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Quasi-interpolation in the Fourier algebra

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Abstract

We derive new convergence results for the Schoenberg operator and more general quasi-interpolation operators. In particular, we prove that natural conditions on the generator function imply convergence of these operators in the Fourier algebra $A(\mathbb{R}^d) = \mathscr{F}L^1(\mathbb{R}^d)$ and in $S_0(\mathbb{R}^d)$, a function space developed by the first author and often used in time-frequency analysis. As a simple yet very useful consequence for applications in Gabor analysis we obtain that piecewise linear interpolation converges in $A(\mathbb{R})$ as well as in $S_0(\mathbb{R})$. Generally, the results presented in this paper are motivated by discretization problems arising in time-frequency analysis and have important consequences in this field.

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1. Introduction

The Schoenberg operator is an important tool for the approximation of continuous functions from uniform samples. Given a continuous function f on $\mathbb{R}^d$, the Schoenberg approximant $Q_h f$, $h > 0$, is a superposition of dilated and shifted versions of a given generator function φ on $\mathbb{R}^d$, using as coefficients the samples of f on the lattice $h\mathbb{Z}^d$,

$$Q_h f(x) := \sum_{k \in \mathbb{Z}^d} f(hk)\varphi(x/h - k), \quad x \in \mathbb{R}^d. \tag{1}$$

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Basic examples, for $d = 1$, are the piecewise linear interpolation and the spline quasi-interpolation; in these examples φ is a B-spline.

The Schoenberg approximation is based on samples of the given function. In a more general type of quasi-interpolation the sampling is replaced by the application of a linear functional, as described next. Let $\mathcal{M}$ denote the Banach space of bounded measures, i.e., bounded linear functionals on C_0, the space of continuous functions vanishing at infinity. The application of $\mu \in \mathcal{M}$ to $f \in C_0$ is written as usual $\int_{\mathbb{R}^d} f(t)\, d\mu(t)$. The Fourier transform $\widehat{\mu}$ is defined by Fourier-Stieltjes integration and it is a bounded, uniformly continuous function [32]. By $f^{[j]}$ we denote the dilation of a function f on $\mathbb{R}^d$ normalized by

$$f^{[j]}(x) := j^{-d} f(x/j), \quad x \in \mathbb{R}^d, \quad j > 0.$$

This notion extends to measures or distributions as usual, $\int_{\mathbb{R}^d} f(t)\, d\mu^{[j]}(t) := \int_{\mathbb{R}^d} f(jt)\, d\mu(t)$. Given $\mu \in \mathcal{M}$, a suitable generator function φ, and $h, j > 0$, the quasi-interpolation operator is defined by

$$Q_{h,j} f(x) := \sum_{k \in \mathbb{Z}^d} \left(\int_{\mathbb{R}^d} f(t + hk)\, d\mu^{[j]}(t) \right) \varphi(x/h - k), \quad x \in \mathbb{R}^d. \tag{2}$$

Note that usually $j = h$, while we allow j to be different from h. If μ is the delta distribution or point measure $\delta \in \mathcal{M}$, then $Q_{h,j}$ is independent of j and reduces to the Schoenberg operator Q_h.

There is by now a quite substantial literature concerning quasi-interpolation in L^2- and L^p-Sobolev spaces, we point to the presentations in the research and survey papers [4,23,24,26]. Our list of references contains a selection of articles in this field most relevant to our approach, with attention to globally supported generators in particular [1,3–12,14,15,22–30,33–37, 40–42]; see also Remark 5.2. Sobolev spaces are the relevant framework for most applications of approximations in shift-invariant spaces. On the other hand, there are standard spaces in harmonic analysis and time-frequency analysis that are not yet included in the theory of quasi-interpolation: the Fourier algebra A and the function space S_0, described below. It is the purpose of the present paper to develop new results that describe convergence $Q_{h,j} f \to f$, as $h, j \to 0$, in A and in S_0. This work is motivated by far-reaching applications in Gabor analysis: our results are one of the crucial technical steps in [31] to derive for the first time the approximation of dual Gabor windows based on the computation of dual Gabor windows on finite groups.

The main results are formulated in Section 2. In Section 3, we derive the approximation results in the Fourier algebra A and Section 4 is devoted to the results for S_0. Section 5 contains additional remarks concerning generators in S_0 and convergence in L^p.

2. Main results

Throughout this paper we assume $1 \leqslant p \leqslant 2$. As usual L^p denotes the Lebesgue space on $\mathbb{R}^d$ with the norm

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^d} |f(t)|^p\, dt \right)^{1/p}.$$

We use the following normalization of the Fourier transform $\mathcal{F}$, for integrable f,

$$\widehat{f}(t) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle t, x \rangle}\, dx, \quad t \in \mathbb{R}^d. \tag{3}$$

The Fourier algebra A is the Banach space consisting of the (inverse) Fourier transforms of the integrable functions. By the Riemann–Lebesgue lemma it is continuously embedded in C_0. Phrased differently, A consists of all $f \in C_0$ whose Fourier transform in the sense of tempered distributions is given by an integrable function,

$$A := \left\{ f \in C_0 \colon \widehat{f} \in L^1 \right\}, \quad \|f\|_A := \|\widehat{f}\|_{L^1}. \tag{4}$$

Note that for general $f \in A$ we cannot compute $\widehat{f}$ by the integral formula above. On the other hand it is not necessary to invoke the distributional version of the Fourier transform, since for $f \in A$ the usual Fejér summability method converges to $\widehat{f}$ in L^1 [32, Section VI.1]. The convolution theorem for L^1 implies that A is a Banach algebra under pointwise multiplication.

By the Hausdorff–Young theorem the Fourier transform extends to a continuous mapping from L^p, $1 \leqslant p \leqslant 2$, into $L^{p'}$, where $2 \leqslant p' \leqslant \infty$ denotes the conjugate exponent, $1/p + 1/p' = 1$. We define for $1 \leqslant p \leqslant 2$,

$$\mathcal{F}L^p := \left\{ f \in L^{p'} \colon \widehat{f} \in L^p \right\}, \quad \|f\|_{\mathcal{F}L^p} := \|\widehat{f}\|_{L^p}, \tag{5}$$

and this class of spaces includes $A = \mathcal{F}L^1$ and by Plancherel's theorem $L^2 = \mathcal{F}L^2$. The Fourier transform $\widehat{f} \in L^p$, for $f \in \mathcal{F}L^p$, can be computed by using Fejér means [32, Section VI.3] as mentioned above for $p = 1$. Many results in harmonic analysis involve the Fourier algebra A and the more general class of spaces $\mathcal{F}L^p$ in an explicit or implicit way [32,39].

