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On the Convergence to a Statistical Equilibrium in the Crystal Coupled to a Scalar Field

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Abstract. We consider the dynamics of a field coupled to a harmonic crystal with n components in dimension d, $d, n \geqslant 1$. The crystal and the dynamics are translation-invariant with respect to the subgroup $\mathbb{Z}^d$ of $\mathbb{R}^d$. The initial data form a random function with a finite mean density of energy which also satisfies a mixing condition of Rosenblatt or Ibragimov–Linnik type. Moreover, initial correlation functions are translation invariant with respect to the discrete subgroup $\mathbb{Z}^d$. We study the distribution μ_t of the solution at times $t \in \mathbb{R}$. The main result is the convergence of μ_t to a Gaussian measure as $t \to \infty$, where μ_∞ is translation invariant with respect to the subgroup $\mathbb{Z}^d$.

1. INTRODUCTION

The paper concerns problems of long-time convergence to an equilibrium distribution in a coupled system which is similar to the Born–Oppenheimer model of a solid state. In [5–7, 10], we have started the convergence analysis for partial differential equations of hyperbolic type in $\mathbb{R}^d$. In [8, 9], we have extended the results to harmonic crystals.

Here we treat a harmonic crystal coupled to a scalar Klein–Gordon field. In this case, the corresponding problem in the unit cell is an infinite-dimensional Schrödinger operator, whereas in [8, 9] (and in [5–7, 10]), it was a finite-dimensional matrix. This situation usually arises in the solid-state problems similar to that for the Schrödinger equation with space-periodic potential [18]. The main novelty in our methods consists in that they yield exact estimates of trace norms for the problem in the unit cell.

We assume that an initial state Y_0 of the coupled system is a random element of a Hilbert phase space $\mathcal{E}$, see Definition 2.4. The distribution of Y_0 is a probability measure μ_0 (with zero mean) satisfying conditions **S1**–**S3**. In particular, the measure μ_0 is invariant with respect to translations by vectors of $\mathbb{Z}^d$. For a given $t \in \mathbb{R}$, we denote by μ_t the probability measure defining the distribution of the solution $Y(t)$ to the dynamical equations with random initial state Y_0. We study the asymptotics of μ_t as $t \to \pm\infty$.

Our main result gives the (weak) convergence of the measures μ_t to a limit measure μ_∞,

$$\mu_t \rightharpoonup \mu_\infty, \quad t \to \infty. \tag{1.1}$$

The measure μ_∞ is Gaussian and translation-invariant with respect to the group $\mathbb{Z}^d$. We give explicit formulas for the covariance of the measure μ_∞. The dynamical group is ergodic and mixing with respect to the limit measure μ_∞. Similar results also hold as $t \to -\infty$ because the dynamics is time-reversible.

Similar results have been established in [1, 24] for one-dimensional chains of harmonic oscillators (with $d = 1$) and in [11, 13, 16, 21] for one-dimensional chains of anharmonic oscillators coupled to

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heat baths. For d-dimensional harmonic crystals, with $d \geqslant 1$, the convergence (1.1) was proved in [8, 9, 17]. The mixing condition was first introduced by R. Dobrushin and Yu. Suhov for an ideal gas in [3]. The condition can replace the (quasi-) ergodic hypothesis when proving the convergence to the equilibrium distribution, and this plays a crucial role in our approach. Developing a Bernshtein-type approach, we have proved the convergence for the wave and Klein–Gordon equations and for harmonic crystals with translation-invariant initial measures [5, 6, 8]. In [7, 9, 10], we have extended the results to two-temperature initial measures. The present paper extends our previous results to the scalar Klein–Gordon field coupled to the nearest neighbor crystal.

Let us outline our main result and the strategy of the proof. (For the formal definitions and statements, see Section 2.) Consider the Hamiltonian system with the following Hamiltonian functional:

$$H(\psi, u, \pi, v) = \frac{1}{2}\int \Big(|\nabla\psi(x)|^2 + |\pi(x)|^2 + m_0^2|\psi(x)|^2\Big)\,dx \tag{1.2}$$
$$+ \frac{1}{2}\sum_{k\in\mathbb{Z}^d}\Big(\sum_{j=1}^{d}|u(k+e_j) - u(k)|^2 + |v(k)|^2 + \nu_0^2|u(k)|^2\Big) + \sum_{k\in\mathbb{Z}^d}\int R(x-k)\cdot u(k)\psi(x)\,dx,$$

involving a real scalar field $\psi(x)$ and its momentum $\pi(x)$, $x \in \mathbb{R}^d$, coupled to a "simple lattice" described by the deviations $u(k) \in \mathbb{R}^n$ of the "atoms" and their velocities $v(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}^d$. The symbol $R(x)$ stands for an $\mathbb{R}^n$-valued function and $e_j \in \mathbb{Z}^d$ for the vector with the coordinates $e_j^i := \delta_j^i$. Taking the variational derivatives of $H(\psi, u, \pi, v)$, we formally obtain the following system for $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$:

$$\begin{cases} \dot\psi(x,t) = \quad (\delta H/\delta\pi) = \pi(x,t), \\ \dot u(k,t) = \quad (\partial H/\partial v) = v(k,t), \\ \dot\pi(x,t) = -(\delta H/\delta\psi) = (\Delta - m_0^2)\psi(x,t) - \sum_{k'\in\mathbb{Z}^d} u(k',t)\cdot R(x-k'), \\ \dot v(k,t) = -(\partial H/\partial u) = (\Delta_L - \nu_0^2)u(k,t) - \int R(x'-k)\psi(x',t)\,dx'. \end{cases} \tag{1.3}$$

Here $m_0, \nu_0 > 0$ and the symbol Δ_L denotes the discrete Laplace operator on the lattice $\mathbb{Z}^d$,

$$\Delta_L u(k) := \sum_{e,|e|=1}(u(k+e) - u(k)).$$

Note that, for $n = d$ and $R(x) = -\nabla\rho(x)$, the interaction term in the Hamiltonian is the linearized Pauli–Fierz approximation of the translation-invariant coupling

$$\sum_k \int \rho(x - k - u(k))\psi(x)\,dx. \tag{1.4}$$

A similar model was analyzed by Born and Oppenheimer [2] as a model of a solid state (coupled Maxwell–Schrödinger equations for electrons in the harmonic crystal; see, e.g., [18]). The traditional analysis of the coupled field-crystal system (1.3) is based on an iterative perturbation procedure using the *adiabatic approximation*. Namely, in the zero approximation, the crystal and the (electron) field are discoupled. The electron field in the crystal defines a slow displacement of nuclei. The displacements give the corresponding contribution to the field via the static Coulombic potentials, which means a non-relativistic approximation, etc. The iterations converge if the motion of the nuclei is *sufficiently slow*, i.e., *the nuclei are rather heavy as compared with the electrons*. A similar procedure applies to the corresponding stationary problem of finding the dispersion relations.

Our analysis of the dispersion relations is a bit different and holds for *small displacements*. Namely, we linearize the translation-invariant coupling (1.4) at the zero displacements of the nuclei and obtain equations (1.3) corresponding to the Pauli–Fierz approximation. On the other hand, we analyze the dispersion relations of the linearized equations without any adiabatic or non-relativistic approximation. We give an exact nonperturbative spectral analysis of the coupled system (1.3).

We study the Cauchy problem for the system (1.3) with the initial data

$$\begin{cases} \psi(x,0) = \psi_0(x), \quad \pi(x,0) = \pi_0(x), \quad x \in \mathbb{R}^d, \\ u(k,0) = u_0(k), \quad v(k,0) = v_0(k), \quad k \in \mathbb{Z}^d. \end{cases} \tag{1.5}$$

Let us write

$$\psi^0 := \psi, \quad \psi^1 := \pi, \quad u^0 := u, \quad u^1 := v,$$

$$Y(t) := (Y^0(t), Y^1(t)) \quad \left| \begin{array}{l} Y^0(t) := (\psi^0(x,t), u^0(k,t)) := (\psi(x,t), u(k,t)), \\ Y^1(t) := (\psi^1(x,t), u^1(k,t)) := (\pi(x,t), v(k,t)). \end{array} \right. \tag{1.6}$$

In other words, $Y(\cdot, t)$ are functions defined on the disjoint union $\mathbb{P} := \mathbb{R}^d \cup \mathbb{Z}^d$,

$$Y^i(t) = Y^i(p,t) := \begin{cases} \psi^i(x,t), & p = x \in \mathbb{R}^d, \\ u^i(k,t), & p = k \in \mathbb{Z}^d, \end{cases} \quad i = 0, 1.$$

In this case, the system (1.3), (1.5) becomes a dynamical problem of the form

$$\dot{Y}(t) = \mathcal{A} Y(t), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \tag{1.7}$$

Here $Y_0 = (\psi_0, u_0, \pi_0, v_0)$ and

$$\mathcal{A} = J\nabla H(Y) = \begin{pmatrix} 0 & 1 \\ -\mathcal{H} & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} -\Delta + m_0^2 & S \\ S^* & -\Delta_L + \nu_0^2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.8}$$

where $Su(x) = \sum_{k \in \mathbb{Z}^d} R(x-k) u(k)$, $S^*\psi(k) = \int_{\mathbb{R}^d} R(x-k)\psi(x)\, dx$, and

$$\langle \psi, Su \rangle_{L^2(\mathbb{R}^d)} = \langle S^*\psi, u \rangle_{[l^2(\mathbb{Z}^d)]^n}, \quad \psi \in L^2(\mathbb{R}^d), \quad u \in [l^2(\mathbb{Z}^d)]^n.$$

We assume that the initial data Y_0 define a random function, and the initial correlation matrix

$$Q_0(p,p') := E\Big(Y_0(p) \otimes Y_0(p')\Big), \quad p, p' \in \mathbb{P},$$

is translation invariant with respect to translations by $\mathbb{Z}^d$, i.e.,

$$Q_0(p+k, p'+k) = Q_0(p,p'), \quad p, p' \in \mathbb{P}, \tag{1.9}$$

for any $k \in \mathbb{Z}^d$. We also assume that the initial mean energy densities are uniformly bounded,

$$e_F(x) := E(|\nabla \psi_0(x)|^2 + |\psi_0(x)|^2 + |\pi_0(x)|^2) \leqslant \bar{e}_F < \infty, \quad \text{a.a. } x \in \mathbb{R}^d, \tag{1.10}$$

$$e_L := E(|u_0(k)|^2 + |v_0(k)|^2) < \infty, \quad k \in \mathbb{Z}^d. \tag{1.11}$$

Finally, we assume that the measure μ_0 satisfies a mixing condition of Rosenblatt- or Ibragimov–Linnik type, which means that

$$Y_0(p) \quad \text{and} \quad Y_0(p') \quad \text{are asymptotically independent as} \quad |p - p'| \to \infty. \tag{1.12}$$

Our main result gives the (weak) convergence (1.1) of μ_t to a limit measure μ_∞, which is a stationary Gaussian probability measure.

Let us comment on the methods of the proof. The key role in our proof is played by the standard reduction of system (1.7) to the Bloch problem on the torus. Namely, we split $x \in \mathbb{R}^d$ in the form $x = k + y$, $k \in \mathbb{Z}^d$, $y \in K_1^d := [0,1]^d$, and apply the Fourier transform $F_{k\to\theta}$ to the solution $Y(k,t) := \Big(\psi(k+y,t), u(k,t), \pi(k+y,t), v(k,t)\Big)$,

$$\tilde{Y}(\theta,t) := F_{k\to\theta} Y(k,t) \equiv \sum_{k \in \mathbb{Z}^d} e^{ik\theta} Y(k,t) = (\tilde{\psi}(\theta,y,t), \tilde{u}(\theta,t), \tilde{\pi}(\theta,y,t), \tilde{v}(\theta,t)), \quad \theta \in \mathbb{R}^d,$$

which is a version of the Bloch–Floquet transform. The functions $\tilde{\psi}$, $\tilde{\pi}$ are periodic with respect to θ and quasi-periodic with respect to y, i.e.,

$$\tilde{\psi}(\theta, y+m, t) = e^{-im\theta}\psi(\theta,y,t), \quad \tilde{\pi}(\theta,y+m,t) = e^{-im\theta}\pi(\theta,y,t), \quad m \in \mathbb{Z}^d.$$

Further, introduce the Zak transform of $Y(\cdot,t)$ (which is also known under the title of Lifshitz–Gelfand–Zak transform) (cf. [18, p. 5]) as

$$\mathcal{Z}Y(\cdot,t) \equiv \tilde{Y}_\Pi(\theta,t) := (\tilde{\psi}_\Pi(\theta,y,t), \tilde{u}(\theta,t), \tilde{\pi}_\Pi(\theta,y,t), \tilde{v}(\theta,t)), \tag{1.13}$$

where $\tilde{\psi}_\Pi(\theta,y,t) := e^{iy\theta}\tilde{\psi}(\theta,y,t)$ and $\tilde{\pi}_\Pi(\theta,y,t) := e^{iy\theta}\tilde{\pi}(\theta,y,t)$ are periodic functions with respect to y (and quasi-periodic with respect to θ). Denote by $T_1^d := \mathbb{R}^d/\mathbb{Z}^d$ the real d-torus and write $\mathcal{R} := T_1^d \cup \{0\}$. Set

$$\tilde{Y}_\Pi(\theta,r,t)\equiv\tilde{Y}_\Pi(\theta,t):=\begin{cases}(\tilde{\psi}_\Pi(\theta,y,t),\tilde{\pi}_\Pi(\theta,y,t)), & r=y\in T_1^d,\\ (\tilde{u}(\theta,t),\tilde{v}(\theta,t)), & r=0.\end{cases}$$

Problem (1.7) is now equivalent to the problem on the unit torus $y\in T_1^d$ with the parameter $\theta\in K^d\equiv[0,2\pi]^d$,

$$\left\{\begin{array}{ll}\dot{\tilde{Y}}_\Pi(\theta,t)=\tilde{\mathcal{A}}(\theta)\tilde{Y}_\Pi(\theta,t), & t\in\mathbb{R}\\ \tilde{Y}_\Pi(\theta,0)=\tilde{Y}_{0\Pi}(\theta)\end{array}\right| \quad \theta\in K^d. \tag{1.14}$$

Here

$$\tilde{\mathcal{A}}(\theta)=\begin{pmatrix}0 & 1\\ -\tilde{\mathcal{H}}(\theta) & 0\end{pmatrix}, \tag{1.15}$$

and $\tilde{\mathcal{H}}(\theta):=\mathcal{Z}\mathcal{H}\mathcal{Z}^{-1}$ is the "Schrödinger operator" on the torus T_1^d,

$$\tilde{\mathcal{H}}(\theta)=\begin{pmatrix}(i\nabla_y+\theta)^2+m_0^2 & \tilde{S}(\theta)\\ \tilde{S}^*(\theta) & \omega_*^2(\theta)\end{pmatrix}, \tag{1.16}$$

where

$$\omega_*^2(\theta):=2(1-\cos\theta_1)+\cdots+2(1-\cos\theta_d)+\nu_0^2, \tag{1.17}$$

$$\Big(\tilde{S}(\theta)\tilde{u}(\cdot)\Big)(\theta,y):=\tilde{R}_\Pi(\theta,y)\cdot\tilde{u}(\theta),\quad \Big(\tilde{S}^*(\theta)\tilde{\psi}_\Pi(\theta,\cdot)\Big)(\theta):=\int_{T_1^d}\tilde{R}_\Pi(-\theta,y)\tilde{\psi}_\Pi(\theta,y)\,dy, \tag{1.18}$$

$$\langle\tilde{\psi}_\Pi(\theta,\cdot),(\tilde{S}(\theta)\tilde{u})(\theta,\cdot)\rangle_{L^2(T_1^d)}=(\tilde{S}^*(\theta)\tilde{\psi}_\Pi)(\theta)\cdot\tilde{u}(\theta),\quad \tilde{\psi}_\Pi(\theta,\cdot)\in H^1(T_1^d),\quad \tilde{u}(\theta)\in\mathbb{C}^n.$$

Then, formally,

$$\tilde{Y}_\Pi(\theta,t)=e^{\tilde{\mathcal{A}}(\theta)t}\tilde{Y}_{0\Pi}(\theta),\quad \theta\in K^d. \tag{1.19}$$

To justify the definition of the exponential, we note that $\tilde{\mathcal{H}}(\theta)$ is a self-adjoint operator with a discrete spectrum. Indeed, if $R=0$, then this follows from elliptic theory and, if $R\neq 0$, then the operators $\tilde{S}(\theta)$ and $\tilde{S}^*(\theta)$ are finite-dimensional for a fixed θ. We assume that $\tilde{\mathcal{H}}(\theta)>0$ (condition **R2**), which corresponds to the hyperbolicity of problem (1.3).

