Scattering of Solitons for the Schrödinger Equation Coupled to a Particle

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Abstract. We establish soliton-like asymptotics for finite energy solutions to the Schrödinger equation coupled to a nonrelativistic classical particle. Any solution with initial state close to the solitary manifold converges to a sum of a travelling wave and an outgoing free wave. The convergence holds in global energy norm. The proof uses spectral theory and the symplectic projection onto the solitary manifold in the Hilbert phase space.

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1. INTRODUCTION

We continue the study of coupled systems of wave fields and particles. In [10], the Klein–Gordon equation coupled to a relativistic particle was considered. Here we extend the result to the Schrödinger equation coupled to a nonrelativistic particle. We prove the long-time convergence to the sum of a soliton and a dispersive wave. The convergence holds in global energy norm for finite-energy solution with initial state close to the solitary manifold.

We consider the Schrödinger wave function $\psi(x)$ in $\mathbb{R}^3$, coupled to a nonrelativistic particle with position q and momentum p, governed by

$$\left\{\begin{aligned} i\dot\psi(x,t) &= -\Delta\psi(x,t)+m^2\psi(x,t)+\rho(x-q(t)),\\ \ddot q(t) &= \frac{1}{2}\int\big[\,\overline{\psi}(x,t)\nabla\rho(x-q(t))+\psi(x,t)\nabla\overline{\rho}(x-q(t))\big]\,dx,\end{aligned}\right. \quad\Bigg|\quad x\in\mathbb{R}^3, \tag{1.1}$$

where $m>0$. Write $\psi_1=\operatorname{Re}\psi,\ \psi_2=\operatorname{Im}\psi,\ \rho_1=\operatorname{Re}\rho,\ \rho_2=\operatorname{Im}\rho$. Then system (1.1) becomes

$$\left\{\begin{aligned} \dot\psi_1(x,t) &= -\Delta\psi_2(x,t)+m^2\psi_2(x,t)+\rho_2(x-q(t)),\\ \dot\psi_2(x,t) &= \Delta\psi_1(x,t)-m^2\psi_1(x,t)-\rho_1(x-q(t)),\\ \ddot q(t) &= \int\big(\psi_1(x,t)\nabla\rho_1(x-q(t))+\psi_2(x,t)\nabla\rho_2(x-q(t))\big)\,dx,\end{aligned}\right. \quad\Bigg|\quad x\in\mathbb{R}^3. \tag{1.2}$$

This is a Hamiltonian system with the Hamiltonian functional

$$\mathcal{H}(\psi_1,\psi_2,q,\dot q)=\frac{1}{2}\int\Big(|\nabla\psi_1(x)|^2+|\nabla\psi_2(x)|^2+m^2|\psi_1(x)|^2+m^2|\psi_2(x)|^2\Big)\,dx$$
$$+\int\Big(\psi_1(x)\rho_1(x-q)+\psi_2(x)\rho_2(x-q)\Big)dx+\frac{1}{2}|\dot q|^2. \tag{1.3}$$

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We consider the Cauchy problem for the Hamiltonian system (1.2), which we write as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R}; \qquad Y(0) = Y_0. \tag{1.4}$$

Here

$$Y(t) = (\psi_1(t), \psi_2(t), q(t), p(t)), \qquad p(t) := \dot{q}(t), \qquad Y_0 = (\psi_{01}, \psi_{02}, q_0, p_0),$$

and all derivatives are understood in the sense of distributions. Below we always deal with column vectors, but often write them as row vectors. System (1.2) is translation invariant and admits the soliton solutions

$$Y_{a,v}(t) = (\psi_{v1}(x - vt - a), \psi_{v2}(x - vt - a), vt + a, v), \tag{1.5}$$

for all $a, v \in \mathbb{R}^3$ with $|v| < 2m$. The states $S_{a,v} := Y_{a,v}(0)$ form the solitary manifold

$$\mathcal{S} := \{S_{a,v} : a, v \in \mathbb{R}^3, |v| < 2m\}. \tag{1.6}$$

Our main result is the soliton asymptotics of type

$$\psi(x,t) \sim \psi_{v_\pm}(x - v_\pm t - a_\pm) + W_0(t)\boldsymbol{\psi}_\pm, \qquad t \to \pm\infty, \tag{1.7}$$

for solutions to (1.1) with initial data close to the solitary manifold $\mathcal{S}$. Here $\psi_{v_\pm} = \psi_{v_\pm 1} + i\psi_{v_\pm 2}$, $W_0(t)$ stands for the dynamical group of the free Schrödinger equation, $\boldsymbol{\psi}_\pm$ are the corresponding *asymptotic scattering states*, and the asymptotics hold *in the global energy norm*, i.e., in the norm of the Sobolev space $H^1(\mathbb{R}^3)$. For the particle trajectory, we prove that

$$\dot{q}(t) \to v_\pm, \qquad q(t) \sim v_\pm t + a_\pm, \qquad t \to \pm\infty. \tag{1.8}$$

The results are established under the following conditions on the complex-valued charge distributions ρ:

$$(1 + |x|)^\beta \rho,\ (1 + |x|)^\beta \nabla\rho,\ (1 + |x|)^\beta \nabla\nabla\rho \in L^2(\mathbb{R}^3), \tag{1.9}$$

with some $\beta > 3/2$. We assume that all “modes” of the wave field are coupled to the particle, this is formalized by the Wiener condition

$$\hat{\rho}(k) = (2\pi)^{-3/2} \int e^{ikx} \rho(x)dx \neq 0 \quad \text{for all} \quad k \in \mathbb{R}^3\,. \tag{1.10}$$

This is an analog of the Fermi Golden Rule: the coupling term $\rho(x - q)$ is not orthogonal to the eigenfunctions e^{ikx} of the continuous spectrum of the linear part of the equation (cf. [4, 21–23]).

Similar results were first proved by Buslaev and Perelman [2, 3] for 1D translation-invariant Schrödinger equation, and extended by Cuccagna [6] for nD case, $n \geqslant 3$. In [10], the Klein–Gordon equation coupled to a particle is considered.

For the proofs of the asymptotics (1.7) and (1.8), we develop the approach of [10] based on the Buslaev and Perelman methods [2, 3], namely, the symplectic orthogonal decomposition of dynamics near the solitary manifold, the time decay for the linearized equation, etc. Our problem differs from that in [10] in the following points.

i) The speed of propagation for the Schrödinger equation is infinite, and solitons exist only for the velocities $|v| < 2m$.
ii) We consider a nonspherically symmetric coupled function $\rho(x)$. In this case, we need additional arguments for the absence of eigenvalues embedded in the continuous spectrum.
iii) We also consider the coupling function $\rho(x)$ with possibly noncompact support. Correspondingly, when proving the time decay for the linearized equation, we use the Jensen–Kato results [14, 15] and the Agmon weighted norms [1].

Remark 1.1. The term m^2 in the Schrödinger equation appears automatically in the nonrelativistic limit of the Klein–Gordon equation and, traditionally, it is removed by a gauge transformation. We keep the term to provide the existence of nonzero solitons.

2. MAIN RESULTS

2.1. *Existence of Dynamics*

To formulate our results precisely, we need some definitions. Introduce a suitable phase space for the Cauchy problem corresponding to (1.2) and (1.3). Let $H^0 = L^2$, and let H^1 be the Sobolev space $H^1 = \{\psi \in L^2 : |\nabla\psi| \in L^2\}$ with the norm

$$\|\psi\|_{H^1} = \|\nabla\psi\|_{L^2} + \|\psi\|_{L^2}.$$

We also introduce the weighted Sobolev spaces H^s_α, $s = 0, 1$, $\alpha \in \mathbb{R}$, with the norms

$$\|\psi\|_{s,\alpha} := \|(1 + |x|)^\alpha \psi\|_{H^s}.$$

Definition 2.1. i) The phase space $\mathcal{E}$ is the real Hilbert space $H^1 \oplus H^1 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\psi_1, \psi_2, q, p)$ with the finite norm

$$\|Y\|_{\mathcal{E}} = \|\psi_1\|_{H^1} + \|\psi_2\|_{H^1} + |q| + |p|.$$

ii) $\mathcal{E}_\alpha$ is the space $H^1_\alpha \oplus H^1_\alpha \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with the norm

$$\|Y\|_\alpha = \|\,Y\|_{\mathcal{E}_\alpha} = \|\psi_1\|_{1,\alpha} + \|\psi_2\|_{1,\alpha} + |q| + |p|.$$

iii) $\mathcal{E}^+$ is the space $H^2 \oplus H^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with the norm

$$\|Y\|^+_{\mathcal{E}} = \|\psi_1\|_{H^2} + \|\psi_2\|_{H^2} + |q| + |p|.$$

For $\psi_j \in L^2$, we have

$$-\frac{1}{2m^2}\|\rho_j\|^2_{L^2} \leqslant \frac{m^2}{2}\|\psi_j\|^2_{L^2} + \langle\psi_j, \rho_j(\cdot - q)\rangle \leqslant \frac{m^2+1}{2}\|\psi_j\|^2_{L^2} + \frac{1}{2}\|\rho_j\|^2_{L^2}. \tag{2.1}$$

Therefore, $\mathcal{E}$ is the space of finite-energy states. The Hamiltonian functional $\mathcal{H}$ is continuous on the space $\mathcal{E}$, and the lower bound in (2.1) implies that the energy (1.3) is bounded below.

System (1.2) looks like the Hamiltonian system

$$\dot{Y} = J\mathcal{D}\mathcal{H}(Y), \qquad J := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad Y = (\psi_1, \psi_2, q, p) \in \mathcal{E}, \tag{2.2}$$

where $\mathcal{D}\mathcal{H}$ is the Fréchet derivative of the Hamiltonian functional (1.3).

Proposition 2.1. *Let* (1.9) *be satisfied. Then the following assertions hold.*

(i) *For every* $Y_0 \in \mathcal{E}$*, the Cauchy problem* (1.4) *has a unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$*.*

(ii) *For every* $t \in \mathbb{R}$*, the mapping* $U(t) : Y_0 \mapsto Y(t)$ *is continuous on* $\mathcal{E}$*.*

(iii) *The energy is conserved, i.e.,*

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \qquad t \in \mathbb{R}. \tag{2.3}$$

Proof. *Step* i). Let us fix an arbitrary $b > 0$ and prove assertions (i)–(iii) for any $Y_0 \in \mathcal{E}$ such that $\|Y_0\|_{\mathcal{E}} \leqslant b$ and for $|t| \leqslant \varepsilon = \varepsilon(b)$, where $\varepsilon(b)$, $\varepsilon(b) > 0$, is sufficiently small. Let us rewrite the Cauchy problem (1.4) as follows:

$$\dot{Y}(t) = F_1(Y(t)) + F_2(Y(t)), \quad t \in \mathbb{R}; \qquad Y(0) = Y_0, \tag{2.4}$$

where $F_1 \colon Y \mapsto ((-\Delta + m^2)\Psi_2, (\Delta - m^2)\Psi_1, 0, 0)$. The Fourier transform provides the existence and the uniqueness of a solution $Y_1(t) \in C(\mathbb{R}, \mathcal{E})$ to the linear problem (2.4) with $F_2 = 0$. Let $U_1(t) \colon Y_0 \mapsto Y_1(t)$ be the corresponding strongly continuous group of bounded linear operators on $\mathcal{E}$. Then (2.4) for $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is equivalent to

$$Y(t) = U_1(t)Y_0 + \int_0^t ds\, U_1(t-s)F_2(Y(s)), \tag{2.5}$$

because $F_2(Y(\cdot)) \in C(\mathbb{R}, \mathcal{E})$ in this case. The last assertion follows from the local Lipschitz continuity of the mapping F_2 in $\mathcal{E}$, namely, for each $b > 0$, there exists a $\varkappa = \varkappa(b) > 0$ such that

$$\|F_2(Y) - F_2(Z)\|_{\mathcal{E}} \leqslant \varkappa \|Y - Z\|_{\mathcal{E}}$$

for all $Y, Z \in \mathcal{E}$ with $\|Y\|_{\mathcal{E}}$, $\|Z\|_{\mathcal{E}} \leqslant b$. Therefore, by the contraction mapping principle, equation (2.5) has a unique local solution $Y(\cdot) \in C([-\varepsilon, \varepsilon], \mathcal{E})$ with ε, $\varepsilon > 0$, depending on b only.

Step ii). Let us now use the energy conservation to ensure the existence of a global solution and the continuity of this solution. Let us first consider a $Y_0 \in \mathcal{E}_c := C_0^\infty \oplus C_0^\infty \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. In this case, we have $Y(t) \in \mathcal{E}^+$ since $U_1(t)Y_0$, $F_2(Y(t)) \in \mathcal{E}^+$ by (1.9). The energy conservation law follows by (2.2) and from the chain rule for the Fréchet derivatives,

$$\frac{d}{dt}\mathcal{H}(Y(t)) = \langle D\mathcal{H}(Y(t)), \dot{Y}(t)\rangle = \langle D\mathcal{H}(Y(t)), JD\mathcal{H}(Y(t))\rangle = 0, \qquad t \in \mathbb{R},$$

since the operator J is skew-symmetric by (2.2), and $D\mathcal{H}(Y(t)) \in L^2 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ for $Y(t) \in \mathcal{E}^+$. Inequality (2.1) implies

$$\mathcal{H} \geqslant \frac{1}{2}\|\nabla\psi\|_{L^2}^2 + \frac{m^2}{4}\|\psi\|_{L^2}^2 + \frac{1}{2}|p|^2 - \frac{1}{m^2}\|\rho\|_{L^2}^2.$$

Hence, by the energy conservation, we have

$$\frac{1}{2}\|\nabla\psi\|_{L^2}^2 + \frac{m^2}{4}\|\psi\|_{L^2}^2 + \frac{1}{2}|p|^2 - \frac{1}{m^2}\|\rho\|_{L^2}^2 \leqslant \mathcal{H}(Y(t)) = \mathcal{H}(Y_0)$$

for $|t| \leqslant \varepsilon$. This implies the *a priori* estimate

$$\|\psi\|_{H^1} + |p| \leqslant B \quad \text{for} \quad |t| \leqslant \varepsilon, \tag{2.6}$$

with B depending on the norm $\|Y_0\|_{\mathcal{E}}$ of the initial data and on $\|\rho\|_{L^2}$ only. An arbitrary initial data $Y_0 \in \mathcal{E}$ can be approximated by initial data in $\mathcal{E}_c$. The corresponding solution exists due to the representation (2.5) by contraction mapping principle, and then (2.6) follows by passing to the limit.

Step iii). Properties (i)-(iii) for arbitrary $t \in \mathbb{R}$ now follow from the same properties for small values of $|t|$ and from the *a priori* bound (2.6).

Let us compute the solitons (1.5). The substitution to (1.1) gives the stationary equations

$$\begin{aligned} -iv\cdot\nabla\psi_v(y) &= (-\Delta+m^2)\psi_v(y)+\rho(y), \\ p=v, \quad 0 &= -\int\big(\overline{\nabla\psi}_v(y)\rho(y)+\nabla\psi_v(y)\overline{\rho}(y)\big)\,dy. \end{aligned} \tag{2.7}$$

The first equation now implies

$$\Lambda\psi_v(y):=[-\Delta+m^2+iv\cdot\nabla]\psi_v(y)=-\rho(y), \qquad y\in\mathbb{R}^3. \tag{2.8}$$

For $|v|<2m$, the operator Λ is an isomorphism $H^4(\mathbb{R}^3)\to H^2(\mathbb{R}^3)$. Hence, it follows from conditions (1.9) that

$$\psi_v(y)=-\Lambda^{-1}\rho(y)\in H^4(\mathbb{R}^3). \tag{2.9}$$

If v is given and $|v|<2m$, then p_v can be found from the second equation in (2.7).

