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On the convergence to statistical equilibrium for harmonic crystals

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We consider the dynamics of a harmonic crystal in d dimensions with n components, d,n arbitrary, $d,n\geqslant 1$, and study the distribution μ_t of the solution at time $t\in\mathbf{R}$. The initial measure μ_0 has a translation-invariant correlation matrix, zero mean, and finite mean energy density. It also satisfies a Rosenblatt—resp. Ibragimov–Linnik type mixing condition. The main result is the convergence of μ_t to a Gaussian measure as $t\to\infty$. The proof is based on the long time asymptotics of the Green's function and on Bernstein's "room-corridors" method. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1571658]

I. INTRODUCTION

Despite considerable efforts, the convergence to equilibrium for a mechanical system has remained as an extremely difficult problem. It has been recognized early on that for an infinitely extended system, possibly on top of local hyperbolicity, the flow of statistical information to infinity serves as a mechanism for relaxation. The two prime examples are the ideal gas and the harmonic crystal. We consider here the latter case. In the harmonic approximation the crystal is characterized by the displacement field $u(x)$, where $x\in\Gamma$, Γ is a regular lattice in $\mathbf{R}^d$, and $u(x)\in\mathbf{R}^n$ with n depending on the number of atoms in the unit cell. The field $u(x)$ is governed by a discrete wave equation. We will consider arbitrary d,n and for notational simplicity set $\Gamma=\mathbf{Z}^d$.

Our motivation to return to a well studied model is to a much wider class of initial measures than before. This project requires novel mathematical techniques. They have been developed for the wave and Klein–Gordon equation on $\mathbf{R}^d$ in Refs. 6–8, but the discrete structure poses extra difficulties.

Let us briefly comment on previous work. In Ref. 14 a general criterion is given which ensures mixing and Bernoulliness of the corresponding mechanical flow. Thereby the convergence to equilibrium is established for initial measures which are absolutely continuous with respect to the canonical Gaussian measure. In Ref. 14 moments of the displacement field are studied. This allows us to reduce the spectral analysis of the Liouvillian flow to the spectral properties of the dynamical group defined on solutions of finite energy. Since the crystal is assumed to be homogeneous, these spectral properties are determined by the dispersion relations $\omega_k(\theta)$, $k=1,\dots,n$. The Liouvillian flow is mixing and even Bernoulli, if, except for crossing points, each $\omega_k(\theta)$ is a real-analytic function which is not identically constant. In particular, the Lebesgue measure of the set $\{\theta\in\mathbf{T}^d\colon \nabla\omega_k(\theta)=0\}$ is equal to zero. In Ref. 20, for the case $d=n=1$, initial

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measures are considered which have distinct temperatures to the left and to the right. In Ref. 2, again $d=n=1$, the convergence to equilibrium is proved for a more general class of initial measures characterized by a mixing condition of Rosenblatt—resp. Ibragimov–Linnik type and which are asymptotically translation-invariant to the left and to the right.

The detailed stationary phase analysis of Ref. 2 does not directly generalize to $d\geqslant 2$. Rather, we have to develop a novel ''cutoff strategy'' which more carefully exploits the mixing condition in Fourier space. This approach allows us to all d within essence the same conditions for the dispersion relations as in Ref. 14. Our extension requires the technique of holomorphic functions of several complex variables.

In parentheses we remark that, for the ideal gas, Dobrushin and Suhov[3] first realized the importance of a mixing condition on the initial measure. In Ref. 9 it is replaced by the condition of finite entropy per unit volume thus establishing convergence whenever the specific particle number, energy, and entropy are finite. No such general result seems to be available for the harmonic crystal.

We outline our main result and strategy of proof. The displacement field $u(x)$ is the deviation of the configuration of crystal atoms from their equilibrium positions. Assuming them to be small and expanding the forces to linear order yields the discrete linear wave equation,

$$\ddot{u}(x,t)=-\sum_{y\in\mathbf{Z}^d}V(x-y)u(y,t); \qquad u|_{t=0}=u_0(x), \quad \dot{u}|_{t=0}=v_0(x), \quad x\in\mathbf{Z}^d. \tag{1.1}$$

Here $u(x,t)=(u_1(x,t),\ldots,u_n(x,t)),u_0=(u_{01},\ldots,u_{0n})\in\mathbf{R}^n$ and correspondingly for v_0. $V(x)$ is the interaction (or force) matrix, $(V_{kl}(x))$, $k,l=1,\ldots,n$. The dynamics (1.1) is invariant under lattice translations.

Let us denote by $Y(t)=(Y^0(t),Y^1(t))\equiv(u(\cdot,t),\dot{u}(\cdot,t))$, $Y_0=(Y_0^0,Y_0^1)\equiv(u_0(\cdot),v_0(\cdot))$. Then (1.1) takes the form of an evolution equation,

$$\dot{Y}(t)=\mathcal{A}Y(t), \quad t\in\mathbf{R}; \quad Y(0)=Y_0. \tag{1.2}$$

Formally, this is the Hamiltonian system since

$$\mathcal{A}Y=J\begin{pmatrix}\mathcal{V} & 0\\ 0 & 1\end{pmatrix}Y=J\,\nabla H(Y), \qquad J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{1.3}$$

Here $\mathcal{V}$ is a convolution operator with the matrix kernel V and H is the Hamiltonian functional,

$$H(Y)=\tfrac{1}{2}\langle v,v\rangle+\tfrac{1}{2}\langle\mathcal{V}u,u\rangle, \quad Y=(u,v), \tag{1.4}$$

where $\langle v,v\rangle=\Sigma_{x\in\mathbf{Z}^d}|v(x)|^2$ and $\langle\mathcal{V}u,u\rangle=\Sigma_{x,y\in\mathbf{Z}^d}(V(x-y)u(y),u(x)),(\cdot\,,\cdot)$ being the real scalar product in the Euclidean space $\mathbf{R}^n$.

We assume that the initial datum Y_0 is a random element of the Hilbert space $\mathcal{H}_\alpha$ of real sequences; see Definition 2.1. Y_0 is distributed according to the probability measure μ_0 of mean zero and satisfying the conditions **S1–S3** below. Given $t\in\mathbf{R}$, denote by μ_t the probability measure for $Y(t)$, the solution to (1.2) with random initial data Y_0. We study the asymptotics of μ_t as $t\to\pm\infty$.

The correlation matrices of the initial data are supposed to be translation-invariant, i.e., for $i,j=0,1$,

$$Q_0^{ij}(x,y):=E\big(Y_0^i(x)\otimes Y_0^j(y)\big)=q_0^{ij}(x-y), \quad x,y\in\mathbf{Z}^d, \tag{1.5}$$

though our methods require in fact much weaker conditions. We also assume that the initial mean ''energy'' density is finite,

$$e_0:=E[|u_0(x)|^2+|v_0(x)|^2]=\mathrm{tr}\,q_0^{00}(0)+\mathrm{tr}\,q_0^{11}(0)<\infty, \quad x\in\mathbf{Z}^d. \tag{1.6}$$

Finally, it is assumed that the measure μ_0 satisfies a mixing condition of a Rosenblatt—resp. Ibragimov–Linnik type, which means that

$$Y_0(x) \text{ and } Y_0(y) \text{ are asymptotically independent as } |x-y|\to\infty. \tag{1.7}$$

Our main result is the (weak) convergence of the measures μ_t on the Hilbert space $\mathcal{H}_\alpha$ with $\alpha < -d/2$,

$$\mu_t \rightharpoondown \mu_\infty \text{ as } t\to\infty. \tag{1.8}$$

μ_∞ is a Gaussian measure on $\mathcal{H}_\alpha$. A similar convergence result holds for $t\to-\infty$. Explicit formulas for the correlation functions of the limit measure μ_∞ are given in (2.18)–(2.22). As an application of the results, we show that the initial "white noise"-correlations provide the limit measure μ_∞ which coincides with the Gibbs canonical measure with the temperature $\sim e_0$. Respectively, μ_∞ is close to the canonical measure if the initial correlations are close to the white noise.

To prove the convergence (1.8) we follow general strategy.[2,4,6,7] There are three steps.

I. The family of measures μ_t, $t\geqslant 0$, is weakly compact in $\mathcal{H}_\alpha$ with $\alpha<-d/2$.

II. The correlation functions converge to a limit, for $i,j=0,1$,

$$Q_t^{ij}(x,y)=\int Y^i(x)\otimes Y^j(y)\,\mu_t(dY)\to Q_\infty^{ij}(x,y) \text{ as } t\to\infty. \tag{1.9}$$

III. The characteristic functionals converge to a Gaussian one,

$$\hat{\mu}_t(\Psi):=\int \exp(i\langle Y,\Psi\rangle)\mu_t(dY)\to\exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\right\} \text{ as } t\to\infty. \tag{1.10}$$

Here $\Psi=(\Psi^0,\Psi^1)\in\mathcal{D}=D\oplus D$, $D=C_0(\mathbf{Z}^d)\otimes\mathbf{R}^n$, where $C_0(\mathbf{Z}^d)$ denotes the space of the real sequences with finite support, $\langle Y,\Psi\rangle=\Sigma_{i=0,1}\Sigma_{x\in\mathbf{Z}^d}(Y^i(x),\Psi^i(x))$ and $\mathcal{Q}_\infty$ is the quadratic form with the matrix kernel $(Q_\infty^{ij}(x,y))_{i,j=0,1}$,

$$\mathcal{Q}_\infty(\Psi,\Psi)=\sum_{i,j=0,1}\sum_{x,y\in\mathbf{Z}^d}(Q_\infty^{ij}(x,y),\Psi^i(x)\otimes\Psi^j(y)). \tag{1.11}$$

Note that (1.1) is the translation-invariant convolution equation and admits a simple structure in the Fourier space. As a consequence, Fourier representation plays a central role in our proofs of properties **I** and **II**. On the other hand, Fourier transform alone does not suffice in proving **III**, since our main condition (1.7) is stated in the coordinate space and its equivalent interpretation in Fourier space is obscure.

Property **I** follows by the method:[22] we prove a uniform bound for the covariance of μ_t and refer to the Prokhorov Theorem. Property **II** is deduced from an analysis of the oscillatory integral representation of the correlation function in Fourier space. An important role is attributed to Lemma 3.1 reflecting the properties of the Fourier transformed correlation functions which is derived from the mixing condition. To prove **III** we exploit the dispersive properties of the dynamics (1.1) in coordinate space. The dispersion follows from a stationary phase method applied to the oscillatory integral representation of the Green's function in Fourier space. The dispersion allows us to represent the solution as a sum of weakly dependent random variables by the Bernstein-type "room-corridor" partition.