Note that in [8, 9], we considered the harmonic crystal without any field. In this case, the operator $\tilde{\mathcal{A}}(\theta)$ is a finite-dimensional matrix.

Let us prove the convergence (1.1) by using the strategy of [5–10] in the following three steps.

I. The family of measures μ_t, $t\geqslant 0$, is weakly compact in an appropriate Fréchet space.

II. The correlation functions converge to a limit,

$$Q_t(p,p'):=\int\Big(Y(p)\otimes Y(p')\Big)\,\mu_t(dY)\to Q_\infty(p,p'),\quad t\to\infty,\quad p,p'\in\mathbb{P}. \tag{1.20}$$

III. The characteristic functionals converge to a Gaussian functional,

$$\hat{\mu}_t(Z):=\int e^{i\langle Y,Z\rangle}\,\mu_t(dY)\to\exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(Z,Z)\Big\},\quad t\to\infty, \tag{1.21}$$

where Z is an arbitrary element of the dual space and $\mathcal{Q}_\infty$ is a quadratic form.

Property **I** follows from the Prokhorov theorem. First, let us prove the uniform bound (2.15) for the mean local energy in μ_t. To this end, we shall show that the operator $\Big(\Omega^i\tilde{q}_t^{ij}(\theta)\Omega^j\Big)_{i,j=0,1}$ is of trace class, where $\tilde{q}_t^{ij}(\theta)$ represents the covariance of the measure μ_t in the Zak transform (see (3.7)) and $\Omega\equiv\Omega(\theta):=\sqrt{\tilde{\mathcal{H}}(\theta)}$. Moreover, we derive the uniform bound

$$\sup_{t\geqslant 0}\ \sup_{\theta\in[0,2\pi]^d}\ \mathrm{tr}\Big(\Omega^i\tilde{q}_t^{ij}(\theta)\Omega^j\Big)_{i,j=0,1}<\infty. \tag{1.22}$$

This implies the compactness of μ_t by the Prokhorov theorem (when applying Sobolev's embedding theorem as in [5]).

To derive property **II**, we study oscillatory integrals in the Zak transform by developing our *cutting strategy* introduced in [8]. Namely, we rewrite (1.20) in the form

$$\mathcal{Q}_t(Z,Z)\to\mathcal{Q}_\infty(Z,Z),\quad t\to\infty, \tag{1.23}$$

where $\mathcal{Q}_t(Z,Z)$ stands for the quadratic correlation form for the measure μ_t. Further, we prove formula (1.23) for $Z\in\mathcal{D}^0$ as follows: by the definition of $\mathcal{D}^0$, the Zak transform $\tilde{Z}_\Pi(\theta)$ vanishes in a neighborhood of a "critical set" $\mathcal{C}\subset K^d$. In particular, the set $\mathcal{C}$ includes all points $\theta\in K^d$ with a degenerate Hessian of $\omega_l(\theta)$ and the points for which the function $\omega_l(\theta)$ is non-smooth. One can cut off the critical set $\mathcal{C}$ by the following two crucial observations: (i) $\operatorname{mes}\mathcal{C}=0$ and (ii) the initial quadratic correlation form is continuous in L^2 due to the mixing condition. The continuity follows from the spatial decay of the correlation functions in accordance with the well-known Schur lemma.

Similarly, we first prove property **III** for $Z\in\mathcal{D}^0$ and then extend it to all points $Z\in\mathcal{D}$. For $Z\in\mathcal{D}^0$, we use a version of the S. N. Bernshtein "room-corridor" technique (cf. [5, 8]). This leads to a representation of the solution as the sum of weakly dependent random variables. Then (1.21) follows from the Central Limit Theorem under a Lindeberg-type condition.

Let us comment on the two main technical novelties of our paper. The first of them is the bound (1.22), which then ensures compactness. We derive formula (1.22) in Section 4 directly from our assumption concerning the finiteness of the mean energy density (1.10), (1.11). The derivation uses the technique of trace class operators [23], which enables us to avoid additional continuity conditions for higher-order derivatives of the correlation functions. An essential ingredient of the proof is the "unitary trick" (4.6), which is a natural consequence of the Hamiltonian structure of system (1.3). The second main novelty is the bound (2.2) for the dynamics in weighted norms. In the Zak transform, the weighted norms become Sobolev norms with negative index. We derive (2.2) in Appendix A, by usng duality arguments, from the corresponding bounds for the derivatives of the exponential (1.19). The bounds for the derivatives follow by differentiating the dynamical equations.

Let us comment on our conditions **E1** and **E2**. The conditions are natural generalizations of similar conditions in [8, 9]. Condition **E1** enables us to apply the stationary phase method to the oscillatory integral representation for the covariance. This helps to prove that the stationary points of the phase functions are non-degenerate.

The paper is organized as follows. In Section 2, we formally state our main result. The compactness (property **I**) is established in Section 4, the convergence (1.20) in Section 6, and the convergence (1.21) in Section 7. In Section 8, mixing properties for the limit measures are proved. Appendix A concerns the dynamics in the Fourier transform, in Appendix B, we analyze the crossing points of the dispersion relations, and in Appendix C, we discuss the covariance in the spectral representation.

2. MAIN RESULTS

2.1. *Notation*

We assume that the initial data Y_0 are given by an element of the real phase space $\mathcal{E}$ defined below.

Definition 2.1. Let $H^{s,\alpha}=H^{s,\alpha}(\mathbb{R}^d)$, $s\in\mathbb{R}$, $\alpha\in\mathbb{R}$, be the Hilbert space of distributions $\psi\in S'(\mathbb{R}^d)$ with finite norm

$$\|\psi\|_{s,\alpha}\equiv\|\langle x\rangle^\alpha\Lambda^s\psi\|_{L^2(\mathbb{R}^d)}<\infty.$$

For $\psi\in D\equiv C_0^\infty(\mathbb{R}^d)$, write $F\psi(\xi)=\int e^{i\xi\cdot x}\psi(x)dx$. Let $\Lambda^s\psi:=F^{-1}_{\xi\to x}(\langle\xi\rangle^s\hat{\psi}(\xi))$ and $\langle x\rangle:=\sqrt{|x|^2+1}$, where $\hat{\psi}:=F\psi$ stands for the Fourier transform of a tempered distribution ψ.

Remark 2.2. For $s=0,1,2,\dots$, the space $H^{s,\alpha}(\mathbb{R}^d)$ is the Hilbert space of real-valued functions $\psi(x)$ with finite norm

$$\sum_{|\gamma|\leqslant s}\int(1+|x|^2)^\alpha|\mathcal{D}^\gamma\psi(x)|^2\,dx<\infty,$$

which is equivalent to $\|\psi\|^2_{s,\alpha}$.

Definition 2.3. Let L^α, $\alpha \in \mathbb{R}$, be the Hilbert space of vector functions $u(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}^d$, with finite norm

$$\|u\|_\alpha^2 \equiv \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^\alpha |u(k)|^2 < \infty.$$

Definition 2.4. Let $\mathcal{E}^{s,\alpha} := H^{1+s,\alpha}(\mathbb{R}^d) \oplus L^\alpha \oplus H^{s,\alpha}(\mathbb{R}^d) \oplus L^\alpha$ be the Hilbert space of vectors $Y \equiv (\psi, u, \pi, v)$ with finite norm

$$\|\!|Y|\!\|_{s,\alpha}^2 = \|\psi\|_{1+s,\alpha}^2 + \|u\|_\alpha^2 + \|\pi\|_{s,\alpha}^2 + \|v\|_\alpha^2.$$

Choose some α, $\alpha < -d/2$. Assume that $Y_0 \in \mathcal{E} := \mathcal{E}^{0,\alpha}$.

Using the standard technique of pseudo-differential operators and Sobolev's theorem (see, e.g. [14]), one can prove that $\mathcal{E}^{0,\alpha} = \mathcal{E} \subset \mathcal{E}^{s,\beta}$ for every $s < 0$ and $\beta < \alpha$, and the embedding is compact.

Definition 2.5. The phase space of problem (1.7) is $\mathcal{E} := \mathcal{E}^{0,\alpha}$, $\alpha < -d/2$.

Introduce the space $H_1^s := H^s(T_1^d) \oplus \mathbb{C}^n$, $s \in \mathbb{R}$, where $H^s(T_1^d)$ stands for the Sobolev space.

We assume that the following conditions hold for the real-valued coupling vector function $R(x)$:
R1. $R \in C^\infty(\mathbb{R}^d)$ and $|R(x)| \leqslant \bar{R}\exp(-\varepsilon|x|)$ for some $\varepsilon > 0$ and some $\bar{R} < \infty$.
R2. The operator $\tilde{\mathcal{H}}(\theta)$ is positive definite for $\theta \in K^d \equiv [0, 2\pi]^d$. This is equivalent to the uniform bound

$$(X^0, \tilde{\mathcal{H}}(\theta)X^0) \geqslant \kappa^2 \|X^0\|_{H_1^1}^2 \quad \text{for} \ X^0 \in H_1^1, \quad \theta \in K^d, \tag{2.1}$$

where $\kappa > 0$ is a constant and $(\cdot\,,\cdot)$ stands for the inner product in H_1^0 (see (3.13)).

Remark 2.6. i) Condition **R2** ensures that the operator $i\tilde{\mathcal{A}}(\theta)$ is self-adjoint with respect to the energy inner product. This corresponds to the hyperbolicity of problem (1.3).

ii) Condition **R2** holds, in particular, if the following condition **R2′** holds (see Remark 9.3):

R2′.
$$\int_{[0,1]^d} \Big| \sum_{k\in\mathbb{Z}^d} R(k+y)\Big|^2 dy < \nu_0^2 m_0^2/2.$$

iii) Condition **R2′** holds for functions R satisfying condition **R1** with $\bar{R}\varepsilon^{-d} \ll 1$.

Proposition 2.7. *Let conditions* **R1** *and* **R2** *hold. Then* (i) *for any $Y_0 \in \mathcal{E}$, there is a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ to the Cauchy problem* (1.7), (ii) *the operator $W(t)\colon Y_0 \mapsto Y(t)$ is continuous in $\mathcal{E}$ for any $t \in \mathbb{R}$,*

$$\sup_{|t|\leqslant T} \|\!|W(t)Y_0|\!\|_{0,\alpha} \leqslant C(T)\|\!|Y_0|\!\|_{0,\alpha} \tag{2.2}$$

if α is even and $\alpha \leqslant -2$.

Proof. (i) *Local existence.* Introduce the matrices

$$\mathcal{A}_0 := \begin{pmatrix} 0 & 1 \\ -\mathcal{H}_0 & 0 \end{pmatrix}, \quad \mathcal{H}_0 = \begin{pmatrix} -\Delta + m_0^2 & 0 \\ 0 & -\Delta_L + \nu_0^2 \end{pmatrix}. \tag{2.3}$$

Then problem (1.3) can be rewritten as the Duhamel integral

$$Y(t) = e^{\mathcal{A}_0 t} Y_0 + \int_0^t e^{\mathcal{A}_0(t-s)} BY(s)\, ds,$$

where

$$BY = \Big(0, 0, -\sum_{k'\in\mathbb{Z}^d} u(k')R(x-k'), -\int_{\mathbb{R}^d} R(x'-k)\psi(x')\,dx'\Big), \quad Y = (\psi, u, \pi, v).$$

Condition **R1** implies that $\|\!|BY(s)|\!\|_{0,\alpha} \leqslant C\|\!|Y(s)|\!\|_{0,\alpha}$. Further, for $0 \leqslant s \leqslant t \leqslant T$, we obtain

$$\|\!|e^{\mathcal{A}_0(t-s)}BY(s)|\!\|_{0,\alpha} \leqslant C(T)\|\!|Y(s)|\!\|_{0,\alpha},$$

(see, e.g., [8]). Hence,

$$\max_{|t|\leqslant T} \|\!|Y(t)|\!\|_{0,\alpha} \leqslant C(T)\|\!|Y_0|\!\|_{0,\alpha} + TC(T)\max_{|s|\leqslant T}\|\!|Y(s)|\!\|_{0,\alpha} \leqslant (T+1)C(T)\max_{|s|\leqslant T}\|\!|Y(s)|\!\|_{0,\alpha}.$$

We choose a $T>0$ so that $(T+1)C(T)<1$. Then the contraction mapping principle implies the existence of a unique solution $Y(t)\in C([0,T];\mathcal{E})$. The *global existence* follows from the bound (2.2).

(ii) The bounds (2.2) are proved in Corollary 9.6. □

Conditions **R1**, **R2** imply that, for a fixed $\theta\in K^d$, the operator $\tilde{\mathcal{H}}(\theta)$ is positive definite and self-adjoint in H_1^0 and its spectrum is discrete. Introduce the Hermitian positive-definite operator

$$\Omega(\theta):=\sqrt{\tilde{\mathcal{H}}(\theta)}>0.$$

Denote by $\omega_l(\theta)>0$ and $F_l(\theta,\cdot)$, $l=1,2,\dots$, the eigenvalues ("Bloch bands") and the orthonormal eigenvectors ("Bloch functions") of the operator $\Omega(\theta)$ in H_1^0, respectively. Note that $F_l(\theta,\cdot)\in H_1^\infty:=C^\infty(T_1^d)\oplus\mathbb{C}^n$ because these are eigenfunctions of the elliptic operator $\tilde{\mathcal{H}}(\theta)$.

As is well known, the functions $\omega_l(\cdot)$ and $F_l(\cdot,r)$ are real-analytic outside the set of the "crossing" points θ_*, where $\omega_l(\theta_*)=\omega_{l'}(\theta_*)$ for some $l\neq l'$. However, the functions are not smooth at the crossing points in general if $\omega_l(\theta)\not\equiv\omega_{l'}(\theta)$. Therefore, we need the following lemma, which is proved in Appendix B.

Lemma 2.8 [26]. *There exists a closed subset* $\mathcal{C}_*\subset K^d$ *such that* (i) *the Lebesgue measure of* $\mathcal{C}_*$ *vanishes,*

$$\operatorname{mes}\mathcal{C}_*=0. \tag{2.4}$$

(ii) *For every point* $\Theta\in K^d\setminus\mathcal{C}_*$ *and* $N\in\mathbb{N}$, *there exists a neighborhood* $\mathcal{O}(\Theta)\subset K^d\setminus\mathcal{C}_*$ *such that each of the functions* $\omega_l(\theta)$ *and* $F_l(\theta,\cdot)$, $l=1,\dots,N$, *can be chosen to be real-analytic on* $\mathcal{O}(\Theta)$.
(iii) *The eigenvalues* $\omega_l(\theta)$ *have constant multiplicity in* $\mathcal{O}(\Theta)$, *i.e., one can enumerate them in such a way that*

$$\omega_1(\theta)\equiv\dots\equiv\omega_{r_1}(\theta),\quad \omega_{r_1+1}(\theta)\equiv\dots\equiv\omega_{r_2}(\theta),\quad \dots, \tag{2.5}$$

$$\omega_{r_\sigma}(\theta)\not\equiv\omega_{r_\nu}(\theta)\quad \textit{if}\quad \sigma\neq\nu,\quad r_\sigma,r_\nu\geqslant 1 \tag{2.6}$$

for any $\theta\in\mathcal{O}(\Theta)$.

Corollary 2.9. *The spectral decomposition holds,*

$$\Omega(\theta)=\sum_{l=1}^{+\infty}\omega_l(\theta)P_l(\theta),\quad \theta\in\mathcal{O}(\Theta), \tag{2.7}$$

where $P_l(\theta)$ *are the orthogonal projectors in* H_1^0 *onto the linear span of* $F_l(\theta,\cdot)$, *and* $P_l(\theta)$ *and* $\omega_l(\theta)$ *depend on* $\theta\in\mathcal{O}(\Theta)$ *analytically.*

Assume that system (1.7) satisfies the following conditions **E1** and **E2**. For every $\Theta\in K^d\setminus\mathcal{C}_*$:

E1. $D_l(\theta)\not\equiv 0$, $l=1,2,\dots$, where $D_l(\theta):=\det\Big(\dfrac{\partial^2\omega_l(\theta)}{\partial\theta_i\partial\theta_j}\Big)_{i,j=1}^d$, $\theta\in\mathcal{O}(\Theta)$ and $\mathcal{O}(\Theta)$ is defined in Lemma 2.8. Write

$$\mathcal{C}_l:=\bigcup_{\Theta\in K^d\setminus\mathcal{C}_*}\{\theta\in\mathcal{O}(\Theta):\ D_l(\theta)=0\},\quad l=1,2,\dots$$

The following lemma is also proved in Appendix B.

