

# Weighted decay for magnetic Schrödinger equation

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## Abstract

We obtain a dispersive long-time decay in weighted norms for solutions of 3D Schrödinger equation with generic magnetic and scalar potentials. The decay extends the results obtained by Jensen and Kato for the Schrödinger equation without magnetic potentials. For the proof we develop the spectral theory of Agmon, Jensen and Kato, extending the high energy decay of the resolvent to the magnetic Schrödinger equation.

*Keywords:* long-time decay, weighted norms, magnetic Schrödinger equation, high energy decay of resolvent.

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# 1 Introduction

We establish a dispersive long time decay for the solutions to 3D magnetic Schrödinger equation

$$i\dot{\psi}(x, t) = H\psi(x, t) := [-i\nabla - A(x)]^2\psi(x, t) + V(x)\psi(x, t), \quad x \in \mathbb{R}^3 \quad (1.1)$$

in weighted norms. For  $s, \sigma \in \mathbb{R}$ , denote by  $\mathcal{H}_\sigma^s = \mathcal{H}_\sigma^s(\mathbb{R}^3)$  the weighted Sobolev spaces introduced by Agmon, [1], with the finite norms

$$\|\psi\|_{\mathcal{H}_\sigma^s} = \|\langle x \rangle^\sigma \langle \nabla \rangle^s \psi\|_{L^2(\mathbb{R}^3)} < \infty, \quad \langle x \rangle = (1 + |x|^2)^{1/2}. \quad (1.2)$$

We will also denote  $L_\sigma^2 = \mathcal{H}_\sigma^0$ . We assume that  $V(x) \in C(\mathbb{R}^3)$ ,  $A_j \in C^2(\mathbb{R}^3)$  are real functions, and for some  $\beta > 3$  and  $\beta_1 > 2$  the bounds hold

$$|V(x)| + |A(x)| + |\nabla A(x)| \leq C\langle x \rangle^{-\beta}, \quad x \in \mathbb{R}^3, \quad (1.3)$$

$$|\nabla \nabla A(x)| \leq C\langle x \rangle^{-\beta_1}, \quad x \in \mathbb{R}^3. \quad (1.4)$$

We restrict ourselves to the “regular case” in the terminology of [8] (or “nonsingular case” in [14]), where the truncated resolvent of the operator  $H$  is bounded at the edge point  $\lambda = 0$  of the continuous spectrum. In other words, the point  $\lambda = 0$  is neither eigenvalue nor resonance for the operator  $H$ ; this holds for *generic potentials*.

Our main result is the following long time decay of the solutions to (1.1): in the regular case,

$$\|\mathcal{P}_c \psi(t)\|_{\mathcal{H}_{-\sigma}^0} = \mathcal{O}(|t|^{-3/2}), \quad t \rightarrow \pm\infty \quad (1.5)$$

for initial data  $\psi_0 = \psi(0) \in \mathcal{H}_\sigma^0$  with  $\sigma > 5/2$  where  $\mathcal{P}_c$  is a Riesz projection onto the continuous spectrum of  $H$ . The decay is desirable for the study of asymptotic stability and scattering for the solutions to nonlinear equations.

Let us comment on previous results in this direction. Asymptotic completeness for the magnetic Schrödinger equation follows by methods of the Birman-Kato theory [15]. Spectral representation for this case has been obtained by Iwatsuka [7] developing the Kuroda approach [11, 12]. The Strichartz estimates for magnetic Schrödinger equation with small potentials were obtained in [3, 5] and with large potentials in [4]. The decay in weighted norms has been obtained first by Jensen and Kato for the Schrödinger equation with scalar potential [8]. However, for the magnetic Schrödinger equation the decay in weighted norms was not obtain up to now.

Finally, let us comment on our approach. We extend methods of Agmon [1], and Jensen and Kato [8], to the magnetic Schrödinger equation. Our main novelties - Theorems 3.7 and 3.8 on high energy decay for the magnetic resolvent, and Lemmas A.2 and A.3 which are extensions of known Agmon’s Lemma A.2 and A.3 from [1] (see also Lemma 4 from [15, p. 442]). Main problem in this extension - presence of the first order derivatives in the perturbation. These derivatives cannot be handle with the perturbation theory like [1, 8] since the corresponding terms do not decay in suitable norms. To avoid the perturbation approach, we apply spectral resolution for magnetic Schrödinger operator in our extension of Lemma A.3 from [1].

Our techniques rely on the D’Ancona-Fanelly magnetic version of the Hardy inequality [2], spectral resolution established by Iwatsuka [7], result of Ionescu and Schlag [6] on absence of singular spectrum, and result of Koch and Tataru [9] on absence of embedded eigenvalues in continuous spectrum. We also apply limiting absorption principle for the magnetic Schrödinger equation. We deduce the principle by a suitable generalization of methods of Agmon [1].

## 2 Free equation

Denote by  $R_0(\omega) = (H_0 - \omega)^{-1}$  the resolvent of the free Schrödinger operator  $H_0 = -\Delta$ . The resolvent is an integral operator with the integral kernel

$$R_0(\omega, x - y) = \exp(i\omega^{1/2}|x - y|)/4\pi|x - y|, \quad \omega \in \mathbb{C} \setminus [0, \infty), \quad \text{Im}\omega^{1/2} > 0. \quad (2.1)$$

Denote by  $\mathcal{L}(B_1, B_2)$  the Banach space of bounded linear operators from a Banach space  $B_1$  to a Banach space  $B_2$ .

Explicit formula (2.1) implies the properties of  $R_0(\omega)$  which are obtained in [1, 8] (see also [10, Appendix A]):

- i)  $R_0(\omega)$  is analytic function of  $\omega \in \mathbb{C} \setminus [0, \infty)$  with the values in  $\mathcal{L}(\mathcal{H}_0^m, \mathcal{H}_0^{m+2})$  for any  $m \in \mathbb{R}$ ;
- ii) The limiting absorption principle holds:

$$R_0(\lambda \pm i\varepsilon) \rightarrow R_0(\lambda \pm i0), \quad \varepsilon \rightarrow 0+, \quad \lambda > 0 \quad (2.2)$$

where the convergence holds in  $\mathcal{L}(\mathcal{H}_\sigma^m, \mathcal{H}_{-\sigma}^{m+2})$  with  $\sigma > 1/2$ ;

- iii) The asymptotics hold for  $\omega \in \mathbb{C} \setminus [0, \infty)$ ,

$$\|R_0(\omega) - R_0(0)\|_{\mathcal{L}(\mathcal{H}_\sigma^m, \mathcal{H}_{-\sigma}^{m+2})} \rightarrow 0, \quad \omega \rightarrow 0, \quad \sigma, \sigma' > 1/2, \quad \sigma + \sigma' > 2 \quad (2.3)$$

$$\|R_0^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{H}_\sigma^m, \mathcal{H}_{-\sigma}^{m+2})} = \mathcal{O}(\omega^{1/2-k}), \quad \omega \rightarrow 0, \quad \sigma > 1/2 + k, \quad k = 1, 2, \dots \quad (2.4)$$

