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On nonlinear wave equations with parabolic potentials

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Abstract. We introduce a new class of piece-wise quadratic potentials for nonlinear wave equations with a kink solutions. The potentials allow an exact description of the spectral properties for the linearized equation at the kink. This description is necessary for the study of the stability properties of the kinks.

In particular, we construct examples of the potentials of Ginzburg–Landau type providing the asymptotic stability of the kinks [6] and [7].

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1. Introduction

Last two decades there was an outstanding activity in the field of asymptotic stability of solitary waves for nonlinear Schrödinger equations [1], [2], [3], [11], [12], [13], [15], [16], [17], and [18], nonlinear Klein-Gordon equations [5] and [14], relativistic Ginzburg–Landau equations [6] and [7], and other Hamiltonian PDEs [8] and [10]. All these results rely on different assumptions on the spectral properties of the corresponding linearized dynamics. On the other hand, the examples were mostly unknown. Here we construct a model nonlinear wave equations, providing various spectral properties: different number of the eigenvalues, absence of the resonances, and Fermi golden rule.

In particular, we construct the examples of relativistic Ginzburg–Landau equations providing all properties assumed in [6] and [7]. The properties imply the asymptotic stability of kinks for real solutions to 1D nonlinear Ginzburg–Landau equations

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R}, \tag{1.1}$$

where $F(\psi) = -U'(\psi)$.

We assume the following conditions.

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Condition U1. For some $K > 3$ and $m > 0$ the potential $U(\psi)$ is smooth even function satisfying

$$(\psi) > 0, \quad \psi \neq a,$$

$$U(\psi) = \frac{m^2}{2}(\psi \mp a)^2 + \mathcal{O}(|\psi \mp a|^K), \quad \psi \to \pm a.$$

(see. Figure 1).

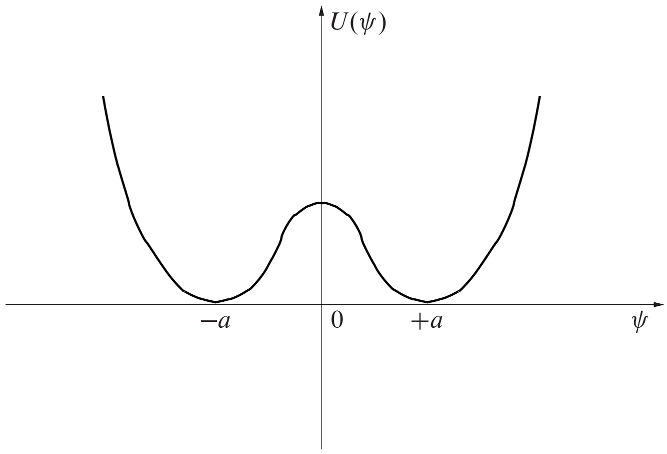

Figure 1. Potential of Ginzburg–Landau type.

The corresponding stationary equation reads

$$s''(x) - U'(s(x)) = 0, \quad x \in \mathbb{R}. \tag{1.2}$$

Constant stationary solutions are: $\psi(x) \equiv 0$ and $\psi(x) \equiv \pm a$. There are also the "kinks", i.e. nonconstant finite energy solutions $s(x)$ to (1.2) such that

$$s(x) \to \pm a, \quad x \to \pm\infty.$$

Condition **U1** implies that $(s(x) \mp a)'' \sim m^2(s(x) \mp a)$ for $x \to \pm\infty$, hence

$$|s(x) \mp a| \sim Ce^{-m|x|}, \quad x \to \pm\infty. \tag{1.3}$$

(see. Figure 2).

Due to relativistic invariance of equation (1.1) the moving kinks

$$s_{q,v}(x,t) = s(\kappa(x - vt - q)), \quad q, v \in \mathbb{R}, |v| < 1, \kappa = 1/\sqrt{1 - v^2}$$

also are the solutions to (1.1). Let us linearize equation (1.1) at the kink $s(x)$. Substituting $\psi(x,t) = s(x) + \varphi(x,t)$, we obtain formally

$$\ddot{\varphi}(x,t) = -H\varphi(x,t) + \mathcal{O}(|\varphi(x,t)|^2),$$

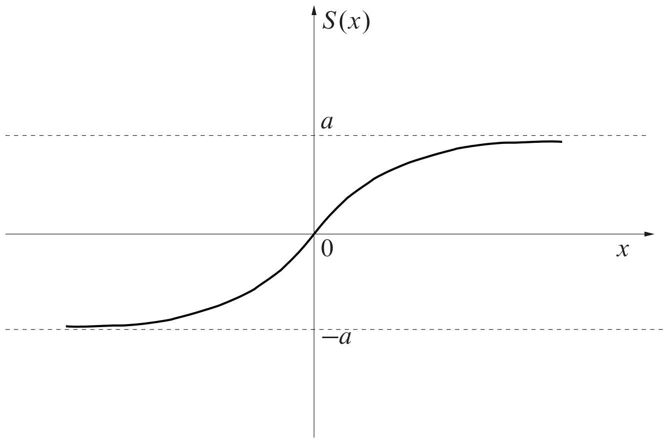

Figure 2. Kink.

where H is the Schrödinger operator

$$H \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + m^2 + W(x)$$

with the potential

$$W(x) = -F'(s(x)) - m^2 = U''(s(x)) - m^2.$$

Condition **U1** and the asymptotics (1.3) imply that

$$|W(x)| = \mathcal{O}(|s(x) \mp a|^{K-1}) \sim Ce^{-(K-1)m|x|}, \quad x \to \pm\infty.$$

The next properties of H hold true.

H1. The continuous spectrum of H is $\sigma_c = [m^2, \infty)$.

