Abstract

We establish soliton-like asymptotics for finite energy solutions to the Dirac equation coupled to a relativistic particle. Any solution with initial state close to the solitary manifold, converges in long time limit to a sum of traveling wave and outgoing free wave. The convergence holds in global energy norm. The proof uses spectral theory and symplectic projection onto solitary manifold in the Hilbert phase space.

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1 Introduction

We prove long time convergence to sum of a soliton and dispersive wave for the Dirac equation coupled to a relativistic particle. The convergence holds in global energy norm for finite energy solutions with initial state close to the solitary manifold. Our main motivation is to develop techniques of Buslaev and Perelman [2, 3] in the context of the Dirac equation. The development is not straightforward because of known peculiarities of the Dirac equation: nonpositivity of the energy, algebra of the Dirac matrices, etc. We expect that the result might be extended to relativistic nonlinear Dirac equations relying on an appropriate development of our techniques.

Let \( \psi(x) \in \mathbb{C}^4 \) be a Dirac spinor field in \( \mathbb{R}^3 \), coupled to a relativistic particle with position \( q \) and momentum \( p \), governed by

\[
\begin{cases}
  i \dot{\psi}(x, t) = [-i\alpha_1 \partial_1 - i\alpha_2 \partial_2 - i\alpha_3 \partial_3 + \beta m] \psi(x, t) + \rho(x - q(t)) \\
  \dot{q}(t) = p(t)/\sqrt{1 + p^2(t)}, \quad \dot{p}(t) = \text{Re} \langle \psi(x, t), \nabla \rho(x - q(t)) \rangle
\end{cases}
\quad x \in \mathbb{R}^3
\tag{1.1}
\]

where \( \rho \in C(\mathbb{R}^3, \mathbb{C}^4) \) and \( \langle \cdot, \cdot \rangle \) stands for the Hermitian scalar product on \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \). Here \( \partial_j = \partial/\partial x_j \), \( \alpha_j \) and \( \beta \) are \( 4 \times 4 \) Dirac matrices. The standard representation for the Dirac matrices \( \alpha_j \) and \( \beta \) (in 2 \times 2 blocks) is

\[
\beta = \alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3
\]

where \( I_2 \) denotes the unit \( 2 \times 2 \) matrix, and

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

The matrices \( \alpha_j \), \( j = 0, 1, 2, 3 \) are Hermitian, and satisfy the anticommutation relations

\[
\alpha_j^* = \alpha_j, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}
\tag{1.2}
\]

We will use the following real orthogonality relations

\[
\beta \psi \cdot \alpha_j \psi = 0, \quad j = 1, 3 \quad \text{and} \quad \alpha_2 \psi \cdot \psi = 0, \quad \psi \in \mathbb{R}^4
\tag{1.3}
\]

Here and below \( " \cdot " \) stands for the Hermitian scalar product on \( \mathbb{C}^n \) with appropriate \( n \). System (1.1) is translation-invariant, and admits soliton solutions

\[
s_{a,v}(t) = (\psi_v(x - vt - a), vt + a, \rho_v), \quad p_v = v/\sqrt{1 - v^2}
\tag{1.4}
\]

for all \( a, v \in \mathbb{R}^3 \) with \( |v| < 1 \). The states \( S_{a,v} := s_{a,v}(0) \) form the solitary manifold

\[
\mathcal{S} := \{ S_{a,v} : a, v \in \mathbb{R}^3, |v| < 1 \}
\]

Our main result is soliton-type asymptotics

\[
\psi(x, t) \sim \psi_{v_\pm}(x - v_\pm t - a_\pm) + W_0(t) \phi_\pm, \quad t \to \pm \infty
\tag{1.5}
\]

for solutions to (1.1) with initial data close to solitary manifold \( \mathcal{S} \). Here \( W_0(t) \) is the dynamical group of the free Dirac equation, \( \phi_\pm \) are the corresponding asymptotic scattering states, and the asymptotics hold in the global norm of the Hilbert space \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \). For the particle trajectory we prove that

\[
\dot{q}(t) \to v_\pm, \quad q(t) \sim v_\pm t + a_\pm, \quad t \to \pm \infty
\tag{1.6}
\]
The results are established under the following conditions on the complex valued charge distribution: for some $\nu > 5/2$

$$(1 + |x|^\nu |\partial^\alpha \rho(x)|) \in L^2(\mathbb{R}^3), \quad |\alpha| \leq 3$$  \hfill (1.7)

We assume that $\rho(x)$ is spherically symmetric for simplicity of calculations. Finally, we assume the Wiener condition for the Fourier transform $\hat{\rho} = (2\pi)^{-3/2} \int e^{ikx} \rho(x) dx$:

$$B(k) = m^2 [\hat{\rho}_1(k)^2 + \hat{\rho}_2(k)^2 - \hat{\rho}_3(k)^2 - \hat{\rho}_4(k)^2] > 0, \quad k \in \mathbb{R}^3$$  \hfill (1.8)

which is nonlinear version of the Fermi Golden Rule in our case (cf. [4, 13, 14, 15]): nonlinear perturbation is not orthogonal to eigenfunctions of continuous spectrum of the linear part. The examples are easily constructed. Namely, let us rewrite (1.8) in the form

$$B(k) = m^2 [\hat{\rho}_1(k)^2 + \hat{\rho}_2(k)^2 - \hat{\rho}_3(k)^2 - \hat{\rho}_4(k)^2] > 0, \quad k \in \mathbb{R}^3$$  \hfill (1.9)

Therefore, we can take e.g. $\rho_1$ constructed in [12], and $\rho_2 = \rho_3 = \rho_4 = 0$.

The system (1.1) describes charged particle interacting with its “own” Dirac field. The asymptotics (1.5)-(1.6) mean asymptotic stability of uniform motion, i.e. “law of inertia”. The stability is caused by “radiative damping”, i.e. radiation of energy to infinity appearing analytically as a local energy decay for solutions to linearized equation. The radiative damping was suggested first by M. Abraham in 1905 in the context of the Maxwell-Lorentz equations, [1].

One could expect asymptotics (1.5) for small perturbations of solitons for relativistic nonlinear Dirac equations and for coupled nonlinear Maxwell-Dirac equations whose solitons were constructed in [6]. Our result models this situation though the relativistic case is still open problem.

Asymptotics of type (1.5)-(1.6) were obtained previously for the Klein-Gordon and Schrödinger equations coupled to the particle [8, 10]. More weak asymptotics of type (1.5) in the local energy norms, and without the dispersive wave, were obtained in [7] and [11] for all finite energy solutions to the Maxwell-Lorentz and wave equations.

Let us comment on our approach. For 1D translation invariant Schrödinger equation, the asymptotics of type (1.5) were proved for the first time by Buslaev and Perelman [2, 3, 4], and extended by Cuccagna [5] to higher dimensions. We prove the asymptotics (1.5)–(1.6) applying approach [8] where general Buslaev and Perelman strategy has been developed for the case of 3D Klein-Gordon equation: i) symplectic orthogonal decomposition of the dynamics near the solitary manifold, ii) modulation equations for the symplectic projection onto the manifold, and iii) time decay in transversal directions, etc (see more details in Introduction [8]). Our main novelties are the following.

I. It is well known that the Hamiltonian of the Dirac equation is nonpositive. Respectively, the energy conservation does not provide a uniform a priori estimate for the solution. We obtain an estimate for $L^2$ norm of the solution using unitarity of free Dirac propagator. This estimate provides existence of global solution.

On the other hand, for linearized equation unitarity of free propagator provides only exponential bounds $\sim e^{\alpha t}$ for $L^2$ norm of the solution. This guarantees existence of global solution and analyticity of the resolvent for $\text{Re}\lambda > \alpha$. However, we need the analyticity for $\text{Re}\lambda > 0$ which we obtain by bifurcation arguments relying on our analysis of the Fourier-Laplace transform (Lemma 14.4).

II. We establish an appropriate decay of the linearized dynamics:
i) We prove time decay $\sim t^{-3/2}$ in weighted norms for a modified free Dirac equation (Lemma 15.1). The proof relies on a “soft version” of strong Huygens principle for the Dirac equation. Namely, the free Dirac propagator is concentrated mainly near the light cone, while contribution of inner zone is a Hilbert-Schmidt operator.

ii) We compute all needed spectral properties of the linearized equation at the soliton in contrast to majority of works in the field, where the corresponding spectral properties are postulated. Namely, we find that under the Wiener condition (1.8), discrete spectrum consists only of one zero point with algebraic multiplicity 6. The multiplicity is totally due to the translation invariance of the system (1.1).

iii) We exactly calculate the symplectic orthogonality conditions (15.18) for initial data of the linearized equation. These conditions are necessary for the proof of the decay. All computations differ significantly from the case of the Klein-Gordon equation [8] because of algebra of the Dirac matrices. An important role play the real orthogonality relations (1.3).

Our paper is organized as follows. In Section 2, we formulate the main result. In Section 3, we introduce symplectic projection onto the solitary manifold. The linearized equation is considered in Sections 4 and 5. In Section 6, we split the dynamics in two components: along the solitary manifold, and in transversal directions. The time decay of transversal component is established in sections 7-10. In Section 11 we prove main result. In Sections 12 - 15 we justify the time decay of the linearized dynamics relying on weighted decay for the free Dirac equation in a moving frame. In Appendices A, B and C we collect some technical calculations.

2 Main results

2.1 Existence of dynamics

We consider the Cauchy problem for the system (1.1) which we write as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R} : \quad Y(0) = Y_0$$

(2.1)

Here $Y(t) = (\psi(t), q(t), p(t))$, $Y_0 = (\psi(0), q_0, p_0)$. We introduce a suitable phase space for equation (2.1). Let $L^2_\alpha$, $\alpha \in \mathbb{R}$ be weighted Agmon spaces with norm $\|\psi\|_\alpha = \|\psi\|_{L^2_\alpha} := \|(1 + |x|)^\alpha \psi\|_{L^2}$.

**Definition 2.1.** i) The phase space $\mathcal{E}$ is the Hilbert space $L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\psi, q, p)$ with finite norm

$$\|Y\|_{\mathcal{E}} = \|\psi\|_0 + |q| + |p| .$$

ii) $\mathcal{E}_\alpha$ is the space $L^2_\alpha \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with finite norm

$$\|Y\|_{\mathcal{E}_\alpha} = \|\psi\|_\alpha + |q| + |p| .$$

**Proposition 2.2.** Let (1.7) hold. Then

(i) For every $Y_0 \in \mathcal{E}$ the Cauchy problem (2.1) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.

(ii) For every $t \in \mathbb{R}$, map $U(t) : Y_0 \mapsto Y(t)$ is continuous on $\mathcal{E}$.

**Proof. Step i)** First, we fix an arbitrary $b > 0$ and prove (i)-(ii) for $Y_0 \in \mathcal{E}$ such that $\|\psi_0\|_0 \leq b$ and $|t| \leq \varepsilon = \varepsilon(b)$ for some sufficiently small $\varepsilon(b) > 0$. We rewrite (2.1) as

$$\dot{Y}(t) = F_1(Y(t)) + F_2(Y(t)), \quad t \in \mathbb{R}, \quad Y(0) = Y_0 ,$$

(2.2)
The Fourier transform provides existence and uniqueness of solution $Y_1(t) \in C(\mathbb{R}, \mathcal{E})$ to (2.2) with $F_2 = 0$. Let $U_1(t) : Y_0 \mapsto Y_1(t)$ be the corresponding strongly continuous group of bounded linear operators on $\mathcal{E}$. Then (2.2) for $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is equivalent to integral Duhamel equation
\[
Y(t) = U_1(t)Y_0 + \int_0^t ds \, U_1(t-s)F_2(Y(s)). \tag{2.3}
\]
Further, the map $F_2 : Y \mapsto F_2(Y)$ is locally Lipschitz continuous in $\mathcal{E}$: for each $b > 0$ there exist a $\varkappa = \varkappa(b) > 0$ such that for all $Y_1 = (\psi_1, q_1, p_1), Y_2 = (\psi_2, q_2, p_2) \in \mathcal{E}$ with $\|\psi_1\|_0, \|\psi_2\|_0 \leq b$,
\[
\|F_2(Y_1) - F_2(Y_2)\|_c \leq \varkappa\|Y_1 - Y_2\|_c.
\]
Therefore, by the contraction mapping principle, equation (2.3) has a unique local solution $Y(\cdot) \in C([-\varepsilon, \varepsilon], \mathcal{E})$ with $\varepsilon > 0$ depending only on $b$.

\textit{Step ii)} Second, we derive \textit{a priori} estimate. Consider $\psi_0 \in C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. Then
\[
\frac{d}{dt}\|\psi\|_0^2 = \int (\psi \cdot \dot{\psi} + \dot{\psi} \cdot \psi) dx = 2 \int \text{Im} (\rho(x-q) \cdot \psi(x)) dx \leq C\|\psi\|_0.
\]
Hence,
\[
\|\psi(t)\|_0 \leq \frac{1}{2}Ct + \|\psi(0)\|_0
\]
Last two equalities (1.1) imply \textit{a priori} estimates for $|\dot{p}|$ and $|\dot{q}|$. The \textit{a priori} estimates for general initial data $\psi_0 \in L^2$ follow by approximating initial data by functions from $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$.

\textit{Step iii)} Properties (i)-(ii) for arbitrary $t \in \mathbb{R}$ now follow from the same properties for small $|t|$ and from \textit{a priori} estimate.

\section*{2.2 Solitary manifold and main result}

Let us compute the solitons (1.4). Substitution to (1.1) gives stationary equations
\[
\begin{align*}
-iv \cdot \nabla \psi_v(y) &= [-i\alpha \cdot \nabla + \beta m] \psi_v(y) + \rho(y) \quad \Bigg| \quad v = p_v / \sqrt{1 + p_v^2}, \quad 0 = \text{Re} \langle \psi_v(y), \nabla \rho(y) \rangle \tag{2.4}
\end{align*}
\]
Applying Fourier transform to first equation in (2.4) we obtain
\[
-(v \cdot k + \alpha \cdot k - \beta m) \hat{\psi}_v(k) = \hat{\rho}(k) .
\]
Then
\[
\hat{\psi}_v(k) = -\frac{(v \cdot k + \alpha \cdot k - \beta m) \hat{\rho}(k)}{(v \cdot k + \alpha \cdot k - \beta m)(v \cdot k - \alpha \cdot k + \beta m)} = \frac{(v \cdot k + \alpha \cdot k - \beta m) \hat{\rho}(k)}{k^2 + m^2 - (v \cdot k)^2} . \tag{2.5}
\]
The soliton is given by the formula
\[
\psi_v(x) = \frac{i\gamma}{4\pi} (v \cdot \nabla + \alpha \cdot \nabla + i\beta m) \int e^{-m|\gamma(y-x)||+(y-x)_\perp|} \rho(y) d^3y / |\gamma(y-x)||+(y-x)_\perp|, \quad p_v = \gamma v = \frac{v}{\sqrt{1 - v^2}} . \tag{2.6}
\]
Here we set \( \gamma = 1/\sqrt{1-v^2} \) and \( x = x_\parallel + x_\perp \), where \( x_\parallel \parallel v \) and \( x_\perp \perp v \) for \( x \in \mathbb{R}^3 \). It remains to prove that last equation of (2.4) holds. Indeed, (2.5) and Parseval identity imply

\[
\text{Re}\langle \psi_v(y), \partial_j \rho(y) \rangle = \text{Re}\langle ik_j \hat{\psi}_v(k), \hat{\rho}(k) \rangle = \text{Re}\int ik_j \frac{(v \cdot k + \alpha \cdot k - \beta m)\hat{\rho}(k) \cdot \hat{\rho}(k)}{k^2 + m^2 - (v \cdot k)^2} dk = 0
\]

since the integrand is pure imaginary by (1.3). Hence, the soliton solution (1.4) exists and is defined uniquely for \( |v| < 1 \) and \( a \in \mathbb{R}^3 \). Denote \( V := \{ v \in \mathbb{R}^3 : |v| < 1 \} \).

**Definition 2.3.** A soliton state is \( S(\sigma) := (\psi_v(x - b), b, v) \), where \( \sigma := (b, v) \) with \( b \in \mathbb{R}^3 \) and \( v \in V \).