- iv) For  $m \in \mathbb{R}$ ,  $k = 1, 2, \dots$  and  $\sigma > k + 1/2$  the asymptotics hold

$$\|R_0^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{H}_\sigma^m, \mathcal{H}_{-\sigma}^{m+l})} = \mathcal{O}(|\omega|^{-\frac{1-l+k}{2}}), \quad \omega \rightarrow \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty) \quad (2.5)$$

where  $l = -1, 0, 1, 2$  for  $k = 0$ , and  $l = -1, 0, 1$  for  $k = 1, 2, \dots$

Denote by  $U_0(t)$  the dynamical group of the free Schrödinger equation. For  $t \in \mathbb{R}$  and  $\psi_0 \in \mathcal{H}_\sigma^0$  with  $\sigma > 1$ , the group  $U_0(t)$  admits the integral representation

$$U_0(t)\psi_0 = \frac{1}{2\pi i} \int_0^\infty e^{-i\omega t} [R_0(\omega + i0) - R_0(\omega - i0)] \Psi_0 \, d\omega \quad (2.6)$$

where the integral converges in the sense of distributions of  $t \in \mathbb{R}$  with the values in  $\mathcal{H}_{-\sigma}^0$ .

Representation (2.6), properties i)-iii), and bounds (2.5) imply the weighted decay (1.5) for the solutions to the free Schrödinger equation (see [8]).

## 3 Perturbed equation

### 3.1 Limiting absorption principle

Here we extend (2.2) to perturbed resolvent  $R(\omega) = (H - \omega)^{-1}$ , where

$$H = H_0 + W, \quad W\psi = (i\nabla \cdot \mathbf{A} + \mathbf{A}^2 + V)\psi + i\mathbf{A} \cdot \nabla\psi. \quad (3.1)$$

**Theorem 3.1.** *Let condition (1.3) hold with  $\beta > 2$ . Then for  $\lambda > 0$ , the convergence holds*

$$R(\lambda \pm i\varepsilon) \rightarrow R(\lambda \pm i0), \quad \varepsilon \rightarrow 0+ \quad (3.2)$$

in  $\mathcal{L}(L^2_\sigma, L^2_{-\sigma})$  with  $\sigma > 1/2$ .

Theorem 3.1 follows from the result of [4] where slightly weaker conditions on potentials are imposed. For the convenience of readers we give the independent proof in our case. For the proof we will use the Born splitting

$$R(\omega) = [1 + R_0(\omega)W]^{-1}R_0(\omega), \quad \omega \in \mathbb{C} \setminus [0, \infty) \quad (3.3)$$

where the operator function  $[1 + R_0(\omega)W]^{-1}$  is meromorphic in  $\mathbb{C} \setminus [0, \infty)$  by the Gohberg-Bleher theorem. The key role in the proofs of (3.2) plays the result on the absence of the embedded eigenvalues in the continuous spectrum which is known due from the paper of Koch and Tataru [9]. We start with the following lemma

**Lemma 3.2.** *i) Let condition (1.3) hold with some  $\beta > 1$ . Then for  $\lambda > 0$ , the operators  $R_0(\lambda \pm i0)W : L^2_{-\sigma} \rightarrow L^2_{-\sigma}$  and  $WR_0(\lambda \pm i0) : L^2_\sigma \rightarrow L^2_\sigma$  are compact for  $\sigma \in (1/2, \beta - 1/2)$ .*

*ii) Let condition (1.3) hold with some  $\beta > 2$ . Then the operators  $R_0(0)W : L^2_{-\sigma} \rightarrow L^2_{-\sigma}$  and  $WR_0(0) : L^2_\sigma \rightarrow L^2_\sigma$  are compact for  $\sigma \in (1/2, \beta - 1/2)$ .*

*Proof.* i) Choose  $\sigma' \in (1/2, \min(\sigma, \beta - \sigma))$ . The operator  $W : L^2_{-\sigma} \rightarrow \mathcal{H}_{\sigma'}^{-1}$  is continuous by (1.3) since  $\sigma + \sigma' < \beta$ . Further,  $R_0(\lambda \pm i0) : \mathcal{H}_{\sigma'}^{-1} \rightarrow \mathcal{H}_{-\sigma'}^1$  is continuous by (2.2) and the embedding  $\mathcal{H}_{-\sigma'}^1 \rightarrow L^2_{-\sigma}$  is compact by the Sobolev embedding theorem. Hence, the operators  $R_0(\lambda \pm i0)W : L^2_{-\sigma} \rightarrow L^2_{-\sigma}$  are compact. The compactness of  $WR_0(\lambda \pm i0) : L^2_\sigma \rightarrow L^2_\sigma$  follows by duality.

ii) Choose sufficiently small  $\varepsilon > 0$  such that

$$\sigma' := \beta - \sigma - \varepsilon > 1/2, \quad \sigma'' := \sigma - \varepsilon > 1/2, \quad \sigma' + \sigma'' = \beta - 2\varepsilon > 2. \quad (3.4)$$

The operator  $W : L^2_{-\sigma} \rightarrow \mathcal{H}_{\sigma'}^{-1}$  is continuous by (1.3) since  $\sigma' + \sigma = \beta - \varepsilon < \beta$ , and the operator  $R_0(0) : \mathcal{H}_{\sigma'}^{-1} \rightarrow \mathcal{H}_{-\sigma''}^1$  is continuous by (2.3). The embedding  $\mathcal{H}_{-\sigma''}^1 \rightarrow L^2_{-\sigma}$  is compact by the Sobolev embedding theorem. Hence, the operator  $R_0(0)W : L^2_{-\sigma} \rightarrow L^2_{-\sigma}$  is compact. The compactness of  $WR_0(0) : L^2_\sigma \rightarrow L^2_\sigma$  follows by duality.  $\square$

Theorem 3.1 will follow from convergence (2.2) and the Born splitting (3.3) if

$$[1 + R_0(\lambda \pm i\varepsilon)W]^{-1} \rightarrow [1 + R_0(\lambda \pm i0)W]^{-1}, \quad \varepsilon \rightarrow +0, \quad \lambda > 0$$

in  $\mathcal{L}(L^2_{-\sigma}, L^2_{-\sigma})$  with  $\sigma > 1/2$ . The convergence holds if and only if the both limiting operators  $1 + R_0(\lambda \pm i0)W : L^2_{-\sigma} \rightarrow L^2_{-\sigma}$  are invertible for  $\lambda > 0$ . The operators are invertible according to the Fredholm theorem by Lemma 3.2 i) and the following lemma.

**Lemma 3.3.** *Let condition (1.3) holds with some  $\beta > 2$ . Then for  $\lambda > 0$  the equations*

$$[1 + R_0(\lambda \pm i0)W]\psi = 0 \quad (3.5)$$

*admit only zero solution in  $L^2_{-1/2-0}$ .*

*Proof.* We adopt general strategy from [1].

*Step i)* We consider the case  $\lambda + i0$  for concreteness. Equality (3.5) implies that

$$(H - \lambda)\psi = (H_0 - \lambda)(1 + R_0(\lambda + i0)W)\psi = 0. \quad (3.6)$$

We will show that if  $\psi \in L^2_{-1/2-0}$  is the solution to (3.5) then  $\psi \in L^2$ , i.e  $\psi$  is the eigenfunction of  $H$  corresponding to the positive eigenvalue  $\lambda > 0$ . However, the embedded eigenvalue is forbidden, and then  $\psi = 0$ .