The function ψ_v can be computed by the Fourier transform. The soliton is given by the formula

$$\psi_v(x)=-\frac{1}{4\pi}\int\frac{e^{-\sqrt{m^2-\frac{v^2}{4}}|x-y|}e^{i\frac{v}{2}(x-y)}\rho(y)d^3y}{|x-y|}. \tag{2.10}$$

Below, in Appendix A, we prove that the last equation in (2.7) also holds. Hence, the soliton solution (1.5) exists and is defined uniquely for any pair (a,v) with $|v|<2m$. Write $V:=\{v\in\mathbb{R}^3: |v|<2m\}$, $\psi_{v1}=\operatorname{Re}\psi_v$, and $\psi_{v2}=\operatorname{Im}\psi_v$.

Definition 2.2. A soliton state is $S(\sigma):=(\psi_{v1}(x-b),\psi_{v2}(x-b),b,v)$, where $\sigma:=(b,v)$ with $b\in\mathbb{R}^3$ and $v\in V$.

Obviously, the soliton solution admits the representation $S(\sigma(t))$, where

$$\sigma(t)=(b(t),v(t))=(vt+a,v). \tag{2.11}$$

Definition 2.3. The *solitary manifold* is the set $\mathcal{S}:=\{S(\sigma):\sigma\in\Sigma:=\mathbb{R}^3\times V\}$.

The main result of our paper is the following theorem.

Theorem 2.1. *Let* (1.9) *and the Wiener condition* (1.10) *hold. Let* $\beta>3/2$ *be the number in* (1.9), *and let* $Y(t)$ *be the solution to the Cauchy problem* (1.4) *with an initial state* Y_0 *that is sufficiently close to the solitary manifold,*

$$p_0<2m, \qquad d_0:=\operatorname{dist}_{\mathcal{E}_\beta}(Y_0,\mathcal{S})\ll 1. \tag{2.12}$$

Then the following asymptotic formulas hold as $t\to\pm\infty$:

$$\dot q(t)=v_\pm+\mathcal{O}(|t|^{-2}), \quad q(t)=v_\pm t+a_\pm+\mathcal{O}(|t|^{-3/2}); \tag{2.13}$$

$$\psi(x,t)=\psi_{v\pm}(x-v_\pm t-a_\pm)+W_0(t)\boldsymbol{\psi}_\pm+r_\pm(x,t) \tag{2.14}$$

with

$$\|r_\pm(t)\|_{H^1}=\mathcal{O}(|t|^{-1/2}). \tag{2.15}$$

It suffices to prove the asymptotic formulas (2.13) and (2.14) as $t\to+\infty$ since system (1.2) is time reversible.

Let us identify the tangent space of $\mathcal{E}$, at every point, with the space $\mathcal{E}$. Consider the symplectic form Ω defined on $\mathcal{E}$ by the rule

$$\Omega = \int d\psi_1(x) \wedge d\psi_2(x)\, dx + dq \wedge dp,$$

i.e.,

$$\Omega(Y_1, Y_2) = \langle Y_1, JY_2 \rangle, \qquad Y_1, Y_2 \in \mathcal{E}, \tag{3.1}$$

where

$$\langle Y_1, Y_2 \rangle := \langle \psi_{11}, \psi_{12} \rangle + \langle \psi_{21}, \psi_{22} \rangle + q_1 q_2 + p_1 p_2$$

and

$$\langle \psi_{11}, \psi_{12} \rangle = \int \psi_{11}(x)\psi_{12}(x)\, dx,$$

etc. It is clear that the form Ω is nondegenerate, i.e.,

$$\Omega(Y_1, Y_2) = 0 \quad \text{for every} \quad Y_2 \in \mathcal{E} \implies Y_1 = 0.$$

Definition 3.1. i) The symbol $Y_1 \nmid Y_2$ means that $Y_1 \in \mathcal{E}$, $Y_2 \in \mathcal{E}$, and Y_1 is symplectic orthogonal to Y_2, i.e., $\Omega(Y_1, Y_2) = 0$.

ii) A projection operator $\mathbf{P} : \mathcal{E} \to \mathcal{E}$ is said to be *symplectic orthogonal* if $Y_1 \nmid Y_2$ for any $Y_1 \in \operatorname{Ker} \mathbf{P}$ and $Y_2 \in \operatorname{Im} \mathbf{P}$.

Consider the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ of the manifold $\mathcal{S}$ at a point $S(\sigma)$. The vectors $\tau_j := \partial_{\sigma_j} S(\sigma)$, where $\partial_{\sigma_j} := \partial_{b_j}$ and $\partial_{\sigma_{j+3}} := \partial_{v_j}$ with $j = 1, 2, 3$, form a basis in $\mathcal{T}_\sigma \mathcal{S}$. In detail,

$$\left.\begin{aligned} \tau_j = \tau_j(v) := \partial_{b_j} S(\sigma) &= (-\partial_j \psi_{v1}(y), -\partial_j \psi_{v2}(y), e_j, 0) \\ \tau_{j+3} = \tau_{j+3}(v) := \partial_{v_j} S(\sigma) &= (\ \partial_{v_j} \psi_{v1}(y),\ \partial_{v_j} \psi_{v2}(y), 0, e_j) \end{aligned}\right| \quad j = 1, 2, 3, \tag{3.2}$$

where $y := x - b$ is the "moving frame coordinate," $e_1 = (1, 0, 0)$, etc. Let us stress that the functions τ_j are always regarded as functions of y rather than those of x.

Formulas (2.10) and conditions (1.9) imply that

$$\tau_j(v) \in \mathcal{E}_\alpha, \quad v \in V, \quad j = 1, \dots, 6, \quad \forall \alpha \leqslant \beta. \tag{3.3}$$

Lemma 3.1. *The matrix with the elements $\Omega(\tau_l(v), \tau_j(v))$ is nondegenerate for any $v \in V$.*

Proof. The elements are computed in Appendix B. As the result, the matrix $\Omega(\tau_l, \tau_j)$ has the form

$$\Omega(v) := (\Omega(\tau_l, \tau_j))_{l,j=1,\dots,6} = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix}, \tag{3.4}$$

where the 3×3-matrix $\Omega^+(v)$ is

$$\Omega^+(v) = K + E. \tag{3.5}$$

Here K is a symmetric 3×3-matrix with the elements

$$K_{ij} = \int \frac{k_j k_l \Big((k^2 + m^2)(|\hat{\psi}_{v1}|^2 + |\hat{\psi}_{v2}|^2) + i(kv)(\hat{\psi}_{v1}\overline{\hat{\psi}}_{v2} - \hat{\psi}_{v2}\overline{\hat{\psi}}_{v1}) \Big)\, dk}{(k^2 + m^2)^2 - (kv)^2}, \tag{3.6}$$

where the "hat" stands for the Fourier transform (cf. (1.10)). The matrix K is the integral of the symmetric nonnegative definite matrix $k \otimes k = (k_i k_j)$ with a nonnegative weight. (The last statement is true since $|\hat{\psi}_{v1} + i\hat{\psi}_{v2}|^2 = |\hat{\psi}_{v1}|^2 + |\hat{\psi}_{v2}|^2 - i(\hat{\psi}_{v1}\overline{\hat{\psi}}_{v2} - \hat{\psi}_{v2}\overline{\hat{\psi}}_{v1}) \geqslant 0$ and $k^2 + m^2 > |(kv)|$ for $|v| < 2m$.) Hence, the matrix K is also nonnegative definite. Since the identity matrix E is positive definite, the matrix $\Omega^+(v)$ is symmetric and positive definite, and hence nondegenerate. Therefore, the matrix $\Omega(\tau_l, \tau_j)$ is also nondegenerate.

Let us introduce the translations

$$T_a\colon (\psi_1(\cdot), \psi_2(\cdot), q, p) \mapsto (\psi_1(\cdot - a), \psi_2(\cdot - a), q + a, p), \qquad a \in \mathbb{R}^3.$$

Note that the manifold $\mathcal{S}$ is invariant with respect to the translations.

Definition 3.2. i) For any $\alpha \in \mathbb{R}$ and $\overline{p} < 2m$, write

$$\mathcal{E}_\alpha(\overline{p}) = \{Y = (\psi_1, \psi_2, q, p) \in \mathcal{E}_\alpha : |p| \leqslant \overline{p}\}.$$

Set $\mathcal{E}(\overline{p}) := \mathcal{E}_0(\overline{p})$.

ii) For any $\overline{v} < 2m$, write

$$\Sigma(\overline{v}) = \{\sigma = (b, v) : b \in \mathbb{R}^3, |v| \leqslant \overline{v}\}.$$

The next lemma shows that, in a small neighborhood of the soliton manifold $\mathcal{S}$, a "symplectic orthogonal projection" onto $\mathcal{S}$ is well defined. The proof is similar to that of Lemma 3.4 in [10].

Lemma 3.2. *Let* (1.9) *hold, and let* $\alpha \in \mathbb{R}$. *Then the following assertions hold.*

i) *There exists a neighborhood* $\mathcal{O}_\alpha(\mathcal{S})$ *of* $\mathcal{S}$ *in* $\mathcal{E}_\alpha$ *and a mapping* $\mathbf{\Pi} : \mathcal{O}_\alpha(\mathcal{S}) \to \mathcal{S}$ *such that* $\mathbf{\Pi}$ *is uniformly continuous in the metric of* $\mathcal{E}_\alpha$ *on the set* $\mathcal{O}_\alpha(\mathcal{S}) \cap \mathcal{E}_\alpha(\overline{p})$ *with* $\overline{p} < 2m$,

$$\mathbf{\Pi}Y = Y \quad \text{for} \quad Y \in \mathcal{S}, \quad \text{and} \quad Y - S \nmid \mathcal{T}_S\mathcal{S}, \quad \text{where} \quad S = \mathbf{\Pi}Y. \tag{3.7}$$

ii) $\mathcal{O}_\alpha(\mathcal{S})$ *is invariant with respect to the translations* T_a *and*

$$\mathbf{\Pi}T_aY = T_a\mathbf{\Pi}Y, \quad \text{for} \quad Y \in \mathcal{O}_\alpha(\mathcal{S}) \quad \text{and} \quad a \in \mathbb{R}^3.$$

iii) *For any* $\overline{p} < 2m$, *there exists a* $\overline{v} < 2m$ *such that the relation*

$$\mathbf{\Pi}Y = S(\sigma)$$

holds with $\sigma \in \Sigma(\overline{v})$ *for any* $Y \in \mathcal{O}_\alpha(\mathcal{S}) \cap \mathcal{E}_\alpha(\overline{p})$.

iv) *For any* $\overline{v} < 2m$, *there exists an* $r_\alpha(\overline{v}) > 0$ *such that*

$$S(\sigma) + Z \in \mathcal{O}_\alpha(\mathcal{S}) \quad \text{if} \quad \sigma \in \Sigma(\overline{v}) \quad \text{and} \quad \|Z\|_\alpha < r_\alpha(\overline{v}).$$

We refer to $\mathbf{\Pi}$ as the *symplectic orthogonal projection onto* $\mathcal{S}$.

Corollary 3.1. *Condition* (2.12) *implies that* $Y_0 = S + Z_0$, *where* $S = S(\sigma_0) = \mathbf{\Pi}Y_0$ *and*

$$\|Z_0\|_\beta \ll 1. \tag{3.8}$$

Proof. Lemma 3.2 implies that $\mathbf{\Pi}Y_0 = S$ is well defined for small $d_0 > 0$. Furthermore, condition (2.12) means that there exists a point $S_1 \in \mathcal{S}$ such that $\|Y_0 - S_1\|_\beta = d_0$. Hence, we have the inclusion $Y_0, S_1 \in \mathcal{O}_\beta(\mathcal{S}) \cap \mathcal{E}_\beta(\overline{p})$ with some $\overline{p} < 2m$, which does not depend on d_0 for sufficiently small d_0. On the other hand, $\mathbf{\Pi}S_1 = S_1$, and hence the uniform continuity of the mapping $\mathbf{\Pi}$ implies that

$$\|S_1 - S\|_\beta \to 0 \quad \text{as} \quad d_0 \to 0.$$

Therefore, for small d_0, we finally have

$$\|Z_0\|_\beta = \|Y_0 - S\|_\beta \leqslant \|Y_0 - S_1\|_\beta + \|S_1 - S\|_\beta \leqslant d_0 + o(1) \ll 1.$$

4. LINEARIZATION ON THE SOLITARY MANIFOLD

Let us consider a solution to the system (1.2) and split it as the sum

$$Y(t) = S(\sigma(t)) + Z(t), \tag{4.1}$$

where $\sigma(t) = (b(t), v(t)) \in \Sigma$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, we can write $Y = (\psi_1, \psi_2, q, p)$ and $Z = (\Psi_1, \Psi_2, Q, P)$. Then (4.1) means that

$$\begin{aligned} \psi_1(x,t) &= \psi_{v(t)1}(x - b(t)) + \Psi_1(x - b(t), t), \qquad q(t) = b(t) + Q(t), \\ \psi_2(x,t) &= \psi_{v(t)2}(x - b(t)) + \Psi_2(x - b(t), t), \qquad p(t) = v(t) + P(t). \end{aligned} \tag{4.2}$$

Substitute (4.2) into (1.2) and linearize the equations in Z. Below we shall choose $S(\sigma(t)) = \mathbf{\Pi} Y(t)$, i.e., $Z(t)$ will be symplectic orthogonal to $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$.

By setting $y = x - b(t)$, which is the "moving frame coordinate," we see from (4.2) and (1.2) that

$$\begin{aligned} \dot\psi_1 &= \dot v \cdot \nabla_v \psi_{v1}(y) - \dot b \cdot \nabla \psi_{v1}(y) + \dot\Psi_1(y,t) - \dot b \cdot \nabla \Psi_1(y,t) \\ &= -\Delta \psi_{v2}(y) + m^2 \psi_{v2}(y) - \Delta \Psi_2(y,t) + m^2 \Psi_2(y,t) + \rho_2(y - Q), \\ \dot\psi_2 &= \dot v \cdot \nabla_v \psi_{v2}(y) - \dot b \cdot \nabla \psi_{v2}(y) + \dot\Psi_2(y,t) - \dot b \cdot \nabla \Psi_2(y,t) \\ &= \Delta \psi_{v1}(y) - m^2 \psi_{v1}(y) + \Delta \Psi_1(y,t) - m^2 \Psi_1(y,t) - \rho_1(y - Q), \\ \dot q &= \dot b + \dot Q = v + P, \\ \dot p &= \dot v + \dot P = -\langle \nabla(\psi_{vj}(y) + \Psi_j(y,t)), \rho_j(y - Q)\rangle. \end{aligned} \tag{4.3}$$

Let us extract the terms linear in Q. Note first that

$$\rho_j(y - Q) = \rho_j(y) - Q \cdot \nabla \rho_j(y) + N_j(Q), \qquad j = 1, 2,$$

where $N_j(Q) = \rho_j(y - Q) - \rho_j(y) + Q \cdot \nabla \rho_j(y)$. Condition (1.9) implies that the bound

$$\|N_j(Q)\|_{0,\beta} \leqslant C_\beta(\overline{Q}) Q^2, \qquad j = 1, 2, \tag{4.4}$$

holds for $N_j(Q)$ uniformly with respect to $|Q| \leqslant \overline{Q}$ for any chosen $\overline{Q}$, where β is the parameter in Theorem 2.1. By using equations (2.7), we obtain from (4.3) the following equations for the components of the vector $Z(t)$:

$$\begin{aligned} \dot\Psi_1(y,t) &= -\Delta \Psi_2(y,t) + m^2 \Psi_2(y,t) + \dot b \cdot \nabla \Psi_1(y,t) - Q \cdot \nabla \rho_2(y) \\ &\qquad + (\dot b - v) \cdot \nabla \psi_{v1}(y) - \dot v \cdot \nabla_v \psi_{v1}(y) + N_2, \\ \dot\Psi_2(y,t) &= \Delta \Psi_1(y,t) - m^2 \Psi_1(y,t) + \dot b \cdot \nabla \Psi_2(y,t) + Q \cdot \nabla \rho_1(y) \\ &\qquad + (\dot b - v) \cdot \nabla \psi_{v2}(y) - \dot v \cdot \nabla_v \psi_{v2}(y) - N_1, \\ \dot Q(t) &= P + (v - \dot b), \\ \dot P(t) &= \langle \Psi_j(y,t), \nabla \rho_j(y)\rangle + \langle \nabla \psi_{vj}(y), Q \cdot \nabla \rho_j(y)\rangle - \dot v + N_4(v, Z), \end{aligned} \tag{4.5}$$

where

$$N_4(v, Z) = -\langle \nabla \psi_{vj}, N_j(Q)\rangle + \langle \nabla \Psi_j, Q \cdot \nabla \rho_j\rangle - \langle \nabla \Psi_j, N_j(Q)\rangle.$$