The function space S_0 is a Fourier-invariant Banach space that is dense in L^2 and whose members are continuous and integrable functions. It is an important space in time-frequency analysis, introduced in [13] and nowadays called Feichtinger's algebra [20,39]. It can be characterized by the integrability of the short-time Fourier transform of its elements; the short-time Fourier transform is a standard representation of f on the time-frequency plane or phase space [20]. For our results it is convenient to view S_0 as a subspace of A, defined by

$$S_0 := \left\{ f \in A \colon \|f\|_{S_0} < \infty \right\}, \quad \|f\|_{S_0} := \int_{\mathbb{R}^d} \|f \cdot T_x g\|_A \, dx, \tag{6}$$

where $T_x g(t) := g(t - x)$ and g is a suitable window function, such as the standard Gaussian window $g(t) := e^{-\pi \|t\|^2}$. An equivalent norm is the discrete version

$$\|f\|'_{S_0} := \sum_{k \in \mathbb{Z}^d} \|f \cdot T_k g\|_A, \tag{7}$$

here g can be any function in S_0 such that $\sum_{k \in \mathbb{Z}^d} g(x - k) = 1$. By using Lemma 5.1 it can be shown that the condition on g can be weakened to $\sum_{k \in \mathbb{Z}^d} g(x - k) \neq 0$, for all $x \in [0, 1]^d$. The discrete norm in fact resembles the first author's original construction of S_0 as the Wiener amalgam space $W(A, \ell^1)$, locally A and globally ℓ^1, see [13,14,19]. The space S_0 is a Banach algebra both under convolution and pointwise multiplication. For various useful characterizations of S_0 and its significance in time-frequency analysis, see [16–21,39] and the original source [13].

Our first main result describes convergence of the quasi-interpolation scheme in the Fourier algebra $A = \mathcal{F}L^1$. The theorem is formulated for convergence in the more general class of $\mathcal{F}L^p$-norms with $1 \leqslant p \leqslant 2$.

Theorem 2.1. *Let* $1 \leqslant p \leqslant 2$, *suppose* $\varphi \in S_0$ *satisfies* $\widehat{\varphi}(k) = \delta_{k,0}$, *for* $k \in \mathbb{Z}^d$, *and let* $\mu \in \mathcal{M}$ *such that* $\widehat{\mu}(0) = 1$. *Then for* $f \in \mathcal{F}L^p \cap A$,

$$\|f - Q_{h,j} f\|_{\mathcal{F}L^p} \to 0 \quad \text{as } h, j \to 0.$$

Observe that the cases $p = 1$ and 2 imply in particular (cf. Remark 5.2):

- If $f \in C_0$ satisfies $\widehat{f} \in L^1$, then $Q_{h,j} f \to f$ uniformly.
- If $f \in L^2$ satisfies $\widehat{f} \in L^1$, then $Q_{h,j} f \to f$ uniformly and in L^2.

Theorem 2.1 is derived in Section 3 as a generalization of the corresponding result for the Schoenberg operator (Corollary 3.4).

The constructions for the Fourier algebra A are essential for developing our results for S_0. The main result is the following.

Theorem 2.2. *Let* $\varphi \in S_0$ *satisfy* $\widehat{\varphi}(k) = \delta_{k,0}$, *for* $k \in \mathbb{Z}^d$. *Let* $\mu \in \mathcal{M}$ *such that* $\widehat{\mu}(0) = 1$. *Then for all* $f \in S_0$,

$$\|f - Q_{h,j} f\|_{S_0} \to 0 \quad \text{as } h, j \to 0.$$

Theorem 2.2 is proved in Section 4 as a generalization of the corresponding result for the Schoenberg approximation (Theorem 4.6). Note that the classic Schoenberg operator can be used since S_0 consists of continuous functions. As a simple but useful consequence we obtain the following corollary.

Corollary 2.3. *Given* $f \in S_0(\mathbb{R})$ *and* $h > 0$, *let* f_h *denote the piecewise linear interpolant to the sequence* $\{(hk, f(hk))\}_{k \in \mathbb{Z}}$. *Then* $\|f - f_h\|_{S_0} \to 0$, *as* $h \to 0$.

Corollary 2.3 follows directly from Theorem 4.6 (Section 4), cf. Section 5.

3. Approximation in the Fourier algebra

Our first topic is convergence in $\mathcal{F}L^p$, $1 \leqslant p \leqslant 2$, with attention to the Fourier algebra $A = \mathcal{F}L^1$. Since in our results for $\mathcal{F}L^p$ we use generators φ from S_0, we list now the following basic properties of S_0 that are relevant for our work. For references see [13,19,20].

- The Fourier transform is an isometry on S_0.
- Translation $f(t) \mapsto f(t-x)$ and modulation $f(t) \mapsto e^{2\pi i \omega t} f(t)$, for $x, \omega \in \mathbb{R}^d$, are isometries on S_0.
- The restriction mapping $f \mapsto (f(k))_{k \in \mathbb{Z}^d}$ is a bounded operator from S_0 onto $\ell^1(\mathbb{Z}^d)$. In fact, there exists $C > 0$ such that for all $f \in S_0$,

$$\sum_{k \in \mathbb{Z}^d} \max_{x \in [0,1]^d} |f(x-k)| \leqslant C \|f\|_{S_0}. \tag{8}$$

We note that for $f \in S_0$, the series $F(x) := \sum_{k \in \mathbb{Z}^d} f(x-k)$, $x \in \mathbb{R}^d$, converges absolutely in x and uniformly on compact sets to a bounded and continuous periodic function. It is also useful to point out the fact that the Poisson summation formula $\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k)$ holds for $f \in S_0$ with absolute convergence of the two series.

3.1. Schoenberg approximation in A

First we note that the Schoenberg approximant $Q_h f$ is indeed well-defined for $f \in A$ and belongs to A. The Fourier transformed version of the Schoenberg operator Q_h is well known and often used in the theory of shift-invariant systems, under various conditions on φ. In the next lemma we also point out that the transition to the Fourier transform works for $\varphi \in S_0$ and $f \in A$ without additional constraints.

Lemma 3.1. *Let $\varphi \in S_0$, and $h > 0$. Then for $f \in A$ the following hold*:

(i) *The Schoenberg approximant $Q_h f$ belongs to A; the series* (1) *converges unconditionally uniformly*.