Let us explain in more detail the main idea for the proof of **III**. First let us consider the case $n=1$ and the nearest neighbor crystal for which the potential energy has the form

$$\frac{1}{2}\sum_{x,y\in\mathbf{Z}^d}(V(x-y)u(y),u(x))=\frac{1}{2}\sum_{x\in\mathbf{Z}^d}\left(\sum_{i=1}^{d}|u(x+e_i)-u(x)|^2+m^2|u(x)|^2\right),\qquad(1.12)$$

where $m\geqslant 0$ and $e_i=(\delta_{i1},\ldots,\delta_{id})$. The solution is represented through the Green's function, $\mathcal{G}(t,x)$,

$$Y(x,t)=\sum_{y\in\mathbf{Z}^d}\mathcal{G}(t,x-y)Y_0(y).\qquad(1.13)$$

The long-time asymptotics of the Green's function is analyzed by the stationary phase method based on the dispersion relation

$$\omega(\theta):=\hat{V}^{1/2}(\theta)=\left(2\sum_{j=1}^{d}(1-\cos\theta_j)+m^2\right)^{1/2},\quad \theta\in\mathbf{T}^d,\qquad(1.14)$$

where $\mathbf{T}^d$ is the real d-torus and $\hat{V}(\theta)$ stands for the Fourier transform of $V(x)$. The main features of ω for $m>0$ are

$$\text{(i)}\ \ \omega(\theta)\neq 0,\ \ \theta\in\mathbf{T}^d,\quad \text{and}\quad \text{(ii)}\ \ \mathrm{mes}\,\mathcal{C}=0,\qquad(1.15)$$

where $\mathcal{C}$ is the *critical set* $\{\theta\in\mathbf{T}^d:\det\mathrm{Hess}\,\omega(\theta)=0\}$ and "mes" stands for the Lebesgue measure in $\mathbf{T}^d$. The Green's function has distinct asymptotic behavior in three zones of (x,t)-space: inside, resp., outside the light cone and in the "buffer zone," which is a small conical neighborhood of the boundary of the light cone. The light cone is determined by the group velocities $\nabla\omega(\theta)$ of the phonons, and its boundary is determined by the group velocities $\nabla\omega(\theta)$ with "critical" $\theta\in\mathcal{C}$, since they correspond to the maximal values of $|\nabla\omega(\theta)|$ with a fixed direction of $\nabla\omega(\theta)$ [cf. (1.16)]. Therefore, the buffer zone is determined by the velocities $\nabla\omega(\theta)$ with the θ from a small neighborhood of the critical set $\mathcal{C}$. The Green's function decays rapidly outside the light cone, as $t^{-d/2}$ inside the light cone except for the buffer zone, and more slowly in the buffer zone; cf. (1.18).

Now let us discuss the general case when $n\geqslant 1$. For $n>1$ an additional important feature occurs. In this case we have n dispersion relations $\omega_k(\theta)$, $k=1,\ldots,n$, which are the eigenvalues of the matrix $\hat{V}^{1/2}(\theta)$. Thus there can be "crossing points" where two or more dispersion relations $\omega_k(\theta)$ coincide which implies that they are not differentiable, in general. In this case the decay of the Green's function generally is slower than $t^{-d/2}$ everywhere in (x,t)-space. We estimate the decay by the stationary phase method, hence we need smooth branches of the dispersion relations $\omega_k(\theta)$ at least locally in θ. We establish the existence of the branches outside a set of the Lebesgue measure zero in $\mathbf{T}^d$ (see Lemma 2.2). For the proof we use the advanced variant of the Weierstrass Preparation Theorem from Ref. 15 and the analytic stratification of analytic sets.[12]

For $n\geqslant 1$ we define the critical set $\mathcal{C}$ as the subset of $\mathbf{T}^d$ which is the union over $k=1,\ldots,n$ of all the points θ either with a nondifferentiable $\omega_k(\theta)$, or with a degenerate Hessian of $\omega_k(\theta)$, or with $\omega_k(\theta)=0$. Lemmas 2.2, 2.3 imply that mes $\mathcal{C}=0$ which plays the central role in all proofs in the paper. The critical set is never empty. For example, let us fix $k=1,\ldots,n$ and consider the point $\theta\in\mathbf{T}^d$ with the maximal group velocity $|\nabla\omega_k(\theta)|>0$. Then $\det\mathrm{Hess}\,\omega_k(\theta)=0$ since $\mathrm{Hess}\,\omega_k(\theta)\,\nabla\omega_k(\theta)=0$:

$$(\mathrm{Hess}\,\omega_k(\theta)\,\nabla\omega_k(\theta))_i=\sum_j\frac{\partial^2\omega_k(\theta)}{\partial\theta_i\,\partial\theta_j}\,\frac{\partial\omega_k(\theta)}{\partial\theta_j}=\frac{1}{2}\,\frac{\partial}{\partial\theta_i}\sum_j\left|\frac{\partial\omega_k(\theta)}{\partial\theta_j}\right|^2=0,\quad i=1,\ldots,d,\qquad(1.16)$$

provided the derivatives exist. Thus even for $d=n=1$ the *uniform in* $x\in\mathbf{R}$ decay of the Green's function is slower than $t^{-1/2}$ since $\omega''(\theta)$ vanishes in some points. To overcome this difficulty, in Ref. 2 it is required that $\omega'''(\theta)\neq 0$ at points with $\omega''(\theta)=0$. Then the uniform decay of the Green's function is $t^{-1/3}$ which suffices in the case $d=1$ together with an additional assumption on the higher moments of the initial measure. In contrast, the critical set and the slow decay of the Green's function do not occur for the Klein–Gordon equation analyzed in Refs. 4, 6.

For $d,n \geqslant 1$ Suhov and Shuhov have proved in Ref. 19 the convergence of the covariance, (1.9), for a *simple singularity* of $\omega_k(\theta)$ (in Arnold's terminology[1]) in the points $\theta \in \mathcal{C}$ with the degenerate Hessian. However, a similar detailed analysis of all degenerate points for $d,n \geqslant 1$ seems to be impossible. We avoid it by a novel "cutoff" strategy which allows us to cover the general case when the Lebesgue measure of the critical set $\mathcal{C}$ is zero. Namely, we choose an $\varepsilon > 0$ and split the Fourier transform of the solution in two components $\hat{Y}(\theta,t) = \hat{Y}_f(\theta,t) + \hat{Y}_g(\theta,t)$ where $\hat{Y}_f(\theta,t) = 0$ outside the ε-neighborhood of the critical set $\mathcal{C}$ while $\hat{Y}_g(\theta,t) = 0$ inside the $\varepsilon/2$-neighborhood of $\mathcal{C}$. First, we use the mixing condition to estimate the contribution from the "critical" component $\hat{Y}_f$: we prove that it is small in the mean, i.e., its dispersion is negligible uniformly in $t \geqslant 0$, if $\varepsilon > 0$ is sufficiently small. This follows from the identity mes $\mathcal{C} = 0$ since the Fourier transforms of the initial correlation functions are absolutely continuous due to the mixing condition. A further step is to develop a Bernstein type argument to prove the Gaussian limit for the main "noncritical" component Y_g. We write it in the form (1.13):

$$Y_g(x,t) = \sum_{y \in \mathbf{Z}^d} \mathcal{G}_g(t,x-y) Y_0(y), \tag{1.17}$$

where $\mathcal{G}_g(t,x-y)$ is the "truncated" Green's function which is defined similarly to $Y_g(x,t)$: its Fourier transform $\hat{\mathcal{G}}_g(t,\theta)$ is zero inside the $\varepsilon/2$-neighborhood of $\mathcal{C}$. Then all the dispersion relations $\omega_k(\theta)$ are smooth and nondegenerate on the support of $\hat{\mathcal{G}}_g(t,\theta)$, hence the truncated Green's function has the standard decay,

$$\mathcal{G}_g(t,x-y) \leqslant \begin{cases} Ct^{-d/2}, & |y-x| \leqslant ct, \\ C_p(|t|+|x-y|+1)^{-p}, & |y-x| \geqslant ct, \end{cases} \tag{1.18}$$

with some $c>0$ and any $p>0$; cf. (5.2), (5.3). Therefore, the representation (1.17) demonstrates that for a fixed $x \in \mathbf{Z}^d$, the main contribution to $Y_g(x,t)$ comes from the section $B_t(x) = \{y \in \mathbf{Z}^d : |y-x| \leqslant ct\}$ of the light cone at time t. The "volume" of the section [i.e., the number of the points $y \in \mathbf{Z}^d \cap B_t(x)$] is $|B_t(x)| \sim t^d$. Therefore, (1.17) becomes, roughly speaking,

$$Y_g(x,t) \sim \frac{\sum_{y \in B_t(x)} Y_0(y)}{\sqrt{|B_t|}}, \quad t \to \infty. \tag{1.19}$$

This implies the Gaussian limit by the Ibragimov–Linnik Central Limit Theorem,[13] since the random values $Y_0(y)$ are weakly dependent because of the mixing condition (1.7).

Remarks 1.1: (i) Physically, the asymptotics (1.18) reflects the isotropic propagation of phonons in the noncritical spectrum. The isotropy provides a "dynamical mixing" which leads to the Gaussian behavior by the statistical mixing condition (1.7). So the convergence to the statistical equilibrium (1.8) is provided by both kinds of the mixing simultaneously: the statistical mixing condition (1.7) and the dynamical mixing (1.18).

(ii) The degree $-d/2$ in (1.18) is related to the energy conservation since the Hamiltonian (1.4) is a quadratic form. Roughly speaking, (1.18) means the "energy diffusion," and the degree $-d/2$ resembles the diffusion kernel.

Finally, let us comment on our conditions concerning the interaction matrix $V(x)$. We assume the conditions **E1–E4** below which in a similar form appear also in Refs. 2, 14. **E1** means the exponential space-decay of the interaction in the crystal. **E2**, resp. **E3**, means that the potential energy is real, resp. non-negative. **E4** eliminates the constant part of the spectrum and ensures that mes $\mathcal{C} = 0$ [cf. (1.15)]. We also introduce a new simple condition **E5** for the case $n>1$ which eliminates the *discrete* part of the spectrum for the covariance dynamics. It can be considerably weakened to the condition **E5**$'$ from Remark 2.10 (iii). For example, the condition **E5**$'$ holds for the canonical Gaussian measures which are considered in Ref. 14. We show that the conditions **E4** and **E5** hold for "almost all" matrix-functions $V(\cdot)$ with the finite range of the interaction.

Furthermore, we do not require that $\omega_k(\theta)\neq 0$, $\theta\in\mathbf{T}^d$: note that $\omega(0)=0$ for the elastic lattice (1.14) in the case $m=0$. Our results hold whenever $\mathrm{mes}\{\theta\in\mathbf{T}^d:\omega_k(\theta)=0\}=0$. To cover this case we impose the new condition **ES** which is roughly speaking necessary and sufficient for the uniform bounds of the covariance. It can be simplified to the stronger condition

$$\|\hat{V}^{-1}(\theta)\|\in L^1(\mathbf{T}^d), \tag{1.20}$$

from Ref. 14, which holds for the elastic lattice (1.14) if either $d\geqslant 3$ or $m>0$. The condition (1.20) is equivalent to **ES** for the canonical Gibbs measures considered in Ref. 14. However, (1.20) does not hold in some particular interesting cases: for instance, for the elastic lattice (1.14) in the case $d=1,2$ and $m=0$, as it is pointed out in Ref. 14.