Obviously, the soliton solution admits representation \( S(\sigma(t)) \), where

\[
\sigma(t) = (t(t), v(t)) = (vt + a, v).
\]

**Definition 2.4.** A solitary manifold is the set \( \mathcal{S} := \{ S(\sigma) : \sigma \in \Sigma := \mathbb{R}^3 \times V \} \).

The main result of our paper is the following theorem.

**Theorem 2.5.** Let (1.7)-(1.8) hold, \( \nu > 5/2 \) be the number from (1.7), and \( Y(t) \) be the solution to the Cauchy problem (2.1) with initial state \( Y_0 \) which is sufficiently close to the solitary manifold:

\[
d_0 := \text{dist}_{\mathcal{C}}(Y_0, \mathcal{S}) \ll 1.
\]

Then the asymptotics hold for \( t \to \pm \infty \),

\[
\begin{align*}
\dot{q}(t) &= v_\pm + O(|t|^{-2}), \quad q(t) = v_\pm t + a_\pm + O(|t|^{-1}) \\
\psi(x, t) &= \psi_{v_\pm}(x - v_\pm t - a_\pm) + W_0(t)\phi_\pm + r_\pm(x, t)
\end{align*}
\]

with

\[
\|r_\pm(t)\|_0 = O(|t|^{-1/2}).
\]

It suffices to prove asymptotics (2.9)-(2.10) for \( t \to +\infty \) since system (1.1) is time reversible.

### 3 Symplectic projection

#### 3.1 Hamiltonian structure

Denote \( \psi_1 = \text{Re} \psi, \psi_2 = \text{Im} \psi, \rho_1 = \text{Re} \rho, \rho_2 = \text{Im} \rho, \alpha_2 = -i\alpha_2 \). Then (1.1) reads

\[
\begin{cases}
\begin{align*}
\dot{\psi}_1(x, t) &= -(\alpha_1 \partial_1 + \alpha_3 \partial_3)\psi_1(x, t) + (\alpha_2 \partial_2 + \beta m)\psi_2(x, t) + \rho_2(x - q(t)) \\
\dot{\psi}_2(x, t) &= -(\alpha_2 \partial_2 + \beta m)\psi_1(x, t) - (\alpha_1 \partial_1 + \alpha_3 \partial_3)\psi_2(x, t) - \rho_1(x - q(t)) \\
\dot{q}(t) &= p(t)/\sqrt{1 + p^2(t)} \\
\dot{p}(t) &= \int (\psi_1(x, t) \cdot \nabla \rho_1(x - q(t)) + \psi_2(x, t) \cdot \nabla \rho_2(x - q(t))) dx
\end{align*}
\end{cases}
\quad x \in \mathbb{R}^3
\]

This is a Hamilton system with the Hamilton functional

\[
\mathcal{H}(\psi_1, \psi_2, q, p) = \frac{1}{2} \int (\psi_1 \cdot (\partial_2 + \beta m) \psi_1 + \psi_2 \cdot (\partial_2 + \beta m) \psi_2 + 2\psi_1 \cdot (\alpha_1 \partial_1 + \alpha_3 \partial_3) \psi_2) dx + \int (\psi_1(x) \cdot \rho_1(x - q) + \psi_2(x) \cdot \rho_2(x - q)) dx + \sqrt{1 + p^2}.
\]
Equation (3.1) can be written as a Hamilton system
\[
\dot{Y} = JD\mathcal{H}(Y), \quad Y = (\psi_1, \psi_2, q, p), \quad J := \begin{pmatrix} 0 & I_4 & 0 & 0 \\ -I_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 \\ 0 & 0 & -I_3 & 0 \end{pmatrix}
\]
where \(D\mathcal{H}\) is the Fréchet derivative with respect to \(\psi_{1k}, \psi_{2k}, k = 1, 2, 3, 4, p \) and \(q\) of the Hamilton functional.

### 3.2 Symplectic projection onto solitary manifold

Let us identify tangent space to \(\mathcal{E}\), at every point, with \(\mathcal{E}\). Consider symplectic form \(\Omega\) defined on \(\mathcal{E}\) by \(\Omega = \int d\psi_1(x) \wedge d\psi_2(x) dx + dq \wedge dp\), i.e.
\[
\Omega(Y^1, Y^2) = \langle Y^1, JY^2 \rangle, \quad Y^j = (\psi_1^j, \psi_2^j, q^j, p^j) \in \mathcal{E}, \quad j = 1, 2
\]
where \(\langle Y^1, Y^2 \rangle := \langle \psi_1^1, \psi_2^1 \rangle + \langle \psi_1^2, \psi_2^2 \rangle + q^1 \cdot q^2 + p^1 \cdot p^2\). It is clear that the form \(\Omega\) is non-degenerate, i.e.
\[
\Omega(Y^1, Y^2) = 0 \quad \text{for every} \quad Y^2 \in \mathcal{E} \implies Y^1 = 0
\]

**Definition 3.1.**

i) \(Y^1 \parallel Y^2\) means that \(Y^1 \in \mathcal{E}, Y^2 \in \mathcal{E}\), and \(Y^1\) is symplectic orthogonal to \(Y^2\), i.e. \(\Omega(Y^1, Y^2) = 0\).

ii) A projection operator \(P : \mathcal{E} \to \mathcal{E}\) is called symplectic orthogonal if \(Y^1 \parallel Y^2\) for \(Y^1 \in \text{Ker} \ P\) and \(Y^2 \in \text{Im} \ P\).

Let us consider tangent space \(T_{S(\sigma)} S\) to the manifold \(S\) at a point \(S(\sigma)\). Vectors \(\tau_j := \partial_{\tau_j} S(\sigma)\), where \(\partial_{\tau_j} := \partial_{\psi_j}\) and \(\partial_{\tau_{j+3}} := \partial_{\psi_j}\) with \(j = 1, 2, 3\), form a basis in \(T_{S(\sigma)} S\). In detail,
\[
\tau_j = \tau_j(v) := \partial_{\psi_j} S(\sigma) = (-\partial_{\psi_1}(y), -\partial_{\psi_2}(y), e_j, 0), \quad \tau_{j+3} = \tau_{j+3}(v) := \partial_{\psi_j} S(\sigma) = (\partial_{\psi_1}(y), \partial_{\psi_2}(y), 0, \partial_{\psi_3}(y)) \quad j = 1, 2, 3
\]
where \(\psi_{e_1} = \text{Re} \psi_1, \psi_{e_2} = \text{Im} \psi_2, y := x - b\) is the “moving frame coordinate”, \(e_1 = (1, 0, 0)\) etc. Formula (2.6) and condition (1.7) imply that
\[
\tau_j(v) \in \mathcal{E}_\alpha, \quad v \in V, \quad j = 1, \ldots, 6, \quad \forall \alpha \in \mathbb{R}.
\]

**Lemma 3.2.** *The matrix with the elements \(\Omega(\tau_i, \tau_j(v))\) is non-degenerate \(\forall v \in V\).*

**Proof.** The elements are computed in Appendix A. As a result, matrix \(\Omega(\tau_i, \tau_j)\) reads
\[
\Omega(v) := (\Omega(\tau_i, \tau_j))_{i, j = 1, \ldots, 6} = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix}
\]
where the \(3 \times 3\)-matrix \(\Omega^+(v)\) equals
\[
\Omega^+(v) = K + (1 - v^2)^{-1/2} E + (1 - v^2)^{-3/2} v \otimes v.
\]
Here \(K\) is a symmetric \(3 \times 3\)-matrix with the elements
\[
K_{ij} = \int dkk_i k_j B(k) \frac{k^2 + m^2 + 3(v \cdot k)^2}{(k^2 + m^2 - (v \cdot k)^2)^3}
\]
where \(B(k) > 0\) is defined in (1.8). The matrix \(K\) is the integral of symmetric nonnegative definite matrix \(k \otimes k = (k_i k_j)\) with a positive weight. Hence, the matrix \(K\) is nonnegative definite. Since unity matrix \(E\) is positive definite, the matrix \(\Omega^+(v)\) is symmetric and positive definite, hence non-degenerate. Then the matrix \(\Omega(\tau_i, \tau_j)\) also is non-degenerate. 

□
Let us introduce translations $T_a : (\psi(\cdot), q, p) \mapsto (\psi(\cdot - a), q + a, p)$, $a \in \mathbb{R}^3$. Note that the manifold $S$ is invariant with respect to the translations. Let us denote $v(p) := p/\sqrt{1 + p^2}$ for $p \in \mathbb{R}^3$.

**Definition 3.3.** i) For any $\alpha \in \mathbb{R}$ and $\varpi < 1$ denote by $E_\alpha(\varpi) = \{ Y = (\psi, q, p) \in E_\alpha : |v(p)| \leq \varpi \}$. We set $E(\varpi) := E_0(\varpi)$.

ii) For any $\tilde{\varpi} < 1$ denote by $\Sigma(\tilde{\varpi}) = \{ \sigma = (b, v) : b \in \mathbb{R}^3, |v| \leq \tilde{\varpi} \}$.

Next Lemma means that in a small neighborhood of the soliton manifold $S$ a “symplectic orthogonal projection” onto $S$ is well-defined.

**Lemma 3.4.** (cf. [8, Lemma 3.4]) Let (1.7) hold, $\alpha \in \mathbb{R}$ and $\varpi < 1$. Then

i) there exists a neighborhood $O_\alpha(S)$ of $S$ in $E_\alpha$ and a map $\Pi : O_\alpha(S) \to S$ such that $\Pi$ is uniformly continuous on $O_\alpha(S) \cap E_\alpha(\varpi)$ in the metric of $E_\alpha$,

$$\Pi Y = Y \quad \text{for} \quad Y \in S, \quad \text{and} \quad Y - S \not\ni T_a S, \quad \text{where} \quad S = \Pi Y . \quad (3.9)$$

ii) $O_\alpha(S)$ is invariant with respect to translations $T_a$, and

$$\Pi T_a Y = T_a \Pi Y, \quad \text{for} \quad Y \in O_\alpha(S) \quad \text{and} \quad a \in \mathbb{R}^3 .$$

iii) For any $\varpi < 1$ there exists a $\tilde{\varpi} < 1$ s.t. $\Pi Y = S(\sigma)$ with $\sigma \in \Sigma(\tilde{\varpi})$ for $Y \in O_\alpha(S) \cap E_\alpha(\varpi)$.

iv) For any $\tilde{\varpi} < 1$ there exists an $r_\alpha(\tilde{\varpi}) > 0$ s.t. $S(\sigma) + Z \in O_\alpha(S)$ if $\sigma \in \Sigma(\tilde{\varpi})$ and $\|Z\|_\alpha < r_\alpha(\tilde{\varpi})$.

We will call $\Pi$ a symplectic orthogonal projection onto $S$.

**Corollary 3.5.** Condition (2.8) implies that $Y_0 = S + Z_0$ where $S = S(\sigma_0) = \Pi Y_0$, and

$$\|Z_0\|_\varpi \ll 1 . \quad (3.10)$$

**Proof.** Lemma 3.4 implies that $\Pi Y_0 = S$ is well defined for small $d_0 > 0$. Furthermore, condition (2.8) means that there exists a point $S_1 \in S$ such that $\|Y_0 - S_1\|_\varpi = d_0$. Hence, $Y_0, S_1 \in O_\varpi(S) \cap E_\varpi(\varpi)$ with a $\varpi < 1$ which does not depend on $d_0$ for sufficiently small $d_0$.

On the other hand, $\Pi S_1 = S_1$, hence the uniform continuity of the map $\Pi$ implies that $\|S_1 - S\|_\varpi \to 0$ as $d_0 \to 0$. Therefore, finally, $\|Z_0\|_\varpi = \|Y_0 - S\|_\varpi \leq \|Y_0 - S_1\|_\varpi + \|S_1 - S\|_\varpi \leq d_0 + o(1) \ll 1$ for small $d_0$. \qed

4 Linearization on solitary manifold

Let us consider a solution to the system (3.1), and split it as the sum

$$Y(t) = S(\sigma(t)) + Z(t) \quad (4.1)$$

where $\sigma(t) = (b(t), v(t)) \in \Sigma$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (\psi, q, p)$ and $Z = (\Psi, Q, P)$. Then (4.1) means that

$$\psi(x, t) = \psi_{v(t)}(x - b(t)) + \Psi(x - b(t), t), \quad q(t) = b(t) + Q(t), \quad p(t) = p_{v(t)} + P(t) . \quad (4.2)$$
Let us substitute (4.2) to (1.1), and linearize the equations in \( Z \). Setting \( y = x - b(t) \) which is “moving frame coordinate”, we obtain that

\[
\dot{\psi} = \dot{v} \cdot \nabla_v \psi_v(y) - \dot{b} \cdot \nabla \psi_v(y) + \dot{\Psi}(y, t) - b \cdot \nabla \Psi(y, t) \\
= [-\alpha \cdot \nabla - i\beta m](\psi_v(y) + \Psi(y, t)) - i\rho(y - Q)
\]

\[
\dot{q} = \dot{b} + \dot{Q} = \frac{p_v + P}{\sqrt{1 + (p_v + P)^2}}
\]

\[
\dot{p} = \dot{v} \cdot \nabla_v p_v + \dot{P} = \text{Re}(\psi_v(y) + \Psi(y, t), \nabla \rho(y - Q))
\]

Let us extract linear terms in \( Q \). First note that \( \rho(y - Q) = \rho(y) - Q \cdot \nabla \rho(y) + N_1(Q) \), \( \nabla \rho(y - Q) = \nabla \rho(y) - \nabla(Q \cdot \nabla \rho(y)) + \tilde{N}_1(Q) \).

Condition (1.7) implies that for \( N_1(Q) \) and \( \tilde{N}_1(Q) \) the bound holds,

\[
||N_1(Q)||_\nu + ||\tilde{N}_1(Q)||_\nu \leq C_\nu(\bar{Q})Q^2
\]

uniformly in \( |Q| \leq \bar{Q} \) for any fixed \( \bar{Q} \). Second, the Taylor expansion gives

\[
\frac{p_v + P}{\sqrt{1 + (p_v + P)^2}} = v + \frac{1}{\gamma}(P - v(\nu \cdot P)) + N_2(v, P)
\]

where \( 1/\gamma = \sqrt{1 - v^2} = (1 + p_v^2)^{-1/2} \), and

\[
|N_2(v, P)| \leq C(\bar{v})P^2
\]

uniformly with respect to \( |v| \leq \bar{v} < 1 \). Using (2.4), we obtain from (4.3) the following equations for components of \( Z(t) \):

\[
\dot{\Psi}(y, t) = [-\alpha \cdot \nabla - i\beta m]\Psi(y, t) + \dot{b} \cdot \nabla \Psi(y, t) + iQ \cdot \nabla \rho(y) \\
+ (\dot{b} - v) \cdot \nabla \psi_v(y) - \dot{v} \cdot \nabla_v \psi_v(y) - iN_1
\]

\[
\dot{Q}(t) = \frac{1}{\gamma}(E - v \otimes v)P + (v - \dot{b}) + N_2
\]

\[
\dot{P}(t) = -\dot{v} \cdot \nabla_v p_v + \text{Re}(\Psi(y, t), \nabla \rho(y)) + \text{Re}(\nabla \psi_v(y), Q \cdot \nabla \rho(y)) + N_3(v, Z)
\]

where \( N_3(v, Z) = -\text{Re}(\nabla \psi_v, N_1(Q)) - \text{Re}(\Psi, \nabla(Q \cdot \nabla \rho)) + \text{Re}(\Psi, \tilde{N}_1(Q)) \). Clearly, \( N_3(v, Z) \) satisfies the following estimate

\[
|N_3(v, Z)| \leq C_\nu(\rho, \tau, \bar{Q})\left[Q^2 + ||\Psi||_{-\nu}||Q||\right]
\]

uniformly in \( |v| \leq \bar{v} \) and \( |Q| \leq \bar{Q} \) for any fixed \( \bar{v} < 1 \). For the vector version \( Z = (\Psi_1, \Psi_2, Q, P) \) with \( \Psi_1 = \text{Re} \Psi, \Psi_2 = \text{Im} \Psi \) we rewrite equations (4.6) as

\[
\dot{Z}(t) = A(t)Z(t) + T(t) + N(t), \quad t \in \mathbb{R}.
\]

Here operator \( A(t) = A_{v,w}(t) \) depends on two parameters, \( v = v(t) \), and \( w = \dot{b}(t) \) and can be written as

\[
A_{v,w} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ Q \\ P \end{pmatrix} = \begin{pmatrix} -\alpha_1 \partial_1 - \alpha_3 \partial_3 + w \cdot \nabla & \dot{\alpha}_2 \partial_2 + \beta m & -\nabla \rho_2 & 0 \\ -(\dot{\alpha}_2 \partial_2 + \beta m) & -\alpha_1 \partial_1 - \alpha_3 \partial_3 + w \cdot \nabla & -\nabla \rho_1 & 0 \\ 0 & 0 & 0 & B_v \\ \langle \cdot, \nabla \rho_1 \rangle & \langle \cdot, \nabla \rho_2 \rangle & \langle \nabla \psi_v, \cdot \nabla \rho_j \rangle & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ Q \\ P \end{pmatrix}
\]
where \( B_v = \frac{1}{2}(E - v \otimes v) \). Furthermore, \( T(t) = T_{v,w}(t) \) and \( N(t) = N(t, \sigma, Z) \) in (4.8) stand for

\[
T_{v,w} = \begin{pmatrix}
(w - v) \cdot \nabla \psi_{v1} - \dot{v} \cdot \nabla \psi_{v1} \\
(w - v) \cdot \nabla \psi_{v2} - \dot{v} \cdot \nabla \psi_{v2} \\
v - w \\
- \dot{v} \cdot \nabla v
\end{pmatrix}, \quad N(\sigma, Z) = \begin{pmatrix}
N_{12}(Z) \\
-N_{11}(Z) \\
N_2(v, Z) \\
N_3(v, Z)
\end{pmatrix}
\]

(4.10)

where \( v = v(t), w = w(t), \sigma = \sigma(t) = \langle b(t), v(t) \rangle \), and \( Z = Z(t) \). Estimates (4.4), (4.5) and (4.7) imply that

\[
\|N(\sigma, Z)\|_r \leq C(\tilde{v}, \tilde{Q})\|Z\|_{-\nu}^2
\]

uniformly in \( \sigma \in \Sigma(\tilde{v}) \) and \( \|Z\|_{-\nu} \leq r_{-\nu}(\tilde{v}) \) for any fixed \( \tilde{v} < 1 \).