Lemma 2.10. *Let conditions* **R1**, **R2** *hold. Then* mes $\mathcal{C}_l=0$, $l=1,2,\dots$

E2. For each $l\neq l'$, the identity $\omega_l(\theta)-\omega_{l'}(\theta)\equiv\text{const}_-$, $\theta\in\mathcal{O}(\Theta)$, cannot hold for any constant $\text{const}_-\neq0$, and the identity $\omega_l(\theta)+\omega_{l'}(\theta)\equiv\text{const}_+$ cannot hold for any constant $\text{const}_+\neq0$.

Condition **E2** could be considerably weakened (cf. [8, Remark 2.10, iii, condition E5′]). Note that conditions **E1** and **E2** hold if $R=0$.

Let us show that conditions **E1** and **E2** hold for "almost all" functions R satisfying conditions **R1**, **R2**. More precisely, consider finitely many coupling functions $R_1,\dots,R_N$ satisfying conditions **R1** and **R2′** and take their linear combinations

$$R_C(x)=\sum_{s=1}^N C_sR_s(x),\quad C=(C_1,\dots,C_N)\in\mathbb{R}^N.$$

For $R_C(x)$, conditions **R1** and **R2′** hold if $\|C\| < \varepsilon$ with a sufficiently small $\varepsilon > 0$. Let

$M_1 := \{C \in B_\varepsilon\colon \text{condition } \mathbf{E1} \text{ holds for } R_C(x)\}$, $M_2 := \{C \in B_\varepsilon\colon \text{condition } \mathbf{E2} \text{ holds for } R_C(x)\}$,

where $B_\varepsilon := \{C \in \mathbb{R}^N : \|C\| < \varepsilon\}$. In Appendix B, we prove the following lemma.

Lemma 2.11. *The sets M_1 and M_2 are dense in some ball B_ε for a sufficiently small $\varepsilon > 0$.*

2.2. *Random Solution. Convergence to Equilibrium*

Let (Ω, Σ, P) be a probability space with expectation E and let $\mathcal{B}(\mathcal{E})$ denote the Borel σ-algebra in $\mathcal{E}$. Assume that $Y_0 = Y_0(\omega, p)$ (see (1.7)) is a measurable random function with values in $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$. In other words, the map $\Omega \times \mathbb{P} \to \mathbb{R}^{2+2n}$ given by the rule $(\omega, p) \mapsto Y_0(\omega, p)$ is measurable with respect to the (completed) σ-algebra $\Sigma \times \mathcal{B}(\mathbb{P})$ and $\mathcal{B}(\mathbb{R}^{2+2n})$. Then $Y(t) = W(t)Y_0$ is also a measurable random function with values in $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ owing to Proposition 2.7. Denote by $\mu_0(dY_0)$ the Borel probability measure in $\mathcal{E}$ defining the distribution of Y_0. Without loss of generality, we can assume $(\Omega, \Sigma, P) = (\mathcal{E}, \mathcal{B}(\mathcal{E}), \mu_0)$ and $Y_0(\omega, p) = \omega(p)$ for $\mu_0(d\omega) \times dp$-almost all point $(\omega, p) \in \mathcal{E} \times \mathbb{P}$.

Definition 2.12. The measure μ_t is a Borel probability measure in $\mathcal{E}$ defining the distribution of $Y(t)$,

$$\mu_t(B) = \mu_0(W(-t)B), \quad \forall B \in \mathcal{B}(\mathcal{E}), \quad t \in \mathbb{R}.$$

Our main objective is to prove the weak convergence of the measures μ_t in the Fréchet spaces $\mathcal{E}^{s,\beta}$ for each $s < 0$, $\beta < \alpha < -d/2$,

$$\mu_t \overset{\mathcal{E}^{s,\beta}}{\rightharpoonup} \mu_\infty \quad \text{as} \quad t \to \infty, \tag{2.8}$$

where μ_∞ is a limit measure on $\mathcal{E} \equiv \mathcal{E}^{0,\alpha}$. This is equivalent to the convergence

$$\int f(Y)\mu_t(dY) \to \int f(Y)\mu_\infty(dY) \quad \text{as} \quad t \to \infty$$

for any bounded continuous functional $f(Y)$ on $\mathcal{E}^{s,\beta}$.

Let $\mathcal{D} = [D_F \oplus D_L]^2$ with $D_F \equiv C_0^\infty(\mathbb{R}^d)$, and let D_L be the set of vector sequences $u(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}^d$, such that $u(k) = 0$ for $k \in \mathbb{Z}^d$ outside a finite set. For a probability measure μ on $\mathcal{E}$, denote by $\hat\mu$ the characteristic functional (Fourier transform)

$$\hat\mu(Z) \equiv \int e^{i\langle Y, Z\rangle}\,\mu(dY), \quad Z \in \mathcal{D}.$$

Here $\langle\cdot\,,\cdot\rangle$ stands for the inner product in $L^2(\mathbb{P}) \otimes \mathbb{R}^N$ with different $N = 1, 2, \ldots,$

$$\langle Y, Z\rangle := \sum_{i=0}^{1} \langle Y^i, Z^i\rangle, \quad Y = (Y^0, Y^1), \quad Z = (Z^0, Z^1),$$

$$\langle Y^i, Z^i\rangle := \int_P Y^i(p)Z^i(p)\,dp \equiv \int_{\mathbb{R}^d} \psi^i(x)\xi^i(x)\,dx + \sum_{k\in\mathbb{Z}^d} u^i(k)\chi^i(k),$$

where $Y^i = (\psi^i, u^i)$, $Z^i = (\xi^i, \chi^i)$. A measure μ is said to be *Gaussian* (with zero expectation) if its characteristic functional has the form

$$\hat\mu(Z) = \exp\Big\{-\frac{1}{2}\mathcal{Q}(Z, Z)\Big\}, \quad Z \in \mathcal{D},$$

where $\mathcal{Q}$ is a real nonnegative quadratic form in $\mathcal{D}$.

Definition 2.13. *The correlation functions of the measure μ_t, $t \in \mathbb{R}$, are defined by*

$$Q_t^{ij}(p, p') \equiv E\Big(Y^i(p, t) \otimes Y^j(p', t)\Big), \quad i, j = 0, 1, \quad p, p' \in \mathbb{P}, \tag{2.9}$$

where E stands for the integral against the measure $\mu_0(dY)$ and the convergence of the integral in (2.9) *is understood in the sense of distributions, namely,*

$$\langle Q_t^{ij}(p, p'), Z_1(p) \otimes Z_2(p')\rangle := E\langle Y^i(p, t), Z_1(p)\rangle\langle Y^j(p', t), Z_2(p')\rangle, \quad Z_1, Z_2 \in D_F \oplus D_L. \tag{2.10}$$

2.3. *Mixing Condition*

Let $O(r)$ be the set of all pairs of open subsets $\mathcal{A}, \mathcal{B} \subset \mathbb{P}$ such that the distance $\rho(\mathcal{A}, \mathcal{B})$ is not less than r, and let $\sigma(\mathcal{A})$ be the σ-algebra in $\mathcal{E}$ generated by the linear functionals $Y \mapsto \langle Y, Z\rangle$ for which $Z \in \mathcal{D}$ and $\operatorname{supp} Z \subset \mathcal{A}$. Define the Ibragimov–Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{E}$ by the formula (cf. [15, Definition 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathcal{A},\mathcal{B})\in O(r)} \sup_{\substack{A\in\sigma(\mathcal{A}),B\in\sigma(\mathcal{B})\\ \mu_0(B)>0}} \frac{|\mu_0(A\cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$

Definition 2.14. A measure μ_0 satisfies the strong uniform Ibragimov–Linnik mixing condition if $\varphi(r) \to 0$ as $r \to \infty$.

Below we specify the rate of the decay of φ (see condition **S3**).

2.4. *Main Theorem*

Assume that the initial measure μ_0 satisfies the following properties **S0**–**S3**:

S0. The measure μ_0 has zero expectation value, $EY_0(p) \equiv 0$, $p \in \mathbb{P}$.

S1. The correlation matrices of μ_0 are invariant with respect to translations in $\mathbb{Z}^d$, i.e., equation (1.9) holds for a.a. $p, p' \in \mathbb{P}$.

S2. The measure μ_0 has a finite mean "energy" density, i.e., equations (1.10), (1.11) hold.

S3. The measure μ_0 satisfies the *strong uniform* Ibragimov–Linnik mixing condition with

$$\int_0^{+\infty} r^{d-1}\varphi^{1/2}(r)dr < \infty. \tag{2.11}$$

Introduce the correlation matrix $Q_\infty(p,p')$ of the limit measure μ_∞. It is translation invariant with respect to translations in $\mathbb{Z}^d$, i.e.,

$$Q_\infty(p,p') = Q_\infty(p+k, p'+k), \quad k \in \mathbb{Z}^d. \tag{2.12}$$

For $Z \in \mathcal{D}$, write

$$\mathcal{Q}_\infty(Z,Z) := \langle Q_\infty(p,p'), Z(p)\otimes Z(p')\rangle = (2\pi)^{-d}\int_{K^d} \Big(\tilde q_\infty(\theta), \tilde Z_\Pi(\theta,\cdot)\otimes \overline{\tilde Z_\Pi(\theta,\cdot)}\Big)d\theta, \tag{2.13}$$

where $\tilde q_\infty(\theta)$ is the operator-valued function given by the rule

$$\tilde q_\infty(\theta) := \sum_{l=1}^{+\infty} P_l(\theta)\frac{1}{2}\begin{pmatrix} \tilde q_0^{00}(\theta) + \tilde{\mathcal{H}}^{-1}(\theta)\tilde q_0^{11}(\theta) & \tilde q_0^{01}(\theta) - \tilde q_0^{10}(\theta) \\ \tilde q_0^{10}(\theta) - \tilde q_0^{01}(\theta) & \tilde{\mathcal{H}}(\theta)\tilde q_0^{00}(\theta) + \tilde q_0^{11}(\theta)\end{pmatrix} P_l(\theta), \tag{2.14}$$

for $\theta \in K^d \setminus \mathcal{C}_*$. Here the symbol $\tilde q_0^{ij}(\theta) = \operatorname{Op}\big(\tilde q_0^{ij}(\theta,r,r')\big)$ stands for the integral operator with the integral kernel $\tilde q_0^{ij}(\theta,r,r')$ (see formula (3.7) with $t=0$), $r,r' \in \mathcal{R}$, and $P_l(\theta)$ is the spectral projection operator introduced in Corollary 2.9.

Theorem A. *Let conditions* **S0**–**S3**, **R1**, **R2**, **E1**, *and* **E2** *hold. Then the following assertions are valid:*

(i) *the convergence* (2.8) *holds for any* $s < 0$ *and* $\beta < -d/2$;

(ii) *the limit measure* μ_∞ *is Gaussian on* $\mathcal{E}$;

(iii) *the characteristic functional of* μ_∞ *is Gaussian,* $\hat\mu_\infty(Z) = \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(Z,Z)\Big\}$, $Z \in \mathcal{D}$;

(iv) *the measure* μ_∞ *is invariant, i.e.,* $[W(t)]^*\mu_\infty = \mu_\infty$, $t \in \mathbb{R}$.

Assertions (i)–(iii) of Theorem A follow from Propositions 2.15 and 2.16 below.

Proposition 2.15. *The family of measures* $\{\mu_t,\ t \in \mathbb{R}\}$ *is weakly compact in* $\mathcal{E}^{s,\beta}$ *with any* $s < 0$ *and* $\beta < \alpha < -d/2$, *and the following bounds hold*:

$$\sup_{t\in\mathbb{R}} E\|W(t)Y_0\|^2_{0,\alpha} \leqslant C(\alpha) < \infty. \tag{2.15}$$

Proposition 2.16. *The convergence* (1.21) *holds for every* $Z \in \mathcal{D}$.

Proposition 2.15 (Proposition 2.16) provides the existence (the uniqueness) of the limit measure μ_∞. They are proved in Sections 4 and 7, respectively.

Theorem A, (iv) follows from (2.8) because the group $W(t)$ is continuous with respect to $\mathcal{E}$ by Proposition 2.7, (ii).

3. CORRELATION MATRICES

To prove the compactness of the family of measures $\{\mu_t\}$, we introduce auxiliary notation and prove necessary bounds for initial correlation matrices. Since $Y^i(p,t) = (\psi^i(x,t), u^i(k,t))$, we can rewrite formula (2.9) as follows:

$$Q_t^{ij}(p,p') = E[Y^i(p,t)\otimes Y^j(p',t)] = \begin{pmatrix} E\Big(\psi^i(x,t)\otimes\psi^j(x',t)\Big) & E\Big(\psi^i(x,t)\otimes u^j(k',t)\Big) \\ E\Big(u^i(k,t)\otimes\psi^j(x',t)\Big) & E\Big(u^i(k,t)\otimes u^j(k',t)\Big)\end{pmatrix}$$

$$\equiv \begin{pmatrix} Q_t^{\psi^i\psi^j}(x,x') & Q_t^{\psi^i u^j}(x,k') \\ Q_t^{u^i\psi^j}(k,x') & Q_t^{u^i u^j}(k,k')\end{pmatrix}, \quad i,j=0,1, \quad t\in\mathbb{R}. \tag{3.1}$$

Let us rewrite the correlation matrices $Q_t^{ij}(p,p')$ by using condition **S1**. Note that the dynamical group $W(t)$ commutes with the translations in $\mathbb{Z}^d$. In this case, condition **S1** implies that

$$Q_t(k+p,k+p') = Q_t(p,p'), \quad t\in\mathbb{R}, \quad k\in\mathbb{Z}^d. \tag{3.2}$$

Let us introduce the splitting $p = k+r$, where $k\in\mathbb{Z}^d$ and $r\in K_1^d\cup 0$. In other words,

$$r = \begin{cases} x-[x]\in K_1^d & \text{if } p = x\in\mathbb{R}^d, \\ 0 & \text{if } p = k\in\mathbb{Z}^d.\end{cases}$$

In this notation, (3.2) implies that

$$Q_t^{ij}(k+r,k'+r') =: q_t^{ij}(k-k',r,r') \equiv \begin{pmatrix} q_t^{\psi^i\psi^j}(k-k'+r,r') & q_t^{\psi^i u^j}(k-k'+r) \\ q_t^{u^i\psi^j}(k'-k+r') & q_t^{u^i u^j}(k-k')\end{pmatrix}. \tag{3.3}$$

Using the Zak transform (1.13), introduce the following matrices (cf. (2.10)):

$$\tilde{Q}_t^{ij}(\theta,r,\theta',r') := E[\tilde{Y}_\Pi^i(\theta,r,t)\otimes\overline{\tilde{Y}_\Pi^j(\theta',r',t)}], \quad \theta,\theta'\in K^d, \quad r,r'\in\mathcal{R}\equiv T_1^d\cup 0, \tag{3.4}$$

where the convergence of the mathematical expectation is understood in the sense of distributions. Namely, write $\tilde{\mathcal{D}} = [\tilde{\mathcal{D}}_F\oplus\tilde{\mathcal{D}}_L]^2$ and $\tilde{\mathcal{D}}_F := C^\infty(K^d\times T_1^d)$, $\tilde{\mathcal{D}}_L := [C^\infty(T^d)]^n$. Then

$$\langle\tilde{Q}_t^{ij}(\theta,r,\theta',r'),\tilde{Z}_\Pi^i(\theta,r)\otimes\overline{\tilde{Z}_\Pi^j(\theta',r')}\rangle = E\langle\tilde{Y}_\Pi^i(\theta,r,t),\tilde{Z}_\Pi^i(\theta,r)\rangle\,\overline{\langle\tilde{Y}_\Pi^j(\theta',r',t),\tilde{Z}_\Pi^j(\theta',r')\rangle} \tag{3.5}$$

for $\tilde{Z}_\Pi = (\tilde{Z}_\Pi^0,\tilde{Z}_\Pi^1)\in\tilde{\mathcal{D}}$. Now (3.2) implies that

$$\tilde{Q}_t^{ij}(\theta,r,\theta',r') = (2\pi)^d\delta(\theta-\theta')\tilde{q}_t^{ij}(\theta,r,r'), \quad \theta,\theta'\in K^d, \quad r,r'\in\mathcal{R}, \quad t\in\mathbb{R}, \tag{3.6}$$

where

$$\tilde{q}_t^{ij}(\theta,r,r') = e^{i(r-r')\theta}\sum_{k\in\mathbb{Z}^d} e^{ik\theta}q_t^{ij}(k,r,r') = \begin{pmatrix} \tilde{q}_t^{\psi^i\psi^j}(\theta,y,y') & \tilde{q}_t^{\psi^i u^j}(\theta,y) \\ \overline{\tilde{q}_t^{u^i\psi^j}}(\theta,y') & \tilde{q}_t^{u^i u^j}(\theta)\end{pmatrix}. \tag{3.7}$$

Recall that $\mathbb{P}$ is the disjoint union $\mathbb{R}^d\cup\mathbb{Z}^d$. For a measurable function $Y(p)$, write

$$\int_{\mathbb{P}} Y(p)\,dp = \int_{\mathbb{R}^d} Y(x)\,dx + \sum_{k\in\mathbb{Z}^d} Y(k). \tag{3.8}$$

Proposition 3.1. *Let conditions* **S0**–**S3** *be satisfied. Then the following assertions hold.*

(i)

$$\int_{\mathbb{P}} |Q_0(p,p')|\,dp \leqslant C < \infty \quad \textit{for any} \quad p'\in\mathbb{P}, \tag{3.9}$$

$$\int_{\mathbb{P}} |Q_0(p,p')|\,dp' \leqslant C < \infty \quad \textit{for any} \quad p\in\mathbb{P}, \tag{3.10}$$

where the constant C *does not depend on* $p,p'\in\mathbb{P}$.