H2. The point $\lambda_0 = 0$ belongs to the discrete spectrum, and corresponding eigenfunction is $s'(x)$.

H3. Since $s'(x) > 0$, the point $\lambda_0 = 0$ is the groundstate, and all remaining discrete spectrum is contained in $(0, m^2]$.

To establish an asymptotic stability of the kinks $s_{q,v}(x, t)$ one need certain spectral properties of H (cf. [6] and [7]).

Condition U2. The edge point $\lambda = m^2$ of the continuous spectrum is neither eigenvalue nor resonance.

Condition U3. The discrete spectrum of H consists of two points: $\lambda_0 = 0$ and $\lambda_1 \in (0, m^2)$ satisfying

$$4\lambda_1 > m^2. \tag{1.4}$$

We assume also a non-degeneracy condition known as "Fermi golden rule" meaning the strong coupling of the nonlinear term to the continuous spectrum. This coupling provides the energy radiation to infinity (cf. condition (10.0.11) in [2] and condition (1.11) in [7]).

Condition U4. The following inequality holds:

$$\int \varphi_{4\lambda_1}(x) F''(s(x)) \varphi^2_{\lambda_1}(x) dx \neq 0.$$

where $\varphi_{4\lambda_1}$ is the nonzero odd solution to $H\varphi_{4\lambda_1} = 4\lambda_1\varphi_{4\lambda_1}$.

Note that the known quartic double well Ginzburg–Landau potential $U_{GL}(\psi) = (\psi^2 - a^2)^2/(4a^2)$ satisfies condition **U1** with $m^2 = 2$ and $K = 3$ as well as conditions **U3** and **U4**. However, there exist the resonance for the corresponding operator H at the edge point $\lambda = m^2$. Hence, the asymptotic stability of the kinks for U_{GL} is the open problem.

Let us note that the result [4] concerns the wave front solution $\psi(x_1 - vt)$ to 3D wave with the potential U_{GL}. The solution has an infinite energy, so it is not a soliton, and its asymptotic stability is provided by the strong dispersion properties of the 3D case. The 1D case requires different arguments [6] and [7].

Our main result is the following theorem.

Theorem 1.1. *There exist potentials $U(\psi)$ satisfying conditions* **U1–U4**.

2. Piece wise parabolic potentials

As a first step, we will consider the class of the potentials which are piece-wise second order polynomials.

$$U_0(\psi) = \begin{cases} \dfrac{1}{2} - \dfrac{b}{2}\psi^2, & |\psi| \leq \gamma, \\[2ex] \dfrac{d}{2}(\psi \mp 1)^2, & \pm\psi \geq \gamma, \end{cases} \tag{2.1}$$

with some constants $b, d > 0$ and $0 < \gamma < 1$. Let us find the parameters $b = b(\gamma)$ and $d = d(\gamma)$ providing $U_0(\psi) \in C^1(\mathbb{R})$. We have

$$U_0(\gamma) = \frac{1}{2} - \frac{b}{2}\gamma^2 = \frac{d}{2}(\gamma - 1)^2 \quad \text{and} \quad U_0'(\gamma) = -b\gamma = d(\gamma - 1).$$

Solving the equations, we obtain

$$b = \frac{1}{\gamma} \quad \text{and} \quad d = \frac{1}{1-\gamma}, \quad 0 < \gamma < 1. \tag{2.2}$$

Then the functions $U_0''(\psi)$ are piece-wise constant with the jumps at the points $\psi = \pm\gamma$. Thus, the potentials $U_0 \in C^1(\mathbb{R})$ form one-dimensional manifold parametrized by $\gamma \in (0, 1)$.

2.1. Kink. Let us solve the equation of type (1.2) for the kink in the case of potential (2.1):

$$s_0''(x) - U_0'(s_0(x)) = 0, \quad x \in \mathbb{R}. \tag{2.3}$$

We search an odd solution to

$$s_0''(x) = \begin{cases} -bs_0(x), & 0 < s_0(x) \le \gamma, \\ d(s_0(x) - 1), & s_0(x) > \gamma. \end{cases}$$

We have

$$s_0(x) = \begin{cases} C \sin \sqrt{b}x, & 0 < x \le q, \\ Ae^{-\sqrt{d}x} + 1, & x > q, \end{cases}$$

where $C > \gamma$, $A < 0$, $q = \dfrac{1}{\sqrt{b}} \arcsin \dfrac{\gamma}{C}$. Equating the values of $s_0(x)$ and its left and right derivatives at $x = q$ we obtain

$$\begin{cases} Ae^{-\sqrt{d}q} + 1 = C \sin \sqrt{b}q = \gamma, \\ -\sqrt{d}Ae^{-\sqrt{d}q} = \sqrt{b}C \cos \sqrt{b}q. \end{cases} \tag{2.4}$$

The first line of (2.4) implies $Ae^{-\sqrt{d}q} = \gamma - 1$. Hence the second line of (2.4) becomes

$$\sqrt{d}(1-\gamma) = \sqrt{b}C \cos \sqrt{b}q.$$

The both side of the last equality is positive. Hence it is equivalent to

$$d(1-\gamma)^2 = b(C^2 - \gamma^2).$$

Substituting (2.2) we obtain $1 - \gamma = C^2/\gamma - \gamma$. Then

$$C = \sqrt{\gamma}, \quad A = (\gamma - 1)e^{\sqrt{\gamma/(1-\gamma)} \arcsin \sqrt{\gamma}}$$

and

$$q = \sqrt{\gamma} \arcsin \sqrt{\gamma}. \tag{2.5}$$

2.2. Linearized equation. Let us linearize equation (1.1) with $F(\psi) = F_0(\psi) = -U_0(\psi)$ at the kink $s_0(x)$ splitting the solution as the sum

$$\psi(t) = s_0 + \varphi(t). \tag{2.6}$$

Substituting (2.6) to (1.1), we obtain

$$\ddot{\varphi}(x,t) = \varphi''(x,t) - U_0'(s_0(x) + \varphi(x,t)) + U_0'(s_0(x)). \tag{2.7}$$