Clearly, the following estimate holds for $N_4(v, Z)$:

$$|N_4(v, Z)| \leqslant C_\beta(\rho, \overline{v}, \overline{Q}) \Big[Q^2 + \|\Psi_1\|_{1,-\beta} |Q| + \|\Psi_2\|_{1,-\beta} |Q| \Big], \tag{4.6}$$

uniformly with respect to $|v| \leqslant \overline{v}$ and $|Q| \leqslant \overline{Q}$ for any chosen $\overline{v} < 2m$. We can represent equations (4.5) as follows:

$$\dot{Z}(t) = A(t)Z(t) + T(t) + N(t), \qquad t \in \mathbb{R}. \tag{4.7}$$

Here the operator $A(t) = A_{v,w}(t)$ depends on two parameters, $v = v(t)$, and $w := \dot{b}(t)$, and can be written in the form

$$A_{v,w}\begin{pmatrix}\Psi_1\\ \Psi_2\\ Q\\ P\end{pmatrix} = \begin{pmatrix} w\cdot\nabla & -(\Delta - m^2) & -\nabla\rho_2\cdot & 0\\ \Delta - m^2 & w\cdot\nabla & \nabla\rho_1\cdot & 0\\ 0 & 0 & 0 & E\\ \langle\,\cdot\,,\nabla\rho_1\rangle & \langle\,\cdot\,,\nabla\rho_2\rangle & \langle\nabla\psi_{vj},\cdot\nabla\rho_j\rangle & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ Q\\ P\end{pmatrix}. \tag{4.8}$$

Further, $T(t) = T_{v,w}(t)$ and $N(t) = N(t,\sigma,Z)$ in (4.7) stand for

$$T_{v,w} = \begin{pmatrix}(w-v)\cdot\nabla\psi_{v1} - \dot{v}\cdot\nabla_v\psi_{v1}\\ (w-v)\cdot\nabla\psi_{v2} - \dot{v}\cdot\nabla_v\psi_{v2}\\ v-w\\ -\dot{v}\end{pmatrix}, \qquad N(\sigma,Z) = \begin{pmatrix}N_2(Z)\\ -N_1(Z)\\ 0\\ N_4(v,Z)\end{pmatrix}, \tag{4.9}$$

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t), v(t))$, and $Z = Z(t)$. Estimates (4.4) and (4.6) imply that

$$\|N(\sigma,Z)\|_\beta \leqslant C(\overline{v},\overline{Q}\,)\|Z\|^2_{-\beta}, \tag{4.10}$$

uniformly in $\sigma \in \Sigma(\overline{v})$ and $\|Z\|_{-\beta} \leqslant r_{-\beta}(\overline{v})$ for any fixed $\overline{v} < 2m$.

Remark 4.1. i) The term $A(t)Z(t)$ on the right-hand side of equation (4.7) is linear in $Z(t)$, and $N(t)$ is a *high-order term* in $Z(t)$. On the other hand, $T(t)$ is a zero-order term that does not vanish at $Z(t) = 0$ since $S(\sigma(t))$, generally, is not a soliton solution if (2.11) fails to hold (though $S(\sigma(t))$ belongs to the solitary manifold).

ii) Formulas (3.2) and (4.9) imply

$$T(t) = -\sum_{l=1}^{3}[(w-v)_l\tau_l + \dot{v}_l\tau_{l+3}], \tag{4.11}$$

and hence $T(t) \in \mathcal{T}_{S(\sigma(t))}\mathcal{S}$, $t \in \mathbb{R}$. This fact suggests the unstable character of the nonlinear dynamics *along the solitary manifold.*

5. LINEARIZED EQUATION

Here we collect some Hamiltonian and spectral properties of the generator (4.8) of the linearized equation. First, let us consider the linear equation

$$\dot{X}(t) = A_{v,w}X(t), \quad t \in \mathbb{R}, \quad v \in V, \quad w \in \mathbb{R}^3. \tag{5.1}$$

Lemma 5.1 (cf. [10]). i) *For any $v \in V$ and $w \in \mathbb{R}^3$, equation* (5.1) *can be represented as the Hamiltonian system* (*cf.* (2.2)),

$$\dot{X}(t) = JD\mathcal{H}_{v,w}(X(t)), \qquad t \in \mathbb{R}, \tag{5.2}$$

where $D\mathcal{H}_{v,w}$ stands for the Fréchet derivative of the Hamiltonian functional,

$$\mathcal{H}_{v,w}(X) = \frac{1}{2}\int\Big[|\nabla\Psi_1|^2 + m^2|\Psi_1|^2 + |\nabla\Psi_2|^2 + m^2|\Psi_2|^2\Big]dy + \int\Psi_2 w\cdot\nabla\Psi_1 dy + \int\rho_j(y)Q\cdot\nabla\Psi_j dy$$
$$+\frac{1}{2}P^2 - \frac{1}{2}\langle Q\cdot\nabla\psi_{vj}(y), Q\cdot\nabla\rho_j(y)\rangle, \qquad X = (\Psi_1,\Psi_2,Q,P) \in \mathcal{E}. \tag{5.3}$$

ii) *The energy conservation law holds for the solutions* $X(t) \in C^1(\mathbb{R}, \mathcal{E}^+)$,

$$\mathcal{H}_{v,w}(X(t)) = \text{const}, \quad t \in \mathbb{R}. \tag{5.4}$$

iii) *The skew-symmetry relation holds,*

$$\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \quad X_1, X_2 \in \mathcal{E}. \tag{5.5}$$

iv) *The operator* $A_{v,w}$ *acts on the vectors* $\tau_j(v)$ *tangent to the solitary manifold as follows:*

$$A_{v,w}[\tau_j(v)] = (w - v) \cdot \nabla \tau_j(v), \;\; A_{v,w}[\tau_{j+3}(v)] = (w - v) \cdot \nabla \tau_{j+3}(v) + \tau_j(v), \;\; j = 1, 2, 3. \tag{5.6}$$

We shall apply Lemma 5.1 mainly to the operator $A_{v,v}$ corresponding to $w = v$. In this case, the linearized equation has the following additional specific features.

Lemma 5.2. *Assume that* $w = v \in V$. *Then the following assertions hold.*

i) *The tangent vectors* $\tau_j(v)$ *with* $j = 1, 2, 3$ *are eigenvectors, and* $\tau_{j+3}(v)$ *are root vectors of the operator* $A_{v,v}$ *that correspond to the zero eigenvalue, i.e.,*

$$A_{v,v}[\tau_j(v)] = 0, \qquad A_{v,v}[\tau_{j+3}(v)] = \tau_j(v), \quad j = 1, 2, 3. \tag{5.7}$$

ii) *The Hamiltonian function* (5.3) *is nonnegative definite since*

$$\mathcal{H}_{v,v}(X) = \frac{1}{2} \int \left|\Lambda^{1/2}(\Psi_1 + i\Psi_2) - \Lambda^{-1/2} Q \cdot \nabla(\rho_1 + i\rho_2)\right|^2 dx + \frac{1}{2} P^2 \geqslant 0. \tag{5.8}$$

Here Λ *stands for the operator* (2.8), *which is symmetric and nonnegative definite in* $L^2(\mathbb{R}^3)$ *for* $|v| < 2m$, *and* $\Lambda^{1/2}$ *is the nonnegative definite square root defined in the Fourier representation.*

Proof. The first statement follows from (5.6) with $w = v$. In order to prove ii), we can rewrite the integral in (5.8) as follows:

$$\begin{aligned} &\frac{1}{2}\langle \Lambda^{1/2}(\Psi_1 + i\Psi_2) - \Lambda^{-1/2} Q \cdot \nabla(\rho_1 + i\rho_2), \Lambda^{1/2}(\Psi_1 + i\Psi_2) - \Lambda^{-1/2} Q \cdot \nabla(\rho_1 + i\rho_2)\rangle \\ &\quad = \frac{1}{2}\langle \Lambda(\Psi_1 + i\Psi_2), \Psi_1 + i\Psi_2\rangle - \langle \Psi_j, Q \cdot \nabla \rho_j\rangle + \frac{1}{2}\langle \Lambda^{-1} Q \cdot \nabla(\rho_1 + i\rho_2), Q \cdot \nabla(\rho_1 + i\rho_2)\rangle \end{aligned} \tag{5.9}$$

since the operator $\Lambda^{1/2}$ is symmetric in $L^2(\mathbb{R}^3)$. All the terms of expression (5.9) can now be identified with the corresponding terms in (5.3) since

$$\begin{aligned} \frac{1}{2}\langle \Lambda(\Psi_1 + i\Psi_2), \Psi_1 + i\Psi_2\rangle &= \frac{1}{2}\langle [-\Delta + m^2 + iv \cdot \nabla](\Psi_1 + i\Psi_2), (\Psi_1 + i\Psi_2)\rangle \\ &= \frac{1}{2}\langle [-\Delta + m^2]\Psi_1, \Psi_1\rangle + \frac{1}{2}\langle [-\Delta + m^2]\Psi_2, \Psi_2\rangle + \langle \Psi_2, v \cdot \nabla \Psi_1\rangle \end{aligned}$$

and we have $\Lambda^{-1}(\rho_1 + i\rho_2) = -(\psi_{v1} + i\psi_{v2})$ by (2.8) and (2.9).

Remark 5.1. For a soliton solution of the system (1.2), we have $\dot{b} = v$ and $\dot{v} = 0$, and hence $T(t) \equiv 0$. Thus, equation(5.1) is the linearization of system (1.2) on a soliton solution. In fact, we linearize (1.2) on a trajectory $S(\sigma(t))$, where $\sigma(t)$ is nonlinear with respect to t, rather than on a soliton solution. We shall show below that $T(t)$ is quadratic in $Z(t)$ if we choose $S(\sigma(t))$ to be the symplectic orthogonal projection of $Y(t)$. In this case, (5.1) is a linearization of (1.2) again.

6. SYMPLECTIC DECOMPOSITION OF DYNAMICS

Here we decompose the dynamics into two components, along the manifold $\mathcal{S}$ and in the transversal direction. Equation (4.7) is obtained without any assumption on $\sigma(t)$ in (4.1). We are going to specify $S(\sigma(t)) := \mathbf{\Pi} Y(t)$. However, in this case, we must know that

$$Y(t) \in \mathcal{O}_{-\beta}(\mathcal{S}), \qquad t \in \mathbb{R}. \tag{6.1}$$

This is true for $t = 0$ by our main assumption (2.12) with sufficiently small $d_0 > 0$. Then we have $S(\sigma(0)) = \mathbf{\Pi} Y(0)$ and $Z(0) = Y(0) - S(\sigma(0))$ are well defined. We shall prove below that (6.1) holds if d_0 is sufficiently small. Let us choose an arbitrary $\overline{v}$ such that $|v(0)| < \overline{v} < 2m$, and let us denote $\delta = \overline{v} - |v(0)|$. Denote by $r_{-\beta}(\overline{v})$ the positive numbers in Lemma 3.2 iv) which correspond to $\alpha = -\beta$. Then $S(\sigma) + Z \in \mathcal{O}_{-\beta}(\mathcal{S})$ if $\sigma = (b, v)$ with $|v| < \overline{v}$ and $\|Z\|_{-\beta} < r_{-\beta}(\overline{v})$. Note that $\|Z(0)\|_{-\beta} < r_{-\beta}(\overline{v})$ if d_0 is sufficiently small. Therefore, $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ and $Z(t) = Y(t) - S(\sigma(t))$ are well defined for the sufficiently small times $t \geqslant 0$ (for which $|v| < \overline{v}$ and $\|Z(t)\|_{-\beta} < r_{-\beta}(\overline{v})$). This argument can be formalized by using the following standard definition.

Definition 6.1. Let t_* be the "exit time," i.e.,

$$t_* = \sup\{t > 0 : \|Z(s)\|_{-\beta} < r_{-\beta}(\overline{v}),\ |v(s) - v(0)| < \delta,\ 0 \leqslant s \leqslant t\}. \tag{6.2}$$

One of our main goals is to prove that $t_* = \infty$ if d_0 is sufficiently small. This would follow if we could show that

$$\|Z(t)\|_{-\beta} < r_{-\beta}(\overline{v})/2, \qquad |v(s) - v(0)| < \delta/2, \qquad 0 \leqslant t < t_*. \tag{6.3}$$

Note that

$$|Q(t)| \leqslant \overline{Q} := r_{-\beta}(\overline{v}), \qquad 0 \leqslant t < t_*. \tag{6.4}$$

Now, by (4.10), the quantity $N(t)$ in (4.7) satisfies the following estimate:

$$\|N(t)\|_{\beta} \leqslant C_{\beta}(\overline{v}) \|Z(t)\|^2_{-\beta}, \qquad 0 \leqslant t < t_*. \tag{6.5}$$

6.1. *Longitudinal Dynamics*: *Modulation Equations*

From now on, we choose the decomposition $Y(t) = S(\sigma(t)) + Z(t)$ for $0 < t < t_*$ by setting $S(\sigma(t)) = \mathbf{\Pi} Y(t)$, which is equivalent to the symplectic orthogonality condition of type (3.7),

$$Z(t) \nmid \mathcal{T}_{S(\sigma(t))} \mathcal{S}, \qquad 0 \leqslant t < t_*. \tag{6.6}$$

This enables us to drastically simplify the asymptotic analysis of the dynamical equations (4.7) for the transversal component $Z(t)$. As the first step, we derive the longitudinal dynamics, i.e., find the "modulation equations" for the parameters $\sigma(t)$. Thus, let us derive a system of ordinary differential equations for the vector $\sigma(t)$. For this purpose, we write (6.6) in the form

$$\Omega(Z(t), \tau_j(t)) = 0, \quad j = 1, \ldots, 6, \quad 0 \leqslant t < t_*, \tag{6.7}$$

where the vectors $\tau_j(t) = \tau_j(\sigma(t))$ span the tangent space $\mathcal{T}_{S(\sigma(t))} \mathcal{S}$. Note that $\sigma(t) = (b(t), v(t))$, where

$$|v(t)| \leqslant \overline{v} < 2m, \qquad 0 \leqslant t < t_*, \tag{6.8}$$

by Lemma 3.2 iii). It would be convenient for us to use some other parameters (c, v) instead of $\sigma = (b, v)$, where

$$c(t) = b(t) - \int_0^t v(\tau) d\tau \quad \text{and} \quad \dot{c}(t) = \dot{b}(t) - v(t) = w(t) - v(t), \qquad 0 \leqslant t < t_*. \tag{6.9}$$

We do not need an explicit form of the equations for (c, v), except for the following statement, whose proof is similar to that of Lemma 6.2 in [10].

Lemma 6.1. *Let $Y(t)$ be a solution to the Cauchy problem* (1.4), *and let* (4.1) *and* (6.7) *hold. Then $(c(t), v(t))$ satisfies the equation*

$$\begin{pmatrix} \dot{c}(t) \\ \dot{v}(t) \end{pmatrix} = \mathcal{N}(\sigma(t), Z(t)), \qquad 0 \leqslant t < t_*, \tag{6.10}$$

where

$$\mathcal{N}(\sigma, Z) = \mathcal{O}(\|Z\|^2_{-\beta}) \tag{6.11}$$

uniformly in $\sigma \in \Sigma(\overline{v})$.