(ii) $\mathcal{D}^{\alpha,\beta}_{y,y'}\tilde q_0^{\psi^i\psi^j}(\theta,y,y')$, $\mathcal{D}^{\alpha}_{y}\tilde q_0^{\psi^i u^j}(\theta,y)$, $\mathcal{D}^{\beta}_{y'}\tilde q_0^{u^i\psi^j}(\theta,y')$, $\tilde q_0^{u^i u^j}(\theta)$ *are uniformly bounded in* $(\theta,y,y')\in K^d\times T_1^d\times T_1^d$, $|\alpha|\leqslant 1-i$, $|\beta|\leqslant 1-j$.

Proof. (i) By [15, Lemma 17.2.3], conditions **S0**, **S2**, and **S3** imply

$$|Q_0(p,p')|\leqslant C\max\{\bar e_F, e_L\}\,\varphi^{1/2}(|p-p'|),\quad p,p'\in\mathbb{P}. \tag{3.11}$$

Hence, the bounds (3.9) and (3.10) follow from (2.11).

(ii) Similarly to (3.11), conditions **S0**, **S2**, and **S3** imply

$$\begin{aligned}|\mathcal{D}^{\alpha,\beta}_{y,y'}\tilde q_0^{\psi^i\psi^j}(\theta,y,y')| &\leqslant \sum_{k\in\mathbb{Z}^d}|\mathcal{D}^{\alpha,\beta}_{y,y'}q_0^{\psi^i\psi^j}(k+y,y')|\leqslant C\sum_{k\in\mathbb{Z}^d}\varphi^{1/2}(|k+y-y'|)\\ &\leqslant C\sum_{k\in\mathbb{Z}^d}\varphi^{1/2}(|k|-2\sqrt{d}\,)\leqslant C<\infty.\end{aligned}$$

Similar arguments imply the other bounds. □

Corollary 3.2. *By the Schur lemma, Proposition* 3.1, (i) *implies that the following bound holds for any* $F,G\in\mathbf{L}^2:=[L^2(\mathbb{P},dp)]^2=[L^2(\mathbb{R}^d)\oplus[l^2(\mathbb{Z}^d)]^n]^2$:

$$|\langle Q_0(p,p'),F(p)\otimes G(p')\rangle|\leqslant C\|F\|_{\mathbf{L}^2}\|G\|_{\mathbf{L}^2}.$$

Corollary 3.3. *The quadratic form* $\mathcal{Q}_\infty(Z,Z)$ *defined in* (2.13)–(2.14) *is continuous in* $\mathbf{L}^2$.

Proof. Formulas (2.13) and (2.14) imply

$$\begin{aligned}\langle Q_\infty(p,p'),Z(p)\otimes Z(p')\rangle &= (2\pi)^{-d}\int_{K^d\setminus\mathcal{C}_*}\Big(\tilde q_\infty(\theta,r,r'),\tilde Z_\Pi(\theta,r)\otimes\overline{\tilde Z_\Pi(\theta,r')}\Big)\,d\theta\\ &= (2\pi)^{-d}\frac12\sum_{l=1}^\infty\int_{K^d\setminus\mathcal{C}_*}\Big(\tilde q_0(\theta,r,r')+\tilde r_0(\theta,r,r'),P_l(\theta)\tilde Z_\Pi(\theta,r)\otimes P_l(\theta)\overline{\tilde Z_\Pi(\theta,r')}\Big)\,d\theta,\end{aligned} \tag{3.12}$$

where $\tilde r_0(\theta,r,r')$ is the integral kernel of the operator $\tilde r_0(\theta):=\begin{pmatrix}\tilde{\mathcal{H}}^{-1}(\theta)\tilde q_0^{11}(\theta) & -\tilde q_0^{10}(\theta)\\ -\tilde q_0^{01}(\theta) & \tilde{\mathcal{H}}(\theta)\tilde q_0^{00}(\theta)\end{pmatrix}$. Here and below, the symbol $(\cdot\,,\cdot)$ stands for the inner product in $H_1^0\equiv H^0(K_1^d)\oplus\mathbb{C}^n$, i.e.,

$$(F,G)=\int_{K_1^d}\overline{F^1}(y)G^1(y)\,dy+\overline{F^2}\cdot G^2,\qquad F=(F^1,F^2),\quad G=(G^1,G^2)\in H_1^0, \tag{3.13}$$

or in $\mathbf{H}^0\equiv[H_1^0]^2$. Consider the terms on the right-hand side of (3.12). Since

$$\sum_{l=1}^\infty\|P_l\tilde Z_\Pi\|^2_{[L^2(K^d\times\mathcal{R})]^2}\leqslant C\|\tilde Z_\Pi\|^2_{[L^2(K^d\times\mathcal{R})]^2}=C'\|Z\|^2_{\mathbf{L}^2}, \tag{3.14}$$

we obtain

$$\sum_{l=1}^\infty\int_{K^d\setminus\mathcal{C}_*}\Big(\tilde q_0^{ij}(\theta,r,r'),P_l(\theta)\tilde Z_\Pi^{i'}(\theta,r)\otimes P_l(\theta)\overline{\tilde Z_\Pi^{j'}(\theta,r')}\Big)\,d\theta\leqslant C\|Z\|^2_{\mathbf{L}^2},\quad i,j,i',j'=0,1, \tag{3.15}$$

by Corollary 3.2. Further, consider $\tilde r_0^{ij}$. For the terms with $\tilde r_0^{01}$ and $\tilde r_0^{10}$, estimate (3.15) holds. We rewrite the term with $\tilde r_0^{00}$ in the form

$$\begin{aligned}&\sum_{l=1}^\infty\int_{K^d\setminus\mathcal{C}_*}\Big(\tilde r_0^{00}(\theta,r,r'),P_l(\theta)\tilde Z_\Pi^0(\theta,r)\otimes P_l(\theta)\overline{\tilde Z_\Pi^0(\theta,r')}\Big)\,d\theta\\ &\qquad=\sum_{l=1}^\infty\int_{K^d\setminus\mathcal{C}_*}\Big(\tilde q_0^{11}(\theta,r,r'),\tilde{\mathcal{H}}^{-1}(\theta)P_l(\theta)\tilde Z_\Pi^0(\theta,r)\otimes P_l(\theta)\overline{\tilde Z_\Pi^0(\theta,r')}\Big)\,d\theta.\end{aligned} \tag{3.16}$$

It follows from estimates (3.14) and (9.7) and Corollary 3.2 that the right-hand side of (3.16) is estimated from above by $C\|Z^0\|^2_{L^2(\mathbb{P})}$. Finally, consider the term with $\tilde r_0^{11}$ and represent it in the form

$$\sum_{l=1}^\infty \int_{K^d\setminus\mathcal{C}_*} \Big(\tilde r_0^{11}(\theta,r,r'), P_l(\theta)\tilde Z^1_\Pi(\theta,r)\otimes P_l(\theta)\overline{\tilde Z^1_\Pi(\theta,r')}\Big)\,d\theta$$
$$= \sum_{l=1}^\infty \int_{K^d\setminus\mathcal{C}_*} \Big(\tilde q_0^{00}(\theta,r,r'), \Omega(\theta)P_l(\theta)\tilde Z^1_\Pi(\theta,r)\otimes \Omega(\theta)P_l(\theta)\overline{\tilde Z^1_\Pi(\theta,r')}\Big)\,d\theta. \quad (3.17)$$

By the bounds (3.14) and (9.6), the right-hand side of (3.17) is bounded by

$$C\int_{K^d}\Big(\|\tilde q_0^{\psi^0\psi^0}(\theta,\cdot,\cdot)\|_{[H^1(T_1^d)]^2}+\|\tilde q_0^{\psi^0u^0}(\theta,\cdot)\|_{H^1(T_1^d)}+\|\tilde q_0^{u^0\psi^0}(\theta,\cdot)\|_{H^1(T_1^d)}+|\tilde q_0^{u^0u^0}(\theta)|\Big)\|\tilde Z^1_\Pi(\theta,\cdot)\|^2_{H^0_1}\,d\theta. \quad (3.18)$$

In turn, (3.18) is estimated by $C\|Z^1\|^2_{L^2(\mathbb{P})}$ by Proposition 3.1, (ii). □

Remark 3.4. The operator $\tilde q_0(\theta)$ in $\mathbf{H}^0$ is nonnegative and self-adjoint. Indeed, for any function $Z\in\mathcal{D}$, we have

$$\int_{K^d}\Big(\tilde q_0(\theta), \tilde Z_\Pi(\theta,\cdot)\otimes\overline{\tilde Z_\Pi(\theta,\cdot)}\Big)\,d\theta = (2\pi)^d E|\langle Y,Z\rangle|^2 \geqslant 0.$$

Hence, $\big(\tilde q_0(\theta), \tilde Z_\Pi(\theta,\cdot)\otimes\overline{\tilde Z_\Pi(\theta,\cdot)}\big)\geqslant 0$, $\theta\in K^d$.

4. COMPACTNESS OF THE SET OF MEASURES OF THE FORM μ_T

Proposition 2.15 follows from the bound (2.15) by the Prokhorov theorem [25, Lemma II.3.1] by using the method of [25, Theorem XII.5.2] because the embedding $\mathcal{E}^{0,\alpha}\subset\mathcal{E}^{s,\beta}$ is compact if $s<0$ and $\alpha>\beta$.

Lemma 4.1. *Let conditions* **S0**–**S3** *hold. Then the bounds* (2.15) *hold for* $\alpha<-d/2$.

Proof. *Step* (i). By condition **S1**,

$$E\|Y(t)\|^2_{0,\alpha} = E\Big[\int_{\mathbb{R}^d}(1+|x|^2)^\alpha\Big(|\psi(x,t)|^2+|\nabla\psi(x,t)|^2+|\pi(x,t)|^2\Big)\,dx$$
$$+\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^\alpha\Big(|u(k,t)|^2+|v(k,t)|^2\Big)\Big]\leqslant C(\alpha,d)e(t), \quad (4.1)$$

where

$$e(t):=E\Big[\int_{K_1^d}\Big(|\psi(y,t)|^2+|\nabla\psi(y,t)|^2+|\pi(y,t)|^2\Big)\,dy+|u(0,t)|^2+|v(0,t)|^2\Big].$$

Denote by C_t the correlation operator of the random function

$$\Big(\psi(y,t)|_{y\in K_1^d}, u(0,t), \pi(y,t)|_{y\in K_1^d}, v(0,t)\Big)\in\mathcal{E}_1:=H^1(K_1^d)\oplus\mathbb{R}^d\oplus H^0(K_1^d)\oplus\mathbb{R}^d.$$

Then $e(t)$ is equal to the trace of the operator C_t. Note that $C_t=\mathrm{Op}\Big(q_t(0,r,r')\Big)$ is an integral operator with the integral kernel $q_t(0,r,r')$ (see (3.7)),

$$q_t(0,r,r')=(2\pi)^{-d}\int_{K^d}e^{-i(r-r')\theta}\tilde q_t(\theta,r,r')\,d\theta,\quad r,r'\in\mathcal{R}_1:=K_1^d\cup\{0\}. \quad (4.2)$$

Denote by $\tilde q_t(\theta):=\mathrm{Op}\Big(\tilde q_t(\theta,r,r')\Big)$ the integral operator with the integral kernel $\tilde q_t(\theta,r,r')$. In this case, (4.2) implies that

$$e(t)=\mathrm{tr}_{\mathcal{E}_1}C_t=(2\pi)^{-d}\int_{K^d}\mathrm{tr}_{\mathcal{E}_1}\Big[e^{-ir\theta}\tilde q_t(\theta)e^{ir'\theta}\Big]\,d\theta. \quad (4.3)$$

Step (ii). Introduce the operator Γ defined by $\Gamma\tilde\psi_\Pi(\theta,y) := (\nabla_y\tilde\psi_\Pi(\theta,y),\tilde\psi_\Pi(\theta,y))$ and the operator Γ_{ex} given by

$$\Gamma_{ex} := \begin{pmatrix} \begin{pmatrix} \Gamma & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & I \end{pmatrix}.$$

Then, since $\tilde q_t(\theta) \geqslant 0$ and the operator $e^{ir\theta}\colon (\psi(y),u,\pi(y),v) \to (e^{iy\theta}\psi,u,e^{iy\theta}\pi,v)$ in $\mathcal{E}_1$ is bounded uniformly with respect to $\theta \in K^d$, we have

$$\operatorname{tr}_{\mathcal{E}_1}\left[e^{-ir\theta}\tilde q_t(\theta)e^{ir'\theta}\right] \leqslant C\operatorname{tr}_{\mathcal{E}_1}\tilde q_t(\theta) = C\operatorname{tr}_{\mathbf{H}^0}\left[\Gamma_{ex}\tilde q_t(\theta)\Gamma_{ex}^*\right]. \tag{4.4}$$

Let us now estimate the trace $\operatorname{tr}_{\mathbf{H}^0}\left[\Gamma_{ex}\tilde q_t(\theta)\Gamma_{ex}^*\right]$. Introduce the matrix-valued self-adjoint operator Ω_{ex} on the space $\mathbf{H}^0$,

$$\Omega_{ex} \equiv \Omega_{ex}(\theta) := \begin{pmatrix} \Omega(\theta) & 0 \\ 0 & I \end{pmatrix},$$

where I stands for the identity operator on H_1^0. Note that $\Omega_{ex}\tilde q_t(\theta)\Omega_{ex} \geqslant 0$ (recall that Ω_{ex} is a self-adjoint operator). Further, $B := (\Gamma_{ex}\Omega_{ex}^{-1})$ is a bounded operator on $\mathbf{H}^0$ since $\Omega^{-1} : H^0(K_1^d)\oplus\mathbb{C}^n = H_1^0 \to H_1^1$, and $\begin{pmatrix} \Gamma & 0 \\ 0 & 1 \end{pmatrix} : H_1^1 \to H_1^0$. Therefore,

$$\operatorname{tr}_{\mathbf{H}^0}[\Gamma_{ex}\tilde q_t(\theta)\Gamma_{ex}^*] = \operatorname{tr}_{\mathbf{H}^0}[B\Omega_{ex}\tilde q_t(\theta)\Omega_{ex}B^*] \leqslant C\operatorname{tr}_{\mathbf{H}^0}[\Omega_{ex}\tilde q_t(\theta)\Omega_{ex}], \tag{4.5}$$

by [23, Theorem 1.6]. Let us now estimate the trace of the operator $\Omega_{ex}\tilde q_t(\theta)\Omega_{ex}$. We first use the formula $\Omega_{ex}G(\theta,t) = U(\theta,t)\Omega_{ex}$, where $G(\theta,t)$ is defined in (11.1) and

$$U(\theta,t) := \begin{pmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{pmatrix}.$$

Hence, by (11.4), we have

$$\Omega_{ex}\tilde q_t(\theta)\Omega_{ex} = \Omega_{ex}G(\theta,t)\tilde q_0(\theta)G^*(\theta,t)\Omega_{ex} = U(\theta,t)\Omega_{ex}\tilde q_0(\theta)\Omega_{ex}U^*(\theta,t).$$

Since $U(t,\theta)$ is a unitary operator on $\mathbf{H}^0$,

$$\operatorname{tr}_{\mathbf{H}^0}[\Omega_{ex}\tilde q_t(\theta)\Omega_{ex}] = \operatorname{tr}_{\mathbf{H}^0}[U(\theta,t)\Omega_{ex}\tilde q_0(\theta)\Omega_{ex}U^*(\theta,t)] = \operatorname{tr}_{\mathbf{H}^0}[\Omega_{ex}\tilde q_0(\theta)\Omega_{ex}] \tag{4.6}$$

by [19, Theorem VI.18, (c)]. Finally, it follows from (4.3)–(4.6) that

$$\sup_{t\in\mathbb{R}} e(t) \leqslant C_1\int_{K^d}\operatorname{tr}_{\mathbf{H}^0}[\Omega_{ex}\tilde q_0(\theta)\Omega_{ex}]\,d\theta. \tag{4.7}$$

Step (iii). Let us now prove that the right-hand side of (4.7) is finite. We use the representation

$$\Omega_{ex}\tilde q_0(\theta)\Omega_{ex}^* = (\Omega_{ex}\Gamma_{ex}^{-1})\Gamma_{ex}\tilde q_0(\theta)\Gamma_{ex}^*(\Omega_{ex}\Gamma_{ex}^{-1})^*,$$

where Γ_{ex}^{-1} stands for the left inverse operator of Γ_{ex}. On the other hand, $\Omega_{ex}\Gamma_{ex}^{-1}$ is a bounded operator in $\mathbf{H}^0$ since $\Omega_{ex}(\theta)$ (Γ_{ex}^{-1}, respectively) is (a finite-dimensional perturbation of) a pseudo-differential operator of order 1 (-1, respectively) on K_1^d. Moreover, $\Omega_{ex}(\theta)$ is uniformly bounded on $\theta \in K^d$. Hence,

$$\operatorname{tr}_{\mathbf{H}^0}[\Omega_{ex}\tilde q_0(\theta)\Omega_{ex}] \leqslant C\operatorname{tr}_{\mathbf{H}^0}[\Gamma_{ex}\tilde q_0(\theta)\Gamma_{ex}^*] = C\operatorname{tr}_{\mathcal{E}_1}[\tilde q_0(\theta)]. \tag{4.8}$$

Finally, by inequalities (4.7) and (4.8) and by condition **S2**, we obtain

$$\begin{aligned}\sup_{t\in\mathbb{R}} e(t) &\leqslant C\int_{K^d}\operatorname{tr}_{\mathcal{E}_1}\tilde q_0(\theta)\,d\theta \\ &\leqslant C_1E\Big[\int_{K_1^d}(|\psi_0(y)|^2+|\nabla\psi_0(y)|^2+|\pi_0(y)|^2)\,dy+|u_0(0)|^2+|v_0(0)|^2\Big] \\ &\leqslant C_1(\bar e_F+e_L)<\infty. \end{aligned}\tag{4.9}$$

Now the bound (2.15) follows from (4.1) and (4.9). □

5. “CUTTING OUT” THE CRITICAL SPECTRUM

Definition 5.1. (i) Introduce the *critical set* $\mathcal{C} := \mathcal{C}_* \cup \Big(\cup_k \mathcal{C}_k\Big)$ (see **E1**).