**Remark 4.1.** i) Term \( A(t)Z(t) \) in right hand side of equation (4.8) is linear in \( Z(t) \), and \( N(t) \) is a high order term in \( Z(t) \). On the other hand, \( T(t) \) is a zero order term which does not vanish at \( Z(t) = 0 \) since \( S(\sigma(t)) \) generally is not a soliton solution if (2.7) does not hold (though \( S(\sigma(t)) \) belongs to the solitary manifold).

ii) Formulas (3.4) and (4.10) imply:

\[
T(t) = -\sum_{l=1}^{3} [(w - v)_t \tau_l + \dot{v}_l \tau_{l+3}]
\]

(4.12)

and hence \( T(t) \in T_{S(\sigma(t))}S, t \in \mathbb{R} \).

### 5 Linearized equation

Here we collect some Hamiltonian and spectral properties of generator (4.9) of the linearized equation. First, let us consider linear equation

\[
\dot{X}(t) = A_{v,w}X(t), \quad t \in \mathbb{R}, \quad v \in V, \quad w \in \mathbb{R}^3
\]

(5.1)

**Lemma 5.1.** (cf. Lemma 5.1 [8]) i) For any \( v \in V \) and \( w \in \mathbb{R}^3 \) equation (5.1) can be written as the Hamilton system (cf. (3.2)),

\[
\dot{X}(t) = J \mathcal{H}_{v,w}(X(t)), \quad t \in \mathbb{R}
\]

(5.2)

where \( D\mathcal{H}_{v,w} \) is the Fréchet derivative with respect to \( \Psi_{1k}, \Psi_{2k}, k = 1, 2, 3, 4, P \) and \( Q \) of the Hamilton functional

\[
\mathcal{H}_{v,w}(X) = \frac{1}{2} \int (\Psi_1 \cdot (\dot{\alpha}_2 \partial_2 + \beta m)\Psi_1 + \Psi_2 \cdot (\dot{\alpha}_2 \partial_2 + \beta m)\Psi_2 + 2 \Psi_1 \cdot (\alpha_1 \partial_1 + \alpha_2 \partial_2)\Psi_2)dy + \int \rho_j(y)Q \cdot \nabla \Psi_j dy + \frac{1}{2} P \cdot B_v P - \frac{1}{2} (Q \cdot \nabla \psi_{vj}(y), Q \cdot \nabla \rho_j(y)), \quad X = (\Psi_1, \Psi_2, Q, P) \in \mathcal{E}.
\]

ii) The skew-symmetry relation holds,

\[
\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \quad X_1 \in \mathcal{E}, \quad X_2 \in H^1(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3.
\]

(5.3)

**Lemma 5.2.** Operator \( A_{v,w} \) acts on tangent vectors \( \tau_j(v) \) to the solitary manifold as follows,

\[
A_{v,w}[\tau_j(v)] = (w - v) \cdot \nabla \tau_j(v), \quad A_{v,w}[\tau_{j+3}(v)] = (w - v) \cdot \nabla \tau_{j+3}(v) + \tau_j(v), j = 1, 2, 3.
\]
Proof. In detail, we have to show that

\[
A_{v,w} \begin{pmatrix}
-\partial_j \psi v_1 \\
-\partial_j \psi v_2 \\
e_j
\end{pmatrix} = \begin{pmatrix}
(v - w) \cdot \nabla \partial_j \psi v_1 \\
(v - w) \cdot \nabla \partial_j \psi v_2 \\
0
\end{pmatrix}.
\]

Indeed, differentiate equations (2.4) in \( b_j \) and \( v_j \), and obtain that derivatives of soliton state in parameters satisfy the following equations,

\[
\begin{align*}
-v \cdot \nabla \partial_j \psi v &= [-\alpha \cdot \nabla - i \beta m] \partial_j \psi v - i \partial_j \rho \\
-\partial_j \psi v - v \cdot \nabla \partial_j \psi v &= [-\alpha \cdot \nabla - i \beta m] \partial_j \psi v \\
\partial_j p_v &= e_j (1 - v^2)^{-1/2} + v \frac{v_j}{(1 - v^2)^{3/2}} \\
0 &= \langle \partial_j \psi v_1, \nabla \rho_1 \rangle + \langle \partial_j \psi v_2, \nabla \rho_2 \rangle
\end{align*}
\]

for \( j = 1, 2, 3 \). Then (5.4) follows from (5.5) by definition of \( A_{v,w} \) in (4.9)

\[ \square \]

Corollary 5.3. Let \( w = v \in V \). Then \( \tau_j(v) \) are eigenvectors, and \( \tau_{j+3}(v) \) are root vectors of operator \( A_{v,w} \), corresponding to zero eigenvalue, i.e.

\[
A_{v,w}[\tau_j(v)] = 0, \ A_{v,w}[\tau_{j+3}(v)] = \tau_j(v), \ j = 1, 2, 3 .
\]

6 Symplectic decomposition of dynamics

Here we decompose the dynamics in two components: along manifold \( S \) and in transversal directions. Equation (4.8) is obtained without any assumption on \( \sigma(t) \) in (4.1). We are going to choose \( S(\sigma(t)) = \Pi Y(t) \), but then we need to know that

\[
Y(t) \in \mathcal{O}_{-\nu}(S), \quad t \in \mathbb{R} .
\]

It is true for \( t = 0 \) by our main assumption (2.8) with sufficiently small \( d_0 > 0 \). Then \( S(\sigma(0)) = \Pi Y(0) \) and \( Z(0) = Y(0) - S(\sigma(0)) \) are well defined. We will prove below that (6.1) holds if \( d_0 \) is sufficiently small. Let us choose an arbitrary \( \tilde{v} \) such that \( |v(0)| < \tilde{v} < 1 \) and let \( \delta = \tilde{v} - |v(0)| \). Denote by \( r_{-\nu}(\tilde{v}) \) the positive numbers from Lemma 3.4 iv) which corresponds to \( \alpha = -\nu \). Then \( S(\sigma) + Z \in \mathcal{O}_{-\nu}(S) \) if \( \sigma = (\tilde{b}, v) \) with \(|v| < \tilde{v}\) and \( \|Z\|_{-\nu} < r_{-\nu}(\tilde{v}) \). Note that \( \|Z(0)\|_{-\nu} < r_{-\nu}(\tilde{v}) \) if \( d_0 \) is sufficiently small. Therefore, \( S(\sigma(t)) = \Pi Y(t) \) and \( Z(t) = Y(t) - S(\sigma(t)) \) are well defined for \( t \geq 0 \) so small that \(|v| < \tilde{v}\) and \( \|Z(t)\|_{-\nu} < r_{-\nu}(\tilde{v}) \). This is formalized by the following standard definition.

Definition 6.1. \( t_* \) is "exit time",

\[
t_* = \sup \{ t > 0 : \|Z(s)\|_{-\nu} < r_{-\nu}(\tilde{v}), \ |v(s) - v(0)| < \delta, \ 0 \leq s \leq t \} .
\]
One of our main goals is to prove that \( t_* = \infty \) if \( d_0 \) is sufficiently small. This would follow if we show that
\[
\|Z(t)\|_{-\nu} < r_{-\nu}(\tilde{v})/2, \quad |v(s) - v(0)| < \delta/2, \quad 0 \leq t < t_* .
\] (6.3)

Note that
\[
|Q(t)| \leq \overline{Q} := r_{-\nu}(\tilde{v}), \quad 0 \leq t < t_* .
\] (6.4)

Now \( N(t) \) in (4.8) satisfies, by (4.11), the following estimate,
\[
\|N(t)\|_{-\nu} \leq C_{\nu}(\tilde{v}) \|Z(t)\|_{-\nu}^2, \quad 0 \leq t < t_* .
\] (6.5)

### 6.1 Modulation equations

From now on we fix the decomposition \( Y(t) = S(\sigma(t)) + Z(t) \) for \( 0 < t < t_* \) by setting \( S(\sigma(t)) = \Pi Y(t) \) which is equivalent to symplectic orthogonality condition of type (3.9),
\[
Z(t) \not\in T_S(\sigma(t)) S, \quad 0 \leq t < t_* .
\] (6.6)

This allows us to simplify drastically asymptotic analysis of dynamical equations (4.8) for the transversal component \( Z(t) \). As a first step, we derive “modulation equations” for the parameters \( \sigma(t) \). For this purpose, we write (6.6) in the form
\[
\Omega(Z(t), \tau_j(t)) = 0, \quad j = 1, \ldots, 6, \quad 0 \leq t < t_* .
\] (6.7)

where vectors \( \tau_j(t) = \tau_j(\sigma(t)) \) span tangent space \( T_{\Sigma(\sigma(t))} S \). Note that \( \sigma(t) = (b(t), v(t)) \), where \( |v(t)| \leq \tilde{v} < 1 \) for \( 0 \leq t < t_* \) by Lemma 3.4 iii). It would be convenient for us to use some other parameters \( (c, v) \) instead of \( \sigma = (b, v) \), where
\[
c(t) = b(t) - \int_0^t v(\tau)d\tau, \quad \dot{c}(t) = \dot{b}(t) - v(t) = w(t) - v(t), \quad 0 \leq t < t_* .
\] (6.8)

The following statement can be proved similar to Lemma 6.2 from [8].

**Lemma 6.2.** Let \( Y(t) \) be a solution to the Cauchy problem (3.1), and (4.1), (6.7) hold. Then
\[
|\dot{c}(t)| + |\dot{v}(t)| \leq C(\tilde{v}) \|Z\|_{-\nu}^2 .
\] (6.9)

### 6.2 Decay for transversal dynamics

In Section 11 we will show that our main Theorem 2.5 can be derived from the following time decay of the transversal component \( Z(t) \):

**Proposition 6.3.** Let all conditions of Theorem 2.5 hold. Then \( t_* = \infty \), and
\[
\|Z(t)\|_{-\nu} \leq \frac{C(\rho, \tilde{v}, d_0)}{(1 + |t|)^{3/2}}, \quad t \geq 0 .
\] (6.10)

We will derive (6.10) in Sections 7-10 from equation (4.8) for the transversal component \( Z(t) \). This equation can be specified using Lemma 6.2. Indeed, the lemma implies that
\[
\|T(t)\|_{-\nu} \leq C(\tilde{v}) \|Z(t)\|_{-\nu}^2, \quad 0 \leq t < t_*
\] (6.11)

by (4.10) since \( w - v = \dot{c} \). Thus (4.8) becomes the equation
\[
\dot{Z}(t) = A(t)Z(t) + \tilde{N}(t), \quad 0 \leq t < t_*
\] (6.12)
where \( A(t) = A_{v(t),w(t)} \), and \( \tilde{N}(t) := T(t) + N(t) \) satisfies the estimate
\[
\|\tilde{N}(t)\|_\nu \leq C(\tilde{v},\Omega)\|Z(t)\|_{2,\nu}^2, \quad 0 \leq t < t_*. \tag{6.13}
\]
In all remaining part of our paper we will analyze mainly basic equation \( 6.12 \) to establish decay \( 6.10 \). We are going to derive the decay using bound \( 6.13 \) and orthogonality condition \( 6.6 \).

Similarly [8] we reduce the problem to analysis of frozen linear equation,
\[
\tilde{X}(t) = A_1X(t), \quad t \in \mathbb{R} \tag{6.14}
\]
where \( A_1 = A_{v_1,v_1} \) with \( v_1 = v(t_1) \) and a fixed \( t_1 \in [0,t_*) \). Then we can apply some methods of scattering theory and then estimate the error by the method of majorants.

Note, that even for the frozen equation \( 6.14 \), decay of type \( 6.10 \) for all solutions does not hold without orthogonality condition of type \( 6.6 \). Namely, by \( 5.6 \) equation \( 6.14 \) admits secular solutions
\[
X(t) = \sum_1^3 C_j\tau_j(v) + \sum_1^3 D_j[\tau_j(v)t + \tau_{j+3}(v)] \tag{6.15}
\]
which arise by differentiation of soliton \( 1.4 \) in the parameters \( a \) and \( v \).

**Remark 6.4.** The solution \( 6.15 \) lies in tangent space \( T_{\sigma_1}\mathcal{S} \) with \( \sigma_1 = (b_1,v_1) \) (for an arbitrary \( b_1 \in \mathbb{R} \)) that suggests an unstable character of the nonlinear dynamics along the solitary manifold.

Further, we will apply the corresponding symplectic orthogonal projection which kills “runaway solutions” \( 6.15 \).

**Definition 6.5.** i) For \( v \in \mathcal{V} \), denote by \( \Pi_v \) symplectic orthogonal projection of \( \mathcal{E} \) onto tangent space \( T_{\sigma}\mathcal{S} \), and \( P_v = I - \Pi_v \).

ii) Denote by \( Z_v = P_v\mathcal{E} \) the space symplectic orthogonal to \( T_{\sigma}\mathcal{S} \) with \( \sigma = (b,v) \).

Note that by linearity,
\[
\Pi_v Z = \sum \Pi_{j\ell}(v)\tau_j(v)\Omega(\tau_l(v),Z), \quad Z \in \mathcal{E} \tag{6.16}
\]
with some smooth coefficients \( \Pi_{j\ell}(v) \). Hence, projector \( \Pi_v \), in variable \( y = x - b \), does not depend on \( b \). Now we have the symplectic orthogonal decomposition
\[
\mathcal{E} = T_{\sigma}\mathcal{S} + Z_v, \quad \sigma = (b,v) \tag{6.17}
\]
and symplectic orthogonality \( 6.6 \) can be written in the following equivalent forms,
\[
\Pi_{v(t)} Z(t) = 0, \quad P_{v(t)} Z(t) = Z(t), \quad 0 \leq t < t_* \tag{6.18}
\]

**Remark 6.6.** The tangent space \( T_{\sigma}\mathcal{S} \) is invariant under operator \( A_{v,v} \) by Lemma 5.3 i), hence the space \( Z_v \) also is invariant by \( 5.3 \): \( A_{v,v}Z \in Z_v \) for sufficiently smooth \( Z \in Z_v \).