*Step ii)* From (3.5) it follows that

$$\psi = R_0(\lambda + i0)f, \quad \text{where } f = -W\psi. \quad (3.7)$$

Moreover, (3.5) also implies that  $\psi \in \mathcal{H}^1_{-1/2-0}$ . Hence,  $f \in L^2_1$  by (1.3) with  $\beta > 3/2$ . In the Fourier transform, equation (3.7) becomes

$$\hat{\psi}(\xi) = \frac{\hat{f}(\xi)}{\xi^2 - \lambda - i0}, \quad \xi \in \mathbb{R}^3$$

where  $\hat{f}$  is a function from the Sobolev space  $\mathcal{H}^1$ .

*Step iii)* Next, we prove that

$$\hat{f}(\xi) \Big|_{S_{\sqrt{\lambda}}} = 0 \quad (3.8)$$

where  $S_{\sqrt{\lambda}} := \{\xi \in \mathbb{R}^3 : |\xi| = \sqrt{\lambda}\}$ . Note that the trace on the sphere  $S_{\sqrt{\lambda}}$  exists, and  $\hat{f}(\xi) \Big|_{S_{\sqrt{\lambda}}} \in \mathcal{H}^{1/2}(S_{\sqrt{\lambda}})$ . Moreover, in the polar coordinates  $r = |\xi| \in [0, \infty)$ ,  $\varphi = \xi/|\xi| \in S_1$ , the map

$$M : [0, \infty) \rightarrow L^2(S_1), \quad M(r) = \hat{f}(r\varphi), \quad \varphi \in S_1$$

is Hölder continuous with the Hölder exponent  $\alpha \in (0, 1/2)$ . This follows from the Sobolev theorem on the traces ([13, Ch. 1]). Define

$$\hat{\psi}_\varepsilon(\xi) = \frac{\hat{f}(\xi)}{\xi^2 - \lambda - i\varepsilon}, \quad \varepsilon > 0.$$

Then both  $\hat{f}, \hat{\psi}_\varepsilon \in L^2(\mathbb{R}^3)$ , hence the Parseval identity implies that

$$\begin{aligned} (\psi_\varepsilon, f) &= (\hat{\psi}_\varepsilon, \hat{f}) \\ &= \int \frac{|\hat{f}(\xi)|^2}{\xi^2 - \lambda - i\varepsilon} d\xi \longrightarrow \frac{i\pi}{2\sqrt{\lambda}} \int_{S_{\sqrt{\lambda}}} |\hat{f}(\xi)|^2 dS(\xi) + \lim_{\delta \rightarrow 0} \int_{\substack{|\xi| = \sqrt{\lambda} \\ |\xi - \sqrt{\lambda}| > \delta}} \frac{|\hat{f}(\xi)|^2}{\xi^2 - \lambda} d\xi, \quad \varepsilon \rightarrow 0+ \end{aligned} \quad (3.9)$$

by the Sokhotsky-Plemelj formula since the map

$$M_1 : [0, \infty) \rightarrow L^1(S_1), \quad M_1(r) = |\hat{f}(r\varphi)|^2, \quad \varphi \in S_1$$

is the Hölder continuous with the same Hölder exponent  $\alpha \in (0, 1/2)$ . On the other hand,

$$(\psi_\varepsilon, f) = (R_0(\lambda + i\varepsilon)f, f) \longrightarrow (\psi, f) = -(\psi, W\psi), \quad \varepsilon \rightarrow 0+ \quad (3.10)$$

since  $R_0(\lambda + i\varepsilon)f \rightarrow \psi$  in  $\mathcal{H}_{-1/2-0}^2$  by (2.2), while  $f \in L_1^2$ . The operator  $W$  is selfadjoint, hence the scalar product  $(\psi, W\psi)$  is real. Comparing (3.9) and (3.10), we conclude that

$$\int_{S_{\sqrt{\lambda}}} |\hat{f}(\xi)|^2 dS(\xi) = 0$$

i.e. (3.8) is proved. Relation (3.8) and the Hölder continuity imply that

$$\hat{\psi}(\xi) = \frac{\hat{f}(\xi)}{\xi^2 - \lambda} \in L_{\text{loc}}^1(\mathbb{R}^3). \quad (3.11)$$

*Step iv)* Finally we prove that

$$\|\hat{\psi}\|_{L^2} \leq C\|\hat{f}\|_{\mathcal{H}^1}. \quad (3.12)$$

For the proof we take any  $\varepsilon \in (0, \sqrt{\lambda}/2)$ , and a cutoff function

$$\zeta(\xi) \in C_0^\infty(\mathbb{R}^3), \quad \zeta(\xi) = \begin{cases} 1, & \|\xi - \sqrt{\lambda}\| < \varepsilon \\ 0, & \|\xi - \sqrt{\lambda}\| > 2\varepsilon \end{cases}$$

By (3.11), we have

$$\|(1 - \zeta(\xi))\hat{\psi}(\xi)\|_{\mathcal{H}^1} = \left\| \frac{1 - \zeta(\xi)}{\xi^2 - \lambda} \hat{f}(\xi) \right\|_{\mathcal{H}^1} \leq C\|\hat{f}\|_{\mathcal{H}^1}.$$

Hence, it remains to estimate the norm of the function  $\zeta(\xi)\hat{\psi}(\xi)$ . Choose a finite partition of unity  $\sum \zeta_j(\xi) = 1$ ,  $\xi \in \text{supp } \zeta$ , with  $\zeta_j \in C_0^\infty(\mathbb{R}^3 \setminus 0)$ . We may assume that in the  $\text{supp } \zeta_j$ , for every fixed  $j$ , there exist the corresponding local coordinates  $\eta_1, \eta_2, \eta_3$  with  $\eta_1 = \xi^2 - \lambda$ . Then, the problem reduces to the estimate

$$\|\varphi(\eta)\|_{L^2} \leq C\|\eta_1\varphi(\eta)\|_{\mathcal{H}^1}$$

taking into account that  $\varphi(\eta) \in L^1(\mathbb{R}^3)$  by (3.11). It suffices to prove the bound

$$\|\phi(x)\|_{L^2} \leq C\|\partial_1\phi(x)\|_{L_1^2} \quad (3.13)$$

for the function  $\phi(x) := F^{-1}\varphi$ , knowing that  $\phi(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  by the Riemann-Lebesgue theorem. Bound (3.13) follows by the Hardy inequality (see [1])

$$\int \phi^2(x_1, x') dx_1 \leq 4 \int |x_1|^2 |\partial_1\phi(x_1, x')|^2 dx_1, \quad \text{a.a. } x' := (x_2, x_3) \in \mathbb{R}^2$$

by integration over  $x' \in \mathbb{R}^2$ . Now (3.12) is proved. Finally, (3.12) can be rewritten as

$$\|\psi\|_{L^2} \leq C\|f\|_{L_1^2}$$

that proves Lemma 3.3. □

Now the proof of Theorem 3.1 is also completed.