6.2. *Decay for the Transversal Dynamics*

In Section 11, we shall show that our main Theorem 2.1 can be derived from the following time decay of the transversal component $Z(t)$.

Proposition 6.1. *Let all conditions of Theorem* 2.1 *hold. Then $t_* = \infty$, and*

$$\|Z(t)\|_{-\beta} \leqslant \frac{C(\rho, \overline{v}, d_0)}{(1 + |t|)^{3/2}}, \qquad t \geqslant 0. \tag{6.12}$$

We shall derive (6.12) in Sections 7–11 from our equation (4.7) for the transversal component $Z(t)$. This equation can be specified by using Lemma 6.1. Indeed, the lemma implies that

$$\|T(t)\|_\beta \leqslant C(\overline{v})\|Z(t)\|^2_{-\beta}, \qquad 0 \leqslant t < t_*, \tag{6.13}$$

by (4.9) since $w - v = \dot{c}$. Thus, equation (4.7) becomes

$$\dot{Z}(t) = A(t)Z(t) + \tilde{N}(t), \qquad 0 \leqslant t < t_*, \tag{6.14}$$

where $A(t) = A_{v(t),w(t)}$, and $\tilde{N}(t) := T(t) + N(t)$ satisfies the estimate

$$\|\tilde{N}(t)\|_\beta \leqslant C\|Z(t)\|^2_{-\beta}, \qquad 0 \leqslant t < t_*. \tag{6.15}$$

In the remaining part of our paper, we mainly analyze the *basic equation* (6.14) to establish the decay (6.12). We are going to derive the decay by using the bound (6.15) and the orthogonality condition (6.6).

First, we reduce the problem to the analysis of the *frozen* linear equation,

$$\dot{X}(t) = A_1 X(t), \qquad t \in \mathbb{R}, \tag{6.16}$$

where A_1 is the operator A_{v_1,v_1} defined by (4.8) with $v_1 = v(t_1)$ and a chosen $t_1 \in [0, t_*)$. We can now apply well-known methods of scattering theory and then estimate the error by the method of majorants.

Note that, even for the frozen equation (6.16), the decay of type (6.12) for all solutions does not hold without the orthogonality condition of type (6.6). Namely, by (5.7), equation (6.16) admits the *secular solutions*

$$X(t) = \sum_1^3 C_j \tau_j(v) + \sum_1^3 D_j[\tau_j(v)t + \tau_{j+3}(v)], \tag{6.17}$$

which also arise by differentiating the soliton (1.5) with respect to the parameters a and v in the moving coordinate $y = x - v_1 t$. Hence, we must consider the orthogonality condition (6.6) to avoid the secular solutions. To this end, we shall apply the corresponding symplectic orthogonal projection that kills the "runaway solutions" (6.17).

Remark 6.1. The solution (6.17) belongs to the tangent space $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$ with $\sigma_1 = (b_1, v_1)$ (for an arbitrary $b_1 \in \mathbb{R}$), which suggests the unstable character of the nonlinear dynamics *along the solitary manifold* (cf. Remark 4.1 iii).

Definition 6.2. i) For $v \in V$, denote by $\mathbf{\Pi}_v$ the symplectic orthogonal projection of $\mathcal{E}$ onto the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ and write $\mathbf{P}_v = \mathbf{I} - \mathbf{\Pi}_v$.

ii) Denote by $\mathcal{Z}_v = \mathbf{P}_v\mathcal{E}$ the space symplectic orthogonal to $\mathcal{T}_{S(\sigma)}\mathcal{S}$ with $\sigma = (b, v)$ (for an arbitrary $b \in \mathbb{R}$).

Note that, by the linearity,

$$\mathbf{\Pi}_v Z = \sum \mathbf{\Pi}_{jl}(v)\tau_j(v)\Omega(\tau_l(v), Z), \qquad Z \in \mathcal{E}, \tag{6.18}$$

with some smooth coefficients $\mathbf{\Pi}_{jl}(v)$. Hence, the projection $\mathbf{\Pi}_v$ does not depend on b in the variable $y = x - b$, and this explains the choice of the subscript in $\mathbf{\Pi}_v$ and $\mathbf{P}_v$.

We now have the symplectic orthogonal decomposition

$$\mathcal{E} = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{Z}_v, \qquad \sigma = (b, v), \tag{6.19}$$

and the symplectic orthogonality (6.6) can be represented in the following equivalent forms,

$$\mathbf{\Pi}_{v(t)} Z(t) = 0, \qquad \mathbf{P}_{v(t)} Z(t) = Z(t), \qquad 0 \leqslant t < t_*. \tag{6.20}$$

Remark 6.2. The tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is invariant under the operator $A_{v,v}$ by Lemma 5.2 i), and hence the space $\mathcal{Z}_v$ is also invariant by (5.5), namely, $A_{v,v}Z \in \mathcal{Z}_v$ for any *sufficiently smooth* $Z \in \mathcal{Z}_v$.

In Sections 12–18 below, we prove the following proposition, which is one of the main ingredients in the proof of (6.12). Let us consider the Cauchy problem for equation (6.16) with $A = A_{v,v}$ for a chosen $v \in V$. Recall that the parameter $\beta > 3/2$ is also chosen.

Proposition 6.2. *Let conditions* (1.9) *and* (1.10) *hold, let* $|v| \leqslant \overline{v} < 2m$, *and let* $X_0 \in \mathcal{E}$. *Then the following assertions hold.*

i) *Equation* (6.16), *with* $A = A_{v,v}$, *admits a unique solution* $e^{At}X_0 := X(t) \in C(\mathbb{R}, \mathcal{E})$ *with the initial condition* $X(0) = X_0$.

ii) *For* $X_0 \in \mathcal{Z}_v \cap \mathcal{E}_\beta$, *the solution* $X(t)$ *has the following decay,*

$$\|e^{At}X_0\|_{-\beta} \leqslant \frac{C_\beta(\overline{v})}{(1 + |t|)^{3/2}}\|X_0\|_\beta, \qquad t \in \mathbb{R}. \tag{6.21}$$

7. FROZEN TRANSVERSAL DYNAMICS

Now let us choose an arbitrary $t_1 \in [0, t_*)$, and rewrite the equation (6.14) in "frozen form,"

$$\dot{Z}(t) = A_1 Z(t) + (A(t) - A_1)Z(t) + \tilde{N}(t), \qquad 0 \leqslant t < t_*, \tag{7.1}$$

where $A_1 = A_{v(t_1),v(t_1)}$ and

$$A(t) - A_1 = \begin{pmatrix} [w(t) - v(t_1)] \cdot \nabla & 0 & 0 & 0 \\ 0 & [w(t) - v(t_1)] \cdot \nabla & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \langle \nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla\rho_j \rangle & 0 \end{pmatrix}.$$

The next trick is important since it enables us to kill the "bad terms" $[w(t) - v(t_1)] \cdot \nabla$ in the operator $A(t) - A_1$.

Definition 7.1. Let us change the variables $(y,t)\mapsto(y_1,t)=(y+d_1(t),t)$, where

$$d_1(t):=\int_{t_1}^t (w(s)-v(t_1))ds, \qquad 0\leqslant t\leqslant t_1. \tag{7.2}$$

Next, let us write

$$Z_1(t):=(\Psi_1(y_1-d_1(t),t),\Psi_2(y_1-d_1(t),t),Q(t),P(t)). \tag{7.3}$$

Then we obtain the final form of the "frozen equation" for the transversal dynamics,

$$\dot Z_1(t)=A_1Z_1(t)+B_1(t)Z_1(t)+\tilde N_1(t), \qquad 0\leqslant t\leqslant t_1, \tag{7.4}$$

where $\tilde N_1(t)=\tilde N(t)$ is expressed in terms of $y=y_1-d_1(t)$, and

$$B_1(t)=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \langle\nabla(\psi_{v(t)j}-\psi_{v(t_1)j}),\nabla\rho_j\rangle & 0\end{pmatrix}.$$

Lemma 7.1 (see [10]). *For $(\Psi_1,\Psi_2,Q,P)\in\mathcal{E}_\alpha$ with any $\alpha\leqslant\beta$, the following estimate holds*:

$$\|(\Psi_1(y_1-d_1),\Psi_2(y_1-d_1),Q,P)\|_\alpha\leqslant\|(\Psi_1,\Psi_2,Q,P)\|_\alpha(1+|d_1|)^{|\alpha|}\ , \qquad d_1\in\mathbb{R}^3. \tag{7.5}$$

Corollary 7.1. *The following bounds hold for* $0\leqslant t\leqslant t_1$:

$$\|\tilde N_1(t)\|_\beta\leqslant\|Z(t)\|_{-\beta}^2(1+|d_1(t)|)^\beta\ , \qquad \|B_1(t)Z_1(t)\|_\beta\leqslant C\|Z(t)\|_{-\beta}\int_t^{t_1}\|Z(\tau)\|_{-\beta}^2d\tau\ . \tag{7.6}$$

8. INTEGRAL INEQUALITY

Equation (7.4) can be represented in the following integral form:

$$Z_1(t)=e^{A_1t}Z_1(0)+\int_0^t e^{A_1(t-s)}[B_1Z_1(s)+\tilde N_1(s)]ds, \qquad 0\leqslant t\leqslant t_1. \tag{8.1}$$

Now let us apply the symplectic orthogonal projection $\mathbf{P}_1:=\mathbf{P}_{v(t_1)}$ to both sides of (8.1). The space $\mathcal{Z}_1:=\mathbf{P}_1\mathcal{E}$ is invariant with respect to e^{A_1t} by Proposition 6.2 ii) (cf. also Remark 6.2). Therefore, $\mathbf{P}_1$ commutes with the group e^{A_1t} and, applying (6.21), we obtain

$$\|\mathbf{P}_1Z_1(t)\|_{-\beta}\leqslant C\frac{\|\mathbf{P}_1Z_1(0)\|_\beta}{(1+t)^{3/2}}+C\int_0^t\frac{\|\mathbf{P}_1[B_1Z_1(s)+\tilde N_1(s)]\|_\beta\ ds}{(1+|t-s|)^{3/2}}.$$

The operator $\mathbf{P}_1=\mathbf{I}-\mathbf{\Pi}_1$ is continuous in $\mathcal{E}_\beta$ by (6.18). Hence, using (7.6), we obtain

$$\|\mathbf{P}_1Z_1(t)\|_{-\beta}\leqslant\frac{C(\overline{d}_1(0))}{(1+t)^{3/2}}\|Z(0)\|_\beta$$
$$+C(\overline{d}_1(t))\int_0^t\frac{1}{(1+|t-s|)^{3/2}}\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|_{-\beta}^2d\tau+\|Z(s)\|_{-\beta}^2\right]ds, \qquad 0\leqslant t\leqslant t_1, \tag{8.2}$$

where $\overline{d}_1(t):=\sup_{0\leqslant s\leqslant t}|d_1(s)|$.

Definition 8.1. Let t_*' be the exit time,

$$t_*'=\sup\{t\in[0,t_*):\overline{d}_1(s)\leqslant 1,\ 0\leqslant s\leqslant t\}. \tag{8.3}$$

Now it follows from (8.2) that, for $t_1<t_*'$,

$$\|\mathbf{P}_1Z_1(t)\|_{-\beta}\leqslant\frac{C}{(1+t)^{3/2}}\|Z(0)\|_\beta$$
$$+C_1\int_0^t\frac{1}{(1+|t-s|)^{3/2}}\left[\|Z(s)\|_{-\beta}\int_s^{t_1}\|Z(\tau)\|_{-\beta}^2d\tau+\|Z(s)\|_{-\beta}^2\right]ds, \qquad 0\leqslant t\leqslant t_1. \tag{8.4}$$

Finally, we are going to replace $\mathbf{P}_1 Z_1(t)$ by $Z(t)$ on the left-hand side of (8.4). We shall prove that this replacement is possible indeed by using the fact $d_0 \ll 1$ in (2.12) again. For the justification, we reduce the exit time further. First, introduce the "majorant"

$$m(t) := \sup_{s\in[0,t]} (1+s)^{3/2} \|Z(s)\|_{-\beta}\ , \qquad t \in [0, t_*). \tag{9.1}$$

Denote by ε a chosen positive number (which will be specified below).

Definition 9.1. Let t_*'' be the exit time,

$$t_*'' = \sup\{t \in [0, t_*') : m(s) \leqslant \varepsilon,\ 0 \leqslant s \leqslant t\}. \tag{9.2}$$

The following important bound (9.3) enables us to replace the norm of $\mathbf{P}_1 Z_1(t)$ on the left-hand side of (8.4) by the norm of $Z(t)$.

Lemma 9.1 (cf. [10]). *For any sufficiently small $\varepsilon > 0$, we have*

$$\|Z(t)\|_{-\beta} \leqslant C \|\mathbf{P}_1 Z_1(t)\|_{-\beta}, \qquad 0 \leqslant t \leqslant t_1, \tag{9.3}$$

for any $t_1 < t_*''$, where C depends on ρ and $\overline{v}$ only.

Proof. Since $|d_1(t)| \leqslant 1$ for $t \leqslant t_1 < t_*'' < t_*'$, it follows from Lemma 7.1 that it suffices to prove the inequality

$$\|Z_1(t)\|_{-\beta} \leqslant 2 \|\mathbf{P}_1 Z_1(t)\|_{-\beta}, \qquad 0 \leqslant t \leqslant t_1. \tag{9.4}$$

Recall that $\mathbf{P}_1 Z_1(t) = Z_1(t) - \mathbf{\Pi}_{v(t_1)} Z_1(t)$. Then estimate (9.4) will follow from

$$\|\mathbf{\Pi}_{v(t_1)} Z_1(t)\|_{-\beta} \leqslant \frac{1}{2} \|Z_1(t)\|_{-\beta}, \qquad 0 \leqslant t \leqslant t_1. \tag{9.5}$$

The symplectic orthogonality (6.20) implies the relation

$$\mathbf{\Pi}_{v(t),1} Z_1(t) = 0, \quad t \in [0, t_1], \tag{9.6}$$

where $\mathbf{\Pi}_{v(t),1} Z_1(t)$ is the term $\mathbf{\Pi}_{v(t)} Z(t)$ expressed in terms of the variable $y_1 = y + d_1(t)$. Hence, (9.5) follows from (9.6) if the difference $\mathbf{\Pi}_{v(t_1)} - \mathbf{\Pi}_{v(t),1}$ is small uniformly in t, i.e.,

$$\|\mathbf{\Pi}_{v(t_1)} - \mathbf{\Pi}_{v(t),1}\| < 1/2, \qquad 0 \leqslant t \leqslant t_1. \tag{9.7}$$

It remains to justify (9.7) for any sufficiently small $\varepsilon > 0$. Formula (6.18) implies the following relation:

$$\mathbf{\Pi}_{v(t),1} Z_1(t) = \sum \mathbf{\Pi}_{jl}(v(t)) \tau_{j,1}(v(t)) \Omega(\tau_{l,1}(v(t)), Z_1(t)), \tag{9.8}$$

where the terms $\tau_{j,1}(v(t))$ are the vectors $\tau_j(v(t))$ expressed via the variables y_1. Since the functions $|d_1(t)| \leqslant 1$ and $\nabla \tau_j$ are smooth and rapidly decaying at infinity, Lemma 7.1 implies that

$$\|\tau_{j,1}(v(t)) - \tau_j(v(t))\|_\beta \leqslant C |d_1(t)|^\beta, \qquad 0 \leqslant t \leqslant t_1, \tag{9.9}$$

for all $j = 1, 2, \ldots, 6$. Furthermore,

$$\tau_j(v(t)) - \tau_j(v(t_1)) = \int_t^{t_1} \dot{v}(s) \cdot \nabla_v \tau_j(v(s)) ds,$$

and therefore,

$$\|\tau_j(v(t)) - \tau_j(v(t_1))\|_\beta \leqslant C \int_t^{t_1} |\dot{v}(s)|ds, \qquad 0 \leqslant t \leqslant t_1. \tag{9.10}$$

Similarly,

$$\mathbf{\Pi}_{jl}(v(t)) - \mathbf{\Pi}_{jl}(v(t_1))| = \left| \int_t^{t_1} \dot{v}(s) \cdot \nabla_v \mathbf{\Pi}_{jl}(v(s))ds \right| \leqslant C \int_t^{t_1} |\dot{v}(s)|ds, \qquad 0 \leqslant t \leqslant t_1, \tag{9.11}$$

since $|\nabla_v \mathbf{\Pi}_{jl}(v(s))|$ is uniformly bounded by (6.8). Hence, the bounds (9.7) will follow from (6.18), (9.8) and (9.9)–(9.11) if we shall prove that $|d_1(t)|$ and the integral on the right-hand side of (9.10) can be made as small as desired by choosing a sufficiently small $\varepsilon > 0$.