Below in section 12-16 we will prove the following proposition which will be one of main ingredients for proving \( 6.10 \). Consider the Cauchy problem for equation \( 6.14 \) with \( A = A_{v,v} \) for a fixed \( v \in \mathcal{V} \). Recall that parameter \( \nu > 5/2 \) is also fixed.

**Proposition 6.7.** Let conditions \((1.7)- (1.8)\) hold, \( |v| \leq \tilde{v} < 1 \), and \( X_0 \in \mathcal{E} \). Then
i) Equation \( 6.14 \), with \( A = A_{v,v} \), admits the unique solution \( e^{At}X_0 := X(t) \in C(\mathbb{R},\mathcal{E}) \) with initial condition \( X(0) = X_0 \).

ii) For \( X_0 \in Z_v \cap \mathcal{E}_v \), the decay holds,
\[
\|e^{At}X_0\|_{-\nu} \leq \frac{C_\nu(\rho,\tilde{v})}{(1+|t|)^{3/2}}\|X_0\|_{\nu}, \quad t \in \mathbb{R}. \tag{6.19}
\]
7 Frozen transversal dynamics

Now let us fix an arbitrary \( t_1 \in [0, t_*] \), and rewrite equation (6.12) in a “frozen form”

\[
\dot{Z}(t) = A_1 Z(t) + (A(t) - A_1) Z(t) + \tilde{N}(t), \quad 0 \leq t < t_* \tag{7.1}
\]

where \( A_1 = A_{v(t_1), v(t_1)} \) and

\[
A(t) - A_1 = \begin{pmatrix}
[w(t) - v(t_1)] \cdot \nabla & 0 & 0 & 0 \\
0 & [w(t) - v(t_1)] \cdot \nabla & 0 & 0 \\
0 & 0 & 0 & B_{v(t) - B_{v_1(t)}} \\
0 & 0 & \langle \nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla \rho_j \rangle & 0
\end{pmatrix}
\]

Next trick allows us to kill the “bad terms” \([w(t) - v(t_1)] \cdot \nabla \) in operator \( A(t) - A_1 \).

**Definition 7.1.** Let us change the variables \((y, t) \mapsto (y_1, t) = (y + d_1(t), t)\), where

\[
d_1(t) := \int_{t_1}^{t} (w(s) - v(t_1)) ds, \quad 0 \leq t \leq t_1 . \tag{7.2}
\]

Next define

\[
Z_1(t) := (\Psi_1(y_1 - d_1(t), t), \Psi_2(y_1 - d_1(t), t), Q(t), P(t)) .
\]

Then we obtain final form of the “frozen equation” for the transversal dynamics

\[
\dot{Z}_1(t) = A_1 Z_1(t) + B_1(t) Z_1(t) + \tilde{N}_1(t), \quad 0 \leq t \leq t_1 \tag{7.3}
\]

where \( \tilde{N}_1(t) = \tilde{N}(t) \) expressed in terms of \( y = y_1 - d_1(t) \), and

\[
B_1(t) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_{v(t) - B_{v_1(t)}} \\
0 & 0 & \langle \nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla \rho_j \rangle & 0
\end{pmatrix}.
\]

Let us estimate the “remainder terms” \( B_1(t) Z_1(t) \) and \( \tilde{N}_1(t) \).

**Lemma 7.2.** The bound holds

\[
\|B_1(t) Z_1(t)\|_\nu \leq C(\tilde{v}) \|Z(t)\|_{-\nu} \int_{t_1}^{t} \|Z(s)\|_{-\nu}^2 ds, \quad 0 \leq t \leq t_1 . \tag{7.4}
\]

**Proof.** Lemma 6.2 implies

\[
|B_{v(t)} - B_{v_1(t)}| \leq \int_{t_1}^{t} \dot{v}(s) \cdot \nabla v B_{v(s)} ds \leq C(\tilde{v}) \int_{t_1}^{t} \|Z(s)\|_{-\nu}^2 ds
\]

\[
|\langle \nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla \rho_j \rangle| \leq C(\tilde{v}) \int_{t_1}^{t} \|Z(s)\|_{-\nu}^2 ds.
\]

Therefore,

\[
\|B_1(t) Z_1(t)\|_\nu = |\langle \nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla \rho_j \rangle| Q_1(t) + |B_{v(t)} - B_{v_1(t)} P_1(t) |
\]

\[
\leq C(\tilde{v})(|Q(t)| + |P(t)|) \int_{t_1}^{t} \|Z(s)\|_{-\nu}^2 ds \leq C(\tilde{v}) \|Z(t)\|_{-\nu} \int_{t_1}^{t} \|Z(s)\|_{-\nu}^2 ds.
\]

\( \square \)
Lemma 7.3. The bounds hold

\[ \| \tilde{N}_1(t) \|_\nu \leq C(\hat{\nu}, \mathcal{Q})(1 + |d_1(t)|)^\nu \| Z(t) \|_{2, \nu}^2, \quad 0 \leq t \leq t_1. \]  

Proof. For any \( \Phi \in L^2_\alpha \) and \( d \in \mathbb{R}^3 \) we have

\[ \| \Phi(y - d) \|_\alpha^2 = \int |\Phi(y - d)|^2(1 + |y|)^{2\alpha} dy = \int |\Phi(y)|^2(1 + |y + d|)^{2\alpha} dy \]

\[ \leq \int |\Phi(y)|^2(1 + |y|)^{2\alpha}(1 + |d|)^{2\alpha} dy \leq (1 + |d|)^{2\alpha}\| \Phi \|_\alpha^2, \quad \alpha \in \mathbb{R}. \]

Hence, bound (7.5) follows. \( \square \)

8 Integral inequality

Equation (7.3) can be written in integral form:

\[ Z_1(t) = e^{A_1 t} Z_1(0) + \int_0^t e^{A_1 (t-s)} [B_1 Z_1(s) + \tilde{N}_1(s)] ds, \quad 0 \leq t \leq t_1 \]  

(8.1)

Now we apply symplectic orthogonal projection \( P_1 := P_{v(t_1)} \) to both sides of (8.1):

\[ P_1 Z_1(t) = e^{A_1 t} P_1 Z_1(0) + \int_0^t e^{A_1 (t-s)} P_1 [B_1 Z_1(s) + \tilde{N}_1(s)] ds. \]

Projector \( P_1 \) commutes with the group \( e^{A_1 t} \) since the space \( Z_1 := P_1 E \) is invariant with respect to \( e^{A_1 t} \) by Remark 6.6. Applying (6.19) we obtain that

\[ \| P_1 Z_1(t) \|_{-\nu} \leq C \frac{\| P_1 Z_1(0) \|_\nu}{(1 + t)^{3/2}} + C \int_0^t \frac{\| P_1 [B_1 Z_1(s) + \tilde{N}_1(s)] \|_\nu}{(1 + |t-s|)^{3/2}} ds. \]

Operator \( P_1 = I - \Pi_1 \) is continuous in \( E_\nu \) by (6.16). Hence, (7.4)-(7.5) imply

\[ \| P_1 Z_1(t) \|_{-\nu} \leq \frac{C(\overline{d}_1(0))}{(1 + t)^{3/2}} \| Z(0) \|_{\nu} \]

\[ + C(\overline{d}_1(t)) \int_0^t \frac{1}{(1 + |t-s|)^{3/2}} \left[ \| Z(s) \|_{-\nu} \int_s^t \| Z(\tau) \|_{2, \nu}^2 d\tau + \| Z(s) \|_{2, \nu}^2 \right] ds, \quad 0 \leq t \leq t_1 \]

where \( \overline{d}_1(t) := \sup_{0 \leq s \leq t} |d_1(s)| \). Let us introduce the “majorant”

\[ m(t) := \sup_{s \in [0,t]} (1 + s)^{3/2} \| Z(s) \|_{-\nu}, \quad t \in [0, t_*). \]  

(8.3)

Now we reduce further the exit time. Denote by \( \varepsilon < 1 \) a fixed positive number which we will specify below.

Definition 8.1. \( t_*' \) is the exit time

\[ t_*' = \sup \{ t \in [0, t_*) : m(s) \leq \varepsilon, \quad 0 \leq s \leq t \}. \]  

(8.4)

To estimate \( d_1(t) \), note that

\[ w(s) - v(t_1) = w(s) - v(s) + v(s) - v(t_1) = \dot{c}(s) + \int_s^{t_1} \dot{v}(\tau) d\tau \]  

(8.5)
by (6.8). Hence, (7.2), Lemma 6.2 and definition (8.3) imply that for \( t_1 < t'_* \)

\[
|d_1(t)| = |\int_{t}^{t_1} (w(s) - v(t_1))ds| \leq \int_{t}^{t_1} \left( |\dot{c}(s)| + \int_{s}^{t_1} |\dot{v}(\tau)|d\tau \right) ds \leq C(\tilde{v})m^2(t_1) \leq C(\tilde{v}), \quad 0 \leq t \leq t_1.
\]

Now we can replace \( C(\tilde{d}_1) \) with \( C(\tilde{v}) \) in (8.2): for \( t_1 < t'_* \)

\[
\|P_1Z_1(t)\|_{-\nu} \leq \frac{C(\tilde{v})}{(1 + t)^{3/2}}\|Z(0)\|_{\nu} + C(\tilde{v}) \int_{0}^{t} \frac{1}{(1 + |t - s|^{3/2})} \left[ \|Z(s)\|_{-\nu} \int_{s}^{t} \|Z(\tau)\|_{-\nu}^2 d\tau + \|Z(s)\|_{-\nu}^2 \right] ds, \quad 0 \leq t \leq t_1.
\]

### 9 Symplectic orthogonality

Finally, we are going to change \( P_1Z_1(t) \) by \( Z(t) \) in the left hand side of (8.7). We will prove that it is possible using again that \( d_0 \ll 1 \) in (2.8).

**Lemma 9.1.** (cf.[8]) For sufficiently small \( \varepsilon > 0 \), we have for \( t_1 < t'_* \)

\[
\|Z(t)\|_{-\nu} \leq C\|P_1Z_1(t)\|_{-\nu}, \quad 0 \leq t \leq t_1,
\]

where \( C \) depends only on \( \rho \) and \( \overline{v} \).

**Proof.** Since \( |d_1(t)| \leq C \) for \( t \leq t_1 < t'_* \) then \( \|Z(t)\|_{-\nu} \leq C\|Z_1(t)\|_{-\nu} \), and it suffices to prove that

\[
\|Z_1(t)\|_{-\nu} \leq 2\|P_1Z_1(t)\|_{-\nu}, \quad 0 \leq t \leq t_1.
\]

Recall that \( P_1Z_1(t) = Z_1(t) - \Pi_{v(t_1)}Z_1(t) \). Then estimate (9.2) will follow from

\[
\|\Pi_{v(t_1)}Z_1(t)\|_{-\nu} \leq \frac{1}{2}\|Z_1(t)\|_{-\nu}, \quad 0 \leq t \leq t_1.
\]

Symplectic orthogonality (6.18) implies

\[
\Pi_{v(t),1}Z_1(t) = 0, \quad t \in [0, t_1],
\]

where \( \Pi_{v(t),1}Z_1(t) \) is \( \Pi_{v(t)}Z(t) \) expressed in terms of variable \( y_1 = y + d_1(t) \). Hence, (9.3) follows from (9.4) if difference \( \Pi_{v(t_1)} - \Pi_{v(t),1} \) is small uniformly in \( t \), i.e.

\[
\|\Pi_{v(t_1)} - \Pi_{v(t),1}\| < 1/2, \quad 0 \leq t \leq t_1.
\]

It remains to justify (9.5) for small enough \( \varepsilon > 0 \). Formula (6.16) implies

\[
\Pi_{v(t),1}Z_1(t) = \sum \Pi_{j1}(v(t))\tau_{j1}(v(t))\Omega(t_{11}(v(t)), Z_1(t)),
\]

where \( \tau_{j1}(v(t)) \) are vectors \( \tau_j(v(t)) \) expressed in variable \( y_1 \). Since \( |d_1(t)| \leq C \) and \( \nabla \tau_j \) are smooth and fast decaying at infinity functions, we have

\[
\|\tau_{j1}(v(t)) - \tau_j(v(t))\|_{\nu} \leq C|d_1(t)|^{\nu} \leq C, \quad 0 \leq t \leq t_1
\]
for all \( j = 1, 2, \ldots, 6 \). Furthermore,

\[
\tau_j(v(t)) - \tau_j(v(t_1)) = \int_t^{t_1} \dot{v}(s) \cdot \nabla v \tau_j(v(s)) ds ,
\]

and therefore

\[
\| \tau_j(v(t)) - \tau_j(v(t_1)) \|_\nu \leq C \int_t^{t_1} |\dot{v}(s)| ds , \quad 0 \leq t \leq t_1 . \tag{9.8}
\]

Similarly,

\[
|\Pi_{jl}(v(t)) - \Pi_{jl}(v(t_1))| = |\int_t^{t_1} \dot{v}(s) \cdot \nabla_{jl} \Pi_{jl}(v(s)) ds| \leq C \int_t^{t_1} |\dot{v}(s)| ds , \quad 0 \leq t \leq t_1 . \tag{9.9}
\]

Hence, bounds (9.5) will follow from (6.16), (6.6) and (9.7)-(9.9) if we establish that integral in the right hand side of (9.8) can be made as small as we please by choosing \( \varepsilon > 0 \) small enough. Indeed,

\[
\int_t^{t_1} |\dot{v}(s)| ds \leq Cm^2(t_1) \int_t^{t_1} \frac{ds}{(1 + s)^3} \leq C\varepsilon^2 , \quad 0 \leq t \leq t_1 . \tag{10.10}
\]

\[\Box\]

10 Decay of transversal component

Here we prove Proposition 6.3.

**Step i)** We fix \( 0 < \varepsilon < 1 \) and \( t'_* = t'_*(\varepsilon) \) for which Lemma 9.1 holds. Then bound of type (8.7) holds with \( \| P_1Z_1(t) \|_{-\nu} \) in the left hand side replaced by \( \| Z(t) \|_{-\nu} \):

\[
\| Z(t) \|_{-\nu} \leq \frac{C}{(1 + t)^{3/2}} \| Z(0) \|_{\nu} + C \int_0^t \frac{1}{(1 + |t - s|)^{3/2}} \left[ \| Z(s) \|_{-\nu} \int_s^{t_1} \| Z(\tau) \|_{-\nu} d\tau + \| Z(s) \|_{-\nu}^2 \right] ds , \quad 0 \leq t \leq t_1 \tag{10.11}
\]

for \( t_1 < t'_* \). This implies an integral inequality for majorant \( m(t) \) defined in (8.3). Namely, multiplying both sides of (10.1) by \( (1 + t)^{3/2} \), and taking supremum in \( t \in [0, t_1] \), we get

\[
m(t_1) \leq C \| Z(0) \|_{\nu} + C \sup_{t \in [0, t_1]} \int_0^t \frac{(1 + t)^{3/2}}{(1 + |t - s|)^{3/2}} \left[ \frac{m(s)}{(1 + s)^{3/2}} \int_s^{t_1} m^2(\tau) d\tau + \frac{m^2(s)}{(1 + s)^3} \right] ds , \quad t \leq t'_*
\]

for \( t \leq t'_* \). Taking into account that \( m(t) \) is a monotone increasing function, we get

\[
m(t_1) \leq C \| Z(0) \|_{\nu} + C \sup_{t \in [0, t_1]} \left[ m^3(t_1) + m^2(t_1) \right] I(t_1) , \quad t_1 \leq t'_*
\]

where

\[
I(t_1) = \sup_{t \in [0, t_1]} \int_0^t \frac{(1 + \tau)^{3/2}}{(1 + |t - s|)^{3/2}} \left[ \frac{1}{(1 + s)^{3/2}} \int_s^{t_1} \frac{d\tau}{(1 + \tau)^3} + \frac{1}{(1 + s)^3} \right] ds \leq \mathcal{T} < \infty .
\]

Therefore, (10.2) becomes

\[
m(t_1) \leq C \| Z(0) \|_{\nu} + C \mathcal{T} \left[ m^3(t_1) + m^2(t_1) \right] , \quad t_1 < t'_* . \tag{10.3}
\]
This inequality implies that \(m(t_1)\) is bounded for \(t_1 < t'_*.\) and moreover,

\[
m(t_1) \leq C_1 \|Z(0)\|_\nu, \quad t_1 < t'_* \tag{10.4}
\]

since \(m(0) = \|Z(0)\|_\nu\) is sufficiently small by (3.10).