**Corollary 3.4.** *Under the conditions of Theorem 3.1, convergence (3.2) holds in  $\mathcal{L}(\mathcal{H}_\sigma^0, \mathcal{H}_{-\sigma}^2)$  with  $\sigma > 1/2$ .*

*Proof.* The operators  $1 + WR_0(\lambda \pm i0) : L_\sigma^2 \rightarrow L_\sigma^2$  are adjoint to  $1 + R_0(\lambda \mp i0)W : L_{-\sigma}^2 \rightarrow L_{-\sigma}^2$ . The operators  $1 + R_0(\lambda \mp i0)W$  are invertible by Lemma 3.3, hence  $1 + WR_0(\lambda \pm i0)$  also are invertible by the Fredholm theorem. Therefore, the corollary follows by the alternative Born splitting

$$R(\omega) = R_0(\omega)[1 + WR_0(\omega)]^{-1}, \quad \omega \in \mathbb{C} \setminus [0, \infty) \quad (3.14)$$

and convergence (2.2).  $\square$

### 3.2 Zero point $\omega = 0$

Here we consider  $R(\omega)$  near  $\omega = 0$ . We set

$$\mathcal{M} = \{\psi \in L_{-1/2-0}^2 : \psi + R_0(0)W\psi = 0\}.$$

The functions  $\psi \in \mathcal{M} \cap L^2$  are the zero eigenfunctions of  $H$  since  $H\psi = H_0(1 + R_0(0)W)\psi = 0$  by splitting (3.3). The functions  $\psi \in \mathcal{M} \setminus L^2$  are called the **zero resonances** of  $H$ .

Our key assumption is (cf. Condition (i) in [14, Theorem 7.2]):

$$\text{Spectral Condition :} \quad \mathcal{M} = 0 \quad (3.15)$$

In the other words, the point  $\omega = 0$  is neither eigenvalue nor resonance for the operator  $H$ . Condition (3.15) holds for *generic*  $W$ .

**Lemma 3.5.** *Let condition (1.3) with a  $\beta > 2$  and Spectral Condition (3.15) hold. Then the discrete spectral set  $\Sigma$  is finite, and for  $\sigma, \sigma' > 1/2$  with  $\sigma + \sigma' > 2$ , the asymptotics hold,*

$$\|R(\omega) - R(0)\|_{\mathcal{L}(\mathcal{H}_\sigma^0, \mathcal{H}_{-\sigma'}^2)} \rightarrow 0, \quad \omega \rightarrow 0, \quad \omega \in \mathbb{C} \setminus [0, \infty) \quad (3.16)$$

where the operator  $R(0) : \mathcal{H}_\sigma^0 \rightarrow \mathcal{H}_{-\sigma'}^2$  is continuous.

*Proof.* It suffices to consider the case

$$1/2 < \sigma, \sigma' < \beta - 1/2, \quad \sigma + \sigma' > 2$$

since asymptotics (3.16) hold then for larger  $\sigma, \sigma'$ . Spectral Condition (3.15) implies that the operators  $[1 + R_0(0)W] : L_{-\sigma}^2 \rightarrow L_{-\sigma}^2$  and  $[1 + WR_0(0)] : L_\sigma^2 \rightarrow L_\sigma^2$  are invertible by Lemma 3.2 ii) and the Fredholm theorem. Then the operator  $[1 + WR_0(\omega)] : L_\sigma^2 \rightarrow L_\sigma^2$  also is invertible and the operator function  $[1 + WR_0(\omega)]^{-1}$  with the values in  $\mathcal{L}(L_\sigma^2, L_\sigma^2)$  is continuous for small  $\omega \in \mathbb{C} \setminus [0, \infty)$ . Therefore, convergence (3.16) holds by (3.14) and (2.3).  $\square$

**Lemma 3.6.** *Let condition (1.3) with a  $\beta > 3$  and Spectral Condition (3.15) hold. Then*

$$\|R(\omega)\|_{\mathcal{L}(\mathcal{H}_\sigma^0, \mathcal{H}_{-\sigma}^2)} = \mathcal{O}(1), \quad \omega \rightarrow 0, \quad \omega \in \mathbb{C} \setminus [0, \infty), \quad \sigma > 1, \quad (3.17)$$

$$\|R^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{H}_\sigma^0, \mathcal{H}_{-\sigma}^2)} = \mathcal{O}(|\omega|^{1/2-k}), \quad \omega \rightarrow 0, \quad \omega \in \mathbb{C} \setminus [0, \infty), \quad \sigma > 1/2 + k, \quad k = 1, 2. \quad (3.18)$$

*Proof.* Bound (3.17) holds by Lemma 3.5. To prove (3.18) with  $k = 1$  we apply the identity

$$R' = (1 - RW)R'_0(1 - WR) = R'_0 - RW R'_0 - R'_0 WR + RW R'_0 WR. \quad (3.19)$$

The relation implies (3.18) with  $k = 1$  and  $\sigma > 3/2$  by (2.4) with  $k = 1$  and (3.17). Namely, for the first term in the RHS of (3.19) this is obvious. Consider the second term. Choosing  $\sigma' \in (3/2, \beta - 3/2)$ , we obtain

$$\|R(\omega)WR'_0(\omega)\psi\|_{\mathcal{H}^2_{-\sigma}} \leq C\|WR'_0(\omega)\psi\|_{L^2_{\sigma'}} \leq C_1\|R'_0(\omega)\psi\|_{\mathcal{H}^1_{\sigma'-\beta}} \leq C_2|\omega|^{-1/2}\|\psi\|_{L^2_{\sigma}}. \quad (3.20)$$

The remaining terms can be estimated similarly. Hence, (3.18) with  $k = 1$  is proved.

For  $k = 2$  we apply the formula:

$$\begin{aligned} R'' &= (1 - RW)R''_0(1 - WR) - 2R'WR'_0(1 - WR) \\ &= R''_0 - RW R''_0 - R''_0 WR + RW R''_0 WR - 2R'WR'_0 + 2R'WR'_0 WR. \end{aligned} \quad (3.21)$$

Bound (3.18) with  $k = 2$  and  $\sigma > 5/2$  for the first term in the RHS of (3.21) follows from (2.4) with  $k = 2$ . The last two terms can be estimated similarly to (3.20) using (3.17) and (3.18) with  $k = 1$ . Consider the remaining terms. Using (3.17) and (2.4) with  $k = 2$ , we obtain that a) for  $\sigma' \in (5/2, \beta - 1/2)$  the bounds hold

$$\|RW R''_0\psi\|_{\mathcal{H}^2_{-\sigma}} \leq C\|WR''_0\psi\|_{L^2_{-\sigma'+\beta}} \leq C_1\|R''_0\psi\|_{\mathcal{H}^1_{-\sigma'}} \leq C_2|\omega|^{-3/2}\|\psi\|_{L^2_{\sigma}},$$

$$\|R''_0 WR\psi\|_{\mathcal{H}^2_{-\sigma}} \leq C|\omega|^{-3/2}\|WR\psi\|_{L^2_{\sigma'}} \leq C_1|\omega|^{-3/2}\|R\psi\|_{\mathcal{H}^1_{\sigma'-\beta}} \leq C_2|\omega|^{-3/2}\|\psi\|_{L^2_{\sigma}}$$

by Lemma 3.5 since  $-\sigma' + \beta > 1/2$  and  $\sigma + \beta - \sigma' > 2$ .