To estimate $d_1(t)$, note that

$$w(s) - v(t_1) = w(s) - v(s) + v(s) - v(t_1) = \dot{c}(s) + \int_s^{t_1} \dot{v}(\tau)d\tau \tag{9.12}$$

by (6.9). Hence, equality (7.2), Lemma 6.1, and the definition in (9.1) imply that

$$\begin{aligned} |d_1(t)| &= \left| \int_{t_1}^t (w(s) - v(t_1))ds \right| \leqslant \int_t^{t_1} \left(|\dot{c}(s)| + \int_s^{t_1} |\dot{v}(\tau)|d\tau \right) ds \\ &\leqslant Cm^2(t_1) \int_t^{t_1} \left(\frac{1}{(1+s)^3} + \int_s^{t_1} \frac{d\tau}{(1+\tau)^3} \right) ds \leqslant Cm^2(t_1) \leqslant C\varepsilon^2, \qquad 0 \leqslant t \leqslant t_1, \end{aligned} \tag{9.13}$$

since $t_1 < t''_*$. Similarly,

$$\int_t^{t_1} |\dot{v}(s)|ds \leqslant Cm^2(t_1) \int_t^{t_1} \frac{ds}{(1+s)^3} \leqslant C\varepsilon^2, \qquad 0 \leqslant t \leqslant t_1. \tag{9.14}$$

10. DECAY OF THE TRANSVERSAL COMPONENT

Here we prove Proposition 6.1.

Step i). Choose an $\varepsilon > 0$ and a $t''_* = t''_*(\varepsilon)$ for which Lemma 9.1 holds. Then a bound of type (8.4) holds with $\|\mathbf{P}_1 Z_1(t)\|_{-\beta}$ replaced by $\|Z(t)\|_{-\beta}$ on the left-hand side,

$$\begin{aligned} \|Z(t)\|_{-\beta} &\leqslant \frac{C}{(1+t)^{3/2}} \|Z(0)\|_\beta \\ &+ C \int_0^t \frac{1}{(1+|t-s|)^{3/2}} \left[\|Z(s)\|_{-\beta} \int_s^{t_1} \|Z(\tau)\|_{-\beta}^2 d\tau + \|Z(s)\|_{-\beta}^2 \right] ds, \qquad 0 \leqslant t \leqslant t_1, \end{aligned} \tag{10.1}$$

for $t_1 < t'_*$. This implies an integral inequality for the majorant $m(t)$ defined in (9.1). Namely, multiplying (10.1) by $(1+t)^{3/2}$ and taking the supremum in $t \in [0, t_1]$, we obtain

$$m(t_1) \leqslant C\|Z(0)\|_\beta + C \sup_{t \in [0,t_1]} \int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}} \left[\frac{m(s)}{(1+s)^{3/2}} \int_s^{t_1} \frac{m^2(\tau)d\tau}{(1+\tau)^3} + \frac{m^2(s)}{(1+s)^3} \right] ds$$

for $t_1 \leqslant t''_*$. Taking into account that $m(t)$ is a monotone increasing function, we see that

$$m(t_1) \leqslant C\|Z(0)\|_\beta + C[m^3(t_1) + m^2(t_1)]I(t_1), \qquad t_1 \leqslant t''_*, \tag{10.2}$$

where

$$I(t_1) = \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}} \left[\frac{1}{(1+s)^{3/2}} \int_s^{t_1} \frac{d\tau}{(1+\tau)^3} + \frac{1}{(1+s)^3}\right] ds \leqslant \overline{I} < \infty.$$

Therefore, (10.2) becomes

$$m(t_1) \leqslant C\|Z(0)\|_\beta + C\overline{I}[m^3(t_1) + m^2(t_1)], \qquad t_1 < t_*''. \tag{10.3}$$

This inequality implies that $m(t_1)$ is bounded for $t_1 < t_*''$ and, moreover,

$$m(t_1) \leqslant C_1\|Z(0)\|_\beta, \qquad t_1 < t_*'', \tag{10.4}$$

since $m(0) = \|Z(0)\|_\beta$ is sufficiently small by (3.8).

Step ii). The constant C_1 in the estimate (10.4) does not depend on t_*, t_*', and t_*'' by Lemma 9.1. We choose a small d_0 in (2.12) such that $\|Z(0)\|_\beta < \varepsilon/(2C_1)$. This is possible by (3.8). In this case, estimate (10.4) implies that $t_*'' = t_*'$, and therefore (10.4) holds for any $t_1 < t_*'$. Then the bound (9.13) holds for any $t < t_*'$. Choose a small ε such that the right-hand side in (9.13) does not exceed one. Then $t_*' = t_*$. Therefore, (10.4) holds for any $t_1 < t_*$, and hence the first inequality in (6.3) also holds if $\|Z(0)\|_\beta$ is sufficiently small by (9.1) and (9.14). Finally, this implies that $t_* = \infty$, and hence we also have $t_*'' = t_*' = \infty$, and (10.4) holds for any $t_1 > 0$ if d_0 is small enough.

11. SOLITON ASYMPTOTICS

Here we prove our main theorem, Theorem 2.1, under the assumption that the decay (6.12) holds. Let us first prove the asymptotics (1.8) for the vector components, and then the asymptotics (1.1) for the fields.

Asymptotics for the vector components. It follows from (4.3) that $\dot{q} = \dot{b} + \dot{Q}$, and from (6.14), (6.15), and (4.8) that $\dot{Q} = P + \mathcal{O}(\|Z\|_{-\beta}^2)$. Thus,

$$\dot{q} = \dot{b} + \dot{Q} = v(t) + \dot{c}(t) + P(t) + \mathcal{O}(\|Z\|_{-\beta}^2). \tag{11.1}$$

Equation (6.10), together with the estimates (6.11) and (6.12), implies that

$$|\dot{c}(t)| + |\dot{v}(t)| \leqslant \frac{C_1(\rho, \overline{v}, d_0)}{(1+t)^3}, \qquad t \geqslant 0. \tag{11.2}$$

Therefore, $c(t) = c_+ + \mathcal{O}(t^{-2})$ and $v(t) = v_+ + \mathcal{O}(t^{-2})$, $t \to \infty$. Since $|P| \leqslant \|Z\|_{-\beta}$, the estimate (6.12) and relations (11.2) and (11.1) imply that

$$\dot{q}(t) = v_+ + \mathcal{O}(t^{-3/2}). \tag{11.3}$$

Similarly,

$$b(t) = c(t) + \int_0^t v(s)ds = v_+t + a_+ + \mathcal{O}(t^{-1}), \tag{11.4}$$

and hence the second part of (1.8) follows,

$$q(t) = b(t) + Q(t) = v_+t + a_+ + \mathcal{O}(t^{-1}), \tag{11.5}$$

since $Q(t) = \mathcal{O}(t^{-3/2})$ by (6.12).

Asymptotics for the fields. We apply the approach developed in [12], see also [10]. For the field part of the solution, $\psi(x,t) = \psi_1(x,t) + i\psi_2(x,t)$, let us define the accompanying soliton field as $\psi_{\mathrm{v}(t)}(x - q(t))$, where we now set $\mathrm{v}(t) = \dot q(t)$, cf. (11.1). In this case, for the difference $z(x,t) = \psi(x,t) - \psi_{\mathrm{v}(t)}(x - q(t))$, we obtain the equation

$$i\dot z(x,t) = (-\Delta + m^2)z(x,t) - i\dot{\mathrm{v}} \cdot \nabla_{\mathrm{v}}\psi_{\mathrm{v}(t)}(x - q(t)).$$

Then

$$z(t) = W_0(t)z(0) - \int_0^t W_0(t-s)[i\dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}}\psi_{\mathrm{v}(s)}(\cdot - q(s))]ds. \tag{11.6}$$

To obtain the asymptotics (2.14), it suffices to prove that $z(t) = W_0(t)\boldsymbol{\psi}_+ + r_+(t)$ for some $\boldsymbol{\psi}_+ \in H^1$ and that $\|r_+(t)\|_{H^1} = \mathcal{O}(t^{-1/2})$. This is equivalent to the relation

$$W_0(-t)z(t) = \boldsymbol{\psi}_+ + r'_+(t), \tag{11.7}$$

where $\|r'_+(t)\|_{H^1} = \mathcal{O}(t^{-1/2})$, since $W^0(t)$ is a unitary group on the Sobolev space $\mathcal{F}$ by the energy conservation for the free Schrödinger equation. Finally, formula (11.7) holds since (11.6) implies that

$$W_0(-t)z(t) = z(0) - \int_0^t W_0(-s)f(s)ds, \qquad f(s) = i\dot{\mathrm{v}}(s) \cdot \nabla_{\mathrm{v}}\psi_{\mathrm{v}(s)}(\cdot - q(s)),$$

where the integral on the right-hand side converges in the Hilbert space $\mathcal{F}$ with rate of convergence $\mathcal{O}(t^{-1/2})$, which holds since $\|W_0(-s)f(s)\|_{H^1} = \mathcal{O}(s^{-3/2})$ by the unitarity of $W_0(-s)$ and by the decay rate $\|f(s)\|_{H^1} = \mathcal{O}(s^{-3/2})$. Let us prove that this rate of decay holds indeed. It suffices to prove that $|\dot{\mathrm{v}}(s)| = \mathcal{O}(s^{-3/2})$, or, equivalently, $|\dot p(s)| = \mathcal{O}(s^{-3/2})$. Substituting (4.2) into the last equation of (1.2) gives

$$\begin{aligned}
\dot p(t) &= \int \left[\psi_{v(t)j}(x - b(t)) + \Psi_j(x - b(t), t)\right] \nabla\rho_j(x - b(t) - Q(t))dx \\
&= \int \psi_{v(t)j}(y)\nabla\rho_j(y)dy + \int \psi_{v(t)j}(y)\left[\nabla\rho_j(y - Q(t)) - \nabla\rho_j(y)\right]dy + \int \Psi_j(y,t)\nabla\rho_j(y - Q(t))dy.
\end{aligned}$$

The first integral on the right-hand side is zero by the stationary equations (2.7). The second integral is $\mathcal{O}(t^{-3/2})$, which follows from conditions (1.9) on ρ and the relation $Q(t) = \mathcal{O}(t^{-3/2})$. Finally, the third integral is of order $\mathcal{O}(t^{-3/2})$ by estimate (6.12). This completes the proof.

12. DECAY FOR THE LINEARIZED DYNAMICS

In the remaining sections, we prove Proposition 6.2 to complete the proof of the main result (Theorem 2.1). Here we discuss the general strategy of proving the proposition. We apply the Fourier–Laplace transform

$$\tilde X(\lambda) = \int_0^\infty e^{-\lambda t}X(t)dt, \qquad \operatorname{Re}\lambda > 0, \tag{12.1}$$

to (6.16). According to Proposition 6.2, we can expect that the solution $X(t)$ will be bounded in the norm $\|\cdot\|_{-\beta}$. Then the integral (12.1) converges and is analytic for $\operatorname{Re}\lambda > 0$. We shall write A and v instead of A_1 and v_1 in the remaining part of the paper. After the Fourier–Laplace transform, (6.16) becomes

$$\lambda\tilde X(\lambda) = A\tilde X(\lambda) + X_0, \quad \operatorname{Re}\lambda > 0. \tag{12.2}$$

Let us stress that (12.2) is equivalent to the Cauchy problem for the functions

$$X(t) \in C_b([0,\infty); \mathcal{E}_{-\beta}).$$

Hence, the solution $X(t)$ is given by

$$\tilde{X}(\lambda) = -(A-\lambda)^{-1}X_0, \qquad \operatorname{Re}\lambda > 0, \tag{12.3}$$

if the resolvent $R(\lambda) = (A-\lambda)^{-1}$ exists for $\operatorname{Re}\lambda > 0$.

Let us comment on our following strategy in proving the decay (6.12). We shall first construct the resolvent $R(\lambda)$ for $\operatorname{Re}\lambda > 0$ and prove that this resolvent is a continuous operator on $\mathcal{E}_{-\beta}$. In this case, $\tilde{X}(\lambda)$ belongs to $\mathcal{E}_{-\beta}$ and is an analytic function for $\operatorname{Re}\lambda > 0$. After this, we must justify that there exists a (unique) function $X(t) \in C([0,\infty);\mathcal{E}_{-\beta})$ satisfying (12.1).

The analyticity of $\tilde{X}(\lambda)$ and the Paley–Wiener arguments (see [16]) should provide the existence of an $\mathcal{E}_{-\beta}$-valued distribution $X(t)$, $t \in \mathbb{R}$, with a support in $[0,\infty)$. Formally,

$$X(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\tilde{X}(i\omega+0)\,d\omega, \qquad t \in \mathbb{R}. \tag{12.4}$$

However, to establish the continuity of $X(t)$ for $t \geqslant 0$, we need an additional bound for $\tilde{X}(i\omega+0)$ for large values of $|\omega|$. Finally, for the time decay of $X(t)$, we need additional information on the smoothness and decay of $\tilde{X}(i\omega+0)$. More precisely, we must prove that the function $\tilde{X}(i\omega+0)$ has the following properties:

i) it is smooth outside $\omega = 0$ and $\omega = \pm\mu$, where $\mu = \mu(v) > 0$;
ii) it decays in a sense as $|\omega| \to \infty$;
iii) it admits the Puiseux expansion at $\omega = \pm\mu$;
iv) it is analytic at $\omega = 0$ if $X_0 \in \mathcal{Z}_v := \mathbf{P}_v\mathcal{E}$ and $X_0 \in \mathcal{E}_\beta$.

Then the decay (6.12) will follow from the Fourier–Laplace representation (12.4).

We shall check properties of type i)–iv) only for the last two components $\tilde{Q}(\lambda)$ and $\tilde{P}(\lambda)$ of the vector

$$\tilde{X}(\lambda) = (\tilde{\Psi}_1(\lambda), \tilde{\Psi}_2(\lambda), \tilde{Q}(\lambda), \tilde{P}(\lambda)).$$

These properties provide the decay (6.12) for the vector components $Q(t)$ and $P(t)$ of the solution $X(t)$. After this, for the field components $\Psi_1(x,t)$ and $\Psi_2(x,t)$, we shall use well-known properties of the free Schrödinger equation.