(ii) Introduce the set $\mathcal{D}^0 \subset \mathcal{D}$ given by

$$\mathcal{D}^0 = \cup_N \mathcal{D}_N, \quad \mathcal{D}_N := \left\{ Z \in \mathcal{D} \;\middle|\; \begin{array}{c} P_l \tilde{Z}_\Pi(\theta,\cdot) = 0 \quad \text{for } \forall l \geqslant N,\, \theta \in K^d, \\ \tilde{Z}_\Pi(\theta,\cdot) = 0 \text{ in a neighborhood of a set } \mathcal{C} \cup \partial K^d \end{array} \right\}. \tag{5.1}$$

Lemma 5.2. *Let* $\lim_{t\to\infty} \mathcal{Q}_t(Z,Z) = \mathcal{Q}_\infty(Z,Z)$ *for any* $Z \in \mathcal{D}^0$. *Then the convergence holds for any* $Z \in \mathcal{D}$.

Proof. First, Definition 2.13 implies that

$$\mathcal{Q}_t(Z,Z) := E|\langle Y(\cdot,t), Z\rangle|^2 = \langle Q_t(p,p'), Z(p)\otimes Z(p')\rangle, \quad Z \in \mathcal{D}. \tag{5.2}$$

Therefore, by (7.2), we have $\mathcal{Q}_t(Z,Z) = \mathcal{Q}_0(Z(\cdot,t), Z(\cdot,t))$, and hence

$$\sup_{t\in\mathbb{R}} |\mathcal{Q}_t(Z,Z)| \leqslant C \sup_{t\in\mathbb{R}} \|Z(\cdot,t)\|^2_{\mathbf{L}^2} \tag{5.3}$$

by Corollary 3.2. By the Parseval identity and by the bound (7.5), we obtain

$$\|Z(\cdot,t)\|^2_{\mathbf{L}^2} = C(d) \int_{K^d} \|e^{\tilde{\mathcal{A}}^T(\theta)t} \tilde{Z}_\Pi(\theta,\cdot)\|^2_{H^0_1\oplus H^0_1}\, d\theta \leqslant C \int_{K^d} \|\tilde{Z}_\Pi(\theta,\cdot)\|^2_{H^0_1\oplus H^1_1}\, d\theta = C\|Z\|^2_{\mathcal{L}} \tag{5.4}$$

uniformly with respect to t. Here $\mathcal{L} := L^2(\mathbb{R}^d) \oplus [l^2(\mathbb{Z}^d)]^n \oplus H^1(\mathbb{R}^d) \oplus [l^2(\mathbb{Z}^d)]^n$. Further, by Lemma 2.8, for any $Z \in \mathcal{D}$, we can find a $Z^N \in \mathcal{D}_N$ such that $\|Z - Z^N\|_{\mathcal{L}} \to 0$ as $N \to \infty$. Namely, $\tilde{Z}^N_\Pi(\theta) = \sum_{l\leqslant N} P_l(\theta)\tilde{Z}_\Pi(\theta)$ if $\tilde{Z}_\Pi(\theta) = 0$ for θ in a neighborhood of $\mathcal{C} \cup \partial K^d$. Finally, the set of such functions Z is dense in $\mathcal{L}$. Then Lemma 5.2 follows from (5.3), (5.4), and Corollary 3.3. □

Lemma 5.3. *The convergence* (1.21) *holds for any* $Z \in \mathcal{D}$ *if it holds for* $Z \in \mathcal{D}^0$.

Proof. This follows immediately from Lemma 5.2 by the Cauchy–Bunyakovskii–Schwartz inequality:

$$\begin{aligned} |\hat{\mu}_t(Z') - \hat{\mu}_t(Z'')| &= \Big| \int \Big(e^{i\langle Y,Z'\rangle} - e^{i\langle Y,Z''\rangle}\Big)\mu_t(dY)\Big| \leqslant \int |e^{i\langle Y,Z'-Z''\rangle} - 1|\mu_t(dY) \\ &\leqslant \int |\langle Y, Z'-Z''\rangle|\mu_t(dY) \leqslant \sqrt{\int |\langle Y, Z'-Z''\rangle|^2 \mu_t(dY)} \\ &= \sqrt{\mathcal{Q}_t(Z'-Z'', Z'-Z'')} \leqslant C\|Z'-Z''\|_{\mathcal{L}}. \quad \square \end{aligned}$$

6. CONVERGENCE OF THE COVARIANCE

Proposition 6.1. *Let conditions* **E1**–**E2**, **R1**–**R3**, *and* **S0**–**S3** *hold. Then, for any* $Z \in \mathcal{D}$,

$$\mathcal{Q}_t(Z,Z) \to \mathcal{Q}_\infty(Z,Z), \quad t \to \infty. \tag{6.1}$$

Proof. By Lemma 5.2, it suffices to prove the convergence (6.1) for $Z \in \mathcal{D}^0$ only. If $Z \in \mathcal{D}$, then $Z \in \mathcal{D}_N$ for some N. Let us apply the Zak transform to the matrix $Q_t(p,p')$,

$$\langle Q_t(p,p'), Z(p)\otimes Z(p')\rangle = (2\pi)^{-2d} \langle \tilde{Q}_t(\theta,\theta',r,r'), \tilde{Z}_\Pi(\theta,r) \otimes \overline{\tilde{Z}}_\Pi(\theta',r')\rangle. \tag{6.2}$$

Further, by Lemma 2.8, we can choose some smooth branches of the functions $F_l(\theta,r)$ and $\omega_l(\theta)$ to apply the stationary phase arguments, which requires some smoothness with respect to θ. Denote by $\operatorname{supp} \tilde{Z}_\Pi$ the closure of the set $\{\theta \in K^d : \tilde{Z}_\Pi(\theta,y) \not\equiv 0, y \in T^d_1\}$. Since $\operatorname{supp} \tilde{Z}_\Pi \cap (\mathcal{C} \cup \partial K^d) = \emptyset$, we can apply Lemma 2.8. Namely, for any point $\Theta \in \operatorname{supp} \tilde{Z}_\Pi$, there is a neighborhood $\mathcal{O}(\Theta) \subset K^d \setminus (\mathcal{C} \cup \partial K^d)$ with the corresponding properties. Hence, $\operatorname{supp} \tilde{Z}_\Pi \subset \cup_{m=1}^M \mathcal{O}(\Theta_m)$, where $\Theta_m \in \operatorname{supp} \tilde{Z}_\Pi$. Therefore, there is a finite partition of unity

$$\sum_{m=1}^M g_m(\theta) = 1, \quad \theta \in \operatorname{supp} \tilde{Z}_\Pi, \tag{6.3}$$

where g_m are nonnegative functions of $C_0^\infty(K^d)$ and $\operatorname{supp} g_m \subset \mathcal{O}(\Theta_m)$. Further, using Definition 5.1, (ii) and the partition (6.3), represent the right-hand side of (6.2) as

$$\langle Q_t(p,p'), Z(p)\otimes Z(p')\rangle = (2\pi)^{-d}\sum_{m=1}^{M}\sum_{l,l'=1}^{N}\langle g_m(\theta) r_{ll'}(t,\theta), A_l(\theta)\otimes \overline{A}_{l'}(\theta)\rangle, \tag{6.4}$$

using formulas (3.6) and (11.6). Here $A_l(\theta) = (F_l(\theta,\cdot), \tilde Z_\Pi(\theta,\cdot))$, $r_{ll'}(t,\theta)$ is the 2×2 matrix

$$r_{ll'}(t,\theta) := \frac{1}{2}\sum_{\pm}\Big\{\cos\big(\omega_l(\theta)\pm\omega_{l'}(\theta)\big)t\,\Big(p_{ll'}(\theta)\mp C_l(\theta)p_{ll'}(\theta)C_{l'}^T(\theta)\Big)$$
$$+\sin\big(\omega_l(\theta)\pm\omega_{l'}(\theta)\big)t\,\Big(C_l(\theta)p_{ll'}(\theta)\pm p_{ll'}(\theta)C_{l'}^T(\theta)\Big)\Big\}, \tag{6.5}$$

where

$$C_l(\theta) := \begin{pmatrix} 0 & \omega_l^{-1}(\theta)\\ -\omega_l(\theta) & 0\end{pmatrix}, \qquad C_l^T(\theta) := \begin{pmatrix} 0 & -\omega_l(\theta)\\ \omega_l^{-1}(\theta) & 0\end{pmatrix}, \tag{6.6}$$

$$p_{ll'}^{ij}(\theta) := \Big(F_l(\theta,\cdot), (\tilde q_0^{ij}(\theta)F_{l'})(\theta,\cdot)\Big), \quad \theta\in\mathcal{O}(\Theta_m), \quad l,l'=1,2,\dots, \quad i,j=0,1, \tag{6.7}$$

and $(\cdot\,,\cdot)$ stands for the inner product in $H_1^0 \equiv H^0(T_1^d)\oplus\mathbb{C}^n$ (see (3.13)) or in $\mathbf{H}^0 \equiv [H_1^0]^2$. By Lemma 2.8, the eigenvalues $\omega_l(\theta)$ and the eigenfunctions $F_l(\theta,r)$ are real-analytic functions in $\theta\in\operatorname{supp} g_m$ for every m: we do not mark the functions by the index m to simplify the notation.

Lemma 6.2. *Let conditions* **S0**–**S3** *hold. Then* $p_{ll'}^{ij}(\theta)\in L^1(\mathcal{O}(\Theta_m))$, $i,j=0,1$, $l,l'=1,2,\dots$ *for each* $m=1,\dots,M$.

Proof. Since $\{F_l(\theta,\cdot)\}$ is an orthonormal basis, by the Cauchy–Bunyakovskii–Schwartz inequlity, we have

$$\Big|\int_{\mathcal{O}(\Theta_m)}|p_{ll'}^{ij}(\theta)|\,d\theta\Big|^2 \leqslant C\int_{\mathcal{O}(\Theta_m)}|p_{ll'}^{ij}(\theta)|^2\,d\theta \leqslant \int_{\mathcal{O}(\Theta_m)}\Big|\Big(F_l(\theta,r),\tilde q_0^{ij}(\theta,r,r')F_{l'}(\theta,r')\Big)\Big|^2\,d\theta$$
$$\leqslant \int_{\mathcal{O}(\Theta_m)}d\theta\int_{\mathcal{R}}dr\int_{\mathcal{R}}|\tilde q_0^{ij}(\theta,r,r')|^2\,dr'. \quad \square$$

Further, let us study the terms in (6.4), which are oscillatory integrals with respect to the variable θ. The identities $\omega_l(\theta)+\omega_{l'}(\theta)\equiv\text{const}_+$ or $\omega_l(\theta)-\omega_{l'}(\theta)\equiv\text{const}_-$ with $\text{const}_\pm\neq 0$ are impossible by condition **E2**. Moreover, the oscillatory integrals with $\omega_l(\theta)\pm\omega_{l'}(\theta)\not\equiv\text{const}$ vanish as $t\to\infty$. Hence, only the integrals with $\omega_l(\theta)-\omega_{l'}(\theta)\equiv 0$ contribute to the limit because the relation $\omega_l(\theta)+\omega_{l'}(\theta)\equiv 0$ would imply the relation $\omega_l(\theta)\equiv\omega_{l'}(\theta)\equiv 0$, which is impossible by **E2**. Let us index the eigenvalues $\omega_l(\theta)$ as in (2.5). Then $\cos\big(\omega_l(\theta)-\omega_{l'}(\theta)\big)t = 1$ for $l,l'\in(r_{\sigma-1},r_\sigma]$, $\sigma=1,2,\dots$ Hence, for $l,l'\in(r_{\sigma-1},r_\sigma]$, we have

$$r_{ll'}(t,\theta) = \frac{1}{2}\Big(p_{ll'}(\theta)+C_l(\theta)p_{ll'}(\theta)C_{l'}^T(\theta)\Big)+\frac{1}{2}\cos 2\omega_l(\theta)t\,\Big(p_{ll'}(\theta)-C_l(\theta)p_{ll'}(\theta)C_{l'}^T(\theta)\Big)$$
$$+\frac{1}{2}\sin 2\omega_l(\theta)t\,\Big(C_l(\theta)p_{ll'}(\theta)+p_{ll'}(\theta)C_{l'}^T(\theta)\Big). \tag{6.8}$$

Therefore,

$$\langle Q_t(p,p'), Z(p)\otimes Z(p')\rangle = (2\pi)^{-d}\sum_{m}\sum_{l,l'=1}^{N}\int g_m(\theta)\Big(M_{ll'}(\theta)+\cdots, A_l(\theta)\otimes\overline{A}_{l'}(\theta)\Big)\,d\theta, \tag{6.9}$$

where $M_{ll'}(\theta) = (M_{ll'}^{ij}(\theta))_{i,j=0}^1$, $l,l'=1,2,\dots$, is the matrix with the continuous entries

$$M_{ll'}^{ij}(\theta) = \chi_{ll'}\frac{1}{2}\Big(F_l(\theta,r),\Big[\tilde q_0(\theta,r,r')+C_l(\theta)\tilde q_0(\theta,r,r')C_l^T(\theta)\Big]^{ij}F_{l'}(\theta,r')\Big); \tag{6.10}$$

here the symbol $\chi_{ll'}$ is given by the rule (see (2.5))

$$\chi_{ll'} := \begin{cases} 1 & \text{if} \quad l,l'\in(r_{\sigma-1},r_\sigma], \quad \sigma=1,2,\dots,\ r_0:=0,\\ 0 & \text{otherwise.}\end{cases} \tag{6.11}$$

Further, for $\theta \in \operatorname{supp} g_m \subset \mathcal{O}(\Theta)$ (see Lemma 2.8), we write

$$\tilde{q}_\infty^{ij}(\theta, r, r') = \sum_{l,l'=1}^{+\infty} F_l(\theta, r) M_{ll'}^{ij}(\theta) \overline{F_{l'}}(\theta, r'), \quad i, j = 0, 1. \tag{6.12}$$

The local representation (6.12) can be expressed globally in the form (2.14). Hence,

$$\langle Q_t(p, p'), Z(p) \otimes Z(p') \rangle = (2\pi)^{-d} \sum_m \int g_m(\theta) \Big(\tilde{q}_\infty(\theta, r, r'), \tilde{Z}_\Pi(\theta, r) \otimes \overline{\tilde{Z}}_\Pi(\theta, r') \Big) \, d\theta + \cdots, \tag{6.13}$$

where the symbol "$\cdots$" stands for the oscillatory integrals which contain $\cos(\omega_l(\theta) \pm \omega_{l'}(\theta))t$ and $\sin(\omega_l(\theta) \pm \omega_{l'}(\theta))t$ with $\omega_l(\theta) \pm \omega_{l'}(\theta) \not\equiv$const. The oscillatory integrals converge to zero by the Lebesgue–Riemann theorem because the integrands in "$\cdots$" are summable, and we have $\nabla(\omega_l(\theta) \pm \omega_{l'}(\theta)) = 0$ on the set of Lebesgue measure zero only. The summability follows from Lemma 6.2 because the functions $A_l(\theta)$ are smooth. The zero-measure condition follows as in (2.4) since $\omega_l(\theta) \pm \omega_{l'}(\theta) \not\equiv$const. This completes the proof of Proposition 6.1. □

7. BERNSHTEIN'S ARGUMENT

7.1. *Oscillatory Representation and Stationary Phase Method*

To prove (1.21), we evaluate $\langle Y(\cdot, t), Z \rangle$ by duality arguments. Namely, introduce the dual space $\mathcal{E}' := H^{-1,-\alpha}(\mathbb{R}^d) \oplus L^{-\alpha} \oplus H^{0,-\alpha}(\mathbb{R}^d) \oplus L^{-\alpha}$ with finite norm

$$(|||Z|||'_{0,-\alpha})^2 := \|\psi\|^2_{-1,-\alpha} + \|\pi\|^2_{0,-\alpha} + \|u\|^2_{-\alpha} + \|v\|^2_{-\alpha}.$$

For $t \in \mathbb{R}$, introduce the "formal adjoint" operator $W'(t)$,

$$\langle W(t)Y, Z \rangle := \langle Y, W'(t)Z \rangle, \quad Y \in \mathcal{E}, \quad Z \in \mathcal{E}', \tag{7.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\mathbb{R}^d) \oplus [l^2(\mathbb{Z}^d)]^n \oplus L^2(\mathbb{R}^d) \oplus [l^2(\mathbb{Z}^d)]^n$.