The main results of our paper are stated in Sec. II: Theorem A in Sec. II D, and its application in Sec. II E 4. The convergence (1.9) and the compactness **I** are established in Sec. III, and the convergence (1.10) in Secs. IV–VIII. Section IX concerns the ergodicity and the mixing properties of the limit measure. In the Appendix we analyze the crossing points of the dispersion relations.

II. MAIN RESULTS

A. Dynamics

We assume that the initial date Y_0 belongs to the phase space $\mathcal{H}_\alpha$, $\alpha\in\mathbf{R}$, defined below.

Definition 2.1: $\mathcal{H}_\alpha$ is the Hilbert space of pairs $Y=(u(x),v(x))$ of $\mathbf{R}^n$-valued functions of $x\in\mathbf{Z}^d$ endowed with the norm

$$\|Y\|_\alpha^2=\sum_{x\in\mathbf{Z}^d}(|u(x)|^2+|v(x)|^2)(1+|x|^2)^\alpha<\infty. \tag{2.1}$$

We impose the following conditions **E1–E5** on the matrix V.

E1 There exist constants $C,\alpha>0$ such that $|V_{kl}(z)|\leqslant Ce^{-\alpha|z|}$, $k,l\in \mathrm{I}_n:=\{1,\ldots,n\}$, $z\in\mathbf{Z}^d$.

Let us denote by $\hat{V}(\theta):=(\hat{V}_{kl}(\theta))_{k,\,l\in\mathrm{I}_n}$, where $\hat{V}_{kl}(\theta)\equiv\Sigma_{z\in\mathbf{Z}^d}V_{kl}(z)e^{iz\theta}$, $\theta\in\mathbf{T}^d$, and $\mathbf{T}^d$ denotes the d-torus $\mathbf{T}^d=\mathbf{R}^d/2\pi\mathbf{Z}^d$.

E2 V is real and symmetric, i.e., $V_{lk}(-z)=V_{kl}(z)\in\mathbf{R}$, $k,l\in\mathrm{I}_n$, $z\in\mathbf{Z}^d$.

The condition implies that $\hat{V}(\theta)$ is a real-analytic Hermitian matrix-function in $\theta\in\mathbf{T}^d$.

E3 The matrix $\hat{V}(\theta)$ is non-negative definite for each $\theta\in\mathbf{T}^d$.

The condition means that Eq. (1.1) is a hyperbolic like wave and Klein–Gordon equations considered in Refs. 6–8. Let us define the Hermitian non-negative definite matrix,

$$\Omega(\theta):=(\hat{V}(\theta))^{1/2}\geqslant 0, \tag{2.2}$$

with the eigenvalues $\omega_k(\theta)\geqslant 0$, $k\in\mathrm{I}_n$, the dispersion relations. For each $\theta\in\mathbf{T}^d$ the Hermitian matrix $\Omega(\theta)$ has the diagonal form in the basis of the orthogonal eigenvectors $\{e_k(\theta):k\in\mathrm{I}_n\}$:

$$\Omega(\theta)=B(\theta)\begin{pmatrix}\omega_1(\theta) & \cdots & 0\\ 0 & \ddots & 0\\ 0 & \cdots & \omega_n(\theta)\end{pmatrix}B^*(\theta), \tag{2.3}$$

where $B(\theta)$ is a unitary matrix. It is well known that the functions $\omega_k(\theta)$ and $B(\theta)$ are real-analytic outside the set of the "crossing" points θ_*: $\omega_k(\theta_*)=\omega_l(\theta_*)$ for some $l\neq k$. However, generally the functions are not smooth at the crossing points if $\omega_k(\theta)\not\equiv\omega_l(\theta)$. Therefore, we need the following lemma which we prove in the Appendix (cf. Ref. 21, Lemma 1.1).

Lemma 2.2: Let the conditions **E1**, **E2** *hold. Then there exists a closed subset* $\mathcal{C}_* \subset \mathbf{T}^d$ *such that we have the following:*

(i) the Lebesgue measure of $\mathcal{C}_*$ *is zero:*

$$\mathrm{mes}\mathcal{C}_* = 0. \tag{2.4}$$

(ii) *For any point* $\Theta \in \mathbf{T}^d \setminus \mathcal{C}_*$ *there exists a neighborhood* $\mathcal{O}(\Theta)$ *such that each dispersion relation* $\omega_k(\theta)$ *and the matrix* $B(\theta)$ *can be chosen as the real-analytic functions in* $\mathcal{O}(\Theta)$.

(iii) *The eigenvalues* $\omega_k(\theta)$ *have constant multiplicity in* $\mathbf{T}^d \setminus \mathcal{C}_*$, *i.e., it is possible to enumerate them so that we have for* $\theta \in \mathbf{T}^d \setminus \mathcal{C}_*$,

$$\omega_1(\theta) \equiv \cdots \equiv \omega_{r_1}(\theta), \;\; \omega_{r_1+1}(\theta) \equiv \cdots \equiv \omega_{r_2}(\theta), \;\; \ldots, \;\; \omega_{r_s+1}(\theta) \equiv \cdots \equiv \omega_n(\theta), \tag{2.5}$$

$$\omega_{r_\sigma}(\theta) \not\equiv \omega_{r_\nu}(\theta) \;\; \text{if} \;\; \sigma \neq \nu, \;\; 1 \leqslant r_\sigma, r_\nu \leqslant r_{s+1} := n. \tag{2.6}$$

(iv) *The spectral decomposition holds,*

$$\Omega(\theta) = \sum_1^{s+1} \omega_{r_\sigma}(\theta) \Pi_\sigma(\theta), \quad \theta \in \mathbf{T}^d \setminus \mathcal{C}_*, \tag{2.7}$$

where $\Pi_\sigma(\theta)$ *is the orthogonal projection in* $\mathbf{R}^n$ *which is real-analytic function of* $\theta \in \mathbf{T}^d \setminus \mathcal{C}_*$.

Below we denote by $\omega_k(\theta)$ the local real-analytic functions from Lemma 2.2 (ii). Our next condition is the following:

E4 $D_k(\theta) \not\equiv 0$, $\forall k \in \mathrm{I}_n$, where $D_k(\theta) := \det(\partial^2 \omega_k(\theta)/\partial\theta_i \, \partial\theta_j)_{i,j=1}^d$, $\theta \in \mathbf{T}^d \setminus \mathcal{C}_*$.

Let us denote $\mathcal{C}_0 := \{\theta \in \mathbf{T}^d : \det \hat{V}(\theta) = 0\}$ and $\mathcal{C}_k := \{\theta \in \mathbf{T}^d \setminus \mathcal{C}_* : D_k(\theta) = 0\}$, $k \in \mathrm{I}_n$. The following lemma is also proved in the Appendix.

Lemma 2.3: Let the conditions **E1–E4** *hold. Then* $\mathrm{mes}\,\mathcal{C}_k = 0$ *for* $k = 0,1,\ldots,n$.

Our last condition on V is the following:

E5 For each $k \neq l$ the identity $\omega_k(\theta) - \omega_l(\theta) \equiv \mathrm{const}_-$, $\theta \in \mathbf{T}^d$ does not hold with $\mathrm{const}_- \neq 0$, and the identity $\omega_k(\theta) + \omega_l(\theta) \equiv \mathrm{const}_+$ does not hold with $\mathrm{const}_+ \neq 0$.

This condition holds trivially in the case $n = 1$.

We show that the conditions **E4** and **E5** hold for "almost all" functions V satisfying the conditions **E1**, **E2**. More precisely, let us fix an arbitrary $N \geqslant 1$ and denote by $\mathcal{R}_N$ the set of the "finite range" interaction matrices V with $V(x) = 0$ for $\max_i |x_i| > N$, and satisfying the condition **E2**. In the Appendix we prove the following lemma.

Lemma 2.4: For any $N \geqslant 1$ *the conditions* **E4** *and* **E5** *hold for the matrix-functions V from an open and dense subset of* $\mathcal{R}_N$.

The following proposition is proved in Ref. 14, p. 150 and Ref. 2, p. 128.

Proposition 2.5: Let **E1** *and* **E2** *hold, and* $\alpha \in \mathbf{R}$. *Then*

(i) *for any* $Y_0 \in \mathcal{H}_\alpha$ *there exists a unique solution* $Y(t) \in C(\mathbf{R}, \mathcal{H}_\alpha)$ *to the Cauchy problem (1.2).*

(ii) *The operator* $U(t): Y_0 \mapsto Y(t)$ *is continuous in* $\mathcal{H}_\alpha$.

Proof: Applying the Fourier transform to (1.2), we obtain

$$\dot{\hat{Y}}(\theta,t) = \hat{\mathcal{A}}(\theta) \hat{Y}(\theta,t), \;\; t \in \mathbf{R}, \quad \hat{Y}(0) = \hat{Y}_0, \tag{2.8}$$

where

$$\hat{\mathcal{A}}(\theta) = \begin{pmatrix} 0 & 1 \\ -\hat{V}(\theta) & 0 \end{pmatrix}, \quad \theta \in \mathbf{T}^d. \tag{2.9}$$

Note that $\hat{Y}(\cdot,t)\in D'(\mathbf{T}^d)$ for $t\in\mathbf{R}$. On the other hand, $\hat{V}(\theta)$ is a smooth function by **E1**. Therefore, the solution $\hat{Y}(\theta,t)$ of (2.8) exists, is unique and admits the representation $\hat{Y}(\theta,t)=\exp(\hat{\mathcal{A}}(\theta)t)\hat{Y}_0(\theta)$. It becomes (1.13) in the coordinate space, where the Green's function $\mathcal{G}(t,z)$ admits the Fourier representation

$$\mathcal{G}(t,z):=F^{-1}_{\theta\to z}[\exp(\hat{\mathcal{A}}(\theta)t)]=(2\pi)^{-d}\int_{\mathbf{T}^d}e^{-iz\theta}\exp(\hat{\mathcal{A}}(\theta)t)d\theta. \tag{2.10}$$

Hence, by the partial integration, $\mathcal{G}(t,z)\sim|z|^{-p}$ as $|z|\to\infty$ for any $p>0$ and bounded $|t|$ since $\hat{\mathcal{A}}(\theta)$ is the smooth function of $\theta\in\mathbf{T}^d$. Therefore, the convolution representation (1.13) implies $Y(t)\in\mathcal{H}_\alpha$. □

B. The convergence to statistical equilibrium

Let (Ω,Σ,P) be a probability space with expectation E and let $\mathcal{B}(\mathcal{H}_\alpha)$ denote the Borel σ-algebra in $\mathcal{H}_\alpha$. We assume that $Y_0=Y_0(\omega,\cdot)$ in (1.2) is a measurable random function with values in $(\mathcal{H}_\alpha,\mathcal{B}(\mathcal{H}_\alpha))$. In other words, for each $x\in\mathbf{Z}^d$ the map $\omega\mapsto Y_0(\omega,x)$ is a measurable map $\Omega\to\mathbf{R}^{2n}$ with respect to the (completed) σ-algebras Σ and $\mathcal{B}(\mathbf{R}^{2n})$. Then $Y(t)=U(t)Y_0$ is again a measurable random function with values in $(\mathcal{H}_\alpha,\mathcal{B}(\mathcal{H}_\alpha))$ owing to Proposition 2.5. We denote by $\mu_0(dY_0)$ a Borel probability measure on $\mathcal{H}_\alpha$ giving the distribution of the Y_0. Without loss of generality, we assume $(\Omega,\Sigma,P)=(\mathcal{H}_\alpha,\mathcal{B}(\mathcal{H}_\alpha),\mu_0)$ and $Y_0(\omega,x)=\omega(x)$ for $\mu_0(d\omega)$—almost all $\omega\in\mathcal{H}_\alpha$ and each $x\in\mathbf{Z}^d$.