By (2.1) we can write equations (2.7) as

$$\ddot{\varphi}(t) = -H_0\varphi(t) + \mathcal{N}(\varphi(t)), \quad t \in \mathbb{R},$$

where

$$H_0 = -\frac{d^2}{dx^2} + W_0(x), \quad W_0(x) = U_0''(s_0(x)) = \begin{cases} -b, & |x| \le q, \\ d, & |x| > q, \end{cases} \tag{2.8}$$

(see Figure 3) and

$$\mathcal{N}(\varphi(t)) = -U_0'(s_0 + \varphi(t)) + U_0'(s_0) + U_0''(s_0).$$

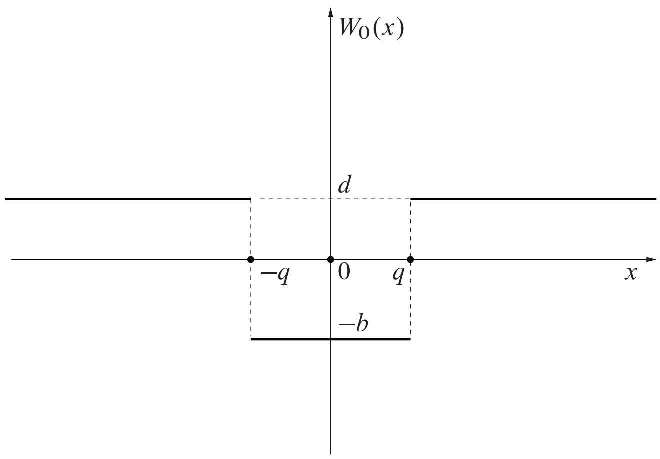

Figure 3. Potential W_0.

The continuous spectrum of H_0 coincides with $[d, \infty)$. The point $\lambda_0 = 0$ is the groundstate since it corresponds to the even positive eigenfunction $\varphi_0(x) = s_0'(x)$:

$$H_0\varphi_0 = -s_0'''(x) + U_0''(s_0(x))s_0'(x) = 0,$$

which follows by differentiation of (2.3). Therefore, the discrete spectrum of H_0 belongs to $[0, d]$, and the next eigenfunction $\varphi_1(x)$ should be odd.

2.3. Odd eigenfunctions. Given an eigenvalue λ, the corresponding eigenfunction $\varphi(x)$ should satisfy the equation

$$\begin{cases} -\varphi''(x) - b\varphi(x) = \lambda\varphi(x), & |x| \le q, \\ -\varphi''(x) + d\varphi(x) = \lambda\varphi(x), & |x| > q. \end{cases} \tag{2.9}$$

Equations (2.9) imply that the odd eigenfunctions have the form

$$\varphi(x) = \begin{cases} B \sin \beta x, & |x| \le q, \\ A \operatorname{sgn} x \; e^{-\alpha|x|}, & |x| > q. \end{cases} \tag{2.10}$$

where $\alpha = \sqrt{d-\lambda} > 0$ and $\beta = \sqrt{b+\lambda} > 0$. Equating the values of the eigenfunction and its left and right derivatives at $x = q$, we obtain

$$\begin{cases} Ae^{-\alpha q} = B \sin \beta q, \\ -A\alpha e^{-\alpha q} = B\beta \cos \beta q, \end{cases} \tag{2.11}$$

where a and b are related as follows

$$\alpha^2 + \beta^2 = b + d.$$

System (2.11) admits nonzero solutions only if its determinant vanishes, that is

$$-\alpha = \beta \cot \beta q.$$

At last, multiplying by q, and denoting $\xi = \beta q$ and $\eta = \alpha q$, we obtain the system of equations

$$-\eta = \xi \cot \xi, \quad \xi^2 + \eta^2 = R^2, \tag{2.12}$$

where $R = q\sqrt{b+d}$ is the radius of the circle. Substituting b, d and q from (2.2) and (2.5) respectively, we obtain

$$R = q\sqrt{\frac{1}{\gamma} + \frac{1}{1-\gamma}} = \frac{q}{\sqrt{\gamma(1-\gamma)}} = \frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}.$$

Finally, the solutions to (2.12) can be found graphically (see Figure 4). Taking into account that $\eta > 0$, we obtain that

$$\left.\begin{array}{r} R \in \left(0, \dfrac{\pi}{2}\right]\text{: system (2.12) has no solution;} \\ R \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]\text{: system (2.12) has one solution;} \\ R \in \left(\dfrac{3\pi}{2}, \dfrac{5\pi}{2}\right]\text{: system (2.12) has two solutions;} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right| \tag{2.13}$$

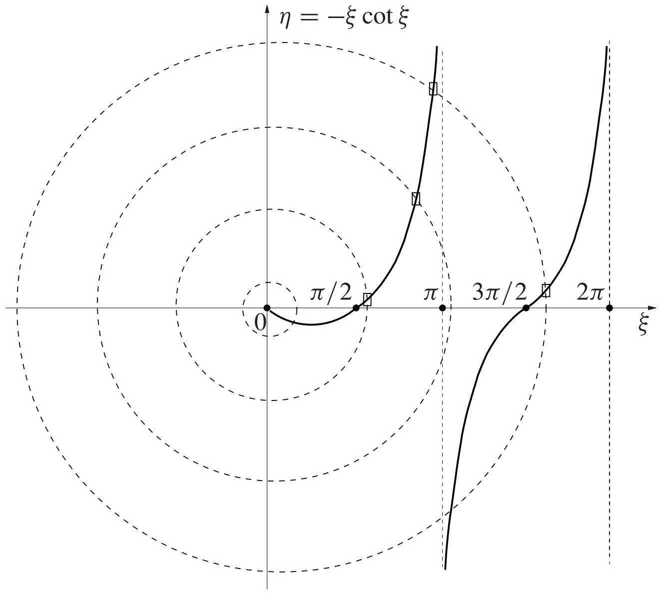

Figure 4. Graphical solution of system (2.12).