13. CONSTRUCTING THE RESOLVENT

Here we construct the resolvent as a bounded operator on $\mathcal{E}_{-\beta}$ for $\operatorname{Re}\lambda > 0$. We shall write $(\Psi_1(y), \Psi_2(y), Q, P)$ instead of $(\tilde{\Psi}_1(y,\lambda), \tilde{\Psi}_2(y,\lambda), \tilde{Q}(\lambda), \tilde{P}(\lambda))$ to simplify the notation. Then (12.2) reads

$$(A-\lambda)\begin{pmatrix}\Psi_1\\ \Psi_2\\ Q\\ P\end{pmatrix} = -\begin{pmatrix}\Psi_{01}\\ \Psi_{02}\\ Q_0\\ P_0\end{pmatrix}.$$

This gives the system of equations

$$\left.\begin{aligned} v\cdot\nabla\Psi_1(y) - (\Delta - m^2)\Psi_2(y) - Q\cdot\nabla\rho_2 - \lambda\Psi_1(y) &= -\Psi_{01}(y),\\ (\Delta - m^2)\Psi_1(y) + v\cdot\nabla\Psi_2(y) + Q\cdot\nabla\rho_1 - \lambda\Psi_2(y) &= -\Psi_{02}(y),\\ P - \lambda Q &= -Q_0,\\ -\langle\nabla\Psi_j(y), \rho_j(y)\rangle + \langle\nabla\psi_{vj}(y), Q\cdot\nabla\rho_j(y)\rangle - \lambda P &= -P_0, \end{aligned}\right|\quad y \in \mathbb{R}^3. \tag{13.1}$$

Step i). Let us study the first two equations. In the Fourier space, they become

$$\begin{aligned} -(ikv+\lambda)\hat{\Psi}_1(k) + (k^2+m^2)\hat{\Psi}_2(k) &= -\hat{\Psi}_{01}(k) - iQk\hat{\rho}_2,\\ -(k^2+m^2)\hat{\Psi}_1(k) - (ikv+\lambda)\hat{\Psi}_2(k) &= -\hat{\Psi}_{02}(k) + iQk\hat{\rho}_1. \end{aligned} \tag{13.2}$$

Let us invert the matrix of the system, obtaining

$$\begin{pmatrix} -(ikv+\lambda) & k^2+m^2 \\ -(k^2+m^2) & -(ikv+\lambda) \end{pmatrix}^{-1} = [(ikv+\lambda)^2+(k^2+m^2)^2]^{-1} \begin{pmatrix} -(ikv+\lambda) & -(k^2+m^2) \\ k^2+m^2 & -(ikv+\lambda) \end{pmatrix}.$$

Taking the inverse Fourier transform, we find the corresponding fundamental solution

$$G_\lambda(y) = \begin{pmatrix} v\cdot\nabla-\lambda & \Delta-m^2 \\ -\Delta+m^2 & v\cdot\nabla-\lambda \end{pmatrix} g_\lambda(y), \tag{13.3}$$

where

$$g_\lambda(y) = F^{-1}_{k\to y}\frac{1}{(k^2+m^2)^2-(kv-i\lambda)^2} = F^{-1}_{k\to y}\frac{1}{(k^2+m^2-kv+i\lambda)(k^2+m^2+kv-i\lambda)}. \tag{13.4}$$

Note that the denominator on the right-hand side of (13.4) does not vanish for $\operatorname{Re}\lambda>0$ and $k\in\mathbb{R}^3$. Moreover, it does not vanish for $\operatorname{Re}\lambda>0$ and $k\in\mathbb{C}^3$ for sufficiently small $|\operatorname{Im}k|$. Therefore, $g_\lambda(y)$ decays exponentially by the Paley–Wiener arguments. Let us compute the entries of the matrix G_λ explicitly,

$$\begin{aligned} G^{11}_\lambda(y) = G^{22}_\lambda(y) &= F^{-1}\frac{-ikv-\lambda}{(k^2+m^2)^2-(kv-i\lambda)^2} \\ &= F^{-1}_{k\to y}\Big(\frac{1/2i}{k^2+m^2-kv+i\lambda}-\frac{1/2i}{k^2+m^2+kv-i\lambda}\Big) = \frac{e^{-\varkappa_+|y|-i\frac{v}{2}y}}{8i\pi|y|}-\frac{e^{-\varkappa_-|y|+i\frac{v}{2}y}}{8i\pi|y|}, \\ G^{21}_\lambda(y) = -G^{12}_\lambda(y) &= F^{-1}\frac{k^2+m^2}{(k^2+m^2)^2-(kv-i\lambda)^2} \\ &= F^{-1}_{k\to y}\Big(\frac{1/2}{k^2+m^2-kv+i\lambda}+\frac{1/2}{k^2+m^2+kv-i\lambda}\Big) = \frac{e^{-\varkappa_+|y|-i\frac{v}{2}y}}{8\pi|y|}+\frac{e^{-\varkappa_-|y|+i\frac{v}{2}y}}{8\pi|y|}, \end{aligned} \tag{13.5}$$

where

$$\varkappa_\pm = \sqrt{m^2-\frac{v^2}{4}\pm i\lambda}, \quad \operatorname{Re}\varkappa_\pm>0. \tag{13.6}$$

This implies the following assertion.

Lemma 13.1. i) *The operator G_λ with the integral kernel $G_\lambda(y-y')$ is continuous as an operator from $H^1(\mathbb{R}^3)\oplus H^1(R^3)$ to $H^2(\mathbb{R}^3)\oplus H^2(R^3)$ for* $\operatorname{Re}\lambda>0$.

ii) *Formulas* (13.5) *and* (13.6) *imply that, for any chosen y, the matrix function $G_\lambda(y)$,* $\operatorname{Re}\lambda>0$, *admits an analytic continuation with respect to λ to the Riemann surface of the algebraic function $\sqrt{\mu^2+\lambda^2}$ with the branching points $\lambda=\pm i\mu$, where $\mu:=m^2-v^2/4$.*

Thus, relations (13.2) and (13.3) imply the convolution representation

$$\begin{aligned} \Psi_1 &= -G^{11}_\lambda*\Psi_{01}-G^{12}_\lambda*\Psi_{02}-(G^{12}_\lambda*\nabla\rho_1)\cdot Q+(G^{11}_\lambda*\nabla\rho_2)\cdot Q, \\ \Psi_2 &= G^{12}_\lambda*\Psi_{01}-G^{11}_\lambda*\Psi_{02}-(G^{11}_\lambda*\nabla\rho_1)\cdot Q-(G^{12}_\lambda*\nabla\rho_2)\cdot Q. \end{aligned} \tag{13.7}$$

Step ii). Let us now proceed to the last two equations (13.1),

$$-\lambda Q+P=-Q_0, \qquad \langle\nabla\psi_{vj},Q\cdot\nabla\rho_j\rangle-\langle\nabla\Psi_j,\rho_j\rangle-\lambda P=-P_0. \tag{13.8}$$

Let us rewrite equations (13.7) in the form $\Psi_j=\Psi_j(Q)+\Psi_j(\Psi_{01},\Psi_{02})$, where

$$\begin{aligned} \Psi_1(\Psi_{01},\Psi_{02}) &= -G^{11}_\lambda*\Psi_{01}-G^{12}_\lambda*\Psi_{02}, &\quad \Psi_1(Q) &= (-G^{12}_\lambda*\nabla\rho_1+G^{11}_\lambda*\nabla\rho_2)\cdot Q, \\ \Psi_2(\Psi_{01},\Psi_{02}) &= G^{12}_\lambda*\Psi_{01}-G^{11}_\lambda*\Psi_{02}, &\quad \Psi_2(Q) &= -(G^{11}_\lambda*\nabla\rho_1+G^{12}_\lambda*\nabla\rho_2)\cdot Q. \end{aligned}$$

Then we have $\langle \nabla \Psi_j, \rho_j \rangle = \langle \nabla \Psi_j(Q), \rho_j \rangle + \langle \nabla \Psi_j(\Psi_{01}, \Psi_{02}), \rho_j \rangle$, and the last equation in (13.8) becomes

$$\langle \nabla \psi_{vj}, Q \cdot \nabla \rho_j \rangle - \langle \nabla \Psi_j(Q), \rho_j \rangle - \lambda P = -P_0 + \langle \nabla \Psi_j(\Psi_{01}, \Psi_{02}), \rho_j \rangle =: -P_0'.$$

Let us first compute the term

$$\langle \nabla \psi_{vj}, Q \cdot \nabla \rho_j \rangle = \sum_{lj} \langle \nabla \psi_{vj}, Q_l \partial_l \rho_j \rangle = \sum_{lj} \langle \nabla \psi_{vj}, \partial_l \rho_j \rangle Q_l.$$

Applying the Fourier transform $F_{y \to k}$, the Parseval identity, and (19.3), we see that

$$\begin{aligned}
\sum_j \langle \partial_i \psi_{vj}, \partial_l \rho_j \rangle &= \sum_j \langle -ik_i \hat{\psi}_{vj}, -ik_l \hat{\rho}_j \rangle \\
&= \Big\langle k_i \frac{-(k^2+m^2)\hat{\rho}_1 + ikv\hat{\rho}_2}{(k^2+m^2)^2 - (kv)^2}, k_l \hat{\rho}_1 \Big\rangle + \Big\langle k_i \frac{-ikv\hat{\rho}_1 - (k^2+m^2)\hat{\rho}_2}{(k^2+m^2)^2 - (kv)^2}, k_l \hat{\rho}_2 \Big\rangle \\
&= -\int \frac{k_i k_l \Big((k^2+m^2)(|\hat{\rho}_1|^2 + |\hat{\rho}_2|^2) + i(kv)(\hat{\rho}_1 \overline{\hat{\rho}}_2 - \hat{\rho}_2 \overline{\hat{\rho}}_1) \Big) dk}{(k^2+m^2)^2 - (kv)^2} =: -L_{il}. \qquad (13.9)
\end{aligned}$$

As a result, $\langle \nabla \psi_{vj}, Q \cdot \nabla \rho_j \rangle = -LQ$, where L is the 3×3 matrix with the matrix elements L_{il}. Let us now compute the term $-\langle \nabla \Psi_j(Q), \rho_j \rangle = \langle \Psi_j(Q), \nabla \rho_j \rangle$. We have

$$\begin{aligned}
\langle \Psi_j(Q), \partial_i \rho_j \rangle &= \sum_l \Big(\langle -G_\lambda^{12} * \partial_l \rho_1 + G_\lambda^{11} * \partial_l \rho_2, \partial_i \rho_1 \rangle - \langle G_\lambda^{11} * \partial_l \rho_1 + G_\lambda^{12} * \partial_l \rho_2, \partial_i \rho_2 \rangle \Big) Q_l \\
&= \sum_l H_{il}(\lambda) Q_l,
\end{aligned}$$

and, by the Parseval identity again,

$$\begin{aligned}
H_{il}(\lambda) :&= \langle -G_\lambda^{12} * \partial_l \rho_1 + G_\lambda^{11} * \partial_l \rho_2, \partial_i \rho_1 \rangle - \langle G_\lambda^{11} * \partial_l \rho_1 + G_\lambda^{12} * \partial_l \rho_2, \partial_i \rho_2 \rangle \\
&= \langle [(k^2+m^2)\hat{\rho}_1 - (ikv+\lambda)\hat{\rho}_2]\hat{g}_\lambda k_l, k_i \hat{\rho}_1 \rangle + \langle [(ikv+\lambda)\hat{\rho}_1 + (k^2+m^2)\hat{\rho}_2]\hat{g}_\lambda k_l, k_i \hat{\rho}_2 \rangle \\
&= \int \frac{k_i k_l \Big((k^2+m^2)(|\hat{\rho}_1|^2 + |\hat{\rho}_2|^2) + (ikv+\lambda)(\hat{\rho}_1 \overline{\hat{\rho}}_2 - \hat{\rho}_2 \overline{\hat{\rho}}_1) \Big) dk}{(k^2+m^2)^2 - (kv - i\lambda)^2}. \qquad (13.10)
\end{aligned}$$

The matrix H is well defined for $\operatorname{Re}\lambda > 0$ since the denominator does not vanish. As a result, $-\langle \nabla \Psi_j(Q), \rho_j \rangle = HQ$, where H is the matrix with the matrix elements H_{il}. Finally, the equations (13.8) become

$$\mathcal{M}(\lambda) \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix}, \quad \text{where} \quad \mathcal{M}(\lambda) = \begin{pmatrix} \lambda E & -E \\ L - H(\lambda) & \lambda E \end{pmatrix}. \qquad (13.11)$$

Assume for a moment that the matrix $\mathcal{M}(\lambda)$ is invertible (later we shall prove that this is the case indeed). Then

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \mathcal{M}^{-1}(\lambda) \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix}, \qquad \operatorname{Re}\lambda > 0. \qquad (13.12)$$

Finally, formula (13.12) and formulas (13.7), where Q is expressed by (13.12), give the expression for the resolvent $R(\lambda) = (A - \lambda)^{-1}$, $\operatorname{Re}\lambda > 0$.

Lemma 13.2. *The matrix function $\mathcal{M}(\lambda)$ $(\mathcal{M}^{-1}(\lambda))$, where $\operatorname{Re}\lambda > 0$, admits an analytic (meromorphic) continuation to the Riemann surface of the function $\sqrt{\mu^2 + \lambda^2}$, $\lambda \in \mathbb{C}$.*

Proof. The analytic continuation of $\mathcal{M}(\lambda)$ exists by Lemma 13.1 ii) and the convolution expressions in (13.10) by (1.9). The inverse matrix is then meromorphic, since it exists for large values of $\operatorname{Re}\lambda$ (which follows from (13.11) since $H(\lambda) \to 0$ as $\operatorname{Re}\lambda \to \infty$ by (13.10)).

14. ANALYTICITY ON THE HALF-PLANE

Here we prove the following assertion.

Proposition 14.1. *The operator-valued function* $R(\lambda)\colon \mathcal{E} \to \mathcal{E}$ *is analytic for* $\operatorname{Re}\lambda > 0$.

Proof. It suffices to prove that the operator $A - \lambda : \mathcal{E} \to \mathcal{E}$ has bounded inverse operator for $\operatorname{Re}\lambda > 0$. Recall that $A = A_{v,v}$, where $|v| < 2m$.

Step i). Let us prove that $\operatorname{Ker}(A - \lambda) = 0$ for $\operatorname{Re}\lambda > 0$. Indeed, assume that the function

$$X_\lambda = (\Psi_{\lambda 1}, \Psi_{\lambda 2}, Q_\lambda, P_\lambda) \in \mathcal{E}$$

satisfies the relation $(A - \lambda)X_\lambda = 0$, i.e., X_λ is a solution to (13.1) with $\Psi_{01} = \Psi_{02} = 0$ and $Q_0 = P_0 = 0$. We must prove that $X_\lambda = 0$.

Let us first show that $P_\lambda = 0$. Indeed, the trajectory $X := X_\lambda e^{\lambda t} \in C(\mathbb{R}, \mathcal{E})$ is the solution to the equation $\dot{X} = AX$, which is equation (5.1) with $w = v$. Then $\mathcal{H}_{v,v}(X(t))$ grows exponentially by (5.8). This growth contradicts the conservation of $\mathcal{H}_{v,v}$. This conservation follows from Lemma 5.1 ii) since $X(t) \in C^1(\mathbb{R}, \mathcal{E}^+)$, which follows from Lemma 13.1 because $(\Psi_{\lambda 1}, \Psi_{\lambda 2})$ satisfies equations (13.7) with $\Psi_{01} = \Psi_{02} = 0$ and $Q = Q_\lambda$.

We now have $\lambda Q_\lambda = P_\lambda = 0$ by the third equation of (13.1), and hence $Q_\lambda = 0$ because $\lambda \neq 0$. Finally, $\Psi_{\lambda 1} = 0$ and $\Psi_{\lambda 2} = 0$ by equations (13.7) with $Q = Q_\lambda = 0$.