b) for  $\sigma' \in (1/2, \beta - 5/2)$  the bound holds

$$\begin{aligned} \|RW R''_0 WR\psi\|_{\mathcal{H}^2_{-\sigma}} &\leq C\|WR''_0 WR\psi\|_{L^2_{\sigma'}} \leq C_1\|R''_0 WR\psi\|_{\mathcal{H}^1_{\sigma'-\beta}} \\ &\leq C_2|\omega|^{-3/2}\|WR\psi\|_{L^2_{-\sigma'+\beta}} \leq C_3|\omega|^{-3/2}\|R\psi\|_{\mathcal{H}^1_{-\sigma'}} \leq C_4|\omega|^{-3/2}\|\psi\|_{L^2_{\sigma}} \end{aligned}$$

by Lemma 3.5 since  $\beta - \sigma' > 5/2$  and  $\sigma + \sigma' > 2$ . Hence, (3.18) with  $k = 2$  is proved.  $\square$

### 3.3 High energy decay

Denote by  $R_A(\omega) = (H_A - \omega)^{-1}$  the resolvent of the operator  $H_A = [-i\nabla - A(x)]^2$ . In Appendix A we will prove the following asymptotics of  $R_A(\omega)$  for large  $\omega$ :

**Theorem 3.7.** *Let  $A_j(x) \in C^2(\mathbb{R}^3)$  are real functions and for some  $\beta > 2$  the bound holds*

$$|A(x)| + |\nabla A(x)| + |\nabla \nabla A(x)| \leq C\langle x \rangle^{-\beta} \quad (3.22)$$

Then for  $\sigma > 1/2$  and  $l = 0; 1$  the asymptotics hold

$$\|R_A(\omega)\|_{\mathcal{L}(\mathcal{H}^0_{\sigma}; \mathcal{H}^l_{-\sigma})} = \mathcal{O}(|\omega|^{-\frac{1-l}{2}}), \quad |\omega| \rightarrow \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty). \quad (3.23)$$



Now we derive the asymptotics of  $R(\omega)$  and its derivatives for large  $\omega$  from (3.23).

**Theorem 3.8.** *Let (1.3) with  $\beta > 3$  and (1.4) with  $\beta > 2$  hold. Then for  $k = 0, 1, 2$ ,  $\sigma > 1/2 + k$ , and  $l = 0, 1$ , the asymptotics hold*

$$\|R^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{H}_\sigma^0; \mathcal{H}_{-\sigma}^l)} = \mathcal{O}(|\omega|^{-\frac{1-l+k}{2}}), \quad |\omega| \rightarrow \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty). \quad (3.24)$$

*Proof.* *Step i)* For  $k = 0$  asymptotics (3.24) follows from the Born splitting

$$R(\omega) = R_A(\omega)[1 + VR_A(\omega)]^{-1}$$

and (3.23), since the norm of  $[1 + VR_A(\omega)]^{-1} : \mathcal{H}_\sigma^0 \rightarrow \mathcal{H}_\sigma^0$  is bounded for large  $\omega \in \mathbb{C} \setminus [0, \infty)$  and  $\sigma \in (1/2, \beta/2]$ .

*Step ii)* For  $k = 1$  we use identity (3.19). The identity implies (3.24) with  $k = 1$  and  $\sigma > 3/2$  by (2.5) with  $k = 1$ , and (3.24) with  $k = 0$ . Indeed, this is obvious for the first term in the RHS of (3.19). Let us consider the second term. Choosing  $\sigma' \in (3/2, \beta - 3/2)$ , we obtain for large  $\omega \in \mathbb{C} \setminus [0, \infty)$

$$\|RW R_0' \psi\|_{\mathcal{H}_{-\sigma}^l} \leq C|\omega|^{-\frac{1-l}{2}} \|W R_0' \psi\|_{\mathcal{H}_\sigma^0} \leq C_1|\omega|^{-\frac{1-l}{2}} \|R_0' \psi\|_{\mathcal{H}_{\sigma'}^1} \leq C_2|\omega|^{-\frac{2-l}{2}} \|\psi\|_{\mathcal{H}_\sigma^0}, \quad l = 0; 1. \quad (3.25)$$

The remaining terms in the RHS of (3.19) can be estimated similarly. Hence, (3.24) with  $k = 1$  and  $\sigma > 3/2$  is proved.

*Step iii)* In the case  $k = 2$  we apply identity (3.21). Asymptotics (3.24) with  $k = 2$  for the first term in the RHS of (3.21) follows from (2.5) with  $k = 2$ . The last two terms can be estimated similarly to (3.25) using (2.5) with  $k = 1$  and (3.24) with  $k = 0; 1$ .

Consider the remaining terms. Using (3.24) with  $k = 0$  and (2.5) with  $k = 2$  and  $l = 0; 1$ , we obtain that

a) for  $\sigma' \in (5/2, \beta - 1/2)$  the bounds hold

$$\|RW R_0'' \psi\|_{\mathcal{H}_{-\sigma}^l} \leq C|\omega|^{-\frac{1-l}{2}} \|W R_0'' \psi\|_{\mathcal{H}_{-\sigma'+\beta}^0} \leq C_1|\omega|^{-\frac{1-l}{2}} \|R_0'' \psi\|_{\mathcal{H}_{-\sigma'}^1} \leq C_2|\omega|^{-\frac{3-l}{2}} \|\psi\|_{\mathcal{H}_\sigma^0},$$

$$\|R_0'' W R \psi\|_{\mathcal{H}_{-\sigma}^l} \leq C|\omega|^{-\frac{3-l}{2}} \|W R \psi\|_{\mathcal{H}_\sigma^0} \leq C_1|\omega|^{-\frac{3-l}{2}} \|R \psi\|_{\mathcal{H}_{\sigma'}^1} \leq C_2|\omega|^{-\frac{3-l}{2}} \|\psi\|_{\mathcal{H}_\sigma^0},$$

b) for  $\sigma' \in (1/2, \beta - 5/2)$  the bound holds

$$\begin{aligned} \|RW R_0'' W R \psi\|_{\mathcal{H}_{-\sigma}^l} &\leq C|\omega|^{-\frac{1-l}{2}} \|W R_0'' W R \psi\|_{\mathcal{H}_\sigma^0} \leq C_1|\omega|^{-\frac{1-l}{2}} \|R_0'' W R \psi\|_{\mathcal{H}_{\sigma'}^1} \\ &\leq C_2|\omega|^{-\frac{3-l}{2}} \|W R \psi\|_{\mathcal{H}_{-\sigma'+\beta}^0} \leq C_3|\omega|^{-\frac{3-l}{2}} \|R \psi\|_{\mathcal{H}_{-\sigma'}^1} \leq C_4|\omega|^{-\frac{3-l}{2}} \|\psi\|_{\mathcal{H}_\sigma^0}. \end{aligned}$$

Hence, (3.24) with  $k = 2$  is proved.  $\square$

## 4 Time Decay

We prove time decay (1.5) follow the methods of [8]. Under conditions (1.3), (1.4) and (3.15) the solution  $\psi(t)$  to (1.1) admits the representation

$$\psi(t) = \sum_{j=1}^N e^{-i\omega_j t} P_j \psi(0) + \frac{1}{2\pi i} \int_0^\infty e^{-i\omega t} [R(\omega + i0) - R(\omega - i0)] \psi(0) d\omega, \quad t \in \mathbb{R} \quad (4.1)$$

for initial state  $\psi(0) \in L_\sigma^2$  with  $\sigma > 1$ . The representation follows from the Cauchy residue theorem, Theorem 3.1, and (3.24) with  $k = 0$ .

