]^d} w_k(x) e^{-2\pi i m \cdot x / b} dx \\ &= (ab)^{-n} \int_{[0,b]^d} \sum_{l \in \mathbb{Z}^n} T_{bl} (T_{k/a} \bar{g} \cdot \widehat{h})(x) e^{-2\pi i m \cdot x / b} dx \\ &= (ab)^{-n} \int_{\mathbb{R}^n} T_{k/a} \bar{g}(x) \widehat{h}(x) e^{-2\pi i m \cdot x / b} dx \\ &= (ab)^{-n} \langle \widehat{h}, M_{-m/b} T_{k/a} \bar{g} \rangle \\ &= (ab)^{-n} \langle h, T_{m/b} M_{k/a} \bar{g} \rangle. \end{aligned} \quad \square$$

**Remark 4.5.**

(i) The Walnut representation of the Gabor frame operator is usually formulated without using the Fourier transform. We note that both descriptions are equivalent, since a Gabor frame operator conjugated with the Fourier transform is again a Gabor frame operator.

(ii) The ordering  $T_{ak}M_{bl}$  as compared to  $M_{bl}T_{ak}$  often used in the literature ensures that the Gabor system is shift-invariant. The commutation relation  $T_{ak}M_{bl} = e^{-2\pi i abl \cdot k} M_{bl}T_{ak}$  shows that spanning, Bessel-sequence, or orthonormal basis properties of Gabor systems are preserved if we exchange the ordering.

By Example 4.4 we have obtained general versions of important results in Gabor analysis. Indeed, suppose that  $G$  and  $H$  are Bessel sequences. Then (4.2) with  $w_k$  given by (4.5) is the Walnut representation, found in [34], see [17, Sections 6.3 and 7.1], or [5, 27]. Secondly, the condition (1.2) with  $w_k$  given by (4.5) is a characterization of dual windows [6, Exercise 8.9]. Third, (4.6) are the Wexler-Raz conditions, found in [36], see [17, Section 7.3], [10, 26].

An alternative approach that also works for non-product time-frequency lattices was developed in [11, 14].

**4.3. Affine systems**

Finally, we apply Corollary 4.2 to affine systems. In dimension  $n = 1$  the dilation by  $a$  is  $D_a f(t) = |a|^{-1/2} f(a^{-1}t)$ , for  $t \in \mathbb{R}$ . Given  $a, b > 0$  and  $g, h \in L^2(\mathbb{R})$ , the families

$$G = \{D_{a^j} T_{bk} g : k \in \mathbb{Z}, j \in \mathbb{Z}\}$$

and  $H = \{D_{a^j} T_{bk} h\}$  are called affine systems. Such systems are investigated in the theory of wavelet frames and bases, see, e.g., [7, 9, 25, 29]. In the following we assume that  $a$  is an integer  $\geq 1$ . Although,  $G$  and  $H$  are not shift-invariant, it is possible to associate a shift-invariant system  $\tilde{G}$  with identical spanning and stability properties, namely the *quasi-affine system* defined by

$$\begin{aligned} \tilde{G} &= \{D_{a^j} T_{bk} g : k \in \mathbb{Z}, j \leq 0\} \cup \{a^{-j/2} T_{bk} D_{a^j} g : k \in \mathbb{Z}, j > 0\} \\ &= \{T_{bk} g_p : k \in \mathbb{Z}, p \in I\}, \end{aligned} \quad (4.7)$$

where

$$I = \{(j, l) : j, l \in \mathbb{Z}, 0 \leq l < a^{-j}\} \quad (4.8)$$

and

$$g_p = \min(1, a^{-j/2}) D_{a^j} T_{bl} g, \quad p = (j, l) \in I. \quad (4.9)$$

The original source for the equivalence of affine and quasi-affine systems is [32] and it is extended in [3, 8, 16, 20, 31]. By  $\tilde{H}$  we denote the quasi-affine system corresponding to  $H$ . The quasi-affine frame type operator  $S = S_{\tilde{G}, \tilde{H}}$  is given by

$$Sf = \sum_{p \in I} \sum_{k \in \mathbb{Z}} \langle f, T_{bk} g_p \rangle T_{bk} h_p, \quad f \in \mathcal{S}(\mathbb{R}). \quad (4.10)$$

We note that if  $G$  and  $H$  are Bessel systems, then so are  $\tilde{G}$ ,  $\tilde{H}$ ; in particular, under this assumption  $S$  is continuous from  $\mathcal{S}(\mathbb{R})$  into  $\mathcal{S}'(\mathbb{R})$ , and the sum over  $I$  converges  $w^*$ -unconditionally.

**Example 4.6.** Assume that the quasi-affine frame type operator  $S$ , given in (4.10), is continuous from  $S(\mathbb{R})$  into  $S'(\mathbb{R})$ , and that the sum over  $I$  converges  $w^*$ -unconditionally.