(ii) *The Fourier transform of $Q_h f$ is given by the formula*

$$\widehat{Q_h f}(t) = \sum_{k \in \mathbb{Z}^d} \widehat{\varphi}(ht) \widehat{f}(t - k/h) \quad a.e.\ t \in \mathbb{R};$$

the series converges absolutely a.e., in fact absolutely in L^1.

Proof. Since all spaces involved are dilation invariant we assume $h = 1$.

(i) Let $I_1 \subsetneq I_2 \subsetneq \cdots$ be subsets of $\mathbb{Z}^d$. The convergence properties of the series $\sum_{k \in \mathbb{Z}^d} f(k) T_k \varphi$ follow since

$$\sum_{k \in \mathbb{Z}^d \setminus I_n} |f(k)\varphi(x-k)| \leqslant \underbrace{\sup_{k \in \mathbb{Z}^d \setminus I_n} |f(k)|}_{\to 0} \underbrace{\sum_{k \in \mathbb{Z}^d \setminus I_n} |\varphi(x-k)|}_{\leqslant C \|\varphi\|_{S_0}}, \quad x \in \mathbb{R}^d,$$

as $n \to \infty$, and $Q_h f$ belongs to A since $\widehat{Q_h f} \in L^1$ by (ii).

(ii) The formula for the Fourier transform is well known and easily verified by using the Poisson summation formula and standard properties of the Fourier transform, cf. [2]. To verify absolute convergence in L^1 note that using (8) and the Fourier invariance of S_0 we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \left|\widehat{\varphi}(t) \widehat{f}(t-k)\right| dt &= \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \left|\widehat{\varphi}(t+k) \widehat{f}(t)\right| dt \\ &\leqslant \underbrace{\sup_{t \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} |\widehat{\varphi}(t+k)|}_{\leqslant C\|\varphi\|_{S_0}} \underbrace{\int_{\mathbb{R}^d} |\widehat{f}(t)|\, dt}_{= \|f\|_A}. \qquad \square \end{aligned}$$

Let $s \geqslant 0$. The Landau big-O symbol $f(h) = O(h^s)$, as $h \to 0$, means there exist $C > 0$ and $\delta > 0$ such that $f(h) \leqslant C\, h^s$ for $|h| \leqslant \delta$. The little-o symbol $f(h) = o(h^s)$, as $h \to 0$, means $h^{-s} f(h) \to 0$, as $h \to 0$.

A characterization of $O(h^s)$-convergence of the Schoenberg operator for smooth functions in L^2 is given in [25]. Several results for $O(h^s)$-convergence in C_0 are found in the literature. Sometimes these are implicitly concerned with the Fourier algebra, e.g., [40], using the fact that A is continuously embedded into C_0. We also note that many important results for the Schoenberg operator are found implicitly in the theory of shift-invariant systems, such as [5,6], mostly in order to verify lower bounds for the approximation order or density order.

While our conditions for $O(h^s)$-convergence in $\mathcal{F}L^p$ are straight-forward from the arguments for $L^2 = \mathcal{F}L^2$ in [25], we are particularly interested in $o(h^s)$-convergence. More precisely, later we elaborate on the case $s = 0$, i.e., convergence without a prescribed rate. Our specific goal is to obtain practicable sufficient conditions on the generator function φ that ensure convergence in the Fourier algebra A, such as formulated in Corollary 3.4.

In our next result we use smooth versions of the $\mathcal{F}L^p$ spaces, By L^p_s denote the weighted L^p-spaces with norm

$$\|f\|_{L^p_s} := \left(\int_{\mathbb{R}^d} \left|f(t)(1+|t|)^s\right|^p dt\right)^{1/p}, \quad s \geqslant 0, \quad 1 \leqslant p \leqslant 2.$$

Define the refined scale of $\mathcal{F}L^p_s$ spaces with parameter $s \geqslant 0$,

$$\mathcal{F}L^p_s := \left\{ f \in L^{p'} : \widehat{f} \in L^p_s \right\}, \quad \|f\|_{\mathcal{F}L^p_s} := \|\widehat{f}\|_{L^p_s}.$$

Note that $L^p_s \subseteq L^p$ and hence $\mathcal{F}L^p_s \subseteq \mathcal{F}L^p$. While s can be seen as a smoothness parameter in a similar way as used in fractional Sobolev spaces, we note that $\mathcal{F}L^p_s$ is not a fractional Sobolev space (except for $p = 2$), since the L^p-norm is taken in the Fourier domain. Thus, the smoothness properties are of a different nature than those usually discussed in the literature.

Given $1 \leqslant p \leqslant 2$, for $\widehat{\varphi} \in L^p$ define

$$\Lambda_p(t) := \left(|1 - \widehat{\varphi}(t)|^p + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\varphi}(t+k)|^p \right)^{1/p}, \quad t \in \mathbb{R}^d.$$

The definition of Λ_p is a suitable modification for general $1 \leqslant p \leqslant 2$ of the case $p = 2$ treated in [25, Theorem 3]. We use the symbol Λ_p in order to indicate the close relation to the function Λ_φ used in the fundamental and important results of [5], cf. the discussion in [25, Section 1].

We exclusively use generators φ from S_0. For this class of generators the function Λ_p is always bounded and continuous. In fact, for $\varphi \in S_0$ the series defining Λ_p converges uniformly on compact sets and

$$\|\Lambda_p\|_{L^\infty} \leqslant 1 + C\|\varphi\|_{S_0}, \tag{9}$$

as follows from the next estimate, based on (8),

$$\left(\sum_{k \in \mathbb{Z}^d} \max_{t \in [0,1]^d} |\widehat{\varphi}(t+k)|^p \right)^{1/p} \leqslant \sum_{k \in \mathbb{Z}^d} \max_{t \in [0,1]^d} |\widehat{\varphi}(t+k)| \leqslant C\|\widehat{\varphi}\|_{S_0} = C\|\varphi\|_{S_0}.$$

Our first key result is the following.