Let us note that $R(0) = 0$ and $R(1) = \infty$, and the radius $R(\gamma)$ is monotone increasing on [0, 1]. Denote by γ_k, $k \in \mathbb{N}$ the solution to the equation

$$\frac{\arcsin\sqrt{\gamma_k}}{\sqrt{1-\gamma_k}} = \frac{k\pi}{2}, \quad k \in \mathbb{N}. \tag{2.14}$$

Numerical calculations give

$$\gamma_1 \sim 0.64643, \quad \gamma_2 \sim 0.8579, \quad \gamma_3 \sim 0.92472, \quad \gamma_4 \sim 0.95359\ldots. \tag{2.15}$$

Further, (2.13) implies that

$$\left.\begin{array}{r}\gamma \in (0,\ \gamma_1]: \text{no nonzero odd eigenfunctions,}\\ \gamma \in (\gamma_1,\ \gamma_3]: \text{one linearly independent odd eigenfunction,}\\ \gamma \in (\gamma_3,\ \gamma_5]: \text{two linearly independent odd eigenfunctions,}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right|$$

In particular, for $\gamma \in (\gamma_1, \gamma_3]$ we have one odd eigenfunction and the corresponding eigenvalue $\lambda_1 \in (0, d)$ reads

$$\begin{aligned} \lambda_1 = \lambda_1(\gamma) = \beta^2 - b &= \frac{\xi^2}{q^2} - b \\ &= \frac{1}{\gamma}\Big(\frac{\xi^2}{\arcsin^2 \sqrt{\gamma}} - 1\Big) = \frac{1}{\gamma}\Big(\frac{\sin^2 \xi}{1-\gamma} - 1\Big), \end{aligned} \tag{2.16}$$

where ξ is the solution to

$$\frac{\xi^2}{\sin^2 \xi} = \frac{\arcsin^2 \sqrt{\gamma}}{1-\gamma}. \tag{2.17}$$

2.4. Even eigenfunctions. Equations (2.9) imply that the even eigenfunctions have the form

$$\varphi(x) = \begin{cases} B \cos \beta x, & |x| \le q, \\ A\, e^{-\alpha |x|}, & |x| > q, \end{cases} \tag{2.18}$$

where $\alpha = \sqrt{d-\lambda} > 0$, $\beta = \sqrt{b+\lambda} > 0$. Equating the values of the eigenfunction and its left and right derivatives at $x = q$, we obtain

$$\begin{cases} Ae^{-\alpha q} = B \cos \beta q, \\ A\alpha e^{-\alpha q} = B\beta \sin \beta q. \end{cases}$$

The system admits nonzero solutions if and only if its determinant vanishes:

$$\alpha = \beta \tan \beta q.$$

Similarly (2.12), we obtain the following equations for $\xi = \beta q$ and $\eta = \alpha q$:

$$\eta = \xi \tan \xi, \quad \xi^2 + \eta^2 = R^2, \tag{2.19}$$

where $R = \dfrac{\arcsin \sqrt{\gamma}}{\sqrt{1-\gamma}}$. The solutions can also be found graphically (see Figure 5).

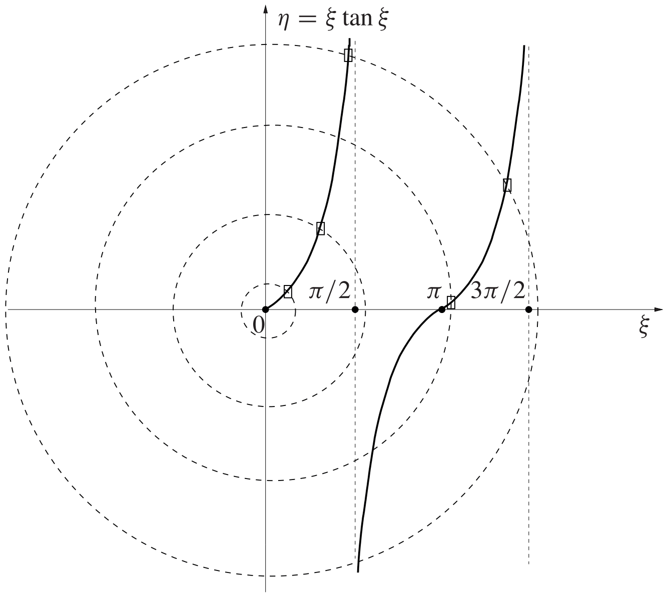

Figure 5. Graphical solution of system (2.19).

We have

$$\left.\begin{array}{r} R \in (0,\ \pi]: \text{system (2.19) has one solution;} \\[2ex] R \in (\pi, 2\pi]: \text{system (2.19) has two solutions;} \\[2ex] \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right| \tag{2.20}$$

Note that for any $\gamma \in (0, 1)$ equation (2.19) has the solution $\xi = \arcsin\sqrt{\gamma} \in (0, \pi/2)$. The solution corresponds to eigenvalue $\lambda = 0$ and the first even eigenfunction. Moreover, (2.20) implies that

$$\left.\begin{array}{r} \gamma \in (0, \gamma_2]: \text{one linearly independent even eigenfunction,} \\[2ex] \gamma \in (\gamma_2, \gamma_4]: \text{two linearly independent even eigenfunctions,} \\[2ex] \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right|$$

where γ_i are defined in (2.14).