(i) Then  $S$  is of the form (4.2) with

$$\begin{aligned} w_k &= b^{-1} \sum_{j \geq 0} a^{-j} (T_{k/b} D_{a^{-j}} \widehat{g}) \cdot (D_{a^{-j}} \widehat{h}) \\ &\quad + b^{-1} \sum_{j < 0} \sum_{l=0}^{a^{-j}-1} (T_{k/b} D_{a^{-j}} \overline{M_{-bl} \widehat{g}}) \cdot (D_{a^{-j}} M_{-bl} \widehat{h}) \end{aligned} \quad (4.11)$$

and  $w^*$ -convergence in  $S'(\mathbb{R}^n)$ .

(ii) Conditions (1.2) for  $S = \text{id}_{S \rightarrow S'}$  are equivalent to the Calderón condition for dual wavelet frames:

$$\sum_{j \in \mathbb{Z}} \widehat{g}(a^j t) \widehat{h}(a^j t) = b \quad \text{a.e.}, \quad (4.12)$$

$$\sum_{j \geq 0} \widehat{g}(a^j(t - m/b)) \widehat{h}(a^j t) = 0 \quad \text{a.e.}, \quad \text{for all } m \in \mathbb{Z} \setminus a\mathbb{Z}. \quad (4.13)$$

Compared to other approaches, the derivation of these conditions based on Theorem 1.2 is fairly easy.

**Proof.** Again the result follows from Corollary 4.2, where  $T_{ak}$  reads  $T_{bk}$ , for  $k \in \mathbb{Z}$ , and  $\{g_j\}_{j \in I}$  takes the special form  $\{g_p\}_{p \in I}$ , described in (4.8), (4.9). The identity (4.11) is obtained by substituting the quasi-affine system (4.7) into the definition of  $w_k$  in (4.3). To verify the Calderón conditions, let

$$w_k^+ = \frac{1}{a^j b} \sum_{j \geq 0} (T_{k/b} D_{a^{-j}} \widehat{g}) \cdot D_{a^{-j}} \widehat{h}$$

and  $w_k^- = w_k - w_k^+$ . Then

$$\begin{aligned} w_k^+(t) &= \frac{1}{a^j b} \sum_{j \geq 0} \left( a^{j/2} \widehat{g}(a^j(t - k/b)) \right) \left( a^{j/2} \widehat{h}(a^j t) \right) \\ &= \frac{1}{b} \sum_{j \geq 0} \widehat{g}(a^j(t - k/b)) \widehat{h}(a^j t). \end{aligned}$$

On the other hand,

$$\begin{aligned} w_k^-(t) &= \frac{a^j}{b} \sum_{j < 0} \sum_{l=0}^{a^{-j}-1} \left( e^{2\pi i b l a^j (t-k/b)} \widehat{g}(a^j(t - k/b)) \right) \left( e^{-2\pi i b l a^j t} \widehat{h}(a^j t) \right) \\ &= \frac{1}{b} \sum_{j < 0} \left( a^j \sum_{l=0}^{a^{-j}-1} e^{-2\pi i l a^j k} \right) \widehat{g}(a^j(t - k/b)) \widehat{h}(a^j t) \\ &= \begin{cases} \frac{1}{b} \sum_{j < 0} \widehat{g}(a^j(t - k/b)) \widehat{h}(a^j t), & \text{if } k \in a^{|j|} \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases} \quad k \in \mathbb{Z}, t \in \mathbb{R}. \end{aligned}$$

In particular, the condition  $w_0 = 1$  is seen to be equivalent to (4.12). Moreover, we find that  $w_m^- = 0$  whenever  $m \in \mathbb{Z} \setminus a\mathbb{Z}$ . Hence,  $w_m = 0$  for all  $m \in \mathbb{Z} \setminus a\mathbb{Z}$  is equivalent to (4.13). It thus remains to show that these relations imply  $w_k = 0$  for all nonzero  $k$ . For this purpose let  $k \in \mathbb{Z} \setminus 0$ , then  $k = a^{-j_0}m$  for some  $j_0 \leq 0$  and  $m \in \mathbb{Z} \setminus a\mathbb{Z}$ . We then compute

$$\begin{aligned} w_k(t) &= w_k^+(t) + \frac{1}{b} \sum_{j=j_0}^{-1} \widehat{g}(a^j(t - k/b)) \widehat{h}(a^j t) \\ &= \frac{1}{b} \sum_{l=0}^{\infty} \widehat{g}(a^l(s - m/b)) \widehat{h}(a^l s), \end{aligned}$$

where we used the substitutions  $l = j - j_0$  and  $s = a^{j_0}t$ . Hence,

$$w_k(t) = w_m(a^{j_0}t),$$

and  $w_m = 0$  implies  $w_k = 0$ . □

Thus, we have obtained a general version of an important result in wavelet theory, the Calderón condition for wavelet frames and bases.

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