Theorem 3.2. *Suppose $1 \leqslant p \leqslant 2$ and $s \geqslant 0$. Let $\varphi \in S_0$ such that Λ_p satisfies*

$$\Lambda_p(t) = \left\{ \begin{array}{c} O(|t|^s) \\ o(|t|^s) \end{array} \right\} \quad \text{as } |t| \to 0. \tag{10}$$

Then for $f \in \mathcal{F}L^p_s \cap A$,

$$\|f - Q_h f\|_{\mathcal{F}L^p} = \left\{ \begin{array}{c} O(h^s) \\ o(h^s) \end{array} \right\} \quad \text{as } h \to 0. \tag{11}$$

Proof. Since $\varphi \in S_0$ and $f \in A$, we calculate by Lemma 3.1(ii), using $|(a+b)/2|^p \leqslant (|a|^p + |b|^p)/2$ for $p \geqslant 1$,

$$\begin{aligned}
&\|\widehat{f} - \widehat{Q_h f}\|_{L^p}^p \\
&= \int_{\mathbb{R}^d} \left| \widehat{f}(t) - \widehat{\varphi}(ht) \sum_{k\in\mathbb{Z}^d} \widehat{f}(t-k/h) \right|^p dt \\
&\leqslant 2^{p-1} \left(\int_{\mathbb{R}^d} |1-\widehat{\varphi}(ht)|^p \cdot \left|\widehat{f}(t)\right|^p dt + \sum_{k\neq 0} \int_{\mathbb{R}^d} |\widehat{\varphi}(ht)|^p \cdot \left|\widehat{f}(t-k/h)\right|^p dt \right) \\
&= 2^{p-1} \left(\int_{\mathbb{R}^d} |1-\widehat{\varphi}(ht)|^p \cdot \left|\widehat{f}(t)\right|^p dt + \sum_{k\neq 0} \int_{\mathbb{R}^d} |\widehat{\varphi}(ht+k)|^p \cdot \left|\widehat{f}(t)\right|^p dt \right) \\
&= 2^{p-1} \int_{\mathbb{R}^d} \left(|1-\widehat{\varphi}(ht)|^p + \sum_{k\neq 0} |\widehat{\varphi}(ht+k)|^p \right) \cdot \left|\widehat{f}(t)\right|^p dt \\
&= 2^{p-1} \int_{\mathbb{R}^d} \left(\Lambda_p(ht)\right)^p \left|\widehat{f}(t)\right|^p dt,
\end{aligned}$$

that is, with $c := 2^{1-1/p}$,

$$\|\widehat{f} - \widehat{Q_h f}\|_{L^p} \leqslant c \|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p}. \tag{12}$$

Now assume that $\Lambda_p(t) = O(|t|^s)$. Since in addition Λ_p is bounded, we have for some $C > 0$,

$$0 \leqslant \Lambda_p(t) \leqslant C|t|^s, \quad t \in \mathbb{R}^d. \tag{13}$$

Therefore, by (12) and (13),

$$\begin{aligned}
\|\widehat{f} - \widehat{Q_h f}\|_{L^p} &\leqslant c\|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p} \\
&\leqslant c h^s \left\|(h\cdot)^{-s}\Lambda_p(h\cdot)\right\|_{L^\infty} \left\|(\cdot)^s \widehat{f}(\cdot)\right\|_{L^p} \\
&\leqslant c h^s \underbrace{\left\|(\cdot)^{-s}\Lambda_p(\cdot)\right\|_{L^\infty}}_{\leqslant C} \underbrace{\left\|(\cdot)^s \widehat{f}(\cdot)\right\|_{L^p}}_{\leqslant \|\widehat{f}\|_{L^p_s}} = O(h^s).
\end{aligned}$$

Next, assume that $\Lambda_p(t) = o(|t|^s)$. In this case we split

$$\begin{aligned}
\|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p} \leqslant\ & \|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p\left(\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} \\
&+ \|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p\left(\mathbb{R}^d \setminus \left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)}.
\end{aligned} \tag{14}$$

We compute

$$\begin{aligned}
&\|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p\left(\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} \\
&\leqslant h^s \left\|(h\cdot)^{-s}\Lambda_p(h\cdot)\right\|_{L^\infty\left(\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} \left\|(\cdot)^s\widehat{f}(\cdot)\right\|_{L^p\left(\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} \\
&\leqslant h^s \underbrace{\left\|(\cdot)^{-s}\Lambda_p(\cdot)\right\|_{L^\infty\left(\left[-\sqrt{h},\sqrt{h}\right]^d\right)}}_{\to 0} \underbrace{\left\|(\cdot)^s\widehat{f}(\cdot)\right\|_{L^p\left(\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)}}_{\leqslant \|\widehat{f}\|_{L^p_s}} = o(h^s) \qquad (15)
\end{aligned}$$

and, using (13),

$$\begin{aligned}
&\|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p\left(\mathbb{R}^d\setminus\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} \\
&\leqslant h^s \left\|(h\cdot)^{-s}\Lambda_p(h\cdot)\right\|_{L^\infty\left(\mathbb{R}^d\setminus\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} \left\|(\cdot)^s\widehat{f}(\cdot)\right\|_{L^p\left(\mathbb{R}^d\setminus\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} \\
&\leqslant h^s \underbrace{\left\|(\cdot)^{-s}\Lambda_p(\cdot)\right\|_{L^\infty\left(\mathbb{R}^d\setminus\left[-\sqrt{h},\sqrt{h}\right]^d\right)}}_{\leqslant C} \underbrace{\left\|(\cdot)^s\widehat{f}(\cdot)\right\|_{L^p\left(\mathbb{R}^d\setminus\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)}}_{\to 0} = o(h^s). \quad (16)
\end{aligned}$$

Hence, combining (12) and (14) with (15) and (16) we obtain

$$\begin{aligned}
\|\widehat{f}-\widehat{Q_h f}\|_{L^p} &\leqslant c\|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p\left(\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} + c\|\Lambda_p(h\cdot)\widehat{f}(\cdot)\|_{L^p\left(\mathbb{R}^d\setminus\left[-1/\sqrt{h},1/\sqrt{h}\right]^d\right)} \\
&= o(h^s)+o(h^s)=o(h^s). \qquad \square
\end{aligned}$$

Remark 3.3. (i) The additional assumption $f \in A$ in (11) cannot be removed. In fact, a function in $\mathcal{F}L^p_s$ is not continuous in general. On the other hand, if the smoothness parameter s is sufficiently large, $s > (1-1/p)d$, then $L^p_s \subset L^1$ and thus $\mathcal{F}L^p_s \subset A$.