Definition 2.6: μ_t is a Borel probability measure in $\mathcal{H}_\alpha$ which gives the distribution of Y(t):

$$\mu_t(B)=\mu_0(U(-t)B),\quad \forall B\in\mathcal{B}(\mathcal{H}_\alpha),\ \ t\in\mathbf{R}. \tag{2.11}$$

Our main goal is to derive the convergence of the measures μ_t as $t\to\infty$. We establish the weak convergence of μ_t in the Hilbert spaces $\mathcal{H}_\alpha$ with $\alpha<-d/2$:

$$\mu_t \overset{\mathcal{H}_\alpha}{\rightharpoonup} \mu_\infty \quad\text{as}\quad t\to\infty, \tag{2.12}$$

where μ_∞ is a limit measure on the space $\mathcal{H}_\alpha$, $\alpha<-d/2$. This means the convergence

$$\int f(Y)\mu_t(dY)\to\int f(Y)\mu_\infty(dY),\quad t\to\infty, \tag{2.13}$$

for any bounded continuous functional f on $\mathcal{H}_\alpha$.

Definition 2.7: The correlation functions of the measure μ_t are defined by

$$Q_t^{ij}(x,y)=E(Y^i(x,t)\otimes Y^j(y,t)),\quad i,j=0,1,\quad x,y\in\mathbf{Z}^d, \tag{2.14}$$

if the expectations on the rhs are finite. Here $Y^i(x,t)$ are the components of the random solution $Y(t)=(Y^0(\cdot,t),Y^1(\cdot,t))$.

For a probability measure μ on $\mathcal{H}_\alpha$ we denote by $\hat{\mu}$ the characteristic functional (Fourier transform),

$$\hat{\mu}(\Psi)=\int \exp(i\langle Y,\Psi\rangle)\,\mu(dY),\quad \Psi\in\mathcal{D}.$$

A measure μ is called Gaussian (of zero mean) if its characteristic functional has the form

$$\hat{\mu}(\Psi)=\exp\{-\tfrac{1}{2}\mathcal{Q}(\Psi,\Psi)\},\quad \Psi\in\mathcal{D}, \tag{2.15}$$

where $\mathcal{Q}$ is a real non-negative quadratic form in $\mathcal{D}$. A measure μ is called translation-invariant if $\mu(T_hB)=\mu(B)$, $B\in\mathcal{B}(\mathcal{H}_\alpha)$, $h\in\mathbf{Z}^d$, where $T_hY(x)=Y(x-h)$, $x\in\mathbf{Z}^d$.

C. The mixing condition

Let $O(r)$ denote the set of all pairs of the subsets $\mathcal{A}, \mathcal{B} \subset \mathbf{Z}^d$ at distance $\mathrm{dist}(\mathcal{A},\mathcal{B}) \geq r$ and $\sigma(\mathcal{A})$ be the σ-algebra in $\mathcal{H}_\alpha$ generated by $Y(x)$ with $x \in \mathcal{A}$. Define the Ibragimov–Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{H}_\alpha$ by (cf. Ref. 13, Definition 17.2.2)

$$\varphi(r) := \sup_{(\mathcal{A},\mathcal{B}) \in O(r)} \sup_{\substack{A \in \sigma(\mathcal{A}), B \in \sigma(\mathcal{B}) \\ \mu_0(B) > 0}} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}. \tag{2.16}$$

Definition 2.8: The measure μ_0 satisfies a strong, uniform Ibragimov–Linnik mixing condition if $\varphi(r) \to 0$ as $r \to \infty$.

Below, we specify the rate of decay of φ (see condition **S3**).

D. Statistical conditions and results

We assume that the initial measure μ_0 satisfies the following conditions **S0–S3**:

S0 μ_0 has zero expectation value, $EY_0(x) \equiv 0$, $x \in \mathbf{Z}^d$.

S1 μ_0 has translation-invariant correlation matrices, i.e., Eq. (1.5) holds for $x, y \in \mathbf{Z}^d$.

S2 μ_0 has a finite mean energy density, i.e., Eq. (1.6) holds.

S3 μ_0 satisfies the strong uniform Ibragimov–Linnik mixing condition with

$$\overline{\varphi} := \int_0^{+\infty} r^{d-1} \varphi^{1/2}(r)\, dr < \infty. \tag{2.17}$$

We will deduce from **S0–S3** that $\hat{q}_0^{ij} \in C(\mathbf{T}^d)$, $i,j = 0,1$ (see Lemma 3.1). This makes sense of our last condition **ES** concerning the initial covariance and the matrix $\Omega(\theta)$. We need it only in the case when $\mathcal{C}_0 \neq \emptyset$, i.e., $\det V(\theta) = 0$ for some points $\theta \in \mathbf{T}^d$:

ES $\|\Omega^{-i}(\theta)\hat{q}_0^{ij}(\theta)\Omega^{-j}(\theta)\| \in L^1(\mathbf{T}^d)$ for $i,j = 0,1$.

This condition follows from **S0–S3** if $i = j = 0$ or $\mathcal{C}_0 = \emptyset$.

Next introduce the correlation matrix of the limit measure μ_∞. It is translation-invariant [cf. (1.5)]:

$$Q_\infty(x,y) = (q_\infty^{ij}(x-y))_{i,j=0,1}. \tag{2.18}$$

In the Fourier transform we have locally outside the critical set $\mathcal{C}_*$ (see Lemma 2.2),

$$\hat{q}_\infty^{ij}(\theta) = B(\theta) M_\infty^{ij}(\theta) B^*(\theta), \quad i,j = 0,1, \tag{2.19}$$

where $B(\theta)$ is the smooth unitary matrix from Lemma 2.2 (ii) and $M_\infty^{ij}(\theta)$ is an $n \times n$-matrix with the smooth entries $(M_\infty^{ij}(\theta))_{kl} = \chi_{kl}(B^*(\theta) M_0^{ij}(\theta) B(\theta))_{kl}$. Here we set [see (2.5)]

$$\chi_{kl} = \begin{cases} 1, & \text{if } \ k,l \in (r_{\sigma-1}, r_\sigma], \ \ \sigma = 1, \ldots, s+1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.20}$$

with $r_0 := 0$, $r_{s+1} := n$, and

$$M_0(\theta) := \frac{1}{2} \begin{pmatrix} \hat{q}_0^{00}(\theta) + \Omega^{-1}(\theta)\, \hat{q}_0^{11}(\theta)\, \Omega^{-1}(\theta) & \hat{q}_0^{01}(\theta) - \Omega^{-1}(\theta)\, \hat{q}_0^{10}(\theta)\, \Omega(\theta) \\ \hat{q}_0^{10}(\theta) - \Omega\, \hat{q}_0^{01}(\theta)\, \Omega^{-1}(\theta) & \hat{q}_0^{11}(\theta) + \Omega(\theta)\, \hat{q}_0^{00}(\theta)\, \Omega(\theta) \end{pmatrix}. \tag{2.21}$$

The local representation (2.19) can be expressed globally as

$$\hat{q}_\infty^{ij}(\theta) = \sum_{\sigma=1}^{s+1} \Pi_\sigma(\theta) M_0^{ij}(\theta) \Pi_\sigma(\theta), \quad \theta \in \mathbf{T}^d \setminus \mathcal{C}_*, \qquad i,j = 0,1, \tag{2.22}$$

where $\Pi_\sigma(\theta)$ is the spectral projection from (2.7).

Remark 2.9: The condition **ES** implies that $(M_0^{ij})_{kl}\in L^1(\mathbf{T}^d)$, $k,l\in \mathrm{I}_n$. Therefore, (2.22) and (2.4) imply that also $(\hat{q}_\infty^{ij})_{kl}\in L^1(\mathbf{T}^d)$, $k,l\in \mathrm{I}_n$.

Theorem A: *Let $d,n\geqslant 1$, $\alpha<-d/2$ and assume that the conditions* **E1–E5**, **S0–S3** *hold. If $\mathcal{C}_0\neq\emptyset$, then we assume also that* **ES** *holds. Then*

(i) the convergence in (2.12) holds.
(ii) The limit measure μ_∞ is a Gaussian translation-invariant measure on $\mathcal{H}_\alpha$.
(iii) The characteristic functional of μ_∞ is the Gaussian,

$$\hat{\mu}_\infty(\Psi)=\exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\},\quad \Psi\in\mathcal{D}, \tag{2.23}$$

where $\mathcal{Q}_\infty$ is the quadratic form defined in (1.11).
(iv) The measure μ_∞ is invariant, i.e., $[U(t)]^*\mu_\infty=\mu_\infty$, $t\in\mathbf{R}$.

Remarks 2.10: (i) In the case $n=1$, the formulas (2.21), (2.22) become

$$\hat{q}_\infty=M_0=\frac{1}{2}\begin{pmatrix}\hat{q}_0^{00}+\omega^{-2}\hat{q}_0^{11} & \hat{q}_0^{01}-\hat{q}_0^{10}\\ \hat{q}_0^{10}-\hat{q}_0^{01} & \hat{q}_0^{11}+\omega^2\hat{q}_0^{00}\end{pmatrix}.$$

(ii) The *uniform* Rosenblatt mixing condition[18] also suffices, together with a higher power >2 in the bound (1.6): there exists $\delta>0$ such that

$$E(|u_0(x)|^{2+\delta}+|v_0(x)|^{2+\delta})<\infty.$$

Then (2.17) requires a modification: $\int_0^{+\infty}r^{d-1}\alpha^p(r)dr<\infty$, with $p=\min(\delta/(2+\delta),1/2)$, where $\alpha(r)$ is the Rosenblatt mixing coefficient defined as in (2.16) but without $\mu_0(B)$ in the denominator. With these modifications, the statements of Theorem A and their proofs remain essentially unchanged.

(iii) The arguments with condition **E5** in Proposition 3.2 [see (3.7)–(3.13) below] demonstrate that the condition could be considerably weakened. Namely, it suffices to assume **E5**$'$. If for some $k\neq l$ we have either $\omega_k(\theta)+\omega_l(\theta)\equiv\mathrm{const}_+\neq 0$ or $\omega_k(\theta)-\omega_l(\theta)\equiv\mathrm{const}_-\neq 0$, then

$$(B^*(\theta)\hat{q}_0^{ij}(\theta)B(\theta))_{kl}=0,\quad \theta\in\mathbf{T}^d,\ i,j=0,1. \tag{2.24}$$

The assertions (i)–(iii) of Theorem A follow from Propositions 2.11 and 2.12.