**Step ii)** The constant \(C_1\) in estimate (10.4) does not depend on \(t_*\) and \(t'_*\) by Lemma 9.1. We choose \(d_0\) in (2.8) so small that \(\|Z(0)\|_\nu < \varepsilon/(2C_1)\). It is possible due to (3.10). Then estimate (10.4) implies that \(t'_* = t_*\) and therefore (10.4) holds for all \(t_1 > 0\) if \(d_0\) is small enough. It complete proof of Proposition 6.3.

### 11 Soliton asymptotics

Here we prove our main Theorem 2.5 under the assumption that decay (6.10) holds. First we will prove asymptotics (2.9) for vector components, and afterwards asymptotics (2.10) for the fields.

**Asymptotics for vector components.** From (4.3) we have \(\dot{q} = \dot{b} + \dot{Q}\), and from (6.12), (6.13), (4.9) it follows that \(\dot{Q} = P + \mathcal{O}(\|Z\|_{\nu}^2)\). Thus,

\[
\dot{q} = \dot{b} + \dot{Q} = v(t) + \dot{c}(t) + P(t) + \mathcal{O}(\|Z\|_{\nu}^2) \tag{11.1}
\]

Bounds (6.9) and (6.10) imply that

\[
|\dot{c}(t)| + |\dot{v}(t)| \leq \frac{C_1(\rho, \overline{\nu}, d_0)}{(1 + t)^3}, \quad t \geq 0. \tag{11.2}
\]

Therefore, \(c(t) = c_+ + \mathcal{O}(t^{-2})\) and \(v(t) = v_+ + \mathcal{O}(t^{-2}), t \to \infty\). Since \(|P| \leq \|Z\|_{-\nu}\), estimate (6.10), and (11.1)-(11.2), imply that

\[
\dot{q}(t) = v_+ + \mathcal{O}(t^{-3/2}), \quad b(t) = c(t) + \int_0^t v(s)ds = v_+ t + a_+ + \mathcal{O}(t^{-1}).
\]

Hence second part of (1.6) follows:

\[
q(t) = b(t) + Q(t) = v_+ t + a_+ + \mathcal{O}(t^{-1})
\]

since \(Q(t) = \mathcal{O}(t^{-3/2})\) by (6.10).

**Asymptotics for fields.** For field part of the solution \(\psi(x, t)\) let us define the “accompanying soliton field” as \(\psi_{\nu(t)}(x - q(t))\), where we define now \(q(t) = \dot{q}(t)\), cf. (11.1). Then for difference \(z(x, t) = \psi(x, t) - \psi_{\nu(t)}(x - q(t))\) we obtain the equation

\[
\dot{z}(x, t) = [\alpha \cdot \nabla - i\beta m]z(x, t) - i\nu \cdot \nabla \psi_{\nu(t)}(x - q(t)).
\]

Then

\[
z(t) = W_0(t)z(0) - \int_0^t W_0(t - s)[i\nu \cdot \nabla \psi_{\nu(s)}(\cdot - q(s))]ds. \tag{11.3}
\]

To obtain asymptotics (2.10) it suffices to prove that \(z(t) = W_0(t)\phi_++r_+ (t)\) with some \(\phi_+ \in L^2_0\) and \(\|r_+(t)\|_0 = \mathcal{O}(t^{-1/2})\). This is equivalent to

\[
W_0(-t)z(t) = \phi_+ + r'_+(t) \tag{11.4}
\]
where \( \|r_2^\ast (t)\|_0 = \mathcal{O}(t^{-1/2}) \) since \( W_0(t) \) is a unitary group in \( L^2_0 \) by charge conservation for the free Dirac equation. Finally, (11.4) holds since (11.3) implies
\[
W_0(-t)z(t) = z(0) - \int_0^t W_0(-s)f(s)ds , \quad f(s) = i\dot{v}(s) \cdot \nabla \psi_{v(s)}(\cdot - q(s))
\]
where the integral in the right hand side converges in \( L^2_0 \) with rate \( \mathcal{O}(t^{-1/2}) \). The latter holds since \( \|W_0(-s)f(s)\|_0 = \mathcal{O}(s^{-3/2}) \) by unitarity of \( W_0(-s) \) and the decay rate \( \|f(s)\|_0 = \mathcal{O}(s^{-3/2}) \). Let us prove this rate of decay. It suffices to prove that \( |\dot{v}(s)| = \mathcal{O}(s^{-3/2}) \), or equivalently \( |\dot{p}(s)| = \mathcal{O}(s^{-3/2}) \). Substitute (4.2) to last equation of (1.1) and obtain
\[
\dot{p}(t) = \Re \int [\dot{\psi}_{v(t)}(x - b(t)) + \Psi(x - b(t), t)] \cdot \nabla \rho(x - b(t) - Q(t))dx
\]
\[
= \Re \int \dot{\psi}_{v(t)}(y) \cdot \nabla \rho(y)dy + \Re \int \psi_{v(t)}(y) \cdot [\nabla \rho(y - Q(t)) - \nabla \rho(y)]dy
\]
\[
+ \Re \int \Psi(y, t) \cdot \nabla \rho(y - Q(t))dy .
\]
First integral in the right hand side is zero by stationary equations (2.4). The second integral is \( \mathcal{O}(t^{-3/2}) \), since \( Q(t) = \mathcal{O}(t^{-3/2}) \), and by conditions (1.7) on \( \rho \). Finally, the third integral is \( \mathcal{O}(t^{-3/2}) \) by estimate (6.10). The proof is complete.

### 12 Decay for linearized dynamics

In remaining sections we prove Proposition 6.7. Applying the Gronwall inequality to frozen linear equation (6.14) we obtain
\[
\|X(t)\|_\varepsilon \leq Ce^{\alpha t}\|X(0)\|_\varepsilon \tag{12.1}
\]
with some \( \alpha > 0 \). Now we can apply the Fourier-Laplace transform
\[
\hat{X}(\lambda) = \int_0^\infty e^{-\lambda t}X(t)dt , \quad \Re \lambda > \alpha \tag{12.2}
\]
to (6.14). Integral (12.2) converges and is analytic for \( \Re \lambda > \alpha \). We will write \( A \) and \( v \) instead of \( A_1 \) and \( v_1 \) in remaining part of the paper. After the Fourier-Laplace transform, equation (6.14) reads
\[
\lambda \hat{X}(\lambda) = A\hat{X}(\lambda) + X_0 , \quad \Re \lambda > \alpha . \tag{12.3}
\]
We will construct the resolvent \( R(\lambda)) = (A - \lambda)^{-1} \) for \( \Re \lambda > 0 \) and prove that it is a continuous operator in \( \mathcal{E}_{-\nu} \). Then \( \hat{X}(\lambda) = -(A - \lambda)^{-1}X_0 \in \mathcal{E}_{-\nu} \) and is an analytic function for \( \Re \lambda > 0 \).

This analyticity and the Paley-Wiener arguments (see [9]) should provide existence of a \( \mathcal{E}_{-\nu} \) - valued distribution \( X(t) \), \( t \in \mathbb{R} \), with a support in \([0, \infty)\). Formally,
\[
X(t) = \Lambda^{-1}\hat{X} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t}\hat{X}(i\omega + 0) d\omega , \quad t \in \mathbb{R} \tag{12.4}
\]
To check continuity of \( X(t) \) for \( t \geq 0 \), we need additionally an asymptotics for \( \hat{X}(i\omega + 0) \) at large \( |\omega| \). Finally, for time decay of \( X(t) \), we need an additional information on smoothness and decay of \( \hat{X}(i\omega + 0) \). More precisely, we should prove that \( \hat{X}(i\omega + 0) \)

\begin{enumerate}
  \item[i)] is smooth outside \( \omega = 0 \) and \( \omega = \pm \mu \), where \( \mu = \mu(v) > 0 \);
  \item[ii)] decays in a certain sense as \( |\omega| \to \infty \);
  \item[iii)] admits the Puiseux expansion at \( \omega = \pm \mu \);
  \item[iv)] is analytic at \( \omega = 0 \) if \( X_0 \in \mathcal{Z}_\nu := \mathcal{P}_\nu \mathcal{E} \) and \( X_0 \in \mathcal{E}_\nu \).
\end{enumerate}

Then decay (6.19) would follow from the Fourier-Laplace representation (12.4).
13 Solving linearized equation

Here we construct the resolvent. By (12.3)

\[
(A - \lambda) \begin{pmatrix}
\Psi_1 \\
\Psi_2 \\
Q \\
P
\end{pmatrix} = - \begin{pmatrix}
\Psi_{01} \\
\Psi_{02} \\
Q_0 \\
P_0
\end{pmatrix}.
\]

It is system of equations

\[
\begin{align*}
(-\alpha_1 \partial_1 - \alpha_3 \partial_3 + v \cdot \nabla - \lambda)\Psi_1 + (\beta m + \alpha_2 \partial_2)\Psi_2 - \hat{Q} \cdot \nabla \rho_2 &= -\Psi_{01} \\
-(\beta m + \alpha_2 \partial_2)\Psi_1 + (-\alpha_1 \partial_1 - \alpha_3 \partial_3 + v \cdot \nabla - \lambda)\Psi_2 + \hat{Q} \cdot \nabla \rho_1 &= -\Psi_{02} \\
B_v \hat{P} - \lambda \hat{Q} &= -Q_0 \\
-\left\langle \nabla \Psi_j, \rho_j \right\rangle + \left\langle \nabla \psi_{vj}, \hat{Q} \cdot \nabla \rho_j \right\rangle - \lambda \hat{P} &= -P_0
\end{align*}
\]

(13.1)

**Step i)** Let us study first two equations. First, we compute matrix integral kernel \(G_L(y - y')\) of the Green operator

\[
G_L = \begin{pmatrix}
-\alpha_1 \partial_1 - \alpha_3 \partial_3 + v \cdot \nabla - \lambda & \beta m + \alpha_2 \partial_2 \\
-\beta m + \alpha_2 \partial_2 & -\alpha_1 \partial_1 - \alpha_3 \partial_3 + v \cdot \nabla - \lambda
\end{pmatrix}^{-1}.
\]

(13.2)

In Fourier space

\[
\hat{G}_L(k) = \begin{pmatrix}
i\alpha_1 k_1 + i\alpha_3 k_3 - iv \cdot k - \lambda & \beta m + \alpha_2 k_2 \\
-\beta m + \alpha_2 k_2 & i\alpha_1 k_1 + i\alpha_3 k_3 - iv \cdot k - \lambda
\end{pmatrix}^{-1}.
\]

To invert the matrix, we solve the system

\[
\begin{align*}
af_1 + bf_2 &= g_1 \\
-bf_1 + af_2 &= g_2
\end{align*}
\]

(13.3)

where \(a = i\alpha_1 k_1 + i\alpha_3 k_3 - iv \cdot k - \lambda\), \(b = \beta m - \alpha_2 k_2\). Multiplying first equation of (13.3) by \(c = -i\alpha_1 k_1 - i\alpha_3 k_3 - iv \cdot k - \lambda\) and the second equation by \(-b\), we obtain

\[
\begin{align*}
ca f_1 + cbf_2 &= cg_1 \\
b^2 f_1 - cbf_2 &= -bg_2
\end{align*}
\]

(13.4)

since \(ba = cb\) by anticommutations (1.2). Further, \(b^2 + ac = k^2 + m^2 + (iv \cdot k + \lambda)^2\). Therefore, summing up equations (13.4), we obtain that

\[
f_1 = \frac{cg_1 - bg_2}{k^2 + m^2 + (iv \cdot k + \lambda)^2}
\]

Similarly, we obtain

\[
f_2 = \frac{bg_1 + cg_2}{k^2 + m^2 + (iv \cdot k + \lambda)^2}
\]

Hence

\[
\hat{G}_L(k) = \frac{1}{k^2 + m^2 + (iv \cdot k + \lambda)^2} \begin{pmatrix}
-i\alpha_1 k_1 - i\alpha_3 k_3 - iv \cdot k - \lambda & \beta m + \alpha_2 k_2 \\
\beta m - \alpha_2 k_2 & -\alpha_1 k_1 - i\alpha_3 k_3 - iv \cdot k - \lambda
\end{pmatrix}
\]

(13.5)
Taking the inverse Fourier transform we obtain

\[ G_\lambda(y) = \left( \begin{array}{ccc} \alpha_1 \partial_1 + \alpha_3 \partial_3 + v \cdot \nabla - \lambda & -\beta m - \alpha_2 \partial_2 \\ \beta m - \alpha_2 \partial_2 & \alpha_1 \partial_1 + \alpha_3 \partial_3 + v \cdot \nabla - \lambda \end{array} \right) g_\lambda(y) \]  

(13.6)

where

\[ g_\lambda(y) = F_{k \to y}^{-1} \frac{1}{k^2 + m^2 + (iv \cdot k + \lambda)^2}, \quad y \in \mathbb{R}^3 \]  

(13.7)

Note that denominator in RHS (13.7) does not vanish for \( \text{Re}\lambda > 0 \) since \(|v| < 1\). This implies

**Lemma 13.1.** Operator \( G_\lambda \) with integral kernel \( G_\lambda(y - y') \), is continuous operator \( L^2_0 \oplus L^2_0 \rightarrow L^2_0 \oplus L^2_0 \) for \( \text{Re}\lambda > 0 \).

From now on we use system of coordinates in \( y \)-space in which \( v = (|v|, 0, 0) \), hence \( v \cdot k = |v| k_1 \). Let us compute the function \( g_\lambda(y) \). One has

\[ k^2 + m^2 + (i|v|k_1 + \lambda)^2 = \frac{1}{\gamma^2}(k_1^2 + k_2^2 + k_3^2 + 2i|v|k_1 \lambda + \lambda^2 + m^2 = \frac{1}{\gamma^2}(k_1 + i\gamma^2|v|\lambda)^2 + k_2^2 + k_3^2 + \gamma^2 \]  

where

\[ \gamma = 1/\sqrt{1 - v^2}, \quad \gamma^2 = \frac{v^2 \lambda^2}{1 - v^2} + \lambda^2 + m^2 = \frac{\lambda^2}{1 - v^2} + m^2 = \gamma^2(\lambda^2 + \mu^2), \quad \mu := m/\gamma. \]  

Hence,

\[ g_\lambda(y) = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{-ikydk}}{\gamma^2(k_1 + i\gamma^2|v|\lambda)^2 + k_2^2 + k_3^2 + \gamma^2} = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{-ikydk}}{k_1^2 + k_2^2 + k_3^2 + \gamma^2} \]  

\[ = \frac{e^{-\gamma|v|\lambda \hat{y}_1}}{(2\pi)^{3/2}} \int \frac{e^{-ikydk}}{k_1^2 + k_2^2 + k_3^2 + \gamma^2} = e^{-\gamma|v|\lambda \hat{y}_1} R_0(\tilde{y}, -\gamma^2) \]  

(13.9)

Here \( \hat{y}_1 = \gamma y_1, \tilde{y} = (\gamma y_1, y_2, y_3) \), and \( R_0(y, \zeta) = (-\Delta - \zeta)^{-1} \). It is well known that \( R_0(y, \zeta) = e^{i\sqrt{\gamma|v|}/4\pi|y|} \). Therefore,

\[ g_\lambda(y) = \frac{e^{-\gamma|\tilde{y}| - \gamma \hat{y}_1}}{4\pi|\tilde{y}|}, \quad \gamma = \gamma \sqrt{\lambda^2 + \mu^2}, \quad \gamma_1 := \gamma |v| \lambda \]  

(13.10)

We choose \( \text{Re}\kappa > 0 \) for \( \text{Re}\lambda > 0 \). Note that for \( 0 < |v| < 1 \)

\[ 0 < \text{Re} \gamma < \text{Re} \gamma, \quad \text{Re} \lambda > 0. \]  

(13.11)

Let us state the result which we have got above.

**Lemma 13.2.** i) The function \( g_\lambda(y) \) decays exponentially in \( y \) for \( \text{Re}\lambda > 0 \).

ii) Formulas (13.10) and (13.8) imply that for every fixed \( y \), function \( g_\lambda(y) \) admits an analytic continuation in \( \lambda \) to the Riemann surface of algebraic function \( \sqrt{\lambda^2 + \mu^2} \) with branching points \( \lambda = \pm i\mu \).