Conclusion. 1) There is one eigenvalue $\lambda_0 = 0$ for $\gamma \in (0, \gamma_1]$. 2) There are two eigenvalues $\lambda_0 = 0$ and $0 < \lambda_1 < d$ for $\gamma \in (\gamma_1, \gamma_2]$. Etc. (See Figure 6.)

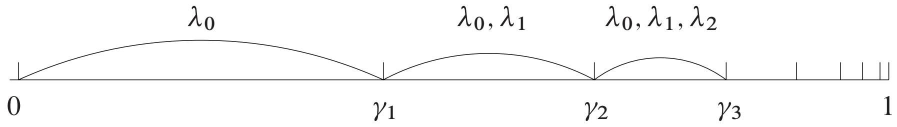

Figure 6. Spectrum.

3. Spectral conditions

We deduce Theorem 1.1 in Section 4 below from the following proposition.

Proposition 3.1. *For any $\gamma \in (\gamma_1, \gamma_2)$ the piece wise parabolic potentials U_0, defined in (2.1), satisfy conditions* **U1**–**U3** *except for the smoothness condition at the points $\psi = \pm\gamma$. Condition* **U4** *holds for any $\gamma \in (\gamma_1, \gamma_2)$ except for one point γ_*.*

Proof. Step i) Obviously, for $U_0(\psi)$ condition **U1** with $a = 1$, $m^2 = d$, and any integer $K \geq 3$ holds except the smoothness at the points $\psi = \pm\gamma$.

Consider condition **U2**. Note that the solutions to (2.12) or (2.19) with $\eta = 0$ and $R = k\pi/2$, $k \in \mathbb{N}$ correspond to $\alpha = 0$ i.e. $\lambda = d$. Then the functions (2.10) or (2.18) with $A \neq 0$ are a nonzero constant for $|x| \geq \gamma$. Hence, the functions are the *resonances* corresponding to the edge point $\lambda = d$ of the continuous spectrum. Thus, the resonances exist only for the discrete set of parameters γ_k, $k \in \mathbb{N}$, defined in (2.14). Evidently, the set has just one limit point 1. Hence, condition **U2** holds for $\gamma \in (0, 1) \setminus \{\cup_{k\in\mathbb{N}}\gamma_k\}$.

Step ii) For any $\gamma \in (\gamma_1, \gamma_2)$ the operator H_0 defined in (2.8) has exactly two eigenvalues $\lambda_0 = 0$ and $\lambda_1 \in (0, d)$. For condition **U3** it remains to verify (1.4) with $m^2 = d$. Namely, due to (2.16) and (2.17) we must prove that for any $\gamma \in (\gamma_1, \gamma_2)$ the following inequality holds:

$$\frac{4}{\gamma}\Big(\frac{\sin^2\xi(\gamma)}{1-\gamma} - 1\Big) > \frac{1}{1-\gamma},$$

where $\xi(\gamma) \in (\pi/2, \pi)$ is the solution to (2.17). After the simple transformations we obtain

$$4\cos^2\xi(\gamma) < 3\gamma,$$

and

$$\frac{\pi}{2} < \xi(\gamma) < \pi - \arccos\frac{\sqrt{3\gamma}}{2}.$$

Since $\dfrac{\xi}{\sin\xi}$ is monotonically increasing function for $\xi \in (\pi/2, \pi)$, then

$$\frac{\pi}{2} < \frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}} < \frac{2(\pi - \arccos\frac{\sqrt{3\gamma}}{2})}{\sqrt{4-3\gamma}}.$$

Finally, we obtain

$$\gamma_1 < \gamma < \alpha,$$

where α is the solution to

$$\frac{\arcsin\sqrt{\alpha}}{\sqrt{1-\alpha}} = \frac{2(\pi - \arccos\frac{\sqrt{3\alpha}}{2})}{\sqrt{4-3\alpha}}.$$

Numerical calculation gives

$$\alpha = 0.921485 > \gamma_2.$$

Therefore, condition **U3** holds for any $\gamma \in (\gamma_1, \gamma_2)$.

Step iii) Finally, consider condition **U4** (Fermi golden rule). The condition can be rewritten as

$$\int U_0'''(s_0(x))\varphi_{4\lambda_1}(x)\varphi_{\lambda_1}^2(x)dx = \int \frac{d}{dx}U_0''(s_0(x))\frac{\varphi_{4\lambda_1}(x)\varphi_{\lambda_1}^2(x)}{s_0'(x)}dx \neq 0. \tag{3.1}$$

By (2.8) we have that $U_0''(s_0(x)) = W_0(x)$ is the piece wise constant function. Hence,

$$\frac{d}{dx}U_0''(s_0(x)) = (b+d)\delta(x-q) - (b+d)\delta(x+q),$$

and (3.1) becomes

$$\varphi_{4\lambda_1}(q)\varphi_{\lambda_1}^2(q) \neq 0.$$

Formula (2.10) yields that $\varphi_{\lambda_1}(q) = Ae^{-\alpha q} \neq 0$. Hence it is sufficient to verify that

$$\varphi_{4\lambda_1}(q) \neq 0.$$

The eigenfunction $\varphi_{4\lambda_1}$ satisfies the equations

$$\begin{cases} -\varphi_{4\lambda_1}''(x) - b\varphi_{4\lambda_1}(x) = 4\lambda_1\varphi_{4\lambda_1}(x), & |x| \le q, \\ -\varphi_{4\lambda_1}''(x) + d\varphi_{4\lambda_1}(x) = 4\lambda_1\varphi_{4\lambda_1}(x), & |x| > q. \end{cases} \tag{3.2}$$

For the odd solution to (3.2) we have

$$\varphi_{4\lambda_1}(q) = \sin\beta q, \quad \beta = \sqrt{b+4\lambda_1} > 0.$$