Proposition 2.11: The family of the measures $\{\mu_t, t\in\mathbf{R}\}$ is weakly compact in $\mathcal{H}_\alpha$ with any $\alpha<-d/2$, and the bounds hold:

$$\sup_{t\geqslant 0}E\|U(t)Y_0\|_\alpha^2<\infty. \tag{2.25}$$

Proposition 2.12: For every $\Psi\in\mathcal{D}$, the convergence (1.10) holds.

Proposition 2.11 ensures the existence of the limit measures of the family $\{\mu_t, t\in\mathbf{R}\}$, while Proposition 2.12 provides the uniqueness. Propositions 2.11 and 2.12 are proved in Secs. III and IV–VIII, respectively.

Theorem A (iv) follows from (2.12) since the group $U(t)$ is continuous in $\mathcal{H}_\alpha$ by Proposition 2.5 (ii).

E. Examples and applications

Let us give the examples of the equations (1.1) and measures μ_0 which satisfy all conditions **E1–E5**, **S0–S3**, and **ES**.

1. Harmonic crystals

All conditions **E1–E5** hold for a one-dimensional (1-D) crystal with $n=1$ considered in Ref. 2. For any $d\geqslant 1$ and $n=1$ consider the simple elastic lattice corresponding to the quadratic form (1.12) with $m\neq 0$. Then $V(x)=F^{-1}_{\theta\to x}\omega^2(\theta)$ with $\omega(\theta)$ defined by (1.14), satisfies **E1–E4** with $\mathcal{C}_*=\emptyset$. In these examples the set $\mathcal{C}_0$ is empty, hence the condition **ES** is superfluous. Condition **E5** holds trivially since $n=1$.

2. Gaussian measures

We consider $n=1$ and construct Gaussian initial measures μ_0 satisfying **S0–S3**. We will define μ_0 by the correlation functions $q_0^{ij}(x-y)$ which are zero for $i\neq j$, while for $i=0,1$,

$$\hat{q}_0^{ii}(\theta):=F_{z\to\theta}q_0^{ii}(z)\in L^1(\mathbf{T}^d),\quad \hat{q}_0^{ii}(\theta)\geqslant 0. \tag{2.26}$$

Then by the Minlos theorem[11] there exists a unique Borel Gaussian measure μ_0 on $\mathcal{H}_\alpha$, $\alpha< -d/2$, with the correlation functions $q_0^{ij}(x-y)$. The measure μ_0 satisfies **S0–S2**. Further, let us provide, in addition to (2.26), that

$$q_0^{ii}(z)=0,\quad |z|\geqslant r_0. \tag{2.27}$$

Then the mixing condition **S3** follows with $\varphi(r)=0$, $r\geqslant r_0$, since for Gaussian random values the orthogonality implies the independence. For example, (2.26) and (2.27) hold if we set $q_0^{ii}(z) =f(z_1)f(z_2)\cdot\cdots\cdot f(z_d)$, where $f(z)=\nu_0-|z|$ for $|z|\leqslant\nu_0$ and $f(z)=0$ for $|z|\geqslant\nu_0$ with $\nu_0 :=[r_0/\sqrt{d}]$ (the integer part). Then by the direct calculation we obtain $\hat{f}(\theta)=(1-\cos\nu_0\theta)/(1-\cos\theta)$, $\theta\in\mathbf{T}^1$, and (2.26) holds. The measure μ_0 is nontrivial if $r_0\geqslant\sqrt{d}$: otherwise $\nu_0=0$, so $q_0^{ij}(z)\equiv 0$, and the measure $\mu_0(dY_0)$ is concentrated at the point $Y_0=0$.

3. Non-Gaussian measures

Let us choose some odd bounded nonconstant functions f^0, $f^1\in C(\mathbf{R})$ and consider a random function $(Y^0(x),Y^1(x))$ with the Gaussian distribution μ_0 from the previous example. Let us define μ_0^* as the distribution of the random function $(f^0(Y^0(x)), f^1(Y^1(x))$. Then **S0–S3** hold for μ_0^* with corresponding mixing coefficients $\varphi^*(r)=0$ for $r\geqslant r_0$. The measure μ_0^* is not Gaussian if the functions f^0, f^1 are bounded and nonconstant.

4. From statistical chaos to the Gibbs measure

Let us consider the initial measures which satisfy **S0–S3**, and with the correlation functions

$$(q_0^{ij})_{kl}(x-y):=E(Y_k^i(x,0)Y_l^j(y,0))=T_i\delta_{ij}\delta_{kl}\delta_{xy},\quad i,j=0,1,\ \ k,l\in\mathrm{I}_n,\quad x,y\in\mathbf{Z}^d, \tag{2.28}$$

where $T_{0,1}\geqslant 0$. These correlations correspond to the "chaos" with the zero correlation radius and uncorrelated components. Such measures exist on $\mathcal{H}_\alpha$ with $\alpha<-d/2$ by the Minlos Theorem:[11] for example, the "white noise" which is the corresponding Gaussian measure. Let us consider the crystal satisfying the conditions **E1–E4** and (1.20). Then also the conditions **E5**$'$, **ES** hold, so Theorem A is applicable [see Remark 2.10 (iii)]: it implies the convergence (2.12) to the Gaussian measure μ_∞ with the covariance (2.21), (2.22).

Additionally, let us assume that $T_0=0$ which physically means that only the initial velocities contribute, and initial deviations are adjusted to zero. Then the formulas (2.21), (2.22) become

$$\hat{q}_\infty(\theta)=M_0(\theta)=\frac{T_1}{2}\begin{pmatrix}\hat{V}^{-1}(\theta) & 0\\ 0 & (\delta_{kl})_{k,l\in\mathrm{I}_n}\end{pmatrix}. \tag{2.29}$$

According to (1.3), this means that the limit measure μ_∞ coincides with the *Gibbs canonical measure* corresponding to the temperature $\sim T_1$. In a more general framework, the limit measure is close to the Gibbs measure if the radius of the initial correlations is small in a suitable scaling limit (cf. Ref. 6, Proposition 4.2).

III. CONVERGENCE OF COVARIANCE AND COMPACTNESS

A. Mixing condition in terms of spectral density

The next Lemma reflects the mixing property in the Fourier transforms $\hat{q}_0^{ij}$ of initial correlation functions q_0^{ij}. Condition **S2** implies that $q_0^{ij}(z)$ is a bounded function. Therefore, its Fourier transform generally belongs to the Schwartz space of tempered distributions.

Lemma 3.1: Let the conditions **S0–S3** *hold. Then* $\hat{q}_0^{ij}\in C(\mathbf{T}^d)$, $i,j=0,1$.

Proof: It suffices to prove that

$$q_0^{ij}(z)\in l^1(\mathbf{Z}^d). \tag{3.1}$$

Conditions **S0–S3** imply by Ref. 13, Lemma 17.2.3 [or Lemma 8.2 (i) below]:

$$|q_0^{ij}(z)|\leqslant Ce_0\varphi^{1/2}(|z|),\quad z\in\mathbf{Z}^d, \tag{3.2}$$

where e_0 is defined by (1.6). Therefore, (2.17) implies (3.1):

$$\sum_{z\in\mathbf{Z}^d}|q_0^{ij}(z)|\leqslant Ce_0\sum_{z\in\mathbf{Z}^d}\varphi^{1/2}(|z|)<\infty.$$

□

B. Oscillatory integral arguments

In this section we uniformly estimate and check the convergence of the correlation matrices of measures μ_t with the help of the Fourier transform. The condition **S1** and the translation-invariant dynamics (1.1) imply that

$$Q_t^{ij}(x,y)\equiv\int Y^i(x)\otimes Y^j(y)\mu_t(dY)=q_t^{ij}(x-y),\quad x,y\in\mathbf{Z}^d. \tag{3.3}$$

Proposition 3.2: (i) The correlation matrices $q_t^{ij}(z)$, $i,j=0,1$, *are uniformly bounded*,

$$\sup_{t\geqslant 0}\sup_{z\in\mathbf{Z}^d}|q_t^{ij}(z)|<\infty. \tag{3.4}$$

(ii) The correlation matrices $q_t^{ij}(z)$, $i,j=0,1$, *converge for each* $z\in\mathbf{Z}^d$, *and*

$$q_t^{ij}(z)\to q_\infty^{ij}(z),\qquad t\to\infty, \tag{3.5}$$

where the functions $q_\infty^{ij}(z)$ *are defined above.*

Proof: For brevity, we prove (3.4) and (3.5) for $i=j=0$. In all other cases the proof of (3.5) is similar. The solution to the Cauchy problem (1.1) is

$$u(x,t)=(2\pi)^{-d}\int_{\mathbf{T}^d}e^{-ix\cdot\theta}(\cos\Omega t\ \hat{Y}_0^0(\theta)+\sin\Omega t\ \Omega^{-1}\hat{Y}_0^1(\theta))d\theta,$$

where $\Omega\equiv\Omega(\theta)$ is the non-negative definite Hermitian matrix defined by (2.2). Furthermore, the translation invariance (1.5) implies that

$$E(\hat{Y}_0^i(\theta)\otimes\hat{Y}_0^j(\theta'))=(2\pi)^d\delta(\theta+\theta')\hat{q}_0^{ij}(\theta),\quad i,j=0,1. \tag{3.6}$$

Hence,

$$q_t^{00}(x-y) := E(u(x,t)\otimes u(y,t))$$
$$= (2\pi)^{-d}\int_{\mathbf{T}^d} e^{-i\theta(x-y)}[\cos\Omega t\ \ \hat{q}_0^{00}(\theta)\cos\Omega t$$
$$+\sin\Omega t\ \ \Omega^{-1}\hat{q}_0^{10}(\theta)\cos\Omega t + \cos\Omega t\ \ \hat{q}_0^{01}(\theta)\Omega^{-1}\sin\Omega t$$
$$+\sin\Omega t\ \ \Omega^{-1}\hat{q}_0^{11}(\theta)\Omega^{-1}\sin\Omega t]d\theta. \tag{3.7}$$

Therefore, the bound (3.4) with $i=j=0$ follows from Lemma 3.1 or condition **ES** if $\mathcal{C}_0 \neq \emptyset$.

Let us check that the convergence (3.5) with $i=j=0$ also follows since the oscillatory integrals in (3.7) tend to zero. Consider for example the last term in the integrand of (3.7). We rewrite it using (2.3), in the form

$$L_0^{11}(\theta,t) := \sin\Omega t\ \ \Omega^{-1}\hat{q}_0^{11}(\theta)\ \ \Omega^{-1}\sin\Omega t = B(\theta)(\sin\omega_k t\ \ A_{kl}^{11}(\theta)\sin\omega_l t)_{k,l\in \mathrm{I}_n}\ B^*(\theta), \tag{3.8}$$

where $A^{11}(\theta) := B^*(\theta)\Omega^{-1}\hat{q}_0^{11}(\theta)\Omega^{-1}B(\theta)$. However, at this moment we have to choose certain smooth branches of the functions $B(\theta)$ and $\omega_k(\theta)$ since we are going to apply the stationary phase arguments which require a smoothness in θ. To make it correctly, we cut off all singularities. First, we define the combined *critical set*,

$$\mathcal{C} := \cup_k \mathcal{C}_k \cup \mathcal{C}_* \cup \mathcal{C}_0. \tag{3.9}$$

Then Lemmas 2.2, 2.3 imply the following lemma.