Step ii). Write $A - \lambda = A_0 + T$, where

$$A_0 = \begin{pmatrix} v\cdot\nabla - \lambda & -(\Delta - m^2) & 0 & 0 \\ \Delta - m^2 & v\cdot\nabla - \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 0 & -\cdot\nabla\rho_2 & 0 \\ 0 & 0 & \cdot\nabla\rho_1 & 0 \\ 0 & 0 & 0 & E \\ \langle\,\cdot\,,\nabla\rho_1\rangle & \langle\,\cdot\,,\nabla\rho_1\rangle & \langle\nabla\psi_{vj}, \cdot\nabla\rho_j\rangle & 0 \end{pmatrix}.$$

The operator T is finite-dimensional, and the operator A_0^{-1} is bounded on $\mathcal{E}$ by Lemma 13.1. Finally, $A - \lambda = A_0(I + A_0^{-1}T)$, where $A_0^{-1}T$ is a compact operator. Since we know that $\operatorname{Ker}(I + A_0^{-1}T) = 0$, the operator $(I + A_0^{-1}T)$ is invertible by Fredholm theory.

Corollary 14.1. *The matrix* $\mathcal{M}(\lambda)$ *of* (13.11) *is invertible for* $\operatorname{Re}\lambda > 0$.

15. REGULARITY ON THE IMAGINARY AXIS

Let us first describe the continuous spectrum of the operator $A = A_{v,v}$ on the imaginary axis. By definition, the continuous spectrum corresponds to $\omega \in \mathbb{R}$ such that the resolvent $R(i\omega + 0)$ is not a bounded operator in $\mathcal{E}$. By formulas (13.7), this is the case if the Green function $G_\lambda(y - y')$ fails to have exponential decay. Thus, $i\omega$ belongs to the continuous spectrum if

$$|\omega| \geqslant \mu = m^2 - v^2/4.$$

By Lemma 13.2, the limit matrix

$$\mathcal{M}(i\omega) := \mathcal{M}(i\omega + 0) = \begin{pmatrix} i\omega E & -E \\ L - H(i\omega + 0) & i\omega E \end{pmatrix}, \qquad \omega \in \mathbb{R}, \tag{15.1}$$

exists, and its entries are continuous functions of $\omega \in \mathbb{R}$ that are smooth for $|\omega| < \mu$ and for $|\omega| > \mu$. Recall that the point $\lambda = 0$ belongs to the discrete spectrum of the operator A by Lemma 5.2 i), and hence $\mathcal{M}(i\omega + 0)$ is (probably) not invertible either at $\omega = 0$.

Proposition 15.1. *Let ρ satisfy condition* (1.9) *and the Wiener condition* (1.10), *and let $|v| < 2m$. Then the limit matrix $\mathcal{M}(i\omega + 0)$ is invertible for $\omega \neq 0$, $\omega \in \mathbb{R}$.*

Proof. Let us consider the three possible cases $0 < |\omega| < \mu$, $\omega = \mu$, and $|\omega| > \mu$, separately. We can assume that $v = (|v|, 0, 0)$. Write $F(\omega) := -L + H(i\omega + 0)$, $M = m^2 + k^2$, $a = |\hat\rho_1|^2 + |\hat\rho_2|^2$, and $b = i(\hat\rho_1\overline{\hat\rho}_2 - \hat\rho_2\overline{\hat\rho}_1)$. Then the entries of the matrix F become

$$\begin{aligned} F_{ij} &= \int k_i k_j \, dk \Bigg[Ma \left(\frac{1}{M^2 - (|v|k_1 + \omega)^2} - \frac{1}{M^2 - (|v|k_1)^2} \right) \\ &\quad + b \left(\frac{|v|k_1 + \omega}{M^2 - (|v|k_1 + \omega)^2} - \frac{|v|k_1}{M^2 - (|v|k_1)^2} \right) \Bigg] \\ &= \int \frac{k_i k_j dk}{2} \Bigg[a \left(\frac{1}{M - |v|k_1 - \omega} + \frac{1}{M + |v|k_1 + \omega} - \frac{1}{M - |v|k_1} - \frac{1}{M + |v|k_1} \right) \\ &\quad + b \left(\frac{1}{M - |v|k_1 - \omega} - \frac{1}{M + |v|k_1 + \omega} - \frac{1}{M - |v|k_1} + \frac{1}{M + |v|k_1} \right) \Bigg]. \end{aligned} \tag{15.2}$$

Since a is even and b is odd, we see that

$$F_{ij} = \frac{1}{2} \int dk_2 dk_3 \int_0^{+\infty} k_i k_j \, dk_1 \Big[af_1 + bf_2 \Big], \tag{15.3}$$

where

$$\begin{aligned} f_1 &:= \frac{1}{M - |v|k_1 - \omega} + \frac{1}{M + |v|k_1 + \omega} + \frac{1}{M + |v|k_1 - \omega} + \frac{1}{M - |v|k_1 + \omega} - \frac{2}{M - |v|k_1} - \frac{2}{M + |v|k_1}, \\ f_2 &:= \frac{1}{M - |v|k_1 - \omega} - \frac{1}{M + |v|k_1 + \omega} + \frac{1}{M - |v|k_1 + \omega} - \frac{1}{M + |v|k_1 - \omega} - \frac{2}{M - |v|k_1} + \frac{2}{M + |v|k_1}. \end{aligned} \tag{15.4}$$

Then, by (15.1),

$$\begin{aligned} \det \mathcal{M}(i\omega) &= \det \begin{pmatrix} i\omega & 0 & 0 & -1 & 0 & 0 \\ 0 & i\omega & 0 & 0 & -1 & 0 \\ 0 & 0 & i\omega & 0 & 0 & -1 \\ -F_{11} & -F_{12} & -F_{13} & i\omega & 0 & 0 \\ -F_{12} & -F_{22} & -F_{23} & 0 & i\omega & 0 \\ -F_{13} & -F_{23} & -F_{33} & 0 & 0 & i\omega \end{pmatrix} \\ &= -\omega^6 - \omega^4 \sum_{j=1}^{3} F_{jj} - \omega^2 \sum_{i<j} (F_{ii}F_{jj} - F_{ij}^2) - \det \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{12} & F_{22} & F_{23} \\ F_{13} & F_{23} & F_{33} \end{pmatrix} \end{aligned} \tag{15.5}$$

since $F_{ij} = F_{ji}$.

I. First, let us consider the case $0 < |\omega| < \mu$. Then the invertibility of $\mathcal{M}(i\omega)$ results from the following assertion.

Lemma 15.1. *For $0 < |\omega| < \mu$, the matrix F is positive definite.*

Proof. First, let us note that all denominators in (15.4) are positive for $|\omega| < \mu = m^2 - v^2/4$ and $|v| < 2m$. Indeed,

$$(m^2 + k^2)^2 - (\omega + |v|k_1)^2 = ((k - v/2)^2 + m^2 - \frac{v^2}{4} - \omega)\Big((k + v/2)^2 + m^2 - \frac{v^2}{4} + \omega\Big) > 0.$$

Second, $f_1 > f_2 \geqslant 0$ if $|v| < 2m$ and $0 < |\omega| \leqslant \mu$. This is proved in Appendix C.

Finally, the Wiener condition implies

$$a \pm b = |\hat\rho_1(k) \mp i\hat\rho_2(k)|^2 > 0, \quad \forall k \in \mathbb{R}^3. \tag{15.6}$$

Therefore, $af_1 + bf_2 > 0$ and (15.3) is the integral of the symmetric nonnegative definite matrix $k \otimes k = (k_i k_j)$ with positive weight. Hence, the matrix F is positive definite.

II. $\omega = \pm\mu$. For example, consider the case $\omega = \mu = m^2 - |v|^2/4$. Then formula (13.10) reads

$$H_{ij}(i\mu) = \int \frac{k_i k_j (Ma - (kv+\mu)b)dk}{\Big((k_1 - \frac{|v|}{2})^2 + k_2^2 + k_3^2\Big)\Big((k_1 + \frac{|v|}{2})^2 + k_2^2 + k_3^2 + 2\mu\Big)}.$$

Now the integrand has a unique singular point. The singularity is integrable, and hence $\det \mathcal{M}(i\omega)$ is also negative by the representations (15.5). Hence, the matrix $\mathcal{M}(i\mu)$ is also invertible.

III. $|\omega| > \mu$. Here we apply other arguments. The invertibility of $\mathcal{M}(i\omega)$ now follows from (15.5) by virtue of the following lemma (cf. [10]).

Lemma 15.2. *If* (1.10) *holds and if* $\omega > \mu$ $(\omega < -\mu)$, *then the matrix* $\operatorname{Im} F(\omega)$ *is negative* (*positive*) *definite.*

Proof. Consider the case $\omega > \mu$ (the case $\omega < -\mu$ can be treated similarly). Let us calculate the imaginary part of F_{ij}. Since $F_{ij} = H_{ij}(i\omega + 0) - L_{ij}$ and L_{ij} is real, we shall consider the value $H_{ij}(i\omega + 0)$ only. For $\varepsilon > 0$, we have

$$\begin{aligned} H_{ij}(i\omega+\varepsilon) = \int \frac{k_i k_j (Ma + (kv+\omega - i\varepsilon)b)dk}{M^2 - (kv + \omega - i\varepsilon)^2} &= \frac{1}{2}\int \frac{k_i k_j (a+b)dk}{M - kv - \omega + i\varepsilon} + \frac{1}{2}\int \frac{k_i k_j (a-b)dk}{M + kv + \omega - i\varepsilon} \\ &= H^1_{ij}(i\omega+\varepsilon) + H^2_{ij}(i\omega+\varepsilon). \end{aligned} \tag{15.7}$$

It suffices to study the first summand in (15.7) only, since the second summand is real for $\varepsilon = 0$. Consider the denominator

$$\hat D_\varepsilon(k) = k^2 + m^2 - kv - \omega + i\varepsilon.$$

Note that $\hat D_0(k) = 0$ on the ellipsoid T_ω given by

$$T_\omega = \Big\{k : \Big|k - \frac{v}{2}\Big| = R := \sqrt{\omega - \mu}\Big\}.$$

Then the Plemelj formula for C^1 functions implies that

$$\operatorname{Im} H^1_{ij}(i\omega + 0) = -\frac{\pi}{2}\int_{T_\omega} \frac{k_i k_j (a+b)}{|\nabla \hat D_0(k)|} dS, \tag{15.8}$$

where dS is the surface area element. Hence, the matrix $\operatorname{Im} H^1(i\omega + 0)$ is negative definite by (15.6).

Now let us prove that the limit matrix $\mathcal{M}(i\omega + 0)$ is invertible. Recall that

$$\mathcal{M}(i\omega+0) = \begin{pmatrix} i\omega E & -E \\ -F(i\omega+0) & i\omega E \end{pmatrix}$$

Then the equation

$$\mathcal{M}(i\omega+0)\begin{pmatrix} Q \\ P \end{pmatrix} = 0.$$

becomes

$$i\omega Q - P = 0, \qquad -FQ + i\omega P = 0. \tag{15.9}$$

Then $(F + \omega^2)Q = 0$, which implies that $Q = 0$ and then $P = 0$, since the matrix $\operatorname{Im} F$ is negative definite for $\omega > \mu$. This completes the proof of Proposition 15.1.

Corollary 15.1. *Proposition* 15.1 *implies that the matrix* $\mathcal{M}^{-1}(i\omega)$ *is smooth with respect to* $\omega \in \mathbb{R}$ *outside the three points* $\omega = 0, \pm\mu$.

Recall that formula (13.12) expresses the Fourier–Laplace transforms $\tilde{Q}(\lambda), \tilde{P}(\lambda)$. Hence, the components are given by the Fourier integral

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \frac{1}{2\pi} \int e^{i\omega t} \mathcal{M}^{-1}(i\omega + 0) \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix} d\omega \tag{16.1}$$

if it converges in the sense of distributions. Corollary 15.1 by itself is insufficient to prove the convergence and decay of the integral. Namely, we need additional information about the regularity of the matrix $\mathcal{M}^{-1}(i\omega)$ at the singular points $\omega = 0, \pm\mu$ and about some bounds at $|\omega| \to \infty$. We shall study these points separately.

I. First consider the points $\pm\mu$.

Lemma 16.1. *The matrix $\mathcal{M}^{-1}(i\omega)$ admits the following Puiseux expansion in a neighborhood of $\pm\mu$: there exists an $\varepsilon_\pm > 0$ such that*

$$\mathcal{M}^{-1}(i\omega) = \sum_{k=0}^{\infty} R_k^{\pm}(\omega \mp \mu)^{k/2}, \qquad |\omega \mp \mu| < \varepsilon_\pm, \quad \omega \in \mathbb{R}. \tag{16.2}$$

Proof. It suffices to prove a similar expansion for $\mathcal{M}(i\omega)$. Then (16.2) holds for $\mathcal{M}^{-1}(i\omega)$ as well, since the matrices $\mathcal{M}(\pm i\mu)$ are invertible. The asymptotics for $\mathcal{M}(i\omega)$ holds by the convolution representation (13.10),

$$H_{ij}(\lambda) = -\langle G_\lambda^{12} * \partial_l\rho_1 + G_\lambda^{11} * \partial_l\rho_2, \partial_i\rho_1\rangle - \langle G_\lambda^{11} * \partial_l\rho_1 + G_\lambda^{12} * \partial_l\rho_2, \partial_i\rho_2\rangle, \tag{16.3}$$

since the entries G_λ^{ij} admit the corresponding Puiseux expansions by formula (13.5).

II. Second, we study the asymptotic behavior of $\mathcal{M}^{-1}(\lambda)$ at infinity. Let us recall that $\mathcal{M}^{-1}(\lambda)$ was originally defined for $\operatorname{Re}\lambda > 0$, and it admits a meromorphic continuation to the Riemann surface of the function $\sqrt{m^2 - v^2/4 + i\lambda}$ (see Lemma 13.2).

Lemma 16.2. *One can find a matrix R_0 and a matrix function $R_1(\omega)$ such that*

$$\mathcal{M}^{-1}(i\omega) = \frac{R_0}{\omega} + R_1(\omega), \qquad |\omega| \geqslant \mu + 1, \qquad \omega \in \mathbb{R},$$

where

$$|\partial_\omega^k R_1(\omega)| \leqslant \frac{C_k}{|\omega|^2}, \qquad |\omega| \geqslant \mu + 1, \qquad \omega \in \mathbb{R}, \tag{16.4}$$

for every $k = 0, 1, 2, ..$

Proof. By the structure (15.1) of the matrix $\mathcal{M}(i\omega)$, it suffices to prove the following estimate for the elements of the matrix $H(i\omega) := H(i\omega + 0)$:

$$|\partial_\omega^k H_{jj}(i\omega)| \leqslant \frac{C_k}{|\omega|}, \qquad \omega \in \mathbb{R}, \quad |\omega| \geqslant \mu + 1, \quad j = 1, 2, 3. \tag{16.5}$$

Note that

$$G_\lambda^{11} * f = \frac{1}{2i}(D_1^{-1}(\lambda)f - D_2^{-1}(\lambda)f), \qquad G_\lambda^{12} * f = \frac{1}{2}(D_1^{-1}(\lambda)f + D_2^{-1}(\lambda)f),$$

where

$$D_1(\lambda) = -\Delta + m^2 - iv\cdot\nabla + i\lambda, \qquad D_2(\lambda) = -\Delta + m^2 + iv\cdot\nabla - i\lambda, \qquad \operatorname{Re}\lambda > 0,$$

and $D_j^{-1}(\lambda)$, $j=1,2$, are bounded operators on $L^2(\mathbb{R}^3)$. Estimate (16.5) immediately follows from a more general bound

$$\|\partial_\omega^k D_j^{-1}(i\omega+0)f\|_{L^2_{-\sigma}} \leqslant \frac{C_k(R)}{|\omega|}\|f\|_{L^2_\sigma}, \qquad \omega\in\mathbb{R}, \quad |\omega|\geqslant \mu+1, \tag{16.6}$$

which holds for $\sigma>3/2$. Namely, by (1.9), formula (16.5) follows from formula (16.6) applied to the functions $f(y)=\partial_l\rho_j(y)\in L^2_\sigma$.

The bound (16.6) was proved in [1, bound (A.2′)] (see also [15, Th. 8.1]).