Thus, from (13.1) and (13.2) we obtain the representation

\[ \tilde{\Psi}_1 = -G^{11}_\lambda \Psi_{01} - G^{12}_\lambda \Psi_{02} - (G^{12}_\lambda \nabla \rho_1) \cdot \dot{Q} + (G^{11}_\lambda \nabla \rho_2) \cdot \dot{Q}, \]  

\[ \tilde{\Psi}_2 = -G^{21}_\lambda \Psi_{01} - G^{22}_\lambda \Psi_{02} - (G^{22}_\lambda \nabla \rho_1) \cdot \dot{Q} + (G^{21}_\lambda \nabla \rho_2) \cdot \dot{Q} \]  

(13.12)
Step ii) Now we proceed to last two equations (13.1):
\[-\lambda \hat{Q} + B \hat{\nabla} \hat{P} = -Q_0, \quad \langle \nabla \psi_{v,j} , \hat{Q} \cdot \nabla \rho_j \rangle - \langle \nabla \hat{\Psi}_j , \rho_j \rangle - \lambda \hat{P} = -P_0 \]  \hspace{1cm} (13.13)
We rewrite equations (13.12) as \( \hat{\Psi}_j = \hat{\Psi}(\hat{Q}) + \hat{\Psi}(\Psi_0) \), where
\[
\hat{\Psi}_1(\Psi_0) = -G^{11}_\lambda \Psi_{01} - G^{12}_\lambda \Psi_{02}, \quad \hat{\Psi}_2(\Psi_0) = -G^{21}_\lambda \Psi_{01} - G^{22}_\lambda \Psi_{02} \]  \hspace{1cm} (13.14)
\[
\hat{\Psi}_1(\hat{Q}) = (-G^{11}_\lambda \nabla \rho_1 + G^{11}_\lambda \nabla \rho_2) \cdot \hat{Q}, \quad \hat{\Psi}_2(\hat{Q}) = (-G^{22}_\lambda \nabla \rho_1 + G^{22}_\lambda \nabla \rho_2) \cdot \hat{Q}. \]  \hspace{1cm} (13.15)
Then \( \langle \nabla \hat{\Psi}_j , \rho_j \rangle = \langle \nabla \hat{\Psi}_j(\hat{Q}) , \rho_j \rangle + \langle \nabla \hat{\Psi}_j(\Psi_0) , \rho_j \rangle \), and last equation (13.13) becomes
\[
\langle \nabla \psi_{v,j} , \hat{Q} \cdot \nabla \rho_j \rangle - \langle \nabla \hat{\Psi}_j(\hat{Q}) , \rho_j \rangle - \lambda \hat{P} = -P_0 + \langle \nabla \hat{\Psi}_j(\Psi_0) , \rho_j \rangle =: -P_0 - \Phi(\lambda) \]
where
\[
\Phi(\lambda) = \langle \hat{\Psi}_j(\Psi_0) , \nabla \rho_j \rangle. \]  \hspace{1cm} (13.16)
First we compute the term
\[
\langle \nabla \psi_{v,j} , \hat{Q} \cdot \nabla \rho_j \rangle = \sum_{ij} \langle \nabla \psi_{v,j} , \hat{Q}_i \partial \rho_j \rangle = \sum_{ij} \langle \nabla \psi_{v,j} , \partial_j \rho_j \rangle \hat{Q}_i. \]  \hspace{1cm} (13.17)
Applying the Fourier transform \( F_{y \rightarrow k} \), we have by the Parseval identity and (A.25) that
\[
\sum_j (\partial_i \psi_{v,j} , \partial_j \rho_j) = \sum_j (\langle -ik_i \hat{\psi}_{v1} , -ik_i \hat{\rho}_1 \rangle - \langle \hat{\psi}_{v2} , -ik_i \hat{\rho}_2 \rangle dk = \int k_i k_l B(k) dk \]
\[
= \int \frac{k_i k_l B(k) dk}{k^2 + m^2 - |v| k_1|^2} = -L_{ii}. \]
As a result, \( \langle \nabla \psi_{v,j} , \hat{Q} \cdot \nabla \rho_j \rangle = -L_{ii} \hat{Q} \), where \( L \) is a \( 3 \times 3 \) matrix with matrix elements \( L_{ii} \).
The matrix \( L \) is diagonal and positive defined by (1.8).
Now we compute the term \( -\langle \nabla \hat{\Psi}_j(\hat{Q}) , \rho_j \rangle = \langle \hat{\Psi}_j(\hat{Q}) , \nabla \rho_j \rangle \). One has
\[
\langle \nabla \hat{\Psi}_j(\hat{Q}) , \partial \rho_j \rangle = \sum_l \left( \langle -G^{12}_\lambda \partial_i \rho_1 + G^{11}_\lambda \partial_i \rho_2 , \partial_i \rho_1 \rangle - \langle G^{22}_\lambda \partial_i \rho_1 - G^{21}_\lambda \partial_i \rho_2 , \partial_i \rho_2 \rangle \right) \hat{Q}_l = \sum_l H_{il}(\lambda) \hat{Q}_l \]
and by the Parseval identity and (1.2)-(1.3) we have
\[
H_{il}(\lambda) : = \langle -G^{12}_\lambda \partial_i \rho_1 + G^{11}_\lambda \partial_i \rho_2 , \partial_i \rho_1 \rangle - \langle G^{22}_\lambda \partial_i \rho_1 - G^{21}_\lambda \partial_i \rho_2 , \partial_i \rho_2 \rangle \\
= \langle \{[\alpha m - \alpha \lambda] k_1 + i k_1 + i k_1 + \lambda \} \hat{\rho}_1 , \{[\alpha \lambda - \alpha m] k_1 + i k_1 + i k_1 + \lambda \} \hat{\rho}_2 \rangle \\
+ \langle \{[\alpha \lambda - \alpha m] k_1 + i k_1 + i k_1 + \lambda \} \hat{\rho}_1 , \{[\alpha m - \alpha \lambda] k_1 + i k_1 + i k_1 + \lambda \} \hat{\rho}_2 \rangle \\
= \int k_i k_l m \frac{\beta \hat{\rho}_1 \cdot \hat{\rho}_1 + \beta \hat{\rho}_2 \cdot \hat{\rho}_2}{k^2 + m^2 - |v| k_1|^2} dk = \int k_i k_l B(k) dk \]
The matrix \( H \) is well defined for \( \text{Re} \lambda > 0 \) since the denominator does not vanish. The matrix \( H \) is diagonal. Indeed, if \( i \neq l \), then at least one of these indices is not equal to one, and the integrand in (13.17) is odd with respect to the corresponding variable. Thus, \( H_{il} = 0 \). As a result, \( \langle \hat{\Psi}_j(\hat{Q}) , \nabla \rho_j \rangle = H_{ii} \hat{Q} \), where \( H \) is the matrix with matrix elements \( H_{ii} \). Finally, (13.13) becomes
\[
M(\lambda) \left( \frac{\hat{Q}}{\hat{P}} \right) = \left( \begin{array}{c} Q_0 \\ P_0 + \Phi(\lambda) \end{array} \right), \text{ where } M(\lambda) = \left( \begin{array}{cc} \lambda E & -B \lambda E \\ L - H(\lambda) \end{array} \right). \]  \hspace{1cm} (13.19)
Assume for a moment that the matrix \( M(\lambda) \) is invertible (later we will prove this). Then we obtain
\[
\left( \frac{\hat{Q}}{\hat{P}} \right) = M^{-1}(\lambda) \left( \begin{array}{c} Q_0 \\ P_0 + \Phi(\lambda) \end{array} \right), \text{ Re } \lambda > 0. \]  \hspace{1cm} (13.20)
Finally, formula (13.20) and formulas (13.12), where \( \hat{Q} \) is expressed from (13.20), give the expression of the resolvent \( R(\lambda) = (A - \lambda)^{-1} \), Re \( \lambda > 0 \).
Lemma 13.3. i) $M(\lambda)$ admits an analytic continuation from complex half-plane $\text{Re}\, \lambda > 0$ to the Riemann surface $\Sigma$ of the function $\sqrt{\mu^2 + \lambda^2}$; ii) $M(\lambda)$ is Hölder continuous on each compact set in $\Sigma$; iii) $M^{-1}(\lambda)$ is meromorphic on $\Sigma$.

Proof. i) The analytic continuation of $M(\lambda)$ exists by Lemma 13.1, conditions (1.7), and last line in (13.18):

$$H_{jj}(\lambda) = \langle mg_\lambda \beta \ast \partial_j \rho_1, \partial_j \rho_1 \rangle + \langle mg_\lambda \beta \ast \partial_j \rho_2, \partial_j \rho_2 \rangle, \quad j = 1, 2, 3$$

(13.21)

since $g_\lambda$ is analytic on $\Sigma$ by (13.10).

ii) The Hölder continuity holds by the same arguments.

iii) The inverse matrix is meromorphic since it exists for large $\text{Re}\, \lambda$. The latter follows from (13.19) since $H(\lambda) \rightarrow 0$, $\text{Re}\, \lambda \rightarrow \infty$, by (13.18).

\[\square\]

14 Inverse matrix

Here we study smoothness of $M^{-1}(\lambda)$ on imaginary axis and in half-plane $\text{Re}\, \lambda > 0$.

14.1 Regularity on imaginary axis

By Lemma 13.3, the limit matrix $M(i \omega) := M(i \omega + 0)$ exists for $\omega \in \mathbb{R}$, and its entries are continuous functions of $\omega \in \mathbb{R}$, smooth for $|\omega| < \mu$ and $|\omega| > \mu$.

Proposition 14.1. Let $\rho$ satisfy conditions (1.7)- (1.8), and $|v| < 1$. Then the matrix $M(i \omega)$ is invertible for $\omega \in \mathbb{R} \setminus 0$.

This proposition follows by methods from [8, Proposition 15.1].

Now let us obtain asymptotics of $M^{-1}(\lambda)$ near singular points $\lambda = 0$ and $\lambda = \pm i \mu$.

I. First we consider the points $\lambda = \pm i \mu$.

Lemma 14.2. The asymptotics hold

$$M^{-1}(\lambda) = \mathcal{O}^\pm + \mathcal{O}((\lambda \mp i \mu)^{\frac{3}{2}}), \quad \partial M^{-1}(\lambda) = \mathcal{O}((\lambda \mp i \mu)^{-\frac{3}{2}}), \quad \partial^2 M^{-1}(\lambda) = \mathcal{O}((\lambda \mp i \mu)^{-\frac{5}{2}})$$

(14.1)

Proof. It suffices to prove similar asymptotics for $M(\lambda)$. Then (14.1) holds also for $M^{-1}(\lambda)$, since the matrices $M(\pm i \mu)$ are invertible. The asymptotics for $M(\lambda)$ hold by convolution representation (13.21) since $g_\lambda$ admits the corresponding asymptotics by (13.10). Namely

$$g_\lambda(y) = \frac{1}{4\pi |y|} + r_\pm(\lambda, y), \quad \lambda \rightarrow \pm i \mu, \quad \text{Re}\, \lambda > 0$$

where

$$r_\pm(\lambda, y) = \mathcal{O}((\lambda \mp i \mu)^{\frac{3}{2}}), \quad \partial_\lambda r_\pm(\lambda, y) = \mathcal{O}((\lambda \mp i \mu)^{-\frac{3}{2}}), \quad \partial^2_\lambda r_\pm(\lambda, y) = \mathcal{O}((1 + |y|)(\lambda \mp i \mu)^{-\frac{5}{2}})$$

Condition (1.7) provides convergence of all integrals arising in $\partial^k_\lambda H_{jj}$.

II. Second, we consider the point $\omega = 0$ which is an isolated pole of a finite degree by Lemma 13.3. In Appendix B we prove that determinant of $M(i \omega)$ can be written as

$$\det M(i \omega) = -\omega^6 \left(1 + \frac{f_{11}(\omega)}{\gamma^3}\right) \left(1 + \frac{f_{22}(\omega)}{\gamma}\right) \left(1 + \frac{f_{33}(\omega)}{\gamma}\right)$$

(14.2)

where $f_{jj}(\omega) \in C^\infty(-\mu, \mu)$ and $f_{jj}(0) > 0$. 
14.2 Behavior at infinity

Here we study asymptotic behavior of $M^{-1}(\lambda)$ at infinity.

**Lemma 14.3.** There exist a matrix $D_0$ and a matrix-function $D_1(\lambda)$, such that

$$M^{-1}(\lambda) = \frac{D_0}{\lambda} + D_1(\lambda), \quad |\lambda| \to \infty, \quad \text{Re} \lambda > 0$$

(14.3)

where

$$|\partial_k^i D_1(\lambda)| \leq \frac{C(k)}{|\lambda|^2}, \quad |\lambda| \to \infty, \quad \text{Re} \lambda > 0, \quad k = 0, 1, 2.$$  

(14.4)

**Proof.** The structure of $M(\lambda)$ provides that it suffices to prove the estimate for $H_{jj}(\lambda)$:

$$|\partial_k^i H_{jj}(\lambda)| \leq C(k), \quad \lambda \in \mathbb{C}, \quad |\lambda| \geq \mu + 1, \quad j = 1, 2, 3, \quad k = 0, 1, 2.$$  

(14.5)

This estimate follows from representation (13.21) and the bounds

$$|g_\lambda(y)| \leq \frac{C_1}{|y|}, \quad |\partial_y g_\lambda(y)| \leq \frac{C_2}{|y|} + C_3, \quad |\partial_y^2 g_\lambda(y)| \leq \frac{C_4}{|y|} + C_5 |y|, \quad \text{Re} \lambda > 0 \quad \Box$$

14.3 Analyticity in half-plane

**Lemma 14.4.** $M^{-1}(\lambda)$ is holomorphic in $\mathbb{C}^+ := \{\lambda \in \mathbb{C} : \text{Re} \lambda > 0\}$.

**Proof.** We apply a bifurcation argument. Namely, we replace $\rho$ by $\varepsilon \rho$ with $\varepsilon \in [0, 1]$, and write $M_\varepsilon(\lambda)$ for the corresponding matrix $M(\lambda)$ with $\varepsilon^2 L$ and $\varepsilon^2 H(\lambda)$ instead of $L$ and $H(\lambda)$. Then (13.19) in the case $\varepsilon = 0$ yields

$$M_0(\lambda) = \begin{pmatrix} \lambda E & -B_e \\ 0 & \lambda E \end{pmatrix}$$

(14.6)

Hence, $M_0^{-1}(\lambda)$ is a holomorphic matrix function for $\lambda \in \mathbb{C}^+$. Let us extend this analyticity to $M_\varepsilon^{-1}(\lambda)$ with $\varepsilon \in (0, 1]$.

**Step i)** Asymptotics of type (14.3)-(14.4) hold for $M_\varepsilon^{-1}(\lambda)$ uniformly in $\varepsilon \in [0, 1]$. Therefore, there exists an $R > 0$ such that $M_\varepsilon^{-1}(\lambda)$ is a holomorphic matrix function of $\lambda \in \mathbb{C}^+$ with $|\lambda| > R$ for all $\varepsilon \in (0, 1]$.

**Step ii)** Similarly, formulas (14.2), (14.6) imply asymptotics

$$\det M_\varepsilon(\lambda) \sim \lambda^6, \quad \lambda \to 0$$

which hold uniformly in $\varepsilon \in [0, 1]$. Therefore, there exists a $\delta_1 > 0$ such that

a) $M_\varepsilon^{-1}(\lambda)$ is holomorphic in the semicircle $\lambda \in \mathbb{C}^+, \quad |\lambda| < 2\delta_1$ for all $\varepsilon \in (0, 1]$;

b) $M_\varepsilon^{-1}(\lambda)$ is bounded on the ring $\mathcal{R}(\delta_1) := \{\lambda \in \mathbb{C}^+ : \delta_1 < |\lambda| < 2\delta_1\}$ uniformly in $\varepsilon \in (0, 1]$.