(ii) The condition on φ in Theorem 3.2 is of a different form than usual versions of the Strang–Fix conditions, which are formulated in terms of derivatives of $\widehat{\varphi}$. Indeed in order to satisfy (10) the Fourier transform $\widehat{\varphi}$ need not be differentiable. For bandlimited φ the condition (10) takes an especially simple form. That is, if $\operatorname{supp}\widehat{\varphi}$ is compact, then (10) reduces to

$$\widehat{\varphi}(t) = 1 + \left\{\begin{array}{c} O(|t|^s) \\ o(|t|^s) \end{array}\right\} \quad \text{and} \quad \widehat{\varphi}(t+k) = \left\{\begin{array}{c} O(|t|^s) \\ o(|t|^s) \end{array}\right\}, \quad \text{for } k \neq 0 \text{ as } |t| \to 0.$$

We obtain the following specialized version of Theorem 3.2 for the case of convergence without a prescribed rate, i.e., the case of $o(h^s)$ convergence with $s=0$.

Corollary 3.4. *Let $1 \leqslant p \leqslant 2$ and suppose $\varphi \in S_0$ satisfies $\widehat{\varphi}(k) = \delta_{k,0}$, for $k \in \mathbb{Z}^d$. Then for $f \in \mathcal{F}L^p \cap A$,*

$$\|f - Q_h f\|_{\mathcal{F}L^p} \to 0 \quad \textit{as } h \to 0.$$

Proof. Since $\varphi \in S_0$ implies that Λ_p is bounded and continuous, the assumption $\widehat{\varphi}(k) = \delta_{k,0}$ implies $\Lambda_p(0) = 0$, so by continuity $\Lambda_p(t) = o(1)$, as $|t| \to 0$. Thus the $o(h^s)$-part of Theorem 3.2 with $s = 0$ implies $Q_h f \to f$ in $\mathcal{F}L^p$. $\square$

3.2. Proof of Theorem 2.1

The proofs of our results for $Q_{h,j}$ make use of the interesting fact that the definition of $Q_{h,j}$ in (2) can be read also in the following way, replacing $\mu(x) \to \mu(-x)$ if necessary,

$$Q_{h,j} f(x) = Q_h(f * \mu^{[j]})(x), \quad x \in \mathbb{R}^d. \tag{17}$$

Here Q_h is the Schoenberg operator and the asterisk denotes the usual convolution of a function with a measure,

$$(f * \mu^{[j]})(x) = \int_{\mathbb{R}^d} f(x-t)\, d\mu^{[j]}(t), \quad x \in \mathbb{R}^d.$$

The next lemma describes the action of approximate units with respect to convolution, on elements from $\mathcal{F}L^p$.

Lemma 3.5. *Let $1 \leqslant p \leqslant 2$, suppose $\mu \in \mathcal{M}$ satisfies $\widehat{\mu}(0) = 1$, and let $f \in \mathcal{F}L^p$. Then $\|f * \mu^{[j]} - f\|_{\mathcal{F}L^p} \to 0$, as $j \to 0$.*

Proof. First, we have $\widehat{\mu^{[j]}}(t) = \widehat{\mu}(jt) \to \widehat{\mu}(0) = 1$, as $j \to 0$, uniformly on compact sets. Secondly, $\widehat{\mu^{[j]}}(\cdot) = \widehat{\mu}(j\cdot)$ is uniformly bounded by $\|\mu\|_{\mathcal{M}} = \int_{\mathbb{R}^d} |d\mu(t)|$. We conclude that $\widehat{f} \in L^p$ implies

$$\widehat{f * \mu^{[j]}} = \widehat{f} \cdot \widehat{\mu^{[j]}} \to \widehat{f} \quad \text{in } L^p \quad \text{as } j \to 0. \qquad \square$$

Next we observe the following boundedness property.

Lemma 3.6. *There exists $C > 0$ such that for $\varphi \in S_0$ and $f \in \mathcal{F}L^p \cap A$, with $1 \leqslant p \leqslant 2$,*

$$\|Q_h f\|_{\mathcal{F}L^p} \leqslant C \|\varphi\|_{S_0} \|f\|_{\mathcal{F}L^p} \quad \textit{for all } h > 0.$$

Proof. Using (9) and (12) we calculate

$$\|\widehat{f} - \widehat{Q_h f}\|_{L^p} \leqslant c \|\Lambda_p\|_{L^\infty} \|\widehat{f}\|_{L^p} \leqslant c(1 + C' \|\varphi\|_{S_0}) \|\widehat{f}\|_{L^p}.$$

The fact that $Q_h f = Q_h^\varphi f$ is bilinear in φ and f thus implies

$$\|\widehat{Q_h f}\|_{L^p} \leqslant C \|\varphi\|_{S_0} \|\widehat{f}\|_{L^p}. \qquad \square$$

Proof of Theorem 2.1. First, we note that (17) and the fact that $f * \mu^{[j]} \in \mathcal{F}L^p$, cf. Lemma 3.5, imply that the convergence properties of the series defining $Q_h f$ (Lemma 3.1) also hold for the series defining $Q_{h,j} f$. In particular the quasi-interpolants belong to $\mathcal{F}L^p$. Combining (17), Corollary 3.4, Lemmas 3.5 and 3.6 we thus obtain

$$\begin{aligned} \|f - Q_{h,j} f\|_{\mathcal{F}L^p} &= \left\| f - Q_h(f * \mu^{[j]}) \right\|_{\mathcal{F}L^p} \\ &\leqslant \|f - Q_h f\|_{\mathcal{F}L^p} + \left\| Q_h(f - f * \mu^{[j]}) \right\|_{\mathcal{F}L^p} \\ &\leqslant \|f - Q_h f\|_{\mathcal{F}L^p} + C \|f - f * \mu^{[j]}\|_{\mathcal{F}L^p} \to 0 \quad \text{as } h, j \to 0. \qquad \square \end{aligned}$$

4. Approximation in S_0

4.1. Schoenberg approximation in S_0

In this section, we describe convergence of the Schoenberg operator in S_0. These results are based on a new approach: We view the Schoenberg approximation as a limiting case of a family of sampling theorems. First, we record that the approximants $Q_h f$ are indeed well-defined and belong to S_0, for $f \in S_0$. Note the stronger convergence properties compared with Lemma 3.1.

Lemma 4.1. *Let $\varphi \in S_0$, and $h > 0$. Then for $f \in S_0$ the following hold*:

(i) *The quasi-interpolant $Q_{h,j} f$ belongs to S_0; the series* (2) *converges absolutely in S_0.*

(ii) *The Fourier transform of $Q_h f$ is given by the formula*

$$\widehat{Q_h f}(t) = \sum_{k \in \mathbb{Z}^d} \widehat{\varphi}(ht) \widehat{f}(t - k/h), \quad t \in \mathbb{R};$$

and the series on the right-hand side converges absolutely in S_0.