III. Finally, consider the point $\omega=0$, which is the most singular. This is an isolated pole of finite degree by Lemma 13.2, and hence the Laurent expansion holds,

$$\mathcal{M}^{-1}(i\omega)=\sum_{k=0}^{n} M_k\omega^{-k-1}+\mathcal{H}(\omega), \qquad |\omega|<\varepsilon_0, \tag{16.7}$$

where M_k are 6×6 complex matrices, $\varepsilon_0>0$, and $\mathcal{H}(\omega)$ is an analytic matrix valued function for complex ω with $|\omega|<\varepsilon_0$.

17. TIME DECAY OF THE VECTOR COMPONENTS

Here we prove the decay (6.12) for the components $Q(t)$ and $P(t)$.

Lemma 17.1 (cf. [10]). *Let $X_0\in\mathcal{Z}_{v,\beta}$. Then $Q(t)$ and $P(t)$ are continuous and*

$$|Q(t)|+|P(t)|\leqslant \frac{C(\rho,\overline{v},d_0)}{(1+|t|)^{3/2}}, \qquad t\geqslant 0. \tag{17.1}$$

Proof. Expansions (16.2), (16.4), and (16.7) imply the convergence of the Fourier integral (16.1) in the sense of distributions to a continuous function of $t\geqslant 0$. Let us prove inequality (17.1). Note first that the condition $X_0\in\mathcal{Z}_{v,\beta}$ implies that the entire trajectory $X(t)$ lies in $\mathcal{Z}_{v,\beta}$. This follows from the invariance of the space $\mathcal{Z}_{v,\beta}$ under the generator $A_{v,v}$ (cf. Remark 6.2). Note that, for X_0 not belonging to $\mathcal{Z}_{v,\beta}$, the components $Q(t)$ and $P(t)$ can contain nondecaying terms that correspond to the singular point $\omega=0$. Indeed, we know that the linearized dynamics admits the secular solutions without decay, see (6.17). The formulas (3.2) give the corresponding components $Q_S(t)$ and $P_S(t)$ of the secular solutions,

$$\begin{pmatrix} Q_S(t)\\ P_S(t)\end{pmatrix}=\sum_1^3 C_j\begin{pmatrix} e_j\\ 0\end{pmatrix}+\sum_1^3 D_j\left[\begin{pmatrix} e_j\\ 0\end{pmatrix}t+\begin{pmatrix} 0\\ e_j\end{pmatrix}\right]. \tag{17.2}$$

We claim that the symplectic orthogonality condition leads to (17.1). Let us split the Fourier integral (16.1) into three terms by using the partition of unity $\zeta_1(\omega)+\zeta_2(\omega)+\zeta_3(\omega)=1$, $\omega\in\mathbb{R}$,

$$\begin{pmatrix} Q(t)\\ P(t)\end{pmatrix}=\frac{1}{2\pi}\int e^{i\omega t}(\zeta_1(\omega)+\zeta_2(\omega)+\zeta_3(\omega))\mathcal{M}^{-1}(i\omega+0)\begin{pmatrix} Q_0\\ P_0'\end{pmatrix}d\omega=I_1(t)+I_2(t)+I_3(t), \tag{17.3}$$

where the functions $\zeta_k(\omega)\in C^\infty(\mathbb{R})$ are supported by

$$\operatorname{supp}\zeta_1\subset\{\omega\in\mathbb{R}:\varepsilon_0/2<|\omega|<\mu+2\},$$
$$\operatorname{supp}\zeta_2\subset\{\omega\in\mathbb{R}:|\omega|>\mu+1\}, \qquad \operatorname{supp}\zeta_3\subset\{\omega\in\mathbb{R}:|\omega|<\varepsilon_0\}. \tag{17.4}$$

Then

i) The function $I_1(t)$ decays like $(1+|t|)^{-3/2}$ by the Puiseux expansion (16.2).

ii) The function $I_2(t)$ decays faster than any power of t due to Proposition 16.2.

iii) Finally, the function $I_3(t)$ generally does not decay if $n \geqslant 0$ in the Laurent expansion (16.7). Namely, the contribution of the analytic function $\mathcal{H}(\omega)$ decays faster than any power of t. On the other hand, the contribution of the Laurent series,

$$\begin{pmatrix} Q_L(t) \\ P_L(t) \end{pmatrix} := \frac{1}{2\pi} \int e^{i\omega t} \zeta_3(\omega) \sum_{k=0}^{n} M_k (\omega - i0)^{-k-1} \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix} d\omega, \qquad t \in \mathbb{R}, \tag{17.5}$$

is a polynomial function of $t \in \mathbb{R}$ of a degree $\leqslant n$, modulo functions decaying faster than any power of t. Let us note that formula (17.2) gives an example of polynomial function arising from (17.5).

We must show that the symplectic orthogonality condition eliminates the polynomial functions. Our main difficulty is that we know nothing about the order n of the pole and about the Laurent coefficients M_k of the matrix $\mathcal{M}^{-1}(i\omega)$ at $\omega = 0$.

Our crucial observation has the following form.

a) The components (17.2) of the secular solutions form a linear space $\mathcal{L}_S$ of dimension $\dim \mathcal{L}_S = 6$.

b) The polynomial functions in (17.5) belong to a linear space $\mathcal{L}_L$ of dimension $\dim \mathcal{L}_L \leqslant 6$ since $(Q_0, P_0') \in \mathbb{R}^6$.

c) $\mathcal{L}_S \subset \mathcal{L}_L$ since any function (17.2) admit a representation of the form (17.5). The validity of this representation follows from the fact that the secular solutions (6.17) can be reproduced by our calculations with the Laplace transform.

Therefore, we can conclude that

$$\mathcal{L}_L = \mathcal{L}_S. \tag{17.6}$$

It remains to note that the secular solutions are forbidden since $X_0 \in \mathcal{Z}_{v,\beta}$. Hence, the polynomial terms in (17.5) vanish, which implies the decay (17.1).

More precisely, we know that $X(t) = \mathbf{P}_v X(t)$ for any $t \in \mathbb{R}$. On the other hand, identity (17.6) implies that $X(t)$ can be corrected by a secular solution $X_S(t)$ such that the corresponding components $Q_\Delta(t)$ and $P_\Delta(t)$ (of the difference $\Delta(t) := X(t) - X_S(t)$) decay. Hence, the components $Q(t)$ and $P(t)$ of $X(t) = \mathbf{P}_v X(t) = \mathbf{P}_v[X(t) - X_S(t)]$ also decay.

18. TIME DECAY OF FIELDS

Here we prove the decay of the field components $\Psi_1(x,t), \Psi_2(x,t)$ corresponding to (6.12). The first two equations of (6.16) can be represented as a single equation,

$$i\dot{\Psi}(t) = (-\Delta + m^2 + iv \cdot \nabla)\Psi - Q(t) \cdot \nabla\rho, \tag{18.1}$$

where $\Psi(t) = \Psi_1(\cdot,t) + i\Psi_2(\cdot,t)$. By Lemma 17.1, we know that Q is continuous function of $t \geqslant 0$ and

$$|Q(t)| \leqslant \frac{C(\rho, \overline{v}, d_0)}{(1+|t|)^{3/2}}, \qquad t \geqslant 0. \tag{18.2}$$

Hence, Proposition 6.2 is reduced now to the following assertion.

Proposition 18.1. i) *Let $Q(t) \in C([0,\infty); \mathbb{R}^3)$ and $\Psi_0 \in H^1_\beta$. Then equation* (18.1) *admits a unique solution $\Psi(t) \in C([0,\infty); H^1_\beta)$ with the initial condition $\Psi(0) = \Psi_0$.*

ii) *If $\Psi_0 \in H^1_\beta$ and if the decay* (18.2) *holds, then the corresponding fields also decay uniformly with respect to v,*

$$\|\Psi(t)\|_{1,-\beta} \leqslant \frac{C(\rho, \overline{v}, d_0, \|\Psi_0\|_{1,\beta})}{(1+|t|)^{3/2}}, \qquad t \geqslant 0, \tag{18.3}$$

for $|v| \leqslant \overline{v}$ with any $\overline{v} \in (0, 2m)$.

Proof. The statements follow from the Duhamel representation

$$\Psi(t) = W(t)\Psi_0 - \int_0^t W(t-s)Q(s)\cdot\nabla\rho\, ds, \qquad t \geqslant 0, \tag{18.4}$$

where $W(t)$ is the dynamical group (propagator) of the free equation

$$i\dot\Psi(t) = (-\Delta + m^2 + iv\cdot\nabla)\Psi(t).$$

Lemma 18.1. *Let* $|v| \leqslant \overline{v}$ *with any* $\overline{v} \in (0, 2m)$. *Then*

$$\|W(t)\Psi_0\|_{1,-\beta} \leqslant C(\overline{v})(1+|t|)^{-3/2}\|\Psi_0\|_{1,\beta}, \qquad t \geqslant 0, \tag{18.5}$$

for any $\Psi_0 \in H^1_\beta$.

Proof. Note that $W(t)\Psi_0 = e^{-i(m^2-|v|^2/4)t}e^{ivx/2}\Phi(t)$, where $\Phi(t)$ is a solution to the free Schrödinger equation

$$i\dot\Phi(t) = -\Delta\Phi(t), \qquad \Phi(0) = e^{ivx/2}\Psi_0.$$

It is well known that $\Phi(t)$ satisfies the estimate $\|\Phi(t)\|_{1,-\beta} \leqslant C(1+|t|)^{-3/2}\|\Phi(0)\|_{1,\beta}$, $t \geqslant 0$ (see, for example, [15]).

Now (18.3) follows from condition (18.2) and from the Duhamel representation (18.4).

19. APPENDIX

A. Solitary waves

Let us verify the last equation in (2.7),

$$0 = \int \big(\nabla\psi_{v1}(y)\rho_1(y) + \nabla\psi_{v2}(y)\rho_2(y)\big)dy. \tag{19.1}$$

After passing to the Fourier representation, we set

$$\hat\psi(k) := (2\pi)^{-3/2}\int e^{ikx}\psi(x)dx.$$

We readily see that

$$-ikv\hat\psi_{v1} + (k^2+m^2)\hat\psi_{v2} = -\hat\rho_2, \qquad (k^2+m^2)\hat\psi_{v1} + ikv\hat\psi_{v2} = -\hat\rho_1. \tag{19.2}$$

Therefore,

$$\hat\psi_{v1}(k) = \frac{-(k^2+m^2)\hat\rho_1(k) + ikv\hat\rho_2(k)}{(k^2+m^2)^2-(kv)^2}, \qquad \hat\psi_{v2}(k) = \frac{-ikv\hat\rho_1(k) - (k^2+m^2)\hat\rho_2(k)}{(k^2+m^2)^2-(kv)^2}. \tag{19.3}$$

By the Parseval identity, formula (19.1) becomes

$$0 = \int k_j\big(\hat\psi_{v1}\overline{\hat\rho}_1 + \hat\psi_{v2}\overline{\hat\rho}_2\big)dk = \int \frac{k_j\big[-(k^2+m^2)(|\hat\rho_1|^2+|\hat\rho_2|^2) + ikv(\hat\rho_2\overline{\hat\rho}_1 - \hat\rho_1\overline{\hat\rho}_2)\big]dk}{(k^2+m^2)^2-(kv)^2},$$

which is true since the integrand is odd.

B. Computing $\Omega(\tau_i, \tau_j)$

Let us justify formulas (3.4)–(3.6) for the matrix Ω. For $j, l = 1, 2, 3$, it follows from (3.2) and (3.1) that

$$\Omega(\tau_j, \tau_l) = \langle \partial_j \psi_{v1}, \partial_l \psi_{v2} \rangle - \langle \partial_j \psi_{v2}, \partial_l \psi_{v1} \rangle, \tag{19.4}$$

$$\Omega(\tau_{j+3}, \tau_{l+3}) = \langle \partial_{v_j} \psi_{v1}, \partial_{v_l} \psi_{v2} \rangle - \langle \partial_{v_j} \psi_{v2}, \partial_{v_l} \psi_{v1} \rangle, \tag{19.5}$$

$$\Omega(\tau_j, \tau_{l+3}) = -\langle \partial_j \psi_{v1}, \partial_{v_l} \psi_{v2} \rangle + \langle \partial_j \psi_{v2}, \partial_{v_l} \psi_{v1} \rangle + e_j \cdot e_l. \tag{19.6}$$

Differentiating (19.2), we obtain

$$\partial_{v_j} \hat{\psi}_{v1} = \frac{k_j kv \hat{\psi}_{v1} - i k_j (k^2 + m^2) \hat{\psi}_{v2}}{(k^2 + m^2)^2 - (kv)^2}, \qquad \partial_{v_j} \hat{\psi}_{v2} = \frac{i k_j (k^2 + m^2) \hat{\psi}_{v1} + k_j kv \hat{\psi}_{v2}}{(k^2 + m^2)^2 - (kv)^2}, \quad j = 1, 2, 3. \tag{19.7}$$

Then, for $j, l = 1, 2, 3$, we see from (19.4) by the Parseval identity that

$$\Omega(\tau_j, \tau_l) = \int k_j k_l \; dk (\hat{\psi}_{v1} \overline{\hat{\psi}}_{v2} - \hat{\psi}_{v2} \overline{\hat{\psi}}_{v1}) = 0, \tag{19.8}$$

since the function $\hat{\psi}_{vc} = \hat{\psi}_{v1} \overline{\hat{\psi}}_{v2} - \hat{\psi}_{v2} \overline{\hat{\psi}}_{v1}$ is odd. Similarly, by (19.5) and (19.7),

$$\Omega(\tau_{j+3}, \tau_{l+3}) = -\int \frac{k_j k_l \big(2i(k^2 + m^2) kv (|\hat{\psi}_{v1}|^2 + |\hat{\psi}_{v2}|^2) - ((k^2 + m^2)^2 + (kv)^2) \hat{\psi}_{vc}\big) dk}{((k^2 + m^2)^2 - (kv)^2)^2} = 0. \tag{19.9}$$

Finally, by (19.6),

$$\Omega(\tau_j, \tau_{l+3}) = \int \frac{k_j k_l \big((k^2 + m^2)(|\hat{\psi}_{v1}|^2 + |\hat{\psi}_{v2}|^2) + i kv \hat{\psi}_{vc}\big) \; dk}{(k^2 + m^2)^2 - (kv)^2} + e_j \cdot e_l. \tag{19.10}$$

This completes the proof of (3.4)–(3.6).

C. Positivity of f_1 and f_2

Here we prove the inequalities used above in the proof of Lemma 15.1,

$$1) \qquad f_1 = \Big(\frac{1}{M - |v|k_1 - \omega} + \frac{1}{M - |v|k_1 + \omega} - \frac{2}{M - |v|k_1}\Big) + \Big(\frac{1}{M + |v|k_1 - \omega} + \frac{1}{M + |v|k_1 + \omega} - \frac{2}{M + |v|k_1}\Big) > 0,$$

$$2) \qquad f_2 = \Big(\frac{1}{M - |v|k_1 - \omega} + \frac{1}{M - |v|k_1 + \omega} - \frac{2}{M - |v|k_1}\Big) - \Big(\frac{1}{M + |v|k_1 - \omega} + \frac{1}{M + |v|k_1 + \omega} - \frac{2}{M + |v|k_1}\Big) \geqslant 0 \tag{19.11}$$

under the conditions $|v| < 2m$, $0 < |\omega| \leqslant \mu = m^2 - v^2/4$. First, let us note that every bracketed expression is positive, since

$$\frac{1}{b - a} + \frac{1}{b + a} - \frac{2}{b} = \frac{2a^2}{(b + a)(b - a)b} > 0$$

if $b - a$, $b + a \geqslant 0$ and $b > 0$, and this immediately implies that $f_1 > 0$. Next, the first summand on the left-hand side of (19.11) is obviously not less than the second summand since $|v|k_1 \geqslant 0$. Therefore, $f_2 \geqslant 0$ and $f_2 < f_1$.

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