**Step iii)** Proposition 14.1 implies, by a continuity argument, that $M_\varepsilon^{-1}(\lambda)$ is bounded on the set $\{i\omega : \omega \in \mathbb{R}, \quad \delta_1 < |\omega| < R + 1\}$ uniformly in $\varepsilon \in (0, 1]$. Furthermore, the Hölder continuity from Lemma 13.3 ii) is obviously uniform in $\varepsilon \in (0, 1]$. Hence, $M_\varepsilon^{-1}(\lambda)$ is holomorphic and bounded on a region

$$\Pi(R, \delta_1, \delta_2) := \{\lambda \in \mathbb{C} : \delta_1 < |\text{Im} \lambda| < R + 1, \quad 0 < \text{Re} \lambda \leq \delta_2\}$$

uniformly in $\varepsilon \in (0, 1]$, where $\delta_2 = \delta(R, \delta_1) > 0$.

**Step iv)** Finally, we consider a closed path $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ lies in the union $\Pi(R, \delta_2) \cup \mathcal{R}(\delta_1)$, and $\Gamma_2$ lies on the semicircle $|\lambda| = R$, $\text{Re} \lambda > 0$. By the arguments above, $M_\varepsilon^{-1}(\lambda)$ is bounded on $\Gamma$ uniformly in $\varepsilon \in [0, 1]$. Therefore, $M_\varepsilon^{-1}(\lambda)$ is holomorphic inside $\Gamma$ for $\varepsilon \in (0, 1]$ as well as for $\varepsilon = 0$. \qed
15 Transversal decay for linearized equation

Here we prove Proposition 6.7. First, we establish decay in weighted norm for the solution to free Dirac equation.

15.1 Weighted decay for free Dirac equation

Denote by $W^\pm_v(t)$ dynamical groups (propagators) of the following “modified” free Dirac equations

$$\partial_t \Psi^\pm(x, t) = [\pm \alpha \cdot \nabla + i \beta m + v \cdot \nabla] \Psi^\pm(x, t). \quad (15.1)$$

Lemma 15.1. For any $\Phi \in L^2_\nu$ with $\nu > 3/2$ the bound holds

$$\|W^\pm_v(t)\Phi\|_\nu \leq \frac{C_\nu(v)\|\Phi\|_\nu}{(1 + |t|)^{3/2}}, \quad t \geq 0. \quad (15.2)$$

Proof. Step i) For concreteness we consider the case “+”. Note, that

$$(\partial_t + \alpha \cdot \nabla + i \beta m + v \cdot \nabla)(\partial_t - \alpha \cdot \nabla - i \beta m + v \cdot \nabla) = \partial_t^2 - \Delta + (v \cdot \nabla)^2 + 2v \cdot \nabla \partial_t m^2).$$

Hence the integral kernel $W^+_v(x - y, t)$ of the operator $W^+_v(t)$ reads

$$W^+_v(z, t) = (\partial_t + \alpha \cdot \nabla + i \beta m + v \cdot \nabla) G_v(z, t), \quad (15.3)$$

where $G_v(z, t)$ is a fundamental solution of the ”modified” Klein-Gordon operator

$$(\partial_t^2 - \Delta + (v \cdot \nabla)^2 + 2v \cdot \nabla \partial_t m^2)G_v(z, t) = \delta(z)\delta(t).$$

Let $G_v(t), t \geq 0$ be the operator with the integral kernel $G_v(x - y, t)$. It is easy to see that

$$[G_v(t)\Phi](x) = [G_0(t)\Phi](x - vt), \quad x \in \mathbb{R}^3, \quad t \geq 0.$$  

Then

$$G_v(z, t) = G_0(z-vt, t) = \frac{\delta(t - |z - vt|)}{4\pi t} \frac{m \theta(t - |z - vt|)}{4\pi} \frac{J_1(m \sqrt{t^2 - |z - vt|^2})}{\sqrt{t^2 - |z - vt|^2}}, \quad t > 0$$

where $J_1$ is the Bessel function of order 1, and $\theta$ is the Hettavside function. Let us fix an arbitrary $\varepsilon \in (|v|, 1)$. Well known asymptotics of the Bessel function imply that

$$|\partial_{t^j} G_v(z, t)|, |\partial_{z^j} G_v(z, t)| \leq C(\varepsilon)(1 + t)^{-3/2}, \quad |z - vt| \leq \varepsilon t, \quad t \geq 1, \quad j = 1, 2, 3. \quad (15.4)$$

Step ii) Consider an arbitrary $t \geq 1$. Denote $\varepsilon_1 = \varepsilon - |v|$. We split the function $\Phi$ in two terms, $\Phi = \Phi_1, t + \Phi_2, t$ such that

$$\|\Phi_1, t\|_{L^2} + \|\Phi_2, t\|_{L^2} \leq C\|\Phi\|_{L^2}, \quad t \geq 1 \quad (15.5)$$

and

$$\Phi_1, t(x) = 0 \quad \text{for} \quad |x| > \frac{\varepsilon_1 t}{2}, \quad \text{and} \quad \Phi_2, t(x) = 0 \quad \text{for} \quad |x| < \frac{\varepsilon_1 t}{4}. \quad (15.6)$$

Estimate (15.2) for $W^+_v(t)\Phi_2, t$ follows by charge conservation for Dirac equation, (15.5) and (15.6):

$$\|W^+_v(t)\Phi_2, t\|_{L^2_\nu} \leq \|W^+_v(t)\Phi_2, t\|_{L^2} = \|\Phi_2, t\|_{L^2} \leq \frac{C(\varepsilon)\|\Phi_2, t\|_{L^2}}{(1 + t)^\nu} \leq \frac{C_1(\varepsilon)\|\Phi\|_{L^2}}{(1 + t)^{3/2}}, \quad t \geq 1 \quad (15.7)$$
Finally, (15.7), (15.8) and (15.12) imply (15.2).

The matrix kernel of the operator $\zeta W^+(t)\Phi_{1,t}$ is equal to

$$W^+_v(x-y,t) = \zeta(|x|/t)W^+_v(x-y,t)\chi_t(y)$$

Since $\zeta(|x|/t) = 0$ for $|x| > \epsilon_1 t/2$ and $\chi_t(y) = 0$ for $|y| > \epsilon_1 t/2$ then $W^+_v(x-y,t) = 0$ for $|x-y| > \epsilon_1 t$. On the other hand, $|x-y| \leq \epsilon t$ implies $|x-y-vt| \leq \epsilon t$, since $\epsilon_1 + |v| = \epsilon$ by definition of $\epsilon_1$. Hence, (15.3) and (15.4) yield

$$|W^+_v(x-y,t)| \leq C(1+t)^{-3/2}, \quad t \geq 1.$$  \hspace{1cm} (15.10)

The norm of the operator $\zeta W^+_v(t)\chi_t : L^2 \rightarrow L^2$ is equivalent to the norm of the operator

$$(x)^{-\nu}\zeta W^+_v(t)\chi_t(y)(y)^{-\nu} : L^2 \rightarrow L^2$$ \hspace{1cm} (15.11)

Therefore, (15.10) implies that operator (15.11) is Hilbert-Schmidt operator since $\nu > 3/2$, and its Hilbert-Schmidt norm does not exceed $C(1+t)^{-3/2}$. Hence, by (15.9)

$$\|\zeta W^+_v(t)\Phi_{1,t}\|_{L^2} \leq C(1+t)^{-3/2}\|\Phi\|_{L^2}, \quad t \geq 1.$$ \hspace{1cm} (15.12)

Finally, (15.7), (15.8) and (15.12) imply (15.2).

\[\square\]

### 15.2 Decay of vector components

Here we establish the decay (6.19) for $Q(t)$ and $P(t)$.

**Lemma 15.2.** Let $X_0 \in \mathbb{Z}_v \cap \mathcal{E}_v$. Then $Q(t)$, $P(t)$ are continuous and

$$|Q(t)| + |P(t)| \leq C_{\nu}(\rho, \delta)(1+|t|)^{-3/2}, \quad t \geq 0.$$ \hspace{1cm} (15.13)

**Proof.** The components $Q(t)$ and $P(t)$ are given by the Fourier integral

$$\left( \begin{array}{c} Q(t) \\ P(t) \end{array} \right) = \frac{1}{2\pi} \int e^{i\omega t} M^{-1}(i\omega) \left( \begin{array}{c} Q_0 \\ P_0 + \Phi(i\omega) \end{array} \right) d\omega$$ \hspace{1cm} (15.14)

where $M = -\Delta + V(x)$, and $V(x)$ is a first order potential.
with $\Phi(i\omega) := \Phi(i\omega + 0)$ where $\Phi(\lambda)$ is defined in (13.16). The integral converges in the sense of distributions to a continuous function of $t \geq 0$ by (14.1), (14.3) and (B.31). Note that the condition $X_0 \in \mathcal{Z}_v$ implies that the whole trajectory $X(t)$ lies in $\mathcal{Z}_v$. This follows from the invariance of the space $\mathcal{Z}_v$ under the generator $A_{v,v}$ (cf. Remark 6.6). If $X_0 \notin \mathcal{Z}_v$, then $Q(t)$ and $P(t)$ may contain non-decaying terms which correspond to singular point $\omega = 0$ since the linearized dynamics admits the secular solutions without decay, see (6.15). We will show that the symplectic orthogonality condition leads to (15.13). We split integral (15.14) into three terms using the partition of unity (6.15). We then sense of distributions to a continuous function of $X$ that the condition $Z$ from the invariance of the space $\mathcal{Z}_v$.

\[ \left( \begin{array}{c} Q(t) \\ P(t) \end{array} \right) = \frac{1}{2\pi} \int e^{i\omega t} (\zeta_1(\omega) + \zeta_2(\omega) + \zeta_3(\omega)) M^{-1}(i\omega + 0) \left( \begin{array}{c} Q_0 \\ P_0 + \Phi(i\omega) \end{array} \right) d\omega = \sum_{j=1}^{3} I_j(t) \]

where the functions $\zeta_j(\omega) \in C^\infty(\mathbb{R})$ are supported by

\[ \begin{align*}
\text{supp } \zeta_1 &\subset \{ \omega \in \mathbb{R} : \varepsilon_0/2 < |\omega| < \mu + 2 \} \\
\text{supp } \zeta_2 &\subset \{ \omega \in \mathbb{R} : |\omega| > \mu + 1 \} \\
\text{supp } \zeta_3 &\subset \{ \omega \in \mathbb{R} : |\omega| < \varepsilon_0 \}
\end{align*} \tag{15.15} \]

i) Let us represent $I_j(t)$, $j = 1, 2$ as

\[ I_j(t) = \frac{1}{2\pi} \int e^{i\omega t} \zeta_j(\omega) \left[ M^{-1}(i\omega) \left( \begin{array}{c} Q_0 \\ P_0 \end{array} \right) + M^{-1}(i\omega) \left( \begin{array}{c} 0 \\ \Phi(i\omega) \end{array} \right) \right] d\omega \]

\[ = s_j(t) \left( \begin{array}{c} Q_0 \\ P_0 \end{array} \right) + s_j(t) \ast \left( \begin{array}{c} 0 \\ f(t) \end{array} \right) \tag{15.16} \]

where

\[ s_j(t) = \Lambda^{-1} \zeta_j(\omega) M^{-1}(i\omega + 0), \quad f(t) = \Lambda^{-1} \Phi(i\omega). \tag{15.17} \]

By (13.14) and (15.1)

\[ \tilde{\Psi}_1(\Psi_0) = -\Lambda \text{Re} W_v^+(t) \Psi_0, \quad \tilde{\Psi}_2(\Psi_0) = -\Lambda \text{Im} W_v^+(t) \Psi_0. \]

Hence, (13.16), (15.17) and Lemma 15.1 imply

\[ |f(t)| = |\text{Re}(W_v^+(t) \Psi_0, \nabla \rho)| \leq C_v(\rho, v)(1 + t)^{-3/2}. \]

Further, the function $s_1(t)$ decays as $(1 + |t|)^{-3/2}$ by (14.1), and $s_2(t)$ decays as $(1 + |t|)^{-2}$ due to Proposition 14.3. Hence, $I_1(t)$ and $I_2(t)$ decay as $(1 + |t|)^{-3/2}$ by (15.16).

iii) Finally, the function $I_3(t)$ decays as $t^{-\infty}$ since

\[ \left( \begin{array}{c} \tilde{Q}(i\omega) \\ \tilde{P}(i\omega) \end{array} \right) = M^{-1}(i\omega) \left( \begin{array}{c} Q_0 \\ P_0 + \Phi(i\omega) \end{array} \right) \in C^\infty(-\mu, \mu) \quad \text{if} \quad X_0 \in \mathcal{Z}_v. \]

Indeed, in Appendix C we prove that the symplectic orthogonality conditions (6.7) at $t = 0$ imply

\[ P_0 + \Phi(0) = 0, \quad B_v^{-1} Q_0 + \Phi'(0) = 0. \tag{15.18} \]

Then

\[ P_0 + \Phi(i\omega) = \Phi(i\omega) - \Phi(0) = i\omega \Upsilon_1(\omega) \]

\[ B_v^{-1} Q_0 + \Upsilon_1(\omega) = \frac{\Phi(i\omega) - \Phi(0)}{i\omega} - \Phi'(0) = i\omega \Upsilon_2(\omega) \]
where $\Upsilon_j(\omega) \in C^\infty(-\mu, \mu)$ by (13.6), (13.14) and (13.16). Therefore, (B.31)-(B.32) imply
\[
\hat{P}(i\omega) = M_{21}(\omega)Q_0 + iM_{22}(\omega)\Upsilon_1(\omega) \in C^\infty(-\mu, \mu)
\]
\[
\hat{Q}(i\omega) = \frac{1}{\omega}M_{11}(\omega)Q_0 + \frac{i}{\omega}M_{12}(\omega)\Upsilon_1(\omega) = \frac{i}{\omega}M_{12}(B_{\nu}^{-1}Q_0 + \Upsilon_1(\omega))
\]
\[= -M_{12}\Upsilon_2(\omega) \in C^\infty(-\mu, \mu) \]
\[\square\]

### 15.3 Decay of fields

Here we prove the decay (6.19) for the field components $\Psi_1(y, t), \Psi_2(y, t)$. First two equations of (6.14) may be written as one equation:
\[
\Psi(y, t) = [-\alpha \cdot \nabla - i\beta m + v \cdot \nabla]\Psi(y, t) - iQ(t) \cdot \nabla \rho(y) , \quad x \in \mathbb{R}^3 , \quad t \in \mathbb{R} \quad (15.19)
\]
where $\Psi(y, t) = \Psi_1(y, t) + i\Psi_2(y, t)$. Applying the Duhamel representation, we obtain
\[
\Psi(t) = W_v^-(t)\Psi_0 - \int_0^t W_v^-(t-s)Q(s) \cdot \nabla \rho \, ds , \quad t \geq 0
\]
where $W_v^-(t)$ is defined in section 15.1. Hence, Lemma 15.1 and the decay of $Q$ from (15.13) yield
\[
\|\Psi(t)||_{-\nu} \leq C_{\nu}(\rho, \tilde{v})\|\Psi_0||_{\nu}(1 + |t|)^{-3/2} , \quad t \geq 0 . \quad (15.20)
\]
It completes the proof of Proposition 6.7.

### A Computing symplectic form

Here we justify formulas (3.6)-(3.8) for the matrix $\Omega$.

1) First, the Parseval identity implies
\[
\Omega(\tau_j, \tau_l) = \langle \partial_j \psi_{v1}, \partial_l \psi_{v2} \rangle - \langle \partial_j \psi_{v2}, \partial_l \psi_{v1} \rangle = \int k_j k_l \, dk (\hat{\psi}_{v1} \cdot \hat{\psi}_{v2} - \hat{\psi}_{v2} \cdot \hat{\psi}_{v1}) = 0 , \quad j, l = 1, 2, 3 .
\]
since the integrand is odd function.