Proof. As in the proof of Lemma 3.1 we assume $h = 1$.

(i) Absolute convergence of the series $\sum_{k \in \mathbb{Z}^d} f(k) T_k \varphi$ follows since

$$\sum_{k \in \mathbb{Z}^d} \| f(k) T_k \varphi \|_{S_0} \leqslant \underbrace{\sum_{k \in \mathbb{Z}^d} | f(k)|}_{\leqslant C \| f \|_{S_0}} \underbrace{\| T_k \varphi \|_{S_0}}_{\| \varphi \|_{S_0}}.$$

(ii) Absolute convergence of the series $\sum_{k \in \mathbb{Z}^d} (\widehat{\varphi} \cdot T_k \widehat{f})$ is in fact a key property of S_0, cf. (7),

$$\sum_{k \in \mathbb{Z}^d} \| \widehat{\varphi} \cdot T_k \widehat{f} \|_{S_0} \leqslant C \| \widehat{\varphi} \|_{S_0} \| \widehat{f} \|_{S_0} = C \| \varphi \|_{S_0} \| f \|_{S_0}. \qquad \square$$

The next lemma is concerned with special classes of generators. The proof makes use of spectral synthesis in the Banach algebra S_0.

Lemma 4.2. *Define*

$$\Phi := \left\{ \varphi \in S_0 \colon \widehat{\varphi}(k) = \delta_{k,0},\ k \in \mathbb{Z}^d \right\} \quad \textit{and}$$

$$\Phi^\circ := \left\{ \varphi \in S_0 \colon \widehat{\varphi}|_{k + [-\varepsilon, \varepsilon]^d} = \delta_{k,0},\ k \in \mathbb{Z}^d,\ \textit{for some } \varepsilon > 0 \right\}.$$

Then Φ° is dense in Φ in the S_0-norm. In fact, Φ is the closure of Φ° in S_0.

Proof. First, let

$$I := \left\{ f \in S_0 \colon f(k) = 0,\ k \in \mathbb{Z}^d \right\} \quad \text{and}$$

$$I^\circ := \left\{ f \in S_0 \colon f|_{k+[-\varepsilon,\varepsilon]^d} = 0,\ k \in \mathbb{Z}^d,\ \text{for some } \varepsilon > 0 \right\}. \tag{18}$$

We show that I° is dense in I in the S_0-norm; in fact, I is the closure of I° in S_0. This observation follows directly from the fact that $\mathbb{Z}^d$ is a set of spectral synthesis for the Fourier algebra [39, Theorem 2.4.16 or Corollary 6.1.8] in combination with the ideal theorem for Segal algebras [39, Theorem 6.2.9]. In fact, let $I^{\circ\circ}$ denote the set of those $f \in I^\circ$ with compact support and let I_1 be the closure of $I^{\circ\circ}$ in S_0. The ideal theorem implies that the closed ideals I and I_1 in S_0 coincide if and only if their closures in A are equal. This is valid, because $\mathbb{Z}^d$ is a closed subgroup of $\mathbb{R}^d$ and hence a set of spectral synthesis, and therefore there is only one closed ideal with cospectrum $\operatorname{cosp}(I) = \mathbb{Z}^d$.

Next, let $f \in A$ such that $\operatorname{supp} f \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ and $f(x) = 1$ for $x \in [-\frac{1}{4}, \frac{1}{4}]^d$, for example, a tensor product trapezoidal function. Since f is compactly supported in A it belongs to S_0 and it allows us to write $\Phi = \mathcal{F}(f + I)$ and $\Phi^\circ = \mathcal{F}(f + I^\circ)$, for I and I° given in (18). Since I is the closure of I° in S_0 and the Fourier transform is an isometry on S_0, we conclude that Φ is the closure of Φ° in S_0. □

Remark 4.3. We note that functions in Φ exist in abundance, see Section 5.

Given a generator $\varphi \in \Phi^\circ$, the approximation $Q_h f \to f$ is, in fact, replaced by a sampling theorem for bandlimited functions, that is, $Q_h f = f$, for sufficiently small $h > 0$.

Lemma 4.4. *Let $\varphi \in \Phi^\circ$ and let $f \in L^1$ such that* $\operatorname{supp} \widehat{f}$ *is compact. Then there exists $h_0 > 0$ such that $Q_h f = f$, for all $h \leqslant h_0$.*

Proof. We have $\widehat{\varphi}|_{k+[-a,a]^d} = \delta_{k,0}$, for some $a > 0$, and $\operatorname{supp} \widehat{f} \subseteq [-r, r]$, for some $r > 0$. Using Lemma 4.1(ii) we thus obtain for $h \leqslant h_0 := a/r$,

$$\begin{aligned}\widehat{Q_h f}(t) &= \widehat{\varphi}(ht) \sum_{k \in \mathbb{Z}^d} \widehat{f}(t - k/h) \\ &= \underbrace{\widehat{\varphi}(ht) \widehat{f}(t)}_{=\widehat{f}(t)} + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \underbrace{\widehat{\varphi}(ht) \widehat{f}(t - k/h)}_{=0} = \widehat{f}(t), \quad t \in \mathbb{R}^d. \qquad \square\end{aligned}$$

The next result is crucial for our approach. It shows the uniform boundedness of the Schoenberg operator on S_0, for generators $\varphi \in S_0$. The proof is based on an atomic decomposition argument, cf. the original construction of S_0 in [13].