**Definition 4.1.** *i)  $X_d := \sum_{j=1}^N P_j L^2$  is the discrete spectral subspace of  $H$  spanned by all eigenfunctions.*

*ii)  $X_c := X_d^\perp$  is the orthogonal to  $X_d$  subspace of the continuous spectrum of  $H$ .*

**Theorem 4.2.** *Let conditions (1.3), (1.4) and (3.15) hold. Then*

$$\|\psi(t)\|_{L_{-\sigma}^2} \leq C \langle t \rangle^{-3/2} \|\psi(0)\|_{L_\sigma^2}, \quad t \in \mathbb{R} \quad (4.2)$$

for any initial state  $\psi(0) \in X_c \cap L_\sigma^2$  with  $\sigma > 5/2$ .

*Proof.* Since  $P_j \psi(0) = 0$ , then (4.1) reduces to

$$\psi(t) = \frac{1}{2\pi i} \int_0^\infty e^{-i\omega t} [R(\omega + i0) - R(\omega - i0)] \psi(0) d\omega, \quad t \in \mathbb{R}. \quad (4.3)$$

To deduce (4.2), introduce the partition of unity  $1 = \zeta_l(\omega) + \zeta_h(\omega)$ ,  $\omega \in \mathbb{R}$ , where

$$\zeta_l \in C_0^\infty(\mathbb{R}), \quad \zeta_l(\omega) = \begin{cases} 1, & |\omega| \leq \varepsilon/2 \\ 0, & |\omega| \geq \varepsilon \end{cases}$$

with a small  $\varepsilon > 0$ . Then (4.3) reads

$$\begin{aligned} \psi(t) = \psi_l(t) + \psi_h(t) &= \frac{1}{2\pi i} \int_0^\infty \zeta_l(\omega) e^{-i\omega t} [R(\omega + i0) - R(\omega - i0)] \psi(0) d\omega \\ &+ \frac{1}{2\pi i} \int_0^\infty \zeta_h(\omega) e^{-i\omega t} [R(\omega + i0) - R(\omega - i0)] \psi(0) d\omega. \end{aligned} \quad (4.4)$$

Integrating twice by parts and using (3.24) with  $k = 2$  we obtain for the “high energy component”  $\psi_h(t)$  the decay

$$\|\psi_h(t)\|_{L_{-\sigma}^2} \leq C \langle t \rangle^{-2} \|\psi(0)\|_{L_\sigma^2}.$$

To estimate the “low energy component”  $\psi_l(t)$  we apply the following lemma of Jensen-Kato [8, Lemma 10.2] to the vector function  $F(\omega) := \zeta_l(\omega) [R(\omega + i0) - R(\omega - i0)] \psi(0)$  with the values in the Banach space  $\mathbf{B} = L_{-\sigma}^2$  with  $\sigma > 5/2$ :

**Lemma 4.3.** *Let  $F \in C(0, a; \mathbf{B})$  satisfy*

$$F(0) = F(a) = 0; \quad \|F''(\omega)\|_{\mathbf{B}} \leq C |\omega|^{-3/2}, \quad \omega \in (0, a). \quad (4.5)$$

*Then*

$$\left\| \int_0^a e^{-i\omega t} F(\omega) d\omega \right\|_{\mathbf{B}} = \mathcal{O}(t^{-3/2}), \quad t \rightarrow \infty. \quad (4.6)$$

All the conditions of Lemma 4.3 are satisfied due to (3.17)-(3.18). Then

$$\|\psi_l(t)\|_{L_{-\sigma}^2} \leq C \langle t \rangle^{-3/2} \|\psi(0)\|_{L_\sigma^2}. \quad (4.7)$$

□

## A Proof of Theorem 3.7

Here we extend the Agmon-Jensen-Kato estimates [1, (A.2')] and [8, (8.1)] to the resolvent  $R_A(\omega)$ . The operator  $H_A$  for  $A(x) \in C^1(\mathbb{R}^3)$  is a symmetric operator in the Hilbert space  $L^2 := L^2(\mathbb{R}^3)$  with the domain  $\mathcal{D} := C_0^\infty(\mathbb{R}^3)$ . Moreover,  $H_A$  is nonnegative, hence it admits the unique selfadjoint extension which is its closure, by the Friedrichs theorem. Denote by  $H_A^{1/2}$  the nonnegative square root of  $H_A$  which is also selfadjoint operator in  $L^2$ , so

$$\|H_A^{1/2}u\| = \|\nabla_A u\|, \quad u \in \mathcal{D}$$

where  $\nabla_A = \nabla - iA$ , and  $\|\cdot\|$  stands for the norm in  $L^2$ .

**Lemma A.1.** *Let  $A(x) \in C(\mathbb{R}^3)$  and  $|A(x)| \leq C\langle x \rangle^{-\beta}$  with a  $\beta \geq 1$ . Then for any  $\sigma \in \mathbb{R}$ , the bounds hold*

$$\|\nabla u\| \leq C_1 \|\nabla_A u\| \leq C_2 \|\nabla u\|, \quad u \in \mathcal{D}. \quad (\text{A.1})$$

*Proof.* We apply the magnetic version of the Hardy inequality ([2]):

$$\|u\|_{L^2_{-1}} \leq 4\|\nabla_A u\|, \quad u \in \mathcal{D}. \quad (\text{A.2})$$

Writing  $\nabla u = (\nabla - iA(x))u + iA(x)u$ , we obtain by (A.2),

$$\|\nabla u\| \leq \|(\nabla - iA(x))u\| + \|Au\| \leq \|\nabla_A u\| + C\|u\|_{L^2_{-1}} \leq C_1 \|\nabla_A u\|.$$

Further,

$$\|\nabla_A u\| = \|(\nabla - iA(x))u\| \leq \|\nabla u\| + \|Au\| \leq \|\nabla u\| + C\|u\|_{L^2_{-1}} \leq C_2 \|\nabla u\|$$

where the last bound follows from (A.2) with  $A(x) \equiv 0$ .  $\square$

We reduce Theorem 3.7 to certain lemmas. The first lemma generalizes Lemma A.2 from [1].