Lemma 3.3: Let conditions **E1–E4** *hold. Then* $\mathrm{mes}\,\mathcal{C}=0$.

Second, fix an $\varepsilon>0$ and choose a finite partition of unity,

$$f(\theta)+g(\theta)=1,\quad g(\theta)=\sum_{m=1}^{M} g_m(\theta),\quad \theta\in\mathbf{T}^d, \tag{3.10}$$

where f, g_m are non-negative functions from $C_0^\infty(\mathbf{T}^d)$, the supports of g_m are sufficiently small and

$$\mathrm{supp}\ \ f\subset\{\theta\in\mathbf{T}^d\colon \mathrm{dist}(\theta,\mathcal{C})<\varepsilon\},\ \ \mathrm{supp} g_m\subset\{\theta\in\mathbf{T}^d\colon \mathrm{dist}(\theta,\mathcal{C})\geq\varepsilon/2\}. \tag{3.11}$$

Now (3.8) can be rewritten as

$$L_0^{11}(\theta,t) = f(\theta)L_0^{11}(\theta,t) + \frac{1}{2}\sum_m\ g_m(\theta)B(\theta)((\cos(\omega_k-\omega_l)t$$
$$-\cos(\omega_k+\omega_l)t)A_{kl}^{11}(\theta))_{k,l\in\mathrm{I}_n}\ B^*(\theta). \tag{3.12}$$

By Lemma 2.2 and the compactness arguments, we can choose the supports of g_m so small that the eigenvalues $\omega_k(\theta)$ and the matrix $B(\theta)$ are real-analytic functions inside the suppg_m for every m: we do not mark the functions by the index m to not overburden the notations.

Let us substitute (3.12) into the last term of (3.7) and analyze the Fourier integrals with f and g_m separately. The integral with f converges to zero uniformly in $t\geq 0$, as $\varepsilon\to 0$. Indeed, by Lemma 3.3 we have

$$\left|\int_{\mathbf{T}^d} e^{-i\theta(x-y)} f(\theta)L_0^{11}(\theta,t)\,d\theta\right| \leq C\int_{\mathrm{dist}(\theta,\mathcal{C})<\varepsilon} \|\Omega^{-1}(\theta)\hat{q}_0^{11}(\theta)\Omega^{-1}(\theta)\|\,d\theta \to 0,\quad \varepsilon\to 0,$$

since the integrand is summable by Lemma 3.1 or condition **ES** if $\mathcal{C}_0\neq\emptyset$.

Below we will prove the convergence for the integrals with g_m. We will deduce the convergence from the fact that the identities $\omega_k(\theta)\pm\omega_l(\theta)\equiv\mathrm{const}_\pm$ with the $\mathrm{const}_\pm\neq 0$ are impossible by the condition **E5**. Furthermore, the oscillatory integrals with $\omega_k(\theta)\pm\omega_l(\theta)\not\equiv\mathrm{const}$ vanish as $t\to\infty$. Hence, only the integrals with $\omega_k(\theta)-\omega_l(\theta)\equiv 0$ contribute to the limit since $\omega_k(\theta)+\omega_l(\theta)\equiv 0$ would imply $\omega_k(\theta)\equiv\omega_l(\theta)\equiv 0$ which is impossible by **E4**. A similar analysis of the three remaining terms in the integrand of (3.7) gives

$$\begin{aligned} q_t^{00}(x-y) &= (2\pi)^{-d}\int_{\mathbf{T}^d} e^{-i\theta(x-y)} f(\theta) L_0^{11}(\theta,t)\,d\theta + (2\pi)^{-d}\sum_m \int_{\mathbf{T}^d} g_m(\theta) \\ &\quad\times e^{-i\theta(x-y)}\left[\frac{1}{2}B(\theta)(\chi_{kl}(A_{kl}^{00}(\theta)+A_{kl}^{11}(\theta)))_{k,l\in \mathrm{I}_n}B^*(\theta)+\cdots\right]d\theta \\ &= (2\pi)^{-d}\int_{\mathbf{T}^d} e^{-i\theta(x-y)} f(\theta) L_0^{11}(\theta,t)\,d\theta + (2\pi)^{-d}\int_{\mathbf{T}^d} g(\theta) e^{-i\theta(x-y)}\hat{q}_\infty^{00}(\theta)\,d\theta + \cdots, \end{aligned} \tag{3.13}$$

according to the notations (2.18)–(2.21). Here $A^{00}(\theta):=B^*(\theta)\hat{q}_0^{00}(\theta)B(\theta)$ and "$\cdots$" stands for the oscillatory integrals which contain $\cos(\omega_k(\theta)\pm\omega_l(\theta))t$ and $\sin(\omega_k(\theta)\pm\omega_l(\theta))t$ with $\omega_k(\theta)\pm\omega_l(\theta)\not\equiv\mathrm{const}$.

The oscillatory integrals converge to zero by the Lebesgue–Riemann Theorem since all the integrands in "$\cdots$" are summable and $\nabla(\omega_k(\theta)\pm\omega_l(\theta))=0$ only on the set of the Lebesgue measure zero. The summability follows from Lemma 3.1 or the condition **ES** since the matrices $B^*(\theta)$ are unitary. The zero measure follows similarly to (2.4) since $\omega_k(\theta)\pm\omega_l(\theta)\not\equiv\mathrm{const}$.

At last, let us prove the convergence (3.5) with $i=j=0$. From the last line of (3.13) we know that $q_t^{00}(x-y)$ is close to the integral with g if ε is small and t is large. Therefore, the limit of $q_t^{00}(x-y)$ as $t\to\infty$ coincides with the limit of the integral as $\varepsilon\to 0$. Finally, this limit coincides with $q_\infty^{00}(x-y)$ since $\hat{q}_\infty^{00}\in L^1(\mathbf{T}^d)$ by Remark 2.9. □

C. Compactness of measures family

Proof of Proposition 2.11: The compactness of the measures family $\{\mu_t, t\in\mathbf{R}\}$ will follow from the bounds (2.25) by the Prokhorov Theorem (Ref. 22, Lemma II.3.1) using the method of Ref. 22, Theorem XII.5.2 since the embedding $\mathcal{H}_\alpha\subset\mathcal{H}_\beta$ is compact if $\alpha>\beta$.

First, the translation invariance (3.3) and Proposition 3.2 (i) imply that for $x\in\mathbf{Z}^d$ we have

$$e_t := \int [|u_0(x)|^2+|v_0(x)|^2]\,\mu_t(dY_0) = \mathrm{tr}\,q_t^{00}(0)+\mathrm{tr}\,q_t^{11}(0)\leqslant\overline{e}<\infty,\quad t\geqslant 0. \tag{3.14}$$

Hence by the definition (2.1) we get for any $\alpha<-d/2$:

$$E\|U(t)Y_0\|_\alpha^2 = e_t\sum_{x\in\mathbf{Z}^d}(1+|x|^2)^\alpha = C(\alpha)e_t\leqslant C(\alpha)\overline{e}<\infty,\quad t\geqslant 0.$$

□

IV. DUALITY ARGUMENT

To prove Theorem A, it remains to check Proposition 2.12. Let us rewrite (1.10) as follows:

$$E\exp\{i\langle Y(t),\Psi\rangle\}\to\hat{\mu}_\infty(\Psi),\quad t\to\infty. \tag{4.1}$$

We will prove it in Secs. V–VIII. In this section we evaluate $\langle Y(t),\Psi\rangle$ by using the following duality arguments. Remember that $Y_0\in\mathcal{H}_\alpha$ with $\alpha<-d/2$. For $t\in\mathbf{R}$ introduce a "formal adjoint" operator $U'(t)$ from space $\mathcal{D}$ to $\mathcal{H}_{-\alpha}$:

$$\langle Y, U'(t)\Psi\rangle = \langle U(t)Y, \Psi\rangle, \quad \Psi \in \mathcal{D}, \quad Y \in \mathcal{H}_\alpha. \tag{4.2}$$

Let us denote by $\Phi(\cdot,t) = U'(t)\Psi$. Then (4.2) can be rewritten as

$$\langle Y(t), \Psi\rangle = \langle Y_0, \Phi(\cdot,t)\rangle, \quad t \in \mathbf{R}. \tag{4.3}$$

The adjoint group $U'(t)$ admits the following convenient description. Lemma 4.1 below display that the action of group $U'(t)$ coincides with the action of $U(t)$, up to the order of the components.

Lemma 4.1: For $\Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$ *we have*

$$\Phi(\cdot,t) := U'(t)\Psi = (\dot\psi(\cdot,t), \psi(\cdot,t)), \tag{4.4}$$

where $\psi(x,t)$ *is the solution of Eq. (1.1) with the initial data* $(u_0, v_0) = (\Psi^1, \Psi^0)$.

Proof: Differentiating (4.2) in t with $Y, \Psi \in \mathcal{D}$, we obtain that $\langle Y, \dot U'(t)\Psi\rangle = \langle \dot U(t)Y, \Psi\rangle$. The group $U(t)$ has the generator $\mathcal{A}$ from (1.3). The generator of $U'(t)$ is the conjugate operator to $\mathcal{A}$:

$$\mathcal{A}' = \begin{pmatrix} 0 & -\mathcal{V} \\ 1 & 0 \end{pmatrix}. \tag{4.5}$$

Hence, the representation (4.4) holds with $\ddot\psi(x,t) = -\sum_{y\in\mathbf{Z}^d} V(x-y)\psi(y,t)$. □

The lemma allows us to construct the oscillatory integral representation for $\Phi(x,t)$. Namely, (4.4), (4.5) imply that in the Fourier representation for $\Phi(\cdot,t) = U'(t)\Psi$ we have

$$\dot{\hat\Phi}(\theta,t) = \hat{\mathcal{A}}^*(\theta)\hat\Phi(\theta,t), \quad \hat\Phi(\theta,t) = \hat{\mathcal{G}}^*(t,\theta)\hat\Psi(\theta).$$

Here we denote [see (2.9)]

$$\hat{\mathcal{A}}^*(\theta) = \begin{pmatrix} 0 & -\hat V(\theta) \\ 1 & 0 \end{pmatrix}, \quad \hat{\mathcal{G}}^*(t,\theta) = e^{\hat{\mathcal{A}}^*(\theta)t} = \begin{pmatrix} \cos\Omega t & -\Omega\sin\Omega t \\ \Omega^{-1}\sin\Omega t & \cos\Omega t \end{pmatrix}, \tag{4.6}$$

with $\Omega \equiv \Omega(\theta) = \Omega^*(\theta)$. Therefore,

$$\Phi(x,t) = (2\pi)^{-d}\int_{\mathbf{T}^d} e^{-i\theta x}\hat{\mathcal{G}}^*(t,\theta)\hat\Psi(\theta)\,d\theta, \quad x \in \mathbf{Z}^d. \tag{4.7}$$

Since $f(\theta) + g(\theta) \equiv 1$ by (3.10), we can split Φ in two components:

$$\begin{aligned}\Phi(x,t) &= (2\pi)^{-d}\int_{\mathbf{T}^d} e^{-i\theta x}\hat{\mathcal{G}}^*(t,\theta)f(\theta)\hat\Psi(\theta)\,d\theta + (2\pi)^{-d}\int_{\mathbf{T}^d} e^{-i\theta x}\hat{\mathcal{G}}^*(t,\theta)g(\theta)\hat\Psi(\theta)\,d\theta \\ &= \Phi_f(x,t) + \Phi_g(x,t), \quad x \in \mathbf{Z}^d,\end{aligned} \tag{4.8}$$

where each function $\Phi_f(x,t)$ and $\Phi_g(x,t)$ admits the representation of type (4.4). By (3.11), the Fourier spectrum of Φ_f is concentrated near the critical set $\mathcal{C}$, while the spectrum of Φ_g is separated from $\mathcal{C}$.