Lemma 4.5. *There exists $C > 0$ such that for $\varphi, f \in S_0$,*

$$\|Q_h f\|_{S_0} \leqslant C \|\varphi\|_{S_0} \|f\|_{S_0} \quad \textit{for } h \leqslant 1.$$

Proof. Let $g \in S_0$ such that $\operatorname{supp} g \subseteq [-1,1]^d$ and $\sum_{k\in\mathbb{Z}^d} T_k g = 1$. For example, a suitable tensor product trapezoidal function belongs to S_0. For $k \in \mathbb{Z}^d$, define $f_k := f \cdot T_k g$ and $\varphi_k := \varphi \cdot T_k g$. Then

$$f = \sum_{k\in\mathbb{Z}^d} f_k \quad \text{and} \quad \varphi = \sum_{k\in\mathbb{Z}^d} \varphi_k$$

with $\operatorname{supp} f_k \subseteq k + [-1,1]^d$ and $\operatorname{supp} \varphi_k \subseteq k + [-1,1]^d$. Since the discrete norm (7) with g as above is an equivalent norm, there exists $C_1 > 0$ such that

$$\sum_{k\in\mathbb{Z}^d} \|f_k\|_A \leqslant C_1 \|f\|_{S_0}, \qquad \sum_{k\in\mathbb{Z}^d} \|\varphi_k\|_A \leqslant C_1 \|\varphi\|_{S_0}. \tag{19}$$

Since φ_l and f_k have compact support, so does $Q_h^{\varphi_l} f_k$, for $k, l \in \mathbb{Z}^d$, indeed

$$\operatorname{supp} Q_h^{\varphi_l} f_k \subseteq \operatorname{supp} f_k + h \operatorname{supp} \varphi_l. \tag{20}$$

Assuming $h \leqslant 1$ we thus have $\operatorname{supp} Q_h^{\varphi_l} f_k \subseteq k + l + [-2,2]^d$. For the subset of functions in S_0 whose supports do not exceed a fixed diameter, the S_0-norm is equivalent to the A-norm. Hence, there exists $C_2 > 0$ such that

$$\|Q_h^{\varphi_l} f_k\|_{S_0} \leqslant C_2 \|Q_h^{\varphi_l} f_k\|_A. \tag{21}$$

Since $Q_h f = Q_h^{\varphi} f$ is bilinear in φ and f, the lemma now follows by combining (19), (21), and the uniform boundedness of Q_h on A (Lemma 3.6 with $p = 1$),

$$\begin{aligned}
\|Q_h^{\varphi_l} f_k\|_{S_0} = \left\| \sum_{k,l\in\mathbb{Z}^d} Q_h^{\varphi_l} f_k \right\|_{S_0} &\leqslant \sum_{k,l\in\mathbb{Z}^d} \|Q_h^{\varphi_l} f_k\|_{S_0} \\
&\leqslant C_2 \sum_{k,l\in\mathbb{Z}^d} \|Q_h^{\varphi_l} f_k\|_A \\
&\leqslant C_2 C' \sum_{k,l\in\mathbb{Z}^d} \|f_k\|_A \|\varphi_l\|_A \\
&\leqslant C_2 C' C_1^2 \|f\|_{S_0} \|\varphi\|_{S_0}. \qquad \square
\end{aligned}$$

Theorem 4.6. *Suppose* $\varphi \in S_0$ *satisfies* $\widehat{\varphi}(k) = \delta_{k,0}$, *for* $k \in \mathbb{Z}^d$. *Then for* $f \in S_0$,

$$\|f - Q_h f\|_{S_0} \to 0 \quad \text{as } h \to 0.$$

Proof. By Lemma 4.5 we have that Q_h, viewed as a bilinear operator

$$Q_h : S_0 \times S_0 \to S_0, \quad (\varphi, f) \mapsto Q_h^{\varphi} f := Q_h f,$$

is uniformly bounded, for $h \leqslant 1$. Now Φ° is dense in Φ by Lemma 4.2 and the set of all $f \in L^1$ with compactly supported Fourier transform is dense in S_0 [13,19]. Therefore, we find $\varphi_1 \in \Phi^\circ$ and $f_1 \in L^1$, $\operatorname{supp} \widehat{f}$ compact, such that for all $h \leqslant 1$,

$$\|f - Q_h^{\varphi} f\|_{S_0} \leqslant \|f_1 - Q_h^{\varphi_1} f_1\|_{S_0} + \varepsilon.$$

Applying now Lemma 4.4 to φ_1 and f_1 we find some $h_0 > 0$ such that the non-trivial term in the above estimate vanishes for $h \leqslant h_0$ and thus the claim is verified. $\square$

4.2. Proof of Theorem 2.2

The following lemma describes the action of approximate units with respect to convolution, on elements from S_0, cf. Lemma 3.5. For $\mu \in L^1$, the result is given in [19, Lemma 3.2.18(i)], see also [32, Example VI.1.14] and [38, §8, Proposition 1]. For the extension to general $\mu \in \mathcal{M}$ proved next we note that it is known that $\|f * \mu\|_{S_0} \leqslant \|f\|_{S_0}\|\mu\|_{\mathcal{M}}$ [38, §4, Proposition 2; 19, Theorem 3.2.3(iii)].

Lemma 4.7. *Let $\mu \in \mathcal{M}$ such that $\widehat{\mu}(0) = 1$ and let $f \in S_0$. Then $\|f * \mu^{[j]} - f\|_{S_0} \to 0$, as $j \to 0$.*

Proof. Let $\varepsilon > 0$. Since μ is a bounded measure, we can decompose $\mu = \mu_0 + \mu_1$ such that $\operatorname{supp} \mu_0 \subseteq I$, for some compact subset $I \subset \mathbb{Z}^d$, and $\|\mu_1\| \leqslant \varepsilon$. Since $\int_{\mathbb{R}^d} d\mu^{[j]}(x) = 1$,

$$
\begin{aligned}
&\|f * \mu^{[j]} - f\|_{S_0} \\
&\quad = \left\| \int_{\mathbb{R}^d} (f(x-\cdot) - f(\cdot))\, d\mu^{[j]}(x) \right\|_{S_0} \\
&\quad \leqslant \left\| \int_{jI} (f(x-\cdot) - f(\cdot))\, d\mu^{[j]}(x) \right\|_{S_0} + \left\| \int_{\mathbb{R}^d \setminus jI} (f(x-\cdot) - f(\cdot))\, d\mu_1^{[j]}(x) \right\|_{S_0} \\
&\quad \leqslant \underbrace{\sup_{x \in jI} \|f(x-\cdot) - f(\cdot)\|_{S_0}}_{\to 0} \underbrace{\int_{jI} d\left|\mu^{[j]}\right|(x)}_{\substack{\int_I d|\mu|(x) \\ \leqslant \|\mu\|_{\mathcal{M}}}} \\
&\qquad + \underbrace{\sup_{x \in \mathbb{R}^d \setminus jI} \|f(x-\cdot) - f(\cdot)\|_{S_0}}_{\leqslant 2\|f\|_{S_0}} \underbrace{\int_{\mathbb{R}^d \setminus jI} d\left|\mu_1^{[j]}\right|(x)}_{\substack{\int_{\mathbb{R}^d \setminus I} d|\mu_1|(x) \\ \leqslant \varepsilon}}. \qquad \square
\end{aligned}
$$

Remark 4.8. The proof of Lemma 4.7 shows that the same arguments apply to general Segal algebras, indeed to arbitrary homogeneous Banach spaces in the sense of [32].