**Lemma A.2.** *Let the conditions of Theorem 3.7 hold. Then for  $\sigma > 1/2$ , the bound holds*

$$\|(-i\nabla - A)\psi\|_{\mathcal{H}^0_{-\sigma}} \leq C(\sigma)\|(H_A - \omega)\psi\|_{\mathcal{H}^0_{\sigma}}, \quad \psi \in \mathcal{D}, \quad \omega \in \mathbb{C}. \quad (\text{A.3})$$

*Proof.* It suffices to estimate each component:

$$\|(-i\nabla_j - A_j(x))\psi\|_{\mathcal{H}^0_{-\sigma}} \leq C(\sigma)\|(H_A - \omega)\psi\|_{\mathcal{H}^0_{\sigma}}, \quad j = 1, 2, 3. \quad (\text{A.4})$$

Consider  $j = 1$  for the concreteness. Applying the gauge transformation  $\psi(x) \mapsto \psi(x)e^{i\Phi(x)}$  with  $\Phi(x) = \int A_1(x)dx_1$ , we reduce the estimate to the case  $A_1(x) = 0$  and  $A'_j(x) = A_j(x) - \nabla_j\Phi(x)$  instead of  $A_j(x)$  for  $j \neq 1$ . By (3.22), for the real functions  $A'_j(x)$  the bound holds

$$|A'_j(x)| + |\nabla A'_j(x)| \leq C\langle x' \rangle^{-\beta}, \quad x' := (x_2, x_3) \in \mathbb{R}^2. \quad (\text{A.5})$$

Note that this is the only place we need the condition (1.4) on the second derivatives of  $A_j$ . Now (A.4) reduce to bound

$$\int \langle x_1 \rangle^{-2\sigma} |\nabla_1 \psi(x)|^2 dx \leq C(\sigma) \int \langle x_1 \rangle^{2\sigma} |(-\nabla_1^2 + \Lambda(x_1) - \omega)\psi(x)|^2 dx. \quad (\text{A.6})$$

where

$$\Lambda(x_1) = \sum_2^3 [-i\nabla_j - A'_j(x)]^2$$

is a nonnegative selfadjoint operator in  $L^2(\mathbb{R}^2)$ . From [6, Theorem 1.3 (e)] it follows that  $\mathcal{H}_{sing}(\Lambda(x_1)) = 0$ . Moreover,  $\mathcal{H}_{pp}(\Lambda(x_1)) = 0$  since eigenvalues  $\lambda_j > 0$  are forbidden by the results of [9], while  $\lambda_j \leq 0$  are absent since the operator is nonnegative definite.

Denote by  $S$  the circle  $\{\theta \in \mathbb{R}^2 : |\theta| = 1\}$  and by  $X$  the Hilbert space  $L^2(S)$ . Since  $L^2(\mathbb{R}^2) = \mathcal{H}_{ac}(\Lambda(x_1))$  then there exists a unitary generalized Fourier transform

$$F(x_1) : L^2(\mathbb{R}^2) \rightarrow L^2([0, \infty), d\lambda, X) \quad (\text{A.7})$$

such that functions  $\psi \in C_0^\infty(\mathbb{R}^2)$  and the operator  $\Lambda(x_1)$  admit the spectral representations

$$F(x_1) : \psi(x_1, \cdot) \mapsto \tilde{\psi}(x_1, \lambda); \quad F(x_1)[\Lambda(x_1)\psi(x_1, \cdot)] = \lambda\tilde{\psi}(x_1, \lambda), \quad \lambda \geq 0. \quad (\text{A.8})$$

The transform exists by [7, Theorem 4.2] since  $\mathcal{H}_{sing}(\Lambda(x_1)) = 0$  and  $\mathcal{H}_{pp}(\Lambda(x_1)) = 0$ .

Now (A.6) is equivalent to bound

$$\int \langle x_1 \rangle^{-2\sigma} \|\nabla_1 \tilde{\psi}(x_1, \lambda)\|_X^2 d\lambda dx_1 \leq C(\sigma) \int \langle x_1 \rangle^{2\sigma} \|(-\nabla_1^2 + \lambda - \omega)\tilde{\psi}(x_1, \lambda)\|_X^2 d\lambda dx_1. \quad (\text{A.9})$$

Finally, (A.9) follows by the Fubini theorem from vector valued version of [1, Lemma A.2] (see also [15, Lemma 4, p. 442]).  $\square$

Next lemma generalizes Lemma A.3 from [1].

**Lemma A.3.** *For any  $s \in \mathbb{R}$ ,  $b > 0$ , and  $\psi \in \mathcal{D}$ , the estimate holds*

$$\|\psi\|_{\mathcal{H}_s^l} \leq C(s, b)|\omega|^{-\frac{1-l}{2}} \left( \|(H_A - \omega)\psi\|_{\mathcal{H}_s^0} + \|(-i\nabla - A)\psi\|_{\mathcal{H}_s^0} \right), \quad \omega \in \mathbb{C}, \quad |\omega| \geq b, \quad l = 0, 1. \quad (\text{A.10})$$

*Proof. Step i)* First consider  $s = 0$ . By (A.1), in this case (A.10) is equivalent to estimate

$$\|\psi\|^2 + \|H_A^{l/2}\psi\|^2 \leq C(b)|\omega|^{-(1-l)} \left( \|(H_A - \omega)\psi\|^2 + \|H_A^{1/2}\psi\|^2 \right), \quad l = 0, 1. \quad (\text{A.11})$$

We will deduce (A.11) from bound

$$(1 + \lambda^{l/2})^2 \leq C(b)|\omega|^{-(1-l)} (|\lambda - \omega|^2 + \lambda), \quad \lambda \geq 0, \quad \omega \in \mathbb{C}, \quad |\omega| \geq b, \quad l = 0, 1. \quad (\text{A.12})$$

In the case  $l = 1$  the bound is trivial, and in the case  $l = 0$  it is evident separately for  $|\lambda - \omega| < |\omega|/2$  and for  $|\lambda - \omega| > |\omega|/2$ .

For the selfadjoint operator  $H_A$  and functions  $\psi \in \mathcal{D}$  the spectral representations of type (A.8) also hold:

$$F : \psi \mapsto \tilde{\psi}(\lambda), \quad F : H_A \psi \mapsto \lambda \tilde{\psi}(\lambda)$$

Here  $F : L^2(\mathbb{R}^3) \rightarrow L^2([0, \infty), d\lambda, Y)$  is a unitary operator,  $Y = L^2(S)$ , and  $S$  is the sphere  $\{\theta \in \mathbb{R}^3 : |\theta| = 1\}$ .

Multiplying both sides of (A.12) by  $\|\tilde{\psi}(\lambda)\|_Y^2$  and integrating in  $\lambda$ , we obtain (A.11).