Write $Z(\cdot, t) = W'(t)Z$. Then formula (7.1) can be rewritten as

$$\langle Y(t), Z \rangle = \langle Y_0, Z(\cdot, t) \rangle, \quad t \in \mathbb{R}. \tag{7.2}$$

The adjoint group $W'(t)$ admits a convenient description.

Lemma 7.1. *The action of the group $W'(t)$ coincides with the action of $W(t)$ up to the order of components. Namely, $W'(t) = \exp(\mathcal{A}^T t)$, where $\mathcal{A}$ is the generator of the group $W(t)$.*

Proof. Differentiating (7.1) with respect to t for $Y, Z \in \mathcal{D}$, we obtain

$$\langle Y, \dot{W}'(t)Z \rangle = \langle \dot{W}(t)Y, Z \rangle. \tag{7.3}$$

The group $W(t)$ has the generator $\mathcal{A}$ (see (1.8)). The generator of $W'(t)$ is the conjugate operator

$$\mathcal{A}' = \begin{pmatrix} 0 & -\mathcal{H}' \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{H}' = \mathcal{H} = \begin{pmatrix} -\Delta + m_0^2 & S \\ S^* & -\Delta_L + \nu_0^2 \end{pmatrix}. \tag{7.4}$$

Hence, $\mathcal{A}' = \mathcal{A}^T$. □

Corollary 7.2. *The following uniform bound holds*:

$$\|e^{\tilde{\mathcal{A}}^T(\theta)t} \tilde{Z}_\Pi(\theta, \cdot)\|_{H_1^0 \oplus H_1^1} \leqslant C \|\tilde{Z}_\Pi(\theta, \cdot)\|_{H_1^0 \oplus H_1^1}, \quad \tilde{Z}_\Pi(\theta, \cdot) \in H_1^0 \oplus H_1^1, \tag{7.5}$$

which can be proved similarly to (9.3).

Applying Lemma 7.1, we can rewrite $Z(t) = W'(t)Z$ as the Zak transform, i.e., $\tilde{Z}_\Pi(\theta, r, t) = \exp\Big(\tilde{\mathcal{A}}^T(\theta)t\Big)\tilde{Z}_\Pi(\theta, r)$. Recall that we can restrict ourselves to elements $Z \in \mathcal{D}_N$ with a fixed index N. Using the partition of unity (6.3), we obtain

$$\begin{aligned} Z(k + r, t) &= (2\pi)^{-d} \sum_{m=1}^M \sum_{l=1}^N \int_{K^d} g_m(\theta) e^{-i(k+r)\theta} G_l^T(\theta, t) F_l(\theta, r) A_l(\theta) \, d\theta \\ &= \sum_{m,\pm} \sum_{l=1}^N \int_{K^d} e^{-i(\theta(k+r) \pm \omega_l(\theta)t)} g_m(\theta) a_l^\pm(\theta) F_l(\theta, r) A_l(\theta) \, d\theta, \quad Z \in \mathcal{D}_N. \end{aligned} \tag{7.6}$$

Here $A_l(\theta) = (F_l(\theta,\cdot), \tilde Z_\Pi(\theta,\cdot))$,

$$G_l(\theta,t) := \begin{pmatrix} \cos\omega_l(\theta)t & \dfrac{\sin\omega_l(\theta)t}{\omega_l(\theta)} \\ -\omega_l(\theta)\sin\omega_l(\theta)t & \cos\omega_l(\theta)t \end{pmatrix}, \qquad \theta \in \operatorname{supp} g_m, \tag{7.7}$$

and $\omega_l(\theta)$ and $a_l^{\pm}(\theta)$ are real-analytic functions in the interior of the set $\operatorname{supp} g_m$ for every m.

Let us derive formula (1.21) by analyzing the propagation of the solution $Z(k+r,t)$ of the form (7.6) in diverse directions $k = vt$ with $v \in \mathbb{R}^d$ and for $r \in \mathcal{R}$. To this end, we apply the stationary phase method to the oscillatory integral (7.6) along the rays $k = vt$, $t > 0$. Then the phase becomes $(\theta v \pm \omega_l(\theta))t$, and its stationary points are the solutions of the equations $v = \mp\nabla\omega_l(\theta)$.

Note that $\tilde Z_\Pi(\theta,r) = 0$ at the points $(\theta,r) \in K^d \oplus \mathcal{R}$ with degenerate Hessian $D_l(\theta)$ (see **E1**). Therefore, the stationary phase method leads to the following two different types of asymptotic behavior of $Z(vt,t)$ as $t \to \infty$.

I. Let the velocity v be inside the light cone, $v = \pm\nabla\omega_l(\theta)$, where $\theta \in \mathcal{O}(\Theta)\setminus\mathcal{C}$. Then

$$Z(vt,t) = \mathcal{O}(t^{-d/2}). \tag{7.8}$$

II. Let the velocity v be outside the light cone, $v \neq \pm\nabla\omega_l(\theta)$, where $\theta \in \mathcal{O}(\Theta)\setminus\mathcal{C}$, $l = 1,\dots,N$. Then

$$Z(vt,t) = \mathcal{O}(t^{-k}), \quad \forall k > 0. \tag{7.9}$$

Lemma 7.3. *The following bounds hold for any fixed $Z \in \mathcal{D}^0$:*

(i)
$$\sup_{p\in\mathbb{P}} |Z(p,t)| \leqslant C\, t^{-d/2}, \tag{7.10}$$

(ii) *for any $k > 0$, there exist numbers $C_k, \gamma > 0$ such that*

$$|Z(p,t)| \leqslant C_k(1+|p|+|t|)^{-k}, \qquad |p| \geqslant \gamma t. \tag{7.11}$$

Proof. Consider $Z(k+r,t)$ along each ray $k = vt$ with an arbitrary $v \in \mathbb{R}^d$. Substituting the related expressions into (7.6), we obtain

$$Z(vt+r,t) = \sum_{m,\pm}\sum_{l=1}^{N} \int_{K^d} e^{-i(\theta v \pm \omega_l(\theta))t} e^{-i\theta r} a_l^{\pm}(\theta) F_l(\theta,r) A_l(\theta)\, d\theta, \quad Z \in \mathcal{D}_N. \tag{7.12}$$

This is a sum of oscillatory integrals with phase functions of the form $\phi_l^{\pm}(\theta) = \theta v \pm \omega_l(\theta)$ and with amplitudes $a_l^{\pm}(\theta)$ that are real-analytic functions of θ in the interiors of the sets $\operatorname{supp} g_m$. Since $\omega_l(\theta)$ is real-analytic, each function $\phi_l^{\pm}$ has at most finitely many stationary points $\theta \in \operatorname{supp} g_m$ (solutions of the equation $v = \mp\nabla\omega_l(\theta)$). The stationary points are nondegenerate for $\theta \in \operatorname{supp} g_m$ by Definition 5.1 and by **E1** since

$$\det\left(\frac{\partial^2\phi_l^{\pm}}{\partial\theta_i\partial\theta_j}\right) = \pm D_l(\theta) \neq 0, \quad \theta \in \operatorname{supp} g_m. \tag{7.13}$$

At last, $\tilde Z_\Pi(\theta,r)$ is smooth because $Z \in \mathcal{D}$. Therefore, we have $Z(vt+r,t) = \mathcal{O}(t^{-d/2})$ according to the standard stationary phase method of [12, 20]. This implies the bounds (7.10) in each cone $|k| \leqslant ct$ with any finite c.

Further, write $\bar v := \max_m \max_{l=1,N} \max_{\theta\in\operatorname{supp} g_m} |\nabla\omega_l(\theta)|$. Then, for $|v| > \bar v$, there are no stationary points in $\operatorname{supp} \tilde Z_\Pi$. Hence, integration by parts (as in [20]) yields $Z(vt+r,t) = \mathcal{O}(t^{-k})$ for any $k > 0$. On the other hand, the integration by parts in (7.6) implies a similar bound, $Z(p,t) = \mathcal{O}\Big((t/|p|)^l\Big)$ for any $l > 0$. Therefore, relation (7.11) follows with any $\gamma > \overline{v}$. This shows that the bounds (7.10) hold everywhere. □

7.2. *"Room-Corridor" Partition*

The remaining constructions in the proof of (1.21) are similar to [5, 8]. However, the proofs are not identical, since here we consider a non-translation-invariant case and a coupled system.

Introduce a "room-corridor" partition of the ball $\{p\in\mathbb{P}:\ |p|\leqslant\gamma t\}$ with γ taken from (7.11). For $t>0$, choose Δ_t and $\rho_t\in\mathbb{N}$. Asymptotic relations between t, Δ_t, and ρ_t are specified below. Set $h_t=\Delta_t+\rho_t$ and

$$a^j=jh_t,\quad b^j=a^j+\Delta_t,\quad j\in\mathbb{Z},\qquad N_t=[(\gamma t)/h_t]. \tag{7.14}$$

The slabs $R_t^j=\{p\in\mathbb{P}:|p|\leqslant N_th_t,\ a^j\leqslant p_d<b^j\}$ are referred to as "rooms," $C_t^j=\{p\in\mathbb{P}:|p|\leqslant N_th_t,\ b^j\leqslant p_d<a^{j+1}\}$ as "corridors," and $L_t=\{p\in\mathbb{P}:|p|>N_th_t\}$ as "tails." Here $p=(p_1,\dots,p_d)$, Δ_t is the width of a room, and ρ_t is that of a corridor. Denote by χ_t^j the indicator of the room R_t^j, by ξ_t^j the indicator of the corridor C_t^j, and by η_t the indicator of the tail L_t. In this case,

$$\sum_t[\chi_t^j(p)+\xi_t^j(p)]+\eta_t(p)=1,\quad p\in\mathbb{P}, \tag{7.15}$$

where the sum $\sum_t$ stands for $\sum_{j=-N_t}^{N_t-1}$. Hence, we obtain the following Bernshtein's type representation:

$$\langle Y_0,Z(\cdot,t)\rangle=\sum_t[\langle Y_0,\chi_t^jZ(\cdot,t)\rangle+\langle Y_0,\xi_t^jZ(\cdot,t)\rangle]+\langle Y_0,\eta_tZ(\cdot,t)\rangle. \tag{7.16}$$

Introduce the random variables r_t^j, c_t^j, and l_t by the formulas

$$r_t^j=\langle Y_0,\chi_t^jZ(\cdot,t)\rangle,\quad c_t^j=\langle Y_0,\xi_t^jZ(\cdot,t)\rangle,\quad l_t=\langle Y_0,\eta_tZ(\cdot,t)\rangle. \tag{7.17}$$

Then relation (7.16) becomes

$$\langle Y_0,Z(\cdot,t)\rangle=\sum_t(r_t^j+c_t^j)+l_t. \tag{7.18}$$

Lemma 7.4. *Let* **S0–S3** *hold and $Z\in\mathcal{D}^0$. The following bounds hold for $t>1$:*

$$E|r_t^j|^2\leqslant C(Z)\ \Delta_t/t\quad\forall j, \tag{7.19}$$

$$E|c_t^j|^2\leqslant C(Z)\ \rho_t/t\quad\forall j, \tag{7.20}$$

$$E|l_t|^2\leqslant C_k(Z)\ t^{-k}\quad\forall k>0. \tag{7.21}$$

Proof. Relation (7.21) follows from (7.11) and Proposition 3.1, (i). We discuss (7.19) only, and relation (7.20) can be studied in a similar way. Let us express $E|r_t^j|^2$ in terms of correlation matrices. Definition (7.17) implies

$$E|r_t^j|^2=\langle Q_0(p,p'),\chi_t^j(p)Z(p,t)\otimes\chi_t^j(y)Z(p',t)\rangle. \tag{7.22}$$

According to (7.10), equation (7.22) yields

$$E|r_t^j|^2\leqslant Ct^{-d}\int\chi_t^j(p)\|Q_0(p,p')\|\,dpdp'=Ct^{-d}\int\chi_t^j(p)\,dp\int\|Q_0(p,p')\|\,dp'\leqslant C\Delta_t/t, \tag{7.23}$$

where $\|Q_0(p,p')\|$ stands for the norm of the matrix $\Big(Q_0^{ij}(p,p')\Big)$. Therefore, (7.23) follows by Corollary 3.2. □

7.3. *Proof of Theorem* A

The remaining part of the proof of the convergence (1.21) uses the Ibragimov–Linnik central limit theorem [15] and the bounds (7.19)–(7.21). For details, see [5, Secs. 8, 9] and [9, Secs. 9, 10].

8. ERGODICITY AND MIXING FOR THE LIMIT MEASURES

The limit measure μ_∞ is invariant by Theorem A, (iv). Let E_∞ be the integral with respect to μ_∞.

Theorem 8.1. *Let the assumptions of Theorem* A *hold. Then $W(t)$ is mixing with respect to the corresponding limit measure μ_∞, i.e., for any $f,g\in L^2(\mathcal{E},\mu_\infty)$, we have*

$$\lim_{t\to\infty}E_\infty f(W(t)Y)g(Y)=E_\infty f(Y)E_\infty g(Y). \tag{8.1}$$

In particular, the group $W(t)$ is ergodic with respect to the measure μ_∞,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(W(t)Y)dt=E_\infty f(Y)\quad(\text{mod }\ \mu_\infty). \tag{8.2}$$

Proof. *Step* (i). Since μ_∞ is Gaussian, the proof of (8.1) reduces to that of the convergence

$$\lim_{t\to\infty} E_\infty\langle W(t)Y,Z\rangle\langle Y,Z^1\rangle = 0 \tag{8.3}$$

for any $Z,Z^1\in\mathcal{D}$. It suffices to prove relation (8.3) for $Z,Z^1\in\mathcal{D}_N$. However, this follows from Corollary 3.3 and Theorem A, (iv).

Step (ii). Let $Z,Z^1\in\mathcal{D}_N$. Applying the Zak transform and the Parseval identity, we obtain

$$\begin{aligned} I(t) &\equiv E_\infty\langle W(t)Y,Z\rangle\langle Y,Z^1\rangle \\ &= E_\infty\langle Y,W'(t)Z\rangle\langle Y,Z^1\rangle = (2\pi)^{-d}\langle\tilde q_\infty(\theta,r,r'),G^*(\theta,t)\tilde Z_\Pi(\theta,r)\otimes\overline{\tilde Z^1_\Pi(\theta,r')}\rangle. \end{aligned} \tag{8.4}$$

Using a finite partition of unity (6.3), and relations (8.4) and (6.12), we see that

$$I(t) = (2\pi)^{-d}\sum_{m=1}^{M}\sum_{l,l'=1}^{N}\int g_m(\theta)G^*_l(\theta,t)A_l(\theta)M_{ll'}(\theta)\overline{A_{l'}(\theta)}\,d\theta. \tag{8.5}$$

Here $A_l(\theta) = (F_l(\theta,\cdot),\tilde Z_\Pi(\theta,\cdot))$ and $A_{l'}(\theta) = (F_l(\theta,\cdot),\tilde Z^1_\Pi(\theta,\cdot))$, and $G_l(t,\theta)$ is defined in (7.7). Similarly to (7.6), we have

$$I(t) = \sum_m\sum_{l,l'=1}^{N}\int g_m(\theta)e^{\pm i\omega_l(\theta)t}a_l^\pm(\theta)A_l(\theta)M_{ll'}(\theta)\overline{A_{l'}(\theta)}\,d\theta. \tag{8.6}$$

Here all the phase functions $\omega_l(\theta)$ and the amplitudes $a_l^\pm(\theta)$ are smooth on $\operatorname{supp} g_m$. Further, the relation $\nabla\omega_l(\theta)=0$ holds on a set of Lebesgue measure zero only. This follows similarly to (2.4) since $\nabla\omega_l(\theta)\not\equiv\text{const}$ by condition **E2**. Hence, $I(t)\to 0$ as $t\to\infty$ by the Lebesgue–Riemann theorem since the functions $M_{ll'}(\theta)$ are continuous. □

Remark. A similar result for wave equations and for harmonic crystals was proved in [4, 8].