V. STANDARD DECAY IN THE NONCRITICAL SPECTRUM

We prove the decay of type (1.18) for the "noncritical" component Φ_g. The function Φ_g can be expanded similarly to (3.12), in the form

$$\Phi_g(x,t) = \sum_m \sum_{\pm,\ k\in I_n} \int_{\mathbf{T}^d} g_m(\theta)e^{-i(\theta x \pm \omega_k(\theta)t)}a_k^{\pm}(\theta)\hat\Psi(\theta)\,d\theta. \tag{5.1}$$

By Lemma 2.2 and the compactness arguments, we can choose the eigenvalues $\omega_k(\theta)$ and the matrices $a_k^{\pm}(\theta)$ as real-analytic functions inside the suppg_m for every m: we do not mark the functions by the index m to not overburden the notations.

Lemma 4.1 means that each component $\Phi_g^i(x,t)$, $i=0,1$, is a solution to Eq. (1.1). To prove (4.1), we analyze the radiative properties of $\Phi_g(x,t)$ in all directions. For this purpose, we apply the stationary phase method to the oscillatory integral (5.1) along the rays $x=vt$, $t>0$. Then the phase becomes $(\theta v \pm \omega_k(\theta))t$, and its stationary points are the solutions to the equations $v = \mp \nabla \omega_k(\theta)$. We collect all necessary asymptotics in the following lemma [cf. (1.18)].

Lemma 5.1: For any fixed $\Psi \in \mathcal{D}$ *and* $g(\theta) \in C_0^\infty(\mathbf{T}^d \setminus \mathcal{C})$ *the following bounds hold:*

(i)

$$\sup_{x \in \mathbf{Z}^d} |\Phi_g(x,t)| \leqslant C t^{-d/2}. \tag{5.2}$$

(ii) *or any* $p>0$ *there exist* $C_p, \gamma_g > 0$ *such that*

$$|\Phi_g(x,t)| \leqslant C_p(|t|+|x|+1)^{-p}, \quad |x| \geqslant \gamma_g t. \tag{5.3}$$

Proof: Consider $\Phi_g(x,t)$ along each ray $x=vt$ with arbitrary $v \in \mathbf{R}^d$. Substituting to (5.1), we get

$$\Phi_g(vt,t) = \sum_m \sum_{\pm,\ k \in \mathrm{I}_n} \int_{\mathbf{T}^d} g_m(\theta) e^{-i(\theta v \pm \omega_k(\theta))t} a_k^{\pm}(\theta) \hat{\Psi}(\theta) d\theta. \tag{5.4}$$

This is a sum of oscillatory integrals with the phase functions $\phi_k^{\pm}(\theta) = \theta v \pm \omega_k(\theta)$ and the amplitudes $a_k^{\pm}(\theta)$ which are real-analytic functions of the θ inside the suppg_m. Since $\omega_k(\theta)$ is real-analytic, each function $\phi_k^{\pm}$ has no more than a finite number of stationary points $\theta \in \mathrm{supp} g_m$, solutions to the equation $v = \mp \nabla \omega_k(\theta)$. The stationary points are nondegenerate for $\theta \in \mathrm{supp} g_m$ by (3.11), (3.9), and **E4** since

$$\det\left(\frac{\partial^2 \phi_k^{\pm}}{\partial \theta_i\, \partial \theta_j} \right) = \pm D_k(\theta) \neq 0, \quad \theta \in \mathrm{supp} g_m. \tag{5.5}$$

At last, $\hat{\Psi}(\theta)$ is smooth since $\Psi \in \mathcal{D}$. Therefore, $\Phi_g(vt,t) = \mathcal{O}(t^{-d/2})$ according to the standard stationary phase method.[10,17] This implies the bounds (5.2) in each cone $|x| \leqslant ct$ with any finite c.

Further, denote by $\overline{v_g} := \max_m \max_{k \in \mathrm{I}_n} \max_{\theta \in \mathrm{supp} g_m} |\nabla \omega_k(\theta)|$. Then for $|v| > \overline{v_g}$ the stationary points do not exist on the supp$g$. Hence, the integration by parts as in Ref. 17 yields $\Phi_g(vt,t) = \mathcal{O}(t^{-p})$ for any $p>0$. On the other hand, the integration by parts in (5.1) implies similar bound $\Phi_g(x,t) = \mathcal{O}((t/|x|)^l)$ for any $l>0$. Therefore, (5.3) follows with any $\gamma_g > \overline{v_g}$. Now the bounds (5.2) follow everywhere. □

VI. CONTRIBUTION OF CRITICAL SET

We are going to prove (4.1). Rewrite it using (4.3):

$$E \exp\{i \langle Y_0, \Phi(\cdot,t) \rangle\} - \hat{\mu}_\infty(\Psi) \to 0, \quad t \to \infty. \tag{6.1}$$

The splitting (4.8) gives $\langle Y_0, \Phi(\cdot,t)\rangle = \langle Y_0, \Phi_f(\cdot,t)\rangle + \langle Y_0, \Phi_g(\cdot,t)\rangle$. Our main argument is that the contribution of $\langle Y_0, \Phi_f(\cdot,t)\rangle$ to (6.1) has a small dispersion. We will deduce this from Lemmas 3.1, 3.3. At first, let us estimate the difference in (6.1) by the triangle inequality:

$$\begin{aligned} |E \exp\{i\langle Y_0, \Phi(\cdot,t)\rangle\} - \hat{\mu}_\infty(\Psi)| \leqslant &|E \exp\{i\langle U(t) Y_0, \Psi\rangle\} - E \exp\{i\langle Y_0, \Phi_g(\cdot,t)\rangle\}| + |\hat{\mu}_\infty(\Psi_g) \\ &- \hat{\mu}_\infty(\Psi)| + |E \exp\{i\langle Y_0, \Phi_g(\cdot,t)\rangle\} - \hat{\mu}_\infty(\Psi_g)| = I + II + III, \end{aligned} \tag{6.2}$$

where $\Psi_g := F^{-1}[g(\theta)\hat{\Psi}(\theta)] = \Phi_g(\cdot,0)$. Let us consider each of the three terms separately.

I. The first term $I=I(\varepsilon,t)$ represents the contribution of the neighborhood of the critical set $\{\theta\in\mathbf{T}^d: \mathrm{dist}(\theta,\mathcal{C})<\varepsilon\}$ and tends to zero as $\varepsilon\to 0$ uniformly in $t\geqslant 0$. Namely, by the Cauchy–Schwartz inequality,

$$I=|Ee^{i\langle Y_0,\Phi(\cdot,t)\rangle}-Ee^{i\langle Y_0,\Phi_g(\cdot,t)\rangle}|\leqslant E|e^{i\langle Y_0,\Phi_f(\cdot,t)\rangle}-1|\leqslant C(E|\langle Y_0,\Phi_f(\cdot,t)\rangle|^2)^{1/2}. \tag{6.3}$$

Using the Parseval identity and (4.8), we get

$$E|\langle Y_0,\Phi_f(\cdot,t)\rangle|^2=(2\pi)^{-2d}E|\langle\hat{Y}_0(\theta),f(\theta)\hat{\Phi}(\theta,t)\rangle|^2$$
$$=(2\pi)^{-2d}\langle E(\hat{Y}_0(\theta)\otimes\overline{\hat{Y}_0(\theta')}),\ f(\theta)f(\theta')\,\hat{\mathcal{G}}^*(t,\theta)\hat{\Psi}(\theta)\otimes\overline{\hat{\mathcal{G}}^*(t,\theta')\,\hat{\Psi}(\theta')}\,\rangle. \tag{6.4}$$

Now take into account that $E(\hat{Y}_0(\theta)\otimes\overline{\hat{Y}_0(\theta')})=(2\pi)^d\delta(\theta-\theta')\hat{q}_0(\theta)$ similarly to (3.6). Then (6.4), (4.6), (3.11) and the bounds $0\leqslant f(\theta)\leqslant 1$ imply

$$E|\langle Y_0,\Phi_f(\cdot,t)\rangle|^2\leqslant C_1\sum_{i,j=0,1}\int_{\mathrm{dist}(\theta,\mathcal{C})<\varepsilon}\|\Omega^{-i}(\theta)\hat{q}_0^{ij}(\theta)\Omega^{-j}(\theta)\|\,d\theta\to 0,\quad \varepsilon\to 0,$$

owing to Lemma 3.3 since the integrand is summable. The summability follows from Lemma 3.1 or condition **ES** if $\mathcal{C}_0\neq\emptyset$.

II. The second term $II=II(\varepsilon)$ tends to zero as $\varepsilon\to 0$. Indeed,

$$\mathcal{Q}_\infty(\Psi_g,\Psi_g)=(2\pi)^{-2d}\sum_{i,j=0}^{1}\int_{\mathbf{T}^d}(\hat{q}_\infty^{ij}(\theta),g(\theta)\hat{\Psi}^i(\theta)\otimes g(\theta)\overline{\hat{\Psi}^j(\theta)})\,d\theta\to\mathcal{Q}_\infty(\Psi,\Psi),$$
$$\varepsilon\to 0,$$

by the Lebesgue Dominated Convergence Theorem since $0\leqslant g(\theta)\leqslant 1$ and $\hat{q}_\infty^{ij}\in L^1(\mathbf{T}^d)$ by Remark 2.9. Hence for the Gaussian measure μ_∞, we get by (2.23),

$$|\hat{\mu}_\infty(\Psi_g)-\hat{\mu}_\infty(\Psi)|=|\exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\Psi_g,\Psi_g)\}-\exp\{-\tfrac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\}|\to 0,\quad \varepsilon\to 0.$$

III. To prove Proposition 2.12, it remains to check that for any fixed $\varepsilon>0$, we have

$$III(\varepsilon,t)=|E\exp\{i\langle Y_0,\Phi_g(\cdot,t)\rangle\}-\hat{\mu}_\infty(\Psi_g)|\to 0,\quad t\to\infty. \tag{6.5}$$

We prove (6.5) in Sec. VIII using the Bernstein arguments of the next section.