Therefore, $\varphi_{4\lambda_1}(q) = 0$ only if $\beta q = k\pi$, $k \in \mathbb{N}$, or

$$\sqrt{1+4\gamma\lambda_1(\gamma)}\arcsin\sqrt{\gamma} = k\pi, \quad k \in \mathbb{N}, \tag{3.3}$$

where $\lambda_1(\gamma)$ is defined in (2.16) and (2.17). Substituting $\lambda_1(\gamma)$ into (3.3) we obtain from (2.16) and (2.17)

$$\begin{cases} \dfrac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}\sqrt{4\sin^2\xi - 3(1-\gamma)} = k\pi, \\ \dfrac{\xi^2}{\sin^2\xi} = \dfrac{\arcsin^2\sqrt{\gamma}}{1-\gamma}. \end{cases} \tag{3.4}$$

For $\gamma \in (\gamma_1, \gamma_2)$ the system has a solution only for $k = 1$ since

$$0 < \frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}\sqrt{4\sin^2\xi - 3(1-\gamma)} < 2\pi, \quad \gamma_1 < \gamma < \gamma_2.$$

Denote

$$\theta = \arcsin\sqrt{\gamma} \in (\pi\sqrt{1-\gamma_1}/2, \pi\sqrt{1-\gamma_2}). \tag{3.5}$$

Then (3.4) with $k = 1$ is equivalent to

$$\begin{cases} 4\xi^2 - 3\theta^2 = \pi^2, \\ \dfrac{\sin\xi}{\xi} = \dfrac{\cos\theta}{\theta}. \end{cases} \tag{3.6}$$

Let us prove that (3.6) has a unique solution. Consider two functions $\theta_1(\xi)$ and $\theta_2(\xi)$, where $\theta_1(\xi) \stackrel{\text{def}}{=} \frac{1}{\sqrt{3}}\sqrt{4\xi^2 - \pi^2}$ and $\theta_2(\xi)$ is the solution of $\frac{\sin\xi}{\xi} = \frac{\cos\theta}{\theta}$. The function $\theta_1(\xi)$ increases for $\xi(\gamma_1) < \xi < \xi(\gamma_2)$, and

$$\theta_1'(\xi) = \frac{1}{\sqrt{3}}\frac{4\xi}{\sqrt{4\xi^2 - \pi^2}} > \frac{1}{\sqrt{3}}\frac{4(\pi/2)}{\sqrt{4(3\pi/4)^2 - \pi^2}} = \frac{4}{\sqrt{15}} > 1 \tag{3.7}$$

for $\xi(\gamma_2) < \xi < \xi(\gamma_2)$, since $\xi(\gamma_1) = \pi/2$ and $\xi(\gamma_2) \sim 2.3137 < 3\pi/4$. On the other hand,

$$\theta_2'(\xi) = \frac{\sin\xi - \xi\cos\xi}{\xi^2}\frac{\theta^2}{\cos\theta + \theta\sin\theta} > 0, \quad \pi/2 < \xi < \xi(\gamma_2).$$

Moreover, by (3.5) and (3.6) we obtain

$$\theta_2'(\xi) = \frac{\theta}{\xi}\frac{\frac{\sin\xi}{\xi} - \cos\xi}{\frac{\cos\theta}{\theta} + \sin\theta} < \frac{\frac{\sin\xi}{\xi} - \cos\xi}{\frac{\sin\xi}{\xi} + \sin\theta} < 1, \quad \pi/2 < \xi < \xi(\gamma_2)$$

since $|\cos\xi| < |\cos\xi(\gamma_2)| < \sqrt{2}/2$, and $\sin\theta = \sqrt{\gamma} > \sqrt{\gamma}_1 > \sqrt{2}/2$ by (2.15). Finally,

$$\theta_2(\pi/2) > \theta_1(\pi/2) = 0, \quad \theta_2(\xi(\gamma_2)) \sim 1.1843 < \theta_1(\xi(\gamma_2)) \sim 1.9616. \tag{3.8}$$

Therefore, (3.7) and (3.8) imply that $\theta_1(\xi) = \theta_2(\xi)$ for a single value $\xi(\gamma_*) \in (\pi/2, \xi(\gamma_2))$ (see Figure 7).

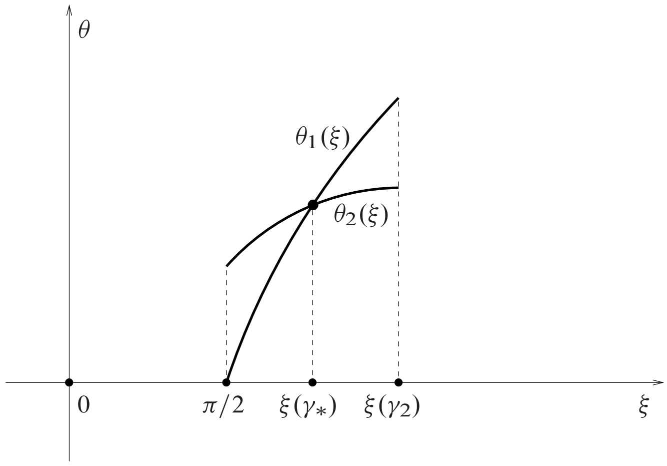

Figure 7. Functions θ_1 and θ_2.

Numerical calculation gives $\gamma_* \sim 0.7925$. Hence, system (3.6) on the interval (γ_1, γ_2) has the only solution $\gamma = \gamma_*$. Thus, the Fermi golden rule holds for any $\gamma \in (\gamma_1, \gamma_2)$ except for the one point γ_*.