9. APPENDIX A: DYNAMICS IN THE BLOCH–FOURIER REPRESENTATION

In this appendix, we prove the bound (2.2). We first construct the exponential $\exp(\tilde{\mathcal{A}}(\theta)t)$ for any chosen $\theta\in K^d\equiv[0,2\pi]^d$ and study its properties. Let us choose $\theta\in K^d$ and $X_0\in\mathbf{H}^1 := H^1_1\oplus H^0_1$, where $H^s_1\equiv H^s(T^d_1)\oplus\mathbb{C}^n$. Introduce the functions $\exp(\tilde{\mathcal{A}}(\theta)t)X_0$ for $X_0\in\mathbf{H}^1$ as the solutions $X(\theta,t)$ to the problem

$$\begin{cases} \dot X(\theta,t) = \tilde{\mathcal{A}}(\theta)X(\theta,t), \quad t\in\mathbb{R}, \\ X(\theta,0) = X_0. \end{cases} \tag{9.1}$$

Proposition 9.1. *For any chosen $\theta\in K^d$, the Cauchy problem* (9.1) *admits a unique solution $X(\theta,t)\in C(\mathbb{R};\mathbf{H}^1)$. Moreover,*

$$X(\theta,t) = e^{\tilde{\mathcal{A}}(\theta)t}X_0, \tag{9.2}$$

and

$$\|X(\theta,t)\|_{\mathbf{H}^1}\leqslant C\|X_0\|_{\mathbf{H}^1}, \tag{9.3}$$

where the constant C does not depend on $\theta\in K^d$ and $t\in\mathbb{R}$.

We prove this proposition in Subsection 9.2.

9.1. *Schrödinger Operator*

Let us first construct solutions $X(\theta,t)$ to problem (9.1) with a chosen parameter $\theta\in K^d$. Write $X(\theta,t) = (X^0(\theta,t),X^1(\theta,t))$, where $X^0(\theta,t) = (\varphi(\theta,t),u(\theta,t))$ and $X^1(\theta,t) = (\phi(\theta,t),v(\theta,t))$. By (9.1) and (1.15), we have $X^1(\theta,t) = \dot X^0(\theta,t)$, and $X^0(\theta,t)$ is a solution to the following Cauchy problem with a chosen parameter $\theta\in K^d$:

$$\begin{cases} \ddot X^0(\theta,t) = -\tilde{\mathcal{H}}(\theta)X^0(\theta,t), \quad t\in\mathbb{R}, \\ \Big(X^0(\theta,t),\dot X^0(\theta,t)\Big)\Big|_{t=0} = (X^0_0,X^1_0) = X_0, \end{cases} \tag{9.4}$$

where $\tilde{\mathcal{H}}(\theta)$ is the "Schrödinger operator" (1.16). Hence, formally,

$$X^0(\theta,t) = \cos\Omega(\theta)t\,X^0_0 + \sin\Omega(\theta)t\,\Omega^{-1}(\theta)X^1_0, \tag{9.5}$$

where $\Omega(\theta) = \sqrt{\tilde{\mathcal{H}}(\theta)} > 0$.

Lemma 9.2. *For $X^0 \in H_1^0$, the following bounds hold*:

$$\|\Omega(\theta)X^0\|_{H_1^{-1}} \leqslant C\|X^0\|_{H_1^0}, \tag{9.6}$$

$$\|\tilde{\mathcal{H}}^{-1}(\theta)X^0\|_{H_1^0} \leqslant C\|X^0\|_{H_1^0}, \tag{9.7}$$

where the constant C does not depend on $\theta \in K^d$.

Proof. (i) Formula (1.16) for $\tilde{\mathcal{H}}(\theta)$ implies that

$$\|\tilde{\mathcal{H}}(\theta)X^0\|_{H_1^{-1}} \leqslant C\|X^0\|_{H_1^1}, \quad X^0 \in H_1^1, \tag{9.8}$$

where the constant C does not depend on $\theta \in K^d$. Hence,

$$\|\Omega(\theta)X^0\|^2_{H_1^0} = (X^0, \tilde{\mathcal{H}}(\theta)X^0) \leqslant \|X^0\|_{H_1^1}\|\tilde{\mathcal{H}}(\theta)X^0\|_{H_1^{-1}} \leqslant C\|X^0\|^2_{H_1^1}. \tag{9.9}$$

Since, $\Omega(\theta) = \Omega^*(\theta)$, the bound (9.9) implies (9.6).

(ii) Condition **R2** implies that

$$\|X^0\|_{H_1^1}\|\tilde{\mathcal{H}}(\theta)X^0\|_{H_1^{-1}} \geqslant (X^0, \tilde{\mathcal{H}}(\theta)X^0) \geqslant \kappa^2\|X^0\|^2_{H_1^1}.$$

Hence, $\|\tilde{\mathcal{H}}(\theta)X^0\|_{H_1^{-1}} \geqslant \kappa^2\|X^0\|_{H_1^1}$. Therefore, $\|\tilde{\mathcal{H}}^{-1}(\theta)X^0\|_{H_1^1} \leqslant \kappa^{-2}\|X^0\|_{H_1^{-1}}$. In particular, (9.7) follows. □

Remark 9.3. Condition **R2′** implies condition **R2**.

Proof. Indeed, for $X^0 = (\varphi(y), u) \in H_1^1$ and $\theta \in K^d$, we have

$$\begin{aligned}
(X^0, \tilde{\mathcal{H}}(\theta)X^0) &= \int_{T_1^d} \Big[\overline{\varphi}(y)[(i\nabla_y + \theta)^2 + m_0^2]\varphi(y) + \overline{\tilde{R}_\Pi(\theta, y)} \cdot \overline{u(\theta)}\varphi(y) \\
&\quad + \tilde{R}_\Pi(\theta, y) \cdot u(\theta)\overline{\varphi}(y)\Big]\, dy + \omega_*^2(\theta)|u(\theta)|^2 \\
&= \int_{T_1^d} \left[|(i\nabla_y + \theta)\varphi(y)|^2 + \frac{m_0^2}{2}\left|\varphi(y) + \frac{2}{m_0^2}\tilde{R}_\Pi(\theta, y)u(\theta)\right|^2\right] dy \\
&\quad + \omega_*^2(\theta)|u(\theta)|^2 - \frac{2}{m_0^2}\int_{T_1^d} |\tilde{R}_\Pi(\theta, y)u(\theta)|^2\, dy + \frac{m_0^2}{2}\int_{T_1^d} |\varphi(y)|^2\, dy \\
&\geqslant \alpha\int_{T_1^d} |\nabla_y\varphi(y)|^2\, dy + \left(\frac{m_0^2}{2} - \beta d2\pi\right)\int_{T_1^d} |\varphi(y)|^2\, dy \\
&\quad + |u(\theta)|^2\Big(\nu_0^2 - \frac{2}{m_0^2}\int_{T_1^d} |\tilde{R}_\Pi(\theta, y)|^2\, dy\Big)
\end{aligned}$$

for some $\beta > 0$ and $\alpha \in (0, \beta/(\beta+1))$. Take $\beta < m_0^2/(4\pi d)$. It remains to prove that

$$\nu_0^2 - \frac{2}{m_0^2}\int_{T_1^d} |\tilde{R}_\Pi(\theta, y)|^2\, dy > 0.$$

With regard to condition **R2′**, the Parseval equality implies

$$\int_{T_1^d} |\tilde{R}_\Pi(\theta, y)|^2 dy = \int_{T_1^d} \Big|\sum_{k\in\mathbb{Z}^d} e^{ik\theta}R(k+y)\Big|^2 dy \leqslant \int_{T_1^d} \Big|\sum_{k\in\mathbb{Z}^d} |R(k+y)|\Big|^2 dy < \nu_0^2 m_0^2/2. \quad \square$$

9.2. *Existence of the Schrödinger Group*

Recall that $\omega_l(\theta) > 0$ ($F_l(\theta, \cdot)$), $l = 1, 2, \ldots$, are the eigenvalues (orthonormal eigenvectors) of the operator $\Omega(\theta)$ in H_1^0. Let us prove the existence of solutions to the Cauchy problem (9.1). We represent $X^0(\theta, t)$ in the form

$$X^0(\theta, t) = \sum_{l=1}^{\infty} A_l(t)F_l(\theta, r), \quad t \in \mathbb{R}, \tag{9.10}$$

where $A_l(t) \equiv A_l(\theta, t)$ is the unique solution of the Cauchy problem

$$\ddot{A}_l(t) = -\omega_l^2(\theta)A_l(t), \quad (A_l(t), \dot{A}_l(t))|_{t=0} = (A_{0l}^0, A_{0l}^1),$$

and $A_{0l}^i \equiv A_{0l}^i(\theta) = (F_l(\theta,\cdot), X_0^i(\cdot))$, $i = 0, 1$. Hence,

$$A_l(t) = \cos\omega_l(\theta)tA_{0l}^0 + \frac{\sin\omega_l(\theta)t}{\omega_l(\theta)}A_{0l}^1. \tag{9.11}$$

By the energy conservation, this yields

$$\frac{|\dot{A}_l(t)|^2}{2} + \omega_l^2(\theta)\frac{|A_l(t)|^2}{2} = \frac{|A_{0l}^1|^2}{2} + \omega_l^2(\theta)\frac{|A_{0l}^0|^2}{2}.$$

Summing up, for $t \in \mathbb{R}$, we obtain

$$\frac{1}{2}\|\dot{X}^0(\theta,t)\|_{H_1^0}^2 + \frac{1}{2}(X^0(\theta,t), \tilde{\mathcal{H}}(\theta)X^0(\theta,t)) = \frac{1}{2}\|X_0^1\|_{H_1^0}^2 + \frac{1}{2}(X_0^0, \tilde{\mathcal{H}}(\theta)X_0^0) \leqslant C\|X_0\|_{\mathbf{H}^1}^2 \tag{9.12}$$

by (9.8). Hence, the solution (9.10) exists and is unique.

Further, relation (9.11) implies (9.5). Finally, the solution to problem (9.1) exists; it is unique and can be represented by (9.2). The bound (9.3) follows from (9.12) and (2.1). □

Now the exponential $\exp\big(\tilde{\mathcal{A}}(\theta)t\big)$ is defined for any chosen value $\theta \in K^d$, and this exponential is a continuous operator in $\mathbf{H}^1$.

9.3. *Smoothness of the Schrödinger Group*

To complete the proof of Proposition 9.1, we must prove the smoothness of the exponential with respect to θ. This is needed to define the product (1.19) of the exponential and the distribution $\tilde{Y}_{0\Pi}(\cdot)$.

Consider the operators $\exp\Big(\tilde{\mathcal{A}}'(\theta)t\Big)$, $t \in \mathbb{R}$, on $\mathbf{H}^{-1} := (\mathbf{H}^1)^* = H_1^{-1} \oplus H_1^0$, where $\tilde{\mathcal{A}}'(\theta)$ is the formal adjoint operator to $\tilde{\mathcal{A}}(\theta)$:

$$(X, \tilde{\mathcal{A}}'(\theta)Z)_{H_1^0} = (\tilde{\mathcal{A}}(\theta)X, Z)_{H_1^0}, \quad X, Z \in C_0^\infty(T_1^d) \times \mathbb{C}^n.$$

Note that

$$\tilde{\mathcal{A}}'(\theta) = \tilde{\mathcal{A}}^T(\theta) = \begin{pmatrix} 0 & -\tilde{\mathcal{H}}(\theta) \\ 1 & 0 \end{pmatrix}. \tag{9.13}$$

Lemma 9.4. *For any $\alpha \geqslant 0$, the following bound holds:*

$$\sup_{|t|\leqslant T}\sup_{\theta\in K^d}\sum_{|\gamma|\leqslant\alpha}\|\mathcal{D}_\theta^\gamma e^{\tilde{\mathcal{A}}'(\theta)t}X_0\|_{\mathbf{H}^{-1}} \leqslant C(T)\|X_0\|_{\mathbf{H}^{-1}}. \tag{9.14}$$

Proof. For $\alpha = 0$, the bound

$$\|e^{\tilde{\mathcal{A}}'(\theta)t}X_0\|_{\mathbf{H}^{-1}} \leqslant C\|X_0\|_{\mathbf{H}^{-1}} \tag{9.15}$$

follows from the bound (9.3) by duality arguments. Consider the case of $\alpha = 1$. Introduce the function $X_\gamma(t) := \mathcal{D}_\theta^\gamma X(\theta,t)$, where $X(\theta,t) = e^{\tilde{\mathcal{A}}'(\theta)t}X_0$. Then

$$\dot{X}_\gamma(t) = \tilde{\mathcal{A}}'(\theta)X_\gamma(t) + [\mathcal{D}_\theta^\gamma\tilde{\mathcal{A}}'(\theta)]X(\theta,t), \quad X_\gamma(0) = 0.$$

Hence,

$$X_\gamma(t) = \int_0^t e^{\tilde{\mathcal{A}}'(\theta)(t-s)}[\mathcal{D}_\theta^\gamma\tilde{\mathcal{A}}'(\theta)]X(\theta,s)\,ds.$$

Therefore, by the bound (9.15),

$$\|X_\gamma(t)\|_{\mathbf{H}^{-1}} \leqslant \int_0^t \|e^{\tilde{\mathcal{A}}'(\theta)(t-s)}[\mathcal{D}_\theta^\gamma\tilde{\mathcal{A}}'(\theta)]X(\theta,s)\|_{\mathbf{H}^{-1}}\,ds \leqslant C\int_0^t \|[\mathcal{D}_\theta^\gamma\tilde{\mathcal{A}}'(\theta)]X(\theta,s)\|_{\mathbf{H}^{-1}}\,ds. \tag{9.16}$$

It follows from (9.13) that

$$[\mathcal{D}_\theta^\gamma \tilde{\mathcal{A}}'(\theta)] = \begin{pmatrix} 0 & -[\mathcal{D}_\theta^\gamma \tilde{\mathcal{H}}(\theta)] \\ 0 & 0 \end{pmatrix}, \qquad [\mathcal{D}_\theta^\gamma \tilde{\mathcal{H}}(\theta)] := \begin{pmatrix} 2(\mathcal{D}_\theta^\gamma \theta)(i\nabla_y + \theta) & [\mathcal{D}_\theta^\gamma \tilde{S}(\theta)] \\ [\mathcal{D}_\theta^\gamma \tilde{S}^*(\theta)] & 2\omega_*(\theta)\mathcal{D}_\theta^\gamma \omega_*(\theta) \end{pmatrix}.$$

Here $[\mathcal{D}_\theta^\gamma \tilde{S}(\theta)]\, u := [\mathcal{D}_\theta^\gamma \tilde{R}_\Pi(\theta,\cdot)]u$, $u \in \mathbb{C}^n$, and $[\mathcal{D}_\theta^\gamma \tilde{S}^*(\theta)]\, \varphi(y) := \int [\mathcal{D}_\theta^\gamma \tilde{R}_\Pi(-\theta, y)]\varphi(y)\, dy$. Hence, if $\theta \in K^d$, then

$$\|[\mathcal{D}_\theta^\gamma \tilde{\mathcal{A}}'(\theta)]X(\theta,s)\|_{\mathbf{H}^{-1}} = \|[\mathcal{D}_\theta^\gamma \tilde{\mathcal{H}}(\theta)]X^1(\theta,s)\|_{H_1^{-1}} \leqslant C\|X^1(\theta,s)\|_{H_1^0} \leqslant C\|e^{\tilde{\mathcal{A}}'(\theta)s}X_0\|_{\mathbf{H}^{-1}} \leqslant C\|X_0\|_{\mathbf{H}^{-1}}, \tag{9.17}$$

by the bound (9.15). Inequalities (9.16) and (9.17) imply the bound (9.14) with $\alpha = 1$. For $\alpha > 1$, the estimate follows by induction. □

9.4. *Dual Group*

Here we complete the proof of the bound (2.2) by duality arguments. Introduce the dual space $\mathcal{E}' := H^{-1,-\alpha}(\mathbb{R}^d) \oplus L^{-\alpha} \oplus H^{0,-\alpha}(\mathbb{R}^d) \oplus L^{-\alpha}$ of functions Z with finite norm

$$(|\!|\!|Z|\!|\!|'_{0,-\alpha})^2 := \|\psi\|^2_{-1,-\alpha} + \|\pi\|^2_{0,-\alpha} + \|u\|^2_{-\alpha} + \|v\|^2_{-\alpha}.$$

For $Z \in \mathcal{E}'$, we have $\tilde{Z}_\Pi(\theta,\cdot) \in H^\alpha(K^d; \mathbf{H}^{-1})$.