2) Second, we consider
\[
\Omega(\tau_{j+3}, \tau_{l+3}) = \langle \partial_{vj} \psi_{v1}, \partial_{vl} \psi_{v2} \rangle - \langle \partial_{vj} \psi_{v2}, \partial_{vl} \psi_{v1} \rangle . \quad (A.21)
\]

Let us derive the formulas for $\psi_{v1}$ and $\psi_{v2}$. First equation of (2.4) implies
\[
[(v \cdot \nabla)^2 - \Delta + m^2]\psi_v = [iv \cdot \nabla + i\alpha \cdot \nabla - \beta m]\rho_1 .
\]
Hence
\[
[(v \cdot \nabla)^2 - \Delta + m^2]\psi_{v1} = -[v \cdot \nabla + \alpha_1 \partial_1 + \alpha_3 \partial_3]\rho_2 - [\alpha_2 \partial_2 + \beta m]\rho_1 ,
\]
\[
[(v \cdot \nabla)^2 - \Delta + m^2]\psi_{v2} = [v \cdot \nabla + \alpha_1 \partial_1 + \alpha_3 \partial_3]\rho_1 - [\alpha_2 \partial_2 + \beta m]\rho_2 .
\]
Applying the Fourier transform, we obtain
\[
\hat{\psi}_{v1} = \frac{[iv \cdot k + i\alpha_1 k_1 + i\alpha_3 k_3]\hat{\rho}_2 + [\alpha_2 k_2 - \beta m]\hat{\rho}_1}{-(v \cdot k)^2 + k^2 + m^2}
\]
\[
\hat{\psi}_{v2} = \frac{[iv \cdot k + i\alpha_1 k_1 + i\alpha_3 k_3]\hat{\rho}_1 + [\alpha_2 k_2 - \beta m]\hat{\rho}_2}{-(v \cdot k)^2 + k^2 + m^2} . \quad (A.22)
\]
Differentiating, we get

$$
\partial_{v_l} \hat{\psi}_{v_1} = \frac{i k_j \dot{\rho}_2}{-(v \cdot k)^2 + k^2 + m^2} + \frac{2 k_j v \cdot k \hat{\psi}_{v_1}}{-(v \cdot k)^2 + k^2 + m^2}
$$

$$
\partial_{v_l} \hat{\psi}_{v_2} = \frac{-i k_j \dot{\rho}_1}{-(v \cdot k)^2 + k^2 + m^2} + \frac{2 k_j v \cdot k \hat{\psi}_{v_2}}{-(v \cdot k)^2 + k^2 + m^2}
$$

(A.23)

Hence, (A.21) implies

$$
\Omega(\tau_{j+3}, \tau_{n+3}) = \int k_j k_l \left[ \dot{\rho}_1 \cdot \dot{\rho}_2 - \dot{\rho}_2 \cdot \dot{\rho}_1 \right] dk + \int 4 k_j k_l (v \cdot k)^2 \left[ \dot{\psi}_{v_1} \cdot \dot{\psi}_{v_2} - \dot{\psi}_{v_2} \cdot \dot{\psi}_{v_1} \right] dk
$$

$$
+ \int \frac{2 i k_j k_l v \cdot k [\dot{\rho}_2 \cdot \dot{\psi}_{v_2} + \dot{\psi}_{v_2} \cdot \dot{\rho}_2 + \dot{\rho}_1 \cdot \dot{\psi}_{v_1} + \dot{\psi}_{v_1} \cdot \dot{\rho}_1]}{(k^2 + m^2 - (v \cdot k)^2)^2} dk = 0
$$

since all integrands are odd functions.

3) Finally, (A.23) implies

$$
\Omega(\tau_j, \tau_{n+3}) = - \langle \partial_{v_l} \hat{\psi}_{v_1}, \partial_{v_l} \hat{\psi}_{v_1} \rangle + \langle \partial_{v_1} \hat{\psi}_{v_1}, \partial_{v_l} \hat{\psi}_{v_1} \rangle + e_j \cdot \partial_{v_1} p_v
$$

$$
= \int k_j k_l \frac{[\dot{\psi}_{v_1} \cdot \dot{\rho}_1 + \dot{\psi}_{v_2} \cdot \dot{\rho}_2] - i k_j k_l \dot{\rho}_2 \cdot \dot{\psi}_{v_1} \cdot [i k_j k_l \dot{\rho}_2 + 2 k_j k_l v \cdot k \dot{\psi}_{v_1}] \rangle dk + e_j \cdot \partial_{v_1} p_v
$$

$$
= \int \frac{k_j k_l [-\dot{\psi}_{v_1} \cdot \dot{\rho}_1 + \dot{\psi}_{v_2} \cdot \dot{\rho}_2] + 2 i k_j k_l v \cdot k [\dot{\psi}_{v_1} \cdot \dot{\psi}_{v_2} - \dot{\psi}_{v_2} \cdot \dot{\psi}_{v_1}]}{(k^2 + m^2 - (v \cdot k)^2)} dk + e_j \cdot \partial_{v_1} p_v
$$

(A.24)

Recall, that $\rho_j(x)$ are even, then $\dot{\rho}_j(k)$ are real. Hence (1.2)-(1.3) and (A.22) imply

$$(k^2 + m^2 - (v \cdot k)^2)\langle \dot{\psi}_{v_1} \cdot \dot{\rho}_1 + \dot{\psi}_{v_2} \cdot \dot{\rho}_2 \rangle = [\alpha_2 k_2 - \beta m] \dot{\rho}_1 \cdot \dot{\rho}_1 + [\alpha_2 k_2 - \beta m] \dot{\rho}_2 \cdot \dot{\rho}_2
$$

$$
+ [i v \cdot k + i \alpha_1 k_1 + i \alpha_3 k_3] \dot{\rho}_2 \cdot \dot{\rho}_1 - [i v \cdot k + i \alpha_1 k_1 + i \alpha_3 k_3] \dot{\rho}_1 \cdot \dot{\rho}_2 = -B \dot{\rho} \cdot \dot{\rho}
$$

(A.25)

$$
(k^2 + m^2 - (v \cdot k)^2)\langle \dot{\psi}_{v_1} \cdot \dot{\psi}_{v_2} - \dot{\psi}_{v_1} \cdot \dot{\psi}_{v_1} \rangle = 2i (k^2 + m^2 - (v \cdot k)^2)^2 \text{Im} \langle \dot{\psi}_{v_1} \cdot \dot{\psi}_{v_2} \rangle
$$

$$
= -2 \beta m \dot{\rho}_1 \cdot [i v \cdot k + i \alpha_1 k_1 + i \alpha_3 k_3] \dot{\rho}_1 - 2 [i v \cdot k + i \alpha_1 k_1 + i \alpha_3 k_3] \dot{\rho}_2 \cdot \beta m \dot{\rho}_2
$$

(A.26)

Substituting (A.25) and (A.26) into the right hand site of (A.24), we obtain

$$
\Omega(\tau_j, \tau_{n+3}) = \int k_j k_l \left( \frac{B(k)}{(k^2 + m^2 - (v \cdot k)^2)^2} + \frac{4 (v \cdot k)^2 B(k)}{(k^2 + m^2 - (v \cdot k)^2)^3} \right) dk + e_j \cdot \partial_{v_1} p_v
$$

that correspond to (3.6) - (3.8).

**B  Computing inverse matrix**

Denote $F(\omega) := -L + H(\omega i + 0)$ which is diagonal. Then by (13.19) for $\omega \in \mathbb{R}$ we obtain

$$
\det M(i \omega) = \det \begin{pmatrix} i \omega E & -B_v \\ -F(\omega) & i \omega E \end{pmatrix} = - \left( \omega^2 + \frac{F_{11}(\omega)}{\gamma^3} \right) \left( \omega^2 + \frac{F_{22}(\omega)}{\gamma^3} \right) \left( \omega^2 + \frac{F_{33}(\omega)}{\gamma^3} \right)
$$

(B.27)

where

$$
F_{jj}(\omega) = \int k_j^2 B dk \left( \frac{1}{m^2 + k^2 - (|v| k_1 + \omega)^2} - \frac{1}{m^2 + k^2 - (|v| k_1)^2} \right), \quad j = 1, 2, 3.
$$

(B.28)
Indeed, formula (B.28) implies that

Here we derive conditions (15.18) from the symplectic orthogonality conditions (6.7). 

C Symplectic orthogonality conditions

Formula (B.27) is obvious since both matrices $F(\omega)$ and $B_v$ are diagonal, hence the matrix $M(i\omega)$ is equivalent to three independent matrices $2 \times 2$. Namely, let us transpose the columns and rows of the matrix $M(i\omega)$ in the order $(142536)$. Then we get the matrix with three $2 \times 2$ blocks on the main diagonal. Therefore, the determinant of $M(i\omega)$ is product of the determinants of these three matrices. Further,

$$
M^{-1}(i\omega) = \begin{pmatrix}
-\frac{i\omega}{\omega^2 + \gamma^2 + F_{11}} & 0 & 0 & -\frac{1}{\omega^2 + \gamma^2 + F_{11}} & 0 & 0 \\
0 & -\frac{i\omega}{\omega^2 + \gamma^2 + F_{22}} & 0 & 0 & -\frac{1}{\omega^2 + \gamma^2 + F_{22}} & 0 \\
0 & 0 & -\frac{i\omega}{\omega^2 + \gamma^2 + F_{33}} & 0 & 0 & -\frac{1}{\omega^2 + \gamma^2 + F_{33}} \\
-\frac{1}{\gamma^2 + F_{11}} & 0 & 0 & 0 & -\frac{1}{\gamma^2 + F_{11}} & 0 \\
0 & -\frac{1}{\gamma^2 + F_{22}} & 0 & 0 & -\frac{1}{\gamma^2 + F_{22}} & 0 \\
0 & 0 & -\frac{1}{\gamma^2 + F_{33}} & 0 & 0 & -\frac{1}{\gamma^2 + F_{33}} 
\end{pmatrix}
$$

(B.29)

where $F_{jj} = F_{jj}(\omega)$. Let us prove that for $\omega \in (-\mu, \mu)$

$$
F_{jj}(\omega) = \omega^2 f_{jj}(\omega), \quad f_{jj}(\omega) \in C^\infty(-\mu, \mu), \quad f_{jj}(0) > 0.
$$

(B.30)

Indeed, formula (B.28) implies that $F_{jj}(0) = 0$. Differentiating (B.28), we obtain

$$
F''_{jj}(0) = 2 \int k^2 B(k) dk \frac{|v| k_1}{(k^2 + m^2 - (|v| k_1)^2} = 0
$$

since integrand is odd function in respect to $k_1$, and

$$
F'''_{jj}(0) = 2 \int k^2 B(k) dk \frac{k^2 + m^2 + 3(|v| k_1)^2}{(k^2 + m^2 - (|v| k_1)^2)^3} > 0.
$$

By (B.30) we can represent the matrices $M^{-1}(i\omega)$ as

$$
M^{-1}(i\omega) = \begin{pmatrix}
\frac{1}{\omega} \mathcal{M}_{11}(\omega) & \frac{1}{\omega} \mathcal{M}_{12}(\omega) \\
\mathcal{M}_{21}(\omega) & \frac{1}{\omega} \mathcal{M}_{22}(\omega)
\end{pmatrix}
$$

(B.31)

where

$$
\mathcal{M}_{11}(\omega) = \mathcal{M}_{22}(\omega) = \begin{pmatrix}
\frac{-i\gamma}{\gamma^2 + F_{11}} & 0 & 0 \\
0 & \frac{i\gamma}{\gamma + F_{22}} & 0 \\
0 & 0 & \frac{i\gamma}{\gamma + F_{33}}
\end{pmatrix}
$$

$$
\mathcal{M}_{12}(\omega) = \begin{pmatrix}
\frac{-1}{\gamma^2 + F_{11}} & 0 & 0 \\
0 & \frac{-1}{\gamma + F_{22}} & 0 \\
0 & 0 & \frac{-1}{\gamma + F_{33}}
\end{pmatrix}, \quad \mathcal{M}_{21}(\omega) = \begin{pmatrix}
\frac{-1}{\gamma^2 + F_{11}} & 0 & 0 \\
0 & \frac{-1}{\gamma + F_{22}} & 0 \\
0 & 0 & \frac{-1}{\gamma + F_{33}}
\end{pmatrix}
$$

where $f_{jj} := f_{jj}(\omega)$, $M_{ij}(\omega) \in C^\infty(-\mu, \mu)$, and

$$
M_{11} = i\mathcal{M}_{12}B_v^{-1}.
$$

(B.32)
On the other hand, by (13.5) formulas (A.22) read
\[ \hat{\psi}_{v1} = -\hat{G}_0^{11} \hat{\rho}_2 + \hat{G}_0^{12} \hat{\rho}_1, \quad \hat{\psi}_{v2} = \hat{G}_0^{11} \hat{\rho}_1 + \hat{G}_0^{12} \hat{\rho}_2. \]

Hence, for \( j = 1, 2, 3 \)
\[
0 = -\Omega(Z_0, \tau_j) = \langle \Psi_{v1}, \partial_j \psi_{v2} \rangle - \langle \Psi_{v2}, \partial_j \psi_{v1} \rangle + P_0 \cdot e_j
= -\langle \Psi_{v1}, ik_j (\hat{G}_0^{11} \hat{\rho}_1 + \hat{G}_0^{12} \hat{\rho}_2) \rangle + \langle \Psi_{v2}, ik_j (\hat{G}_0^{12} \hat{\rho}_1 - \hat{G}_0^{11} \hat{\rho}_2) \rangle + P_0 \cdot e_j = (\Phi(0) + P_0)_j
\]
since \((\hat{G}_0^{11})^* = -\hat{G}_0^{11}, (\hat{G}_0^{12})^* = \hat{G}_0^{12}\). Hence the first condition (15.18) follows. Further,
\[
\partial_{\lambda} \hat{G}_0^{11}\big|_{\lambda=0} = \frac{-1 - 2\imath v \cdot k \hat{G}_0^{11}}{k^2 + m^2 - (v \cdot k)^2}, \quad \partial_{\lambda} \hat{G}_0^{12}\big|_{\lambda=0} = \frac{-2\imath v \cdot k \hat{G}_0^{12}}{k^2 + m^2 - (v \cdot k)^2}.
\]
Then (13.14) and (13.16) imply for \( j = 1, 2, 3 \)
\[
(\Phi'(0))_j = -\left( \frac{\hat{\Psi}_{v1} + 2\imath v \cdot k (\hat{G}_0^{11} \hat{\Psi}_{v1} + \hat{G}_0^{12} \hat{\Psi}_{v2})}{k^2 + m^2 - (v \cdot k)^2}, ik_j \hat{\rho}_1 \right) - \left( \frac{\hat{\Psi}_{v2} + 2\imath v \cdot k (\hat{G}_0^{11} \hat{\Psi}_{v2} - \hat{G}_0^{12} \hat{\Psi}_{v1})}{k^2 + m^2 - (v \cdot k)^2}, ik_j \hat{\rho}_2 \right)
\]
On the other hand, from (A.22) and (A.23) it follows that for \( j = 1, 2, 3 \)
\[
\partial_{v_j} \hat{\psi}_{v1} = \frac{ik_j \hat{\rho}_2 + 2k_j v \cdot k (-\hat{G}_0^{11} \hat{\rho}_2 + \hat{G}_0^{12} \hat{\rho}_1)}{k^2 + m^2 - (v \cdot k)^2}, \quad \partial_{v_j} \hat{\psi}_{v2} = \frac{-ik_j \hat{\rho}_1 + 2k_j v \cdot k (\hat{G}_0^{11} \hat{\rho}_1 + \hat{G}_0^{12} \hat{\rho}_2)}{k^2 + m^2 - (v \cdot k)^2}
\]
Hence,
\[
0 = \Omega(Z_0, \tau_{j+3}) = \langle \Psi_{v1}, \partial_{v_j} \psi_{v2} \rangle - \langle \Psi_{v2}, \partial_{v_j} \psi_{v1} \rangle + Q_0 \cdot \partial_{v_j} p_v
= \langle \Psi_{v1}, -ik_j \hat{\rho}_1 + 2k_j v \cdot k (\hat{G}_0^{11} \hat{\rho}_1 + \hat{G}_0^{12} \hat{\rho}_2) \rangle - \langle \Psi_{v2}, ik_j \hat{\rho}_2 + 2k_j v \cdot k (\hat{G}_0^{12} \hat{\rho}_1 - \hat{G}_0^{11} \hat{\rho}_2) \rangle + Q_0 \cdot \partial_{v_j} p_v = (\Phi'(0) + B_v^{-1} Q_0)_j, \quad j = 1, 2, 3
\]
since \( Q_0 \cdot \partial_{v_j} p_v = Q_0 \cdot B_v^{-1} e_j = B_v^{-1} Q_0 \cdot e_j \). Hence the second condition (15.18) follows.

References