VII. BERNSTEIN'S "ROOMS-CORRIDORS" PARTITION

Our proof of (6.5) is similar to the case of the continuous Klein–Gordon equation in $\mathbf{R}^d$:[6] all the integrals over $\mathbf{R}^d$ become the series over $\mathbf{Z}^d$, etc. Another novelty in the proofs is the following: in the case of the Klein–Gordon equation we have $\Phi(x,t)=0$ for $|x|\geqslant t+c(\Psi)$, while for the discrete crystal we have (5.3) instead.

Let us introduce a "room-corridor" partition of the ball $\{x\in\mathbf{Z}^d:\ |x|\leqslant\gamma_g t\}$ with γ_g from (5.3). For $t>0$ we choose below $\Delta_t,\rho_t\in\mathbf{N}$ (we will specify the asymptotic relations between t, Δ_t, and ρ_t). Let us set $h_t=\Delta_t+\rho_t$ and

$$a^j=jh_t,\quad b^j=a^j+\Delta_t,\quad j\in\mathbf{Z},\quad N_t=[(\gamma_g t)/h_t]. \tag{7.1}$$

We call the slabs $R_t^j=\{x\in\mathbf{Z}^d:|x|\leqslant N_t h_t,\ a^j\leqslant x_d<b^j\}$ the "rooms," $C_t^j=\{x\in\mathbf{Z}^d:|x|\leqslant N_t h_t,\ b^j\leqslant x_d<a^{j+1}\}$ the "corridors" and $L_t=\{x\in\mathbf{Z}^d:|x|>N_t h_t\}$ the "tails." Here $x=(x_1,\ldots,x_d)$, Δ_t is the width of a room, and ρ_t is that of a corridor. Let us denote by χ_t^j the indicator of the room R_t^j, ξ_t^j that of the corridor C_t^j, and η_t that of the tail L_t. Then

$$\sum_t [\chi_t^j(x)+\xi_t^j(x)]+\eta_t(x)=1,\ x\in\mathbf{Z}^d, \tag{7.2}$$

where the sum Σ_t stands for $\Sigma_{j=-N_t}^{N_t-1}$. Hence we get the following Bernstein's type representation:

$$\langle Y_0,\Phi_g(\cdot,t)\rangle=\sum_t [\langle Y_0,\chi_t^j\Phi_g(\cdot,t)\rangle+\langle Y_0,\xi_t^j\Phi_g(\cdot,t)\rangle]+\langle Y_0,\eta_t\Phi_g(\cdot,t)\rangle. \tag{7.3}$$

Let us introduce the random variables r_t^j, c_t^j, l_t by

$$r_t^j=\langle Y_0,\chi_t^j\Phi_g(\cdot,t)\rangle,\quad c_t^j=\langle Y_0,\xi_t^j\Phi_g(\cdot,t)\rangle,\ l_t=\langle Y_0,\eta_t\Phi_g(\cdot,t)\rangle. \tag{7.4}$$

Then (7.3) becomes

$$\langle Y_0,\Phi_g(\cdot,t)\rangle=\sum_t (r_t^j+c_t^j)+l_t. \tag{7.5}$$

Lemma 7.1: Let **S0–S3** *hold. The following bounds hold for* $t>1$:

$$E|r_t^j|^2\leqslant C(\Psi_g)\ \Delta_t/t,\ \ \forall j, \tag{7.6}$$

$$E|c_t^j|^2\leqslant C(\Psi_g)\ \ \rho_t/t,\ \ \forall j, \tag{7.7}$$

$$E|l_t|^2\leqslant C_p(\Psi_g)(1+t)^{-p},\ \ \forall p>0. \tag{7.8}$$

Proof: We discuss (7.6), and (7.7), (7.8) can be done in a similar way [the proof of (7.8) additionally uses (5.3)]. Express $E|r_t^j|^2$ in the correlation matrices. Definition (7.4) implies that

$$E|r_t^j|^2=\langle\chi_t^j(x)\chi_t^j(y)q_0(x-y),\Phi_g(x,t)\otimes\Phi_g(y,t)\rangle. \tag{7.9}$$

According to (5.2), Eq. (7.9) implies that

$$E|r_t^j|^2\leqslant Ct^{-d}\sum_{x,y}\chi_t^j(x)\|q_0(x-y)\|=Ct^{-d}\sum_x\chi_t^j(x)\ \sum_z\|q_0(z)\|\leqslant C\Delta_t/t, \tag{7.10}$$

where $\|q_0(z)\|$ stands for the norm of a matrix $(q_0^{ij}(z))$. Therefore, (7.10) follows as $\|q_0(\cdot)\|\in l^1(\mathbf{Z}^d)$ by (3.1). □

VIII. IBRAGIMOV–LINNIK CENTRAL LIMIT THEOREM

In this section we prove the convergence (6.5). As was said, we use a version of the Central Limit Theorem developed by Ibragimov and Linnik.[13] If $\mathcal{Q}_\infty(\Psi_g,\Psi_g)=0$, (6.5) is obvious. Indeed, $|E\exp(i\langle Y_0,\Phi_g(\cdot,t)\rangle)-1|\leqslant E|\langle Y_0,\Phi_g(\cdot,t)\rangle|\leqslant(E\langle Y_0,\Phi_g(\cdot,t)\rangle^2)^{1/2}=(\mathcal{Q}_t(\Psi_g,\Psi_g))^{1/2}$, where $\mathcal{Q}_t(\Psi_g,\Psi_g)\to\mathcal{Q}_\infty(\Psi_g,\Psi_g)=0$, as $t\to\infty$. Thus, we may assume that for a given $\Psi\in\mathcal{D}$,

$$\mathcal{Q}_\infty(\Psi_g,\Psi_g)\neq 0. \tag{8.1}$$

Let us choose $0<\delta<1$ and

$$\rho_t\sim t^{1-\delta},\qquad \Delta_t\sim\frac{t}{\log t},\qquad t\to\infty. \tag{8.2}$$

Lemma 8.1: The following limit holds true:

$$N_t\left(\varphi(\rho_t)+\left(\frac{\rho_t}{t}\right)^{1/2}\right)+N_t^2\left(\varphi^{1/2}(\rho_t)+\frac{\rho_t}{t}\right)\to 0,\quad t\to\infty. \tag{8.3}$$

Proof: The function $\varphi(r)$ is nonincreasing; hence by (2.17),

$$r^d\varphi^{1/2}(r)=d\int_0^r s^{d-1}\varphi^{1/2}(r)\,ds\le d\int_0^r s^{d-1}\varphi^{1/2}(s)\,ds\le C\overline{\varphi}<\infty. \tag{8.4}$$

Furthermore, (8.2) implies that $h_t=\Delta_t+\rho_t\sim t/\log t$, $t\to\infty$. Therefore, $N_t\sim t/h_t\sim\log t$. Then (8.3) follows by (8.4) and (8.2). □

Proof of (6.5): By the triangle inequality,

$$\begin{aligned}|E\exp\{i\langle Y_0,\Phi_g(\cdot,t)\rangle\}-\hat{\mu}_\infty(\Psi_g)|&\le\left|E\exp\{i\langle Y_0,\Phi_g(\cdot,t)\rangle\}-E\exp\left\{i\sum_t r_t^j\right\}\right|\\&\quad+\left|\exp\left\{-\frac{1}{2}\sum_t E|r_t^j|^2\right\}-\exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi_g,\Psi_g)\right\}\right|\\&\quad+\left|E\exp\left\{i\sum_t r_t^j\right\}-\exp\left\{-\frac{1}{2}\sum_t E|r_t^j|^2\right\}\right|\\&\equiv I_1+I_2+I_3.\end{aligned} \tag{8.5}$$

We are going to show that all the summands I_1, I_2, I_3 tend to zero as $t\to\infty$.

Step (i): Equation (7.5) implies

$$I_1=\left|E\exp\left\{i\sum_t r_t^j\right\}\left(\exp\left\{i\sum_t c_t^j+il_t\right\}-1\right)\right|\le C\sum_t E|c_t^j|+E|l_t|\le C\sum_t (E|c_t^j|^2)^{1/2}+(E|l_t|^2)^{1/2}. \tag{8.6}$$

From (8.6), (7.7), (7.8), and (8.3) we obtain that

$$I_1\le C_p t^{-p}+CN_t(\rho_t/t)^{1/2}\to 0,\quad t\to\infty. \tag{8.7}$$

Step (ii): By the triangle inequality,

$$\begin{aligned}I_2&\le\frac{1}{2}\left|\sum_t E|r_t^j|^2-\mathcal{Q}_\infty(\Psi_g,\Psi_g)\right|\\&\le\frac{1}{2}|\mathcal{Q}_t(\Psi_g,\Psi_g)-\mathcal{Q}_\infty(\Psi_g,\Psi_g)|\\&\quad+\frac{1}{2}\left|E\left(\sum_t r_t^j\right)^2-\sum_t E|r_t^j|^2\right|+\frac{1}{2}\left|E\left(\sum_t r_t^j\right)^2-\mathcal{Q}_t(\Psi_g,\Psi_g)\right|\\&\equiv I_{21}+I_{22}+I_{23},\end{aligned} \tag{8.8}$$

where $\mathcal{Q}_t$ is the quadratic form with the matrix kernel $(Q_t^{ij}(x,y))$. (3.5) implies that $I_{21}\to 0$. As for I_{22}, we first obtain that

$$I_{22}\le\sum_{\substack{j\ne k\\ |j|,|k|\le N_t}}|Er_t^j r_t^k|. \tag{8.9}$$

The next lemma is the corollary of Ref. 13, Lemma 17.2.3.

Lemma 8.2: Let $\mathcal{A}$, $\mathcal{B}$ be the subsets of $\mathbf{Z}^d$ with the distance $\operatorname{dist}(\mathcal{A},\mathcal{B})\geqslant r>0$, *and let ξ, η be random variables on the probability space $(\mathcal{H}_\alpha,\mathcal{B}(\mathcal{H}_\alpha),\mu_0)$. Let ξ be measurable with respect to the σ-algebra $\sigma(\mathcal{A})$, and η with respect to the σ-algebra $\sigma(\mathcal{B})$. Then*

(i) $|E\xi\eta-E\xi E\eta|\leqslant Cab\ \varphi^{1/2}(r)$ *if* $(E|\xi|^2)^{1/2}\leqslant a$ *and* $(E|\eta|^2)^{1/2}\leqslant b$;

(ii) $|E\xi\eta-E\xi E\eta|\leqslant Cab\ \varphi(r)$ *if* $|\xi|\leqslant a$ *and* $|\eta|\leqslant b$, *a.s.*

We apply Lemma 8.2 to deduce that $I_{22}\to 0$ as $t\to\infty$. Note that $r_t^j=\langle Y_0(x),\chi_t^j(x)\Phi_g(\cdot,t)\rangle$ is measurable with respect to the σ-algebra $\sigma(R_t^j)$. The distance between the different rooms R_t^j is greater or equal to ρ_t according to (7.1). Then (8.9) and (7.6), **S3** imply