*Step ii)* Now consider an arbitrary  $s \in \mathbb{R}$ . For an  $\varepsilon > 0$ , denote  $\rho_\varepsilon(x) = (1 + |\varepsilon x|^2)^{1/2}$ . Then (A.10) is equivalent to the estimate:

$$\sum_{|\alpha| \leq l} \|\rho_\varepsilon^s \partial^\alpha \psi\| \leq C(s, b) |\omega|^{-\frac{1-l}{2}} \left( \|\rho_\varepsilon^s (H_A - \omega) \psi\| + \|\rho_\varepsilon^s (-i\nabla - A) \psi\| \right), \quad |\omega| \geq b, \quad (\text{A.13})$$

since for any fixed  $\varepsilon > 0$  the weighted norm with  $\rho_\varepsilon(x)$  is equivalent to the weighted norm with  $\rho_1(x)$  defined in (1.2). We apply (A.11) to  $\rho_\varepsilon^s(x)\psi(x)$  and obtain

$$\sum_{|\alpha| \leq l} \|\partial^\alpha [\rho_\varepsilon^s \psi]\| \leq C(b) |\omega|^{-\frac{1-l}{2}} \left( \|(H_A - \omega)[\rho_\varepsilon^s \psi]\| + \|(-i\nabla - A)[\rho_\varepsilon^s \psi]\| \right), \quad l = 0, 1. \quad (\text{A.14})$$

To deduce (A.13) from (A.14), we should commute the multipliers  $\rho_\varepsilon$  with differential operators. For example, consider the commutators

$$\partial^\alpha (\rho_\varepsilon^s \psi) - \rho_\varepsilon^s \partial^\alpha \psi = \sum_{0 \leq \gamma_j \leq \alpha_j, |\gamma| \geq 1} C_{\alpha, \gamma} \partial^\gamma \rho_\varepsilon^s \cdot \partial^{\alpha - \gamma} \psi. \quad (\text{A.15})$$

The commutators are small and their contributions are negligible for small  $\varepsilon$ . Namely,

$$|\nabla_j \rho_\varepsilon^s(x)| = \left| \frac{s}{2} (1 + |\varepsilon x|^2)^{s/2-1} 2\varepsilon^2 x_j \right| \leq \frac{|s|}{2} (1 + |\varepsilon x|^2)^{s/2-1} \varepsilon (1 + \varepsilon^2 x_j^2) \leq \varepsilon C \rho_\varepsilon^s(x)$$

where  $C = C(s)$ . Similarly, we have

$$|\partial^\alpha \rho_\varepsilon^s(x)| \leq \varepsilon^{|\alpha|} C \rho_\varepsilon^s(x), \quad x \in \mathbb{R}^3, \quad 0 \leq |\alpha| \leq 2.$$

Hence, (A.15) implies that

$$\|\partial^\alpha (\rho_\varepsilon^s \psi) - \rho_\varepsilon^s \partial^\alpha \psi\| \leq \varepsilon C_1 \sum_{|\gamma| \leq |\alpha| - 1} \|\rho_\varepsilon^s \partial^\gamma \psi\|. \quad (\text{A.16})$$

Therefore,

$$\|(H_A - \omega)(\rho_\varepsilon^s \psi) - \rho_\varepsilon^s (H_A - \omega) \psi\| \leq \varepsilon C_2 \sum_{|\gamma| \leq 1} \|\rho_\varepsilon^s \partial^\gamma \psi\|, \quad (\text{A.17})$$

$$\|(-i\nabla - A)(\rho_\varepsilon^s \psi) - \rho_\varepsilon^s (-i\nabla - A) \psi\| \leq \varepsilon C_3 \|\rho_\varepsilon^s \psi\|. \quad (\text{A.18})$$

*Step iii)* Now we can prove (A.13). First, we prove it for  $l = 0$ . Applying (A.14), we obtain by (A.17) and (A.18) that

$$\begin{aligned} \|\rho_\varepsilon^s \psi\| &\leq C(b) \frac{1}{\sqrt{|\omega|}} \left( \|(H_A - \omega)(\rho_\varepsilon^s \psi)\| + \|(-i\nabla - A)(\rho_\varepsilon^s \psi)\| \right) \\ &\leq C(b) \frac{1}{\sqrt{|\omega|}} \left( \|\rho_\varepsilon^s (H_A - \omega) \psi\| + \varepsilon C_2 (\|\rho_\varepsilon^s \psi\| + \|\rho_\varepsilon^s (-i\nabla - A) \psi\|) \right. \\ &\quad \left. + \|\rho_\varepsilon^s (-i\nabla - A) \psi\| + \varepsilon C_1 \|\rho_\varepsilon^s \psi\| \right) \\ &\leq C(b) \frac{1}{\sqrt{|\omega|}} \left( \|\rho_\varepsilon^s (H_A - \omega) \psi\| + \|\rho_\varepsilon^s (-i\nabla - A) \psi\| \right) + \frac{1}{\sqrt{\omega}} \varepsilon C_1(b) \|\rho_\varepsilon^s \psi\|. \end{aligned}$$

Choosing  $\varepsilon > 0$  small enough such that  $\varepsilon C_1(b)/\sqrt{|\omega|} < 1$ , we obtain

$$\|\rho_\varepsilon^s \psi\| \leq C_2(b) \frac{1}{\sqrt{|\omega|}} \left( \|\rho_\varepsilon^s (H_A - \omega) \psi\| + \|\rho_\varepsilon^s (-i\nabla - A) \psi\| \right).$$

Hence, (A.13) with  $l = 0$  follows.

Finally, we prove (A.13) for  $l = 1$ . Applying (A.14), we obtain by (A.16) with  $|\alpha| = 1$  and (A.17), (A.18), that

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|\rho_\varepsilon^s \partial^\alpha \psi\| &\leq \sum_{|\alpha| \leq 1} \|\partial^\alpha (\rho_\varepsilon^s \psi)\| + \varepsilon C_1 \|\rho_\varepsilon^s \psi\| \\ &\leq C(b) \left( \|(H_A - \omega)(\rho_\varepsilon^s \psi)\| + \|(-i\nabla - A)(\rho_\varepsilon^s \psi)\| \right) + \varepsilon C_1 \|\rho_\varepsilon^s \psi\| \\ &\leq C(b) \left( \|\rho_\varepsilon^s (H_A - \omega) \psi\| + \|\rho_\varepsilon^s (-i\nabla - A) \psi\| \right) + \varepsilon C_3(b) \sum_{|\alpha| \leq 1} \|\rho_\varepsilon^s \partial^\alpha \psi\|. \end{aligned}$$

Choosing  $\varepsilon > 0$  small enough, we obtain

$$\sum_{|\alpha| \leq 1} \|\rho_\varepsilon^s \partial^\alpha \psi\| \leq C_4(b) \left( \|\rho_\varepsilon^s (H_A - \omega) \psi\| + \|\rho_\varepsilon^s (-i\nabla - A) \psi\| \right)$$

that implies (A.13) with  $l = 1$ . Lemma A.3 is proved.  $\square$

**Proof of Theorem 3.7** Combining (A.10) with  $s = -\sigma$  and (A.3), we obtain for all  $\psi \in \mathcal{D}$

$$\begin{aligned} \|\psi\|_{\mathcal{H}_{-\sigma}^l} &\leq C(\sigma, b) |\omega|^{-\frac{1-l}{2}} \left( \|(H_A - \omega) \psi\|_{\mathcal{H}_{-\sigma}^0} + C(\sigma) \|(H_A - \omega) \psi\|_{\mathcal{H}_\sigma^0} \right) \\ &\leq C_1(\sigma, b) |\omega|^{-\frac{1-l}{2}} \|(H_A - \omega) \psi\|_{\mathcal{H}_\sigma^0} \end{aligned}$$

and then Theorem 3.7 is proved.

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