Lemma 9.5. *Let α be even and let $\alpha \leqslant -2$. Then*

$$\sup_{|t|\leqslant T} |\!|\!|W'(t)Z|\!|\!|'_{0,-\alpha} \leqslant C(T)|\!|\!|Z|\!|\!|'_{0,-\alpha}. \tag{9.18}$$

Proof. Note first that

$$(|\!|\!|Z|\!|\!|'_{0,-\alpha})^2 \sim \sum_{|\gamma|\leqslant -\alpha} \int_{K^d} \|\mathcal{D}_\theta^\gamma \tilde{Z}_\Pi(\theta,\cdot)\|^2_{\mathbf{H}^{-1}}\, d\theta \tag{9.19}$$

for any $Z \in \mathcal{E}'$. Indeed,

$$\begin{aligned} \|u\|^2_{-\alpha} &= \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{-2\alpha}|u(k)|^2 = C\int_{T^d} |(1-\Delta_\theta)^{-\alpha/2}\tilde{u}(\theta)|^2\, d\theta, \\ \|\psi\|^2_{-1,-\alpha} &= \|\langle x\rangle^{-\alpha}\Lambda^{-1}\psi(x)\|^2_{L^2} \sim \|\Lambda^{-1}\langle x\rangle^{-\alpha}\psi(x)\|^2_{L^2} = C\|(1+|\xi|^2)^{-1/2}(1-\Delta_\xi)^{-\alpha/2}\hat{\psi}(\xi)\|^2_{L^2} \\ &\sim \sum_{m\in\mathbb{Z}^d} \int_{K^d} (1+|2\pi m+\theta|^2)^{-1/2}|(1-\Delta_\theta)^{-\alpha/2}\hat{\psi}(2\pi m+\theta)|^2\, d\theta \\ &\sim \sum_{m\in\mathbb{Z}^d} \int_{K^d} (1+|m|^2)^{-1/2}|(1-\Delta_\theta)^{-\alpha/2}\hat{\psi}(2\pi m+\theta)|^2\, d\theta \\ &\sim \int_{K^d} \|(1-\Delta_\theta)^{-\alpha/2}\tilde{\psi}_\Pi(\theta,\cdot)\|^2_{H^{-1}(T_1^d)}\, d\theta. \end{aligned} \tag{9.20}$$

Hence, by Lemma 9.4 and by (9.19),

$$\begin{aligned} (|\!|\!|W'(t)Z|\!|\!|'_{0,-\alpha})^2 &\sim \sum_{|\gamma|\leqslant -\alpha} \int_{K^d} \left\|\mathcal{D}_\theta^\gamma \left(e^{\tilde{\mathcal{A}}'(\theta)t}\tilde{Z}_\Pi(\theta,\cdot)\right)\right\|^2_{\mathbf{H}^{-1}} d\theta \\ &\leqslant C(t)\sum_{|\gamma|\leqslant -\alpha} \int_{K^d} \left\|\mathcal{D}_\theta^\gamma \tilde{Z}_\Pi(\theta,\cdot)\right\|^2_{\mathbf{H}^{-1}} d\theta \sim C(t)\left(|\!|\!|Z|\!|\!|'_{0,-\alpha}\right)^2. \qquad \square \end{aligned}$$

Corollary 9.6. *The bound* (2.2) *follows from* (9.18) *by the duality considerations.*

10. APPENDIX B: CROSSING POINTS

10.1. *Proof of Lemmas* 2.8 *and* 2.10

Let us prove Lemma 2.8. For any chosen $\theta \in \mathbb{R}^d$, the Schrödinger operator $\tilde{\mathcal{H}}(\theta)$ admits the spectral resolution

$$\tilde{\mathcal{H}}(\theta) = \sum_{l=1}^{\infty} \lambda_l(\theta) P_l(\theta),$$

where $0 < \lambda_1(\theta) \leqslant \lambda_2(\theta) \leqslant \dots$, and $P_l(\theta)$ are one-dimensional orthogonal projectors in H_1^0. Further, let us take an arbitrary point $\Theta \in \mathbb{R}^d$ and a number $\Lambda \in (\lambda_M(\Theta), \lambda_{M+1}(\Theta))$, where $M \geqslant N$ and $\lambda_M(\Theta) < \lambda_{M+1}(\Theta)$. Then $\Lambda \neq \lambda_l(\theta)$ for $\theta \in \mathcal{O}(\Theta)$ if $\mathcal{O}(\Theta)$ is a sufficiently small neighborhood of Θ. Write $\tilde{\mathcal{H}}^\Lambda(\theta) = \sum_{\lambda_l(\theta)<\Lambda} \lambda_l(\theta) P_l(\theta)$, $P^\Lambda(\theta) = \sum_{\lambda_l(\theta)<\Lambda} P_l(\theta)$, $\theta \in \mathcal{O}(\Theta)$.

Further, let us choose a contour Γ_Λ (in the complex plane $\mathbb{C}$) surrounding the interval $(0, \Lambda)$. In this case, by the Cauchy theorem,

$$\tilde{\mathcal{H}}^\Lambda(\theta) = \int_{\Gamma_\Lambda} \frac{\lambda d\lambda}{\tilde{\mathcal{H}}(\theta) - \lambda}, \quad P^\Lambda(\theta) = \int_{\Gamma_\Lambda} \frac{d\lambda}{\tilde{\mathcal{H}}(\theta) - \lambda} \theta \in \mathcal{O}(\Theta).$$

Finally, by (1.16)–(1.18) and by condition **R1**, the mapping $\tilde{\mathcal{H}}(\theta)$ is an analytic operator-valued function in $\theta \in \mathcal{O}_c(\Theta)$, where $\mathcal{O}_c(\Theta)$ is a complex neighborhood of $\mathcal{O}(\Theta)$. Therefore, the same integrals converge for $\theta \in \mathcal{O}_c(\Theta)$, and the functions $P^\Lambda(\theta)$ and $\tilde{\mathcal{H}}^\Lambda(\theta)$ are analytic in a smaller neighborhood $\mathcal{O}_c'(\Theta)$. Reducing $\mathcal{O}_c'(\Theta)$ again, we can choose a basis $e_1(\theta), \dots, e_M(\theta)$ in the space $R^\Lambda(\theta) := P^\Lambda(\theta) H_1^0$, where the functions $e_l(\theta)$ depend analytically on $\theta \in \mathcal{O}_c(\Theta)$. For example, it suffices to choose an arbitrary basis $e_1(\Theta), \dots, e_M(\Theta)$ and set $e_l(\theta) = P^\Lambda(\theta) e_l(\Theta)$. The operator $\tilde{\mathcal{H}}(\theta)$ on the invariant space $R^\Lambda(\theta)$ can be identified with the corresponding matrix

$$\tilde{\mathcal{H}}^\Lambda(\theta) = \left(\tilde{\mathcal{H}}_{kl}(\theta)\right)_{k,l=0,\dots,M}, \tag{10.1}$$

which depends analytically on $\theta \in \mathcal{O}_c(\Theta)$. Therefore, the eigenvalues $\lambda_1(\theta), \dots, \lambda_M(\theta)$ and the eigenvectors $F_1(\theta), \dots, F_M(\theta)$ of this matrix can be chosen as real-analytic functions of $\theta \in \mathcal{O}_r(\Theta) \setminus C_*^\Lambda$, where $\mathcal{O}_r(\Theta) := \mathcal{O}_c(\Theta) \cap \mathbb{R}^d$ and C_*^Λ is a subset of $\mathbb{R}^d$ of Lebesgue measure zero. This can be proved by using the methods of [8, Appendix]. It remains to pass to the limit as $\Lambda \to \infty$ and define $C_* := \cup_1^\infty C_*^\Lambda$. Finally, $\omega_l(\theta) := \sqrt{\lambda_l(\theta)}$. After this, relations (2.5) and (2.6) follow as in [8, Appendix]. □

Lemma 2.10 can be proved in a similar way.

10.2. *Proof of Lemma* 2.11

First let us show that conditions **E1**, **E2** hold for $R_C(x) \equiv 0$ corresponding to $C_1 = \dots = C_N = 0$. Indeed, in this case, relation (1.16) becomes

$$\tilde{\mathcal{H}}(\theta) := \begin{pmatrix} (i\nabla_y + \theta)^2 + m_0^2 & 0 \\ 0 & \omega_*^2(\theta) \end{pmatrix}.$$

Therefore, $\omega_l(\theta)$ are equal to either $\omega_*(\theta)$ or $\sqrt{(2\pi k + \theta)^2 + m_0^2}$, $k \in \mathbb{Z}^d$. Namely, $\omega_*(\theta)$ corresponds to the eigenvectors $(0, u)$ with an arbitrary $u \in \mathbb{R}^n$. The square root corresponds to the eigenvectors $F_k(\theta, y) = (e^{-2\pi i k \cdot y}, 0)$ with $k \in \mathbb{Z}^d$. It can readily be seen that conditions **E1** and **E2** hold in this case.

Further, choose an arbitrary $l = 1, 2, \dots$, a point $\Theta \in \mathbb{R}^d \setminus \mathcal{C}_*$, and a bound $\Lambda \in (\lambda_M(\theta), \lambda_{M+1}(\theta))$ as above (with $M \geqslant l$). The function $R_C(x)$ and the corresponding operator $\tilde{\mathcal{H}}_C^\Lambda(\theta)$ depend analytically on $(\theta, C) \in \mathbb{C}^d \times \mathbb{C}^N$. Moreover, $R_C(x)$ satisfies conditions **R1** and **R2**$'$ for $C \in B_\varepsilon$ with a sufficiently small $\varepsilon > 0$. Therefore, as in the proof of Lemma 2.8, the corresponding eigenvalues $\omega_l(\theta, C)$, $l = 1, \dots, M$, are also analytic functions of (θ, C) in the domain $\mathcal{M}_l(\Theta) = \mathcal{O}_c \setminus \mathcal{C}$, where $\mathcal{O}_c$ is a complex neighborhood of $\mathcal{O}_r(\Theta) \times B_\varepsilon$, and $\mathcal{C}$ is a proper analytic subset of $\mathcal{O}_c$. Hence, the corresponding determinant $D_l(\theta, C)$ is an analytic function of $\mathcal{M}_l(\Theta)$. Further, $\mathcal{M}_l(\Theta)$ is an open connected set since $\mathcal{C}$ is a proper analytic subset. Therefore, $D_l(\theta, C) \not\equiv 0$ on $\mathcal{M}_l(\Theta)$ since $D_l(\theta, 0) \not\equiv 0$, $\theta \in \mathbb{R}^d$. Further, introduce the set $M_{1l} = \{C \in B_\varepsilon : D_l(\theta, C) \not\equiv 0\}$. The set $B_\varepsilon \setminus M_{1l}$ cannot contain any open ball, since otherwise $D_l(\theta, C) \equiv 0$. Hence, M_{1l} is an open dense set in B_ε. It remains to note that $M_1 = \cap_l M_{1l}$ is thus a dense subset of B_ε. For M_2, the proof is similar. □

11. APPENDIX C: COVARIANCE IN THE SPECTRAL REPRESENTATION

Introduce the matrix-valued operator

$$G(\theta,t):=e^{\tilde{\mathcal{A}}(\theta)t}=\begin{pmatrix}\cos\Omega(\theta)t & \sin\Omega(\theta)t\,\Omega^{-1}(\theta)\\ -\Omega(\theta)\sin\Omega(\theta)t & \cos\Omega(\theta)t\end{pmatrix}. \tag{11.1}$$

Note that we can represent the matrix $G(\theta,t)$ in the form

$$G(\theta,t)=\cos\Omega(\theta)t\,I+\sin\Omega(\theta)t\,C(\theta), \tag{11.2}$$

where I stands for the unit matrix, and

$$C(\theta):=\begin{pmatrix}0 & \Omega^{-1}(\theta)\\ -\Omega(\theta) & 0\end{pmatrix}.$$

In this case, the solution of (1.14) has the form $\tilde{Y}_\Pi(\theta,r,t)=G(\theta,t)\tilde{Y}_{0\Pi}(\theta,r)$, $r\in\mathcal{R}$. Using (11.2) and (3.4), we obtain

$$\begin{aligned}\tilde{Q}_t(\theta,r,\theta',r')&=E[\tilde{Y}_\Pi(\theta,r,t)\otimes\overline{\tilde{Y}_\Pi(\theta',r',t)}]\\&=\cos\Omega(\theta)t\;\tilde{Q}_0(\theta,r,\theta',r')\;\cos\Omega(\theta')t\\&\quad+\sin\Omega(\theta)t\;C(\theta)\tilde{Q}_0(\theta,r,\theta',r')C^T(\theta')\;\sin\Omega(\theta')t\\&\quad+\cos\Omega(\theta)t\;\tilde{Q}_0(\theta,r,\theta',r')C^T(\theta')\;\sin\Omega(\theta')t\\&\quad+\sin\Omega(\theta)t\;C(\theta)\tilde{Q}_0(\theta,r,\theta',r')\;\cos\Omega(\theta')t.\end{aligned} \tag{11.3}$$

By (3.6), we see that

$$\begin{aligned}\tilde{q}_t(\theta)=G(\theta,t)\tilde{q}_0(\theta)G^*(\theta,t)&=\cos\Omega(\theta)t\;\tilde{q}_0(\theta)\;\cos\Omega(\theta)t\\&+\cos\Omega(\theta)t\;\tilde{q}_0(\theta)C^T(\theta)\;\sin\Omega(\theta)t+\sin\Omega(\theta)t\;C(\theta)\tilde{q}_0(\theta)\;\cos\Omega(\theta)t\\&+\sin\Omega(\theta)t\;C(\theta)\tilde{q}_0(\theta)C^T(\theta)\;\sin\Omega(\theta)t,\end{aligned} \tag{11.4}$$

where $\tilde{q}_t(\theta)$ is the integral operator with the kernel $\tilde{q}_t(\theta,r,r')$ defined by (3.7).

For the simplicity of our manipulations, we assume now that the set of "crossing" points θ_* is empty, i.e., $\omega_l(\theta)\neq\omega_{l'}(\theta)$ for any $l,l'\in\mathbb{N}$, and the functions $\omega_l(\theta)$ and $F_l(\theta,r)$ are real-analytic. (Otherwise we need a partition of unity (6.3).) Consider the first term on the right-hand side of (11.4) and represent it in the form

$$\begin{aligned}&\cos\Omega(\theta)t\;\tilde{q}_0(\theta)\;\cos\Omega(\theta)t\\&\quad=\sum_{l,l'}F_l(\theta,r)\Big(\cos\omega_l(\theta)t\;p_{ll'}(\theta)\cos\omega_{l'}(\theta)t\Big)\overline{F_{l'}}(\theta,r')\\&\quad=\sum_{l,l'}F_l(\theta,r)\frac{1}{2}\Big[\cos(\omega_l(\theta)-\omega_{l'}(\theta))t+\cos(\omega_l(\theta)+\omega_{l'}(\theta))t\Big]p_{ll'}(\theta)\overline{F_{l'}}(\theta,r'),\end{aligned} \tag{11.5}$$

where $p_{ll'}(\theta)=\big(p^{ij}_{ll'}(\theta)\big)^1_{i,j=0}=\big(F_l(\theta,\cdot),(\tilde{q}^{ij}_0(\theta)F_{l'})(\theta,\cdot)\big)^1_{i,j=0}$ ($p^{ij}_{ll'}(\theta)$ are introduced in (6.7)). Similarly, we can rewrite the remaining three terms on the right-hand side of (11.4). Finally,

$$\tilde{q}^{ij}_t(\theta,r,r'):=\sum_{l,l'=1}^{\infty}F_l(\theta,r)r^{ij}_{ll'}(t,\theta)\otimes\overline{F_{l'}}(\theta,r'), \tag{11.6}$$

where $r_{ll'}(t,\theta)=(r^{ij}_{ll'}(t,\theta))^1_{ij=0}$ are defined in (6.5).

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