Conclusion. The potential $U_0(\psi)$ satisfies conditions **U1**–**U4** except for the smoothness at the points $\psi = \pm\gamma$ for any $\gamma \in (\gamma_1, \gamma_*) \cup (\gamma_*, \gamma_2)$. ☐

4. Smooth potentials

We deduce Theorem 1.1 from Proposition 3.1 by an approximation of the potential U_0 with a smooth functions satisfying conditions **U1**–**U4**. Namely, let $h(\psi) \in C_0^\infty(\mathbb{R})$ be an even mollifying function with the following properties:

$$h(\psi) \geq 0, \quad \operatorname{supp} h \subset [-1, 1], \quad \int h(\psi) d\psi = 1.$$

For $\varepsilon \in (0, 1]$ we set

$$\widetilde{U}_\varepsilon(\psi) \stackrel{\text{def}}{=} \frac{1}{\varepsilon} \int h\Big(\frac{\psi - \psi'}{\varepsilon}\Big) U_0(\psi') d\psi'. \tag{4.1}$$

Evidently, $\widetilde{U}_\varepsilon(\psi) \geq 0$ is a smooth, even function, symmetric with respect to the points $\psi = \pm 1$ in some neighborhoods of these points. In addition we have

$$\widetilde{U}_\varepsilon(\psi) - U_0(\psi) = \begin{cases} \mu_\varepsilon > 0, & |\psi| \geq \gamma + \varepsilon, \\ -\nu_\varepsilon < 0, & |\psi| \leq \gamma - \varepsilon, \end{cases}$$

where $\mu_\varepsilon, \nu_\varepsilon = \mathcal{O}(\varepsilon^2)$. Let us set

$$U_\varepsilon(\psi) = \widetilde{U}_\varepsilon(\psi) - \mu_\varepsilon. \tag{4.2}$$

Then

$$U_\varepsilon(\psi) = \begin{cases} U_0(\psi), & |\psi| \geq \gamma + \varepsilon, \\ U_0(\psi) - \mu_\varepsilon - \nu_\varepsilon, & |\psi| \leq \gamma - \varepsilon. \end{cases} \tag{4.3}$$

Obviously,

$$\sup_{\psi \in \mathbb{R}} |U_\varepsilon(\psi) - U_0(\psi)| \leq C\varepsilon \tag{4.4}$$

with some constant C. Moreover,

$$U_\varepsilon'''(\psi) \leq 0 \text{ for } \psi \leq 0 \quad \text{and} \quad U_\varepsilon'''(\psi) \geq 0 \text{ for } \psi \geq 0. \tag{4.5}$$

The corresponding kink is an odd solution to the equation

$$s_\varepsilon''(x) - U_\varepsilon'(s_\varepsilon(x)) = 0, \quad x \in \mathbb{R}.$$

The equation can be integrated using the "energy conservation"

$$\frac{|s_\varepsilon'(x)|^2}{2} - U_\varepsilon(s_\varepsilon(x)) = \text{const}, \quad x \in \mathbb{R}$$

with const $= 0$:

$$\int_0^{s_\varepsilon(x)} \frac{ds}{\sqrt{2U_\varepsilon(s)}} = x, \quad x \in \mathbb{R}. \tag{4.6}$$

Hence, $s_\varepsilon(x)$ is a monotone increasing function, and

$$s_\varepsilon(x) \to \pm 1, \quad x \to \pm\infty.$$

Moreover, (4.3), (4.4), and (4.6) imply that

$$\sup_{x \in \mathbb{R}} |s_\varepsilon(x) - s_0(x)| \leq C_1 \varepsilon.$$

Therefore,

$$||s_\varepsilon(x)| - \gamma| \geq \varepsilon, \quad ||x| - q| \geq \delta,$$

where

$$\delta \longrightarrow 0 \quad \text{as } \varepsilon \to 0. \tag{4.7}$$

Hence, for the linear potential $W_\varepsilon(x) \overset{\text{def}}{=} U_\varepsilon''(s_\varepsilon(x))$ we obtain

$$W_\varepsilon(x) = W_0(x), \quad ||x| - q| \geq \delta. \tag{4.8}$$

Further, (4.5) implies that

$$W_\varepsilon'(x) \leq 0 \text{ for } x \leq 0 \quad \text{and} \quad W_\varepsilon'(x) \geq 0 \text{ for } x \geq 0.$$

Therefore,

$$|W_\varepsilon(x) - W_0(x)| \le b + d, \quad x \in \mathbb{R}. \tag{4.9}$$

(see Figure 8).

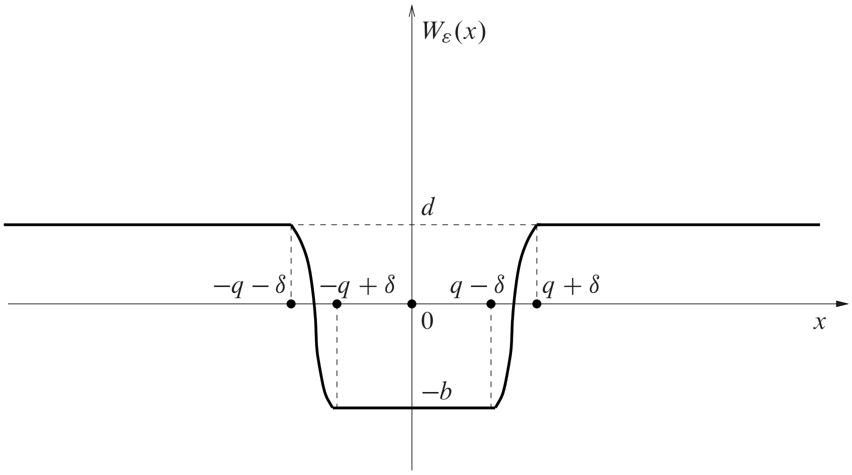

Figure 8. Potential W_ε.