Proof of Theorem 2.2. We note that (17) and the fact that $\|f * \mu^{[j]}\|_{S_0} \leqslant \|f\|_{S_0}\|\mu\|_{\mathcal{M}}$ imply that the convergence properties of the series defining $Q_h f$ (Lemma 4.1) also hold for the series defining $Q_{h,j} f$, for fixed $h, j > 0$. In particular the quasi-interpolants belong to S_0. Using (17) we obtain by combining Lemma 4.5, Theorem 4.6, and Lemma 4.7,

$$
\begin{aligned}
\|f - Q_{h,j} f\|_{S_0} &= \|f - Q_h(f * \mu^{[j]})\|_{S_0} \\
&\leqslant \|f - Q_h f\|_{S_0} + \|Q_h(f - f * \mu^{[j]})\|_{S_0} \\
&\leqslant \|f - Q_h f\|_{S_0} + C\|f - f * \mu^{[j]}\|_{S_0} \to 0 \quad \text{as } h, j \to 0. \qquad \square
\end{aligned}
$$

5. Further remarks

In our results we exclusively use generators φ from S_0. For many practical applications in harmonic analysis this is a mild restriction on φ, satisfied, for example, by any typical classic summation kernel [18]. On the other hand it helps to avoid technicalities that otherwise arise frequently in approximation theory. For example, the following two forms of basic Strang–Fix conditions

$$\widehat{\varphi}(k) = \delta_{k,0}, \quad k \in \mathbb{Z}^d, \quad \text{and} \quad \sum_{k \in \mathbb{Z}^d} \varphi(x-k) = 1, \quad x \in \mathbb{R}^d, \tag{22}$$

are truly equivalent for $\varphi \in S_0$. The standard examples of φ that satisfy (22) are tensor products of the symmetric B-splines defined by $\widehat{B_n}(t) = (\sin \pi t / \pi t)^{n+1}$, $t \in \mathbb{R}$. For $n \geqslant 1$, these functions belong to S_0. Thus, our results include the case of spline approximation and piecewise linear approximation on $\mathbb{R}$ in particular, cf. Corollary 2.3.

There is a well-known useful procedure to construct functions that satisfy (22). We finally point out that this technique works in S_0 without any additional hypothesis [14, Theorem 3.6(c)]. In order to demonstrate the simplification of the standard arguments in the S_0 setting we include the proof.

Lemma 5.1. *Given $\psi \in S_0$, let $\Psi(x) := \sum_{k \in \mathbb{Z}^d} \psi(x-k)$; the series converges absolutely in x and uniformly on compact sets. Suppose $\Psi(x) \neq 0$, for all $x \in [0,1]^d$, and let*

$$\varphi(x) := \psi(x)/\Psi(x), \quad x \in \mathbb{R}^d.$$

Then the function φ, which satisfies the conditions in (22), *indeed belongs to S_0.*

Proof. For $\psi \in S_0$, the periodization $\Psi(x)$, viewed as a function on the fundamental domain $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \cong [0,1)^d$, belongs to the Fourier algebra, i.e., $\Psi \in A(\mathbb{T}^d)$. In fact, its Fourier series is $\widehat{\Psi}(k) = \widehat{\psi}(k)$, $k \in \mathbb{Z}^d$, and belongs to ℓ^1. By assumption we have $\Psi(t) \neq 0$, for all $t \in \mathbb{T}^d$, so Wiener's lemma [32,39] states that the reciprocal function $\Psi(t)^{-1}$, $t \in \mathbb{T}^d$, also belongs to $A(\mathbb{T}^d)$. That is, we have the Fourier series expansion

$$\Psi(t)^{-1} = \sum_{k \in \mathbb{Z}^d} a_k e^{2\pi i k t}, \quad t \in \mathbb{T}^d, \tag{23}$$

with $a = (a_k)_{k \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ and $\|\Psi^{-1}\|_{A(\mathbb{T}^d)} = \|a\|_{\ell^1}$. Thus, $\Psi(x)^{-1}$ viewed as a periodic function of $x \in \mathbb{R}^d$, admits the expansion

$$\Psi(x)^{-1} = \sum_{k \in \mathbb{Z}^d} a_k e^{2\pi i k x}, \quad x \in \mathbb{R}^d. \tag{24}$$

Since the series in (23) converges in $A(\mathbb{T}^d)$, hence uniformly on $\mathbb{T}^d$, we obtain that the series in (24) converges uniformly on $\mathbb{R}^d$. Using (24) and the modulation invariance of S_0 we conclude that

$$\begin{aligned} \|\varphi\|_{S_0} = \|\Psi^{-1} \cdot \psi\|_{S_0} &= \left\| \sum_{k \in \mathbb{Z}^d} a_k e^{2\pi i k x} \psi(x) \right\|_{S_0} \\ &\leqslant \sum_{k \in \mathbb{Z}^d} |a_k| \|e^{2\pi i k x} \psi(x)\|_{S_0} \\ &= \sum_{k \in \mathbb{Z}^d} |a_k| \|\psi\|_{S_0} = \|a\|_{\ell^1} \|\psi\|_{S_0} = \|\Psi^{-1}\|_{A(\mathbb{T}^d)} \|\psi\|_{S_0} < \infty. \quad \square \end{aligned}$$

Remark 5.2. The Hausdorff–Young theorem states that $\|f\|_{L^{p'}} \leqslant \|f\|_{\mathcal{F}L^p}$, for $1 \leqslant p \leqslant 2$, where $2 \leqslant p' \leqslant \infty$ denotes the conjugate exponent, $1/p + 1/p' = 1$. As a consequence our results for convergence in the $\mathcal{F}L^p$-norm also imply convergence in the $L^{p'}$-norm, such as illustrated after Theorem 2.1. We also note that convergence in S_0 implies convergence in the L^p-norm, for any $1 \leqslant p \leqslant \infty$. On the other hand, for convergence in all of L^p it is required to exclude the Schoenberg operator since it involves pointwise evaluations. A suitable setting for approximation in L^p are quasi-interpolation operators with μ from $L^{p'}$, see [8,14,27,29,36] for some results of this kind. In contrast, the goal of our paper was to describe approximation techniques in the function spaces A and in S_0, which play a central role in harmonic analysis and time-frequency analysis. Both A and S_0 are spaces of continuous functions and therefore the Schoenberg operator can be used.

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