As a result, denoting $w_\varepsilon(x) = W_\varepsilon(x) - W_0(x)$, we obtain

$$\|w_\varepsilon\|_{L^2(\mathbb{R})} \longrightarrow 0, \quad \varepsilon \to 0 \tag{4.10}$$

by (4.7), (4.8), and (4.9).

Lemma 4.1. *The eigenvalues of the Schrödinger operator*

$$H_\varepsilon = -\frac{d^2}{dx^2} + W_\varepsilon(x)$$

converge to the ones of H_0 as $\varepsilon \to 0$.

Proof. The eigenvalues of H_0 and H_ε are the poles of the resolvents $R_0(\omega) = (H_0 - \omega)^{-1}$ and $R_\varepsilon(\omega) = (H_\varepsilon - \omega)^{-1}$ respectively. Hence, the lemma follows from (4.10) due to the relation

$$R_\varepsilon(\omega) = (H_0 - \omega + w_\varepsilon)^{-1} = R_0(\omega)(1 + w_\varepsilon R_0(\omega))^{-1}. \tag{4.11}$$

□

Proof of Theorem 1.1. Consider the potential $U(\psi) = U_\varepsilon(\psi)$ defined in eq. (4.1) and eq. (4.2). Let us prove that there exist $\varepsilon_0 > 0$ such that for any $\gamma \in (\gamma_1, \gamma_2) \setminus \gamma_*$, and $0 < \varepsilon < \varepsilon_0$ the potential U_ε satisfies conditions **U1–U4**.

Step i) Condition **U1** with $a = 1$, $m^2 = d$, and any integer $K \geq 3$ obviously holds.

Step ii) For any $\sigma \in \mathbb{R}$, and $s = 0, 1, 2, \ldots$ denote by $\mathcal{H}^s_\sigma = \mathcal{H}^s_\sigma(\mathbb{R})$ the weighted Sobolev spaces with the finite norms

$$\|\psi\|_{\mathcal{H}^s_\sigma} = \sum_{k=0}^{s} \|(1+|x|)^\sigma \psi^{(k)}\|_{L^2(\mathbb{R})} < \infty.$$

By [9], Theorem 7.2, the absence of the resonance at the point $\omega = d$ for the Schrödinger operator H is equivalent to the boundedness of the corresponding resolvent $R(\omega)\colon \mathcal{H}^0_\sigma \to \mathcal{H}^2_{-\sigma}$ at $\omega = d$ for any $\sigma > 1/2$. Hence, the resolvent $R_0(d)\colon \mathcal{H}^0_\sigma \to \mathcal{H}^2_{-\sigma}$ is bounded by Proposition 3.1. Further, (4.8) and (4.9) imply

$$\|w_\varepsilon\|_{\mathcal{H}^0_{-\sigma} \to \mathcal{H}^0_\sigma} \longrightarrow 0, \quad \varepsilon \to 0$$

Hence, for sufficiently small ε the operator $R_\varepsilon(d)\colon \mathcal{H}^0_\sigma \to \mathcal{H}^2_{-\sigma}$ is bounded by (4.11). Then condition **U2** holds for U_ε.

Step iii) Lemma 4.1 implies that for $\gamma \in (\gamma_1, \gamma_2)$ and sufficiently small ε the operator H_ε has exactly two eigenvalues $\lambda_0 = 0$ and $0 < \lambda_1(\varepsilon) < d$. Moreover, $\lambda_1(\varepsilon) \to \lambda_1(0) = \lambda_1$ as $\varepsilon \to 0$ and then $4\lambda_1(\varepsilon) > d$ for sufficiently small ε. Hence, condition **U3** holds.

Step iv) It remains to check condition **U4**. Consider arbitrary $\gamma \in (\gamma_1, \gamma_2) \setminus \gamma_*$. Denote $\varphi^\varepsilon_{\lambda_1(\varepsilon)}$ and $\varphi^\varepsilon_{4\lambda_1(\varepsilon)}$ the corresponding odd eigenfunctions of H_ε. Then we have

$$\begin{aligned}
&\int U'''_\varepsilon(s_\varepsilon(x))\varphi^\varepsilon_{4\lambda_1(\varepsilon)}(x)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(x))^2 dx \\
&\quad= \int\limits_{|x-q|\leq\delta} \frac{d}{dx} W_\varepsilon(x) \frac{\varphi^\varepsilon_{4\lambda_1(\varepsilon)}(x)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(x))^2}{s'_\varepsilon(x)} dx \\
&\quad= \sum_\pm d \frac{\varphi^\varepsilon_{4\lambda_1(\varepsilon)}(\pm q+\delta)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(\pm q+\delta))^2}{s'_\varepsilon(\pm q+\delta)} \\
&\qquad+ \sum_\pm b \frac{\varphi^\varepsilon_{4\lambda_1(\varepsilon)}(\pm q-\delta)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(\pm q-\delta))^2}{s'_\varepsilon(\pm q-\delta)} \\
&\qquad- \int\limits_{|x-q|\leq\delta} W_\varepsilon(x) \frac{d}{dx} \frac{\varphi_{4\lambda_1^\varepsilon(\varepsilon)}(x)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(x))^2}{s'_\varepsilon(x)} dx \\
&\quad\xrightarrow[\varepsilon\to 0]{} 2(d+b)\frac{\varphi_{4\lambda_1}(q)\varphi^2_{\lambda_1}(q)}{s'_0(q)} = \int U'''_0(s_0(x))\varphi_{4\lambda_1}(x)\varphi^2_{\lambda_1}(x)dx \neq 0
\end{aligned}$$

since $\delta \to 0$ as $\varepsilon \to 0$. Hence, **U4** holds for sufficiently small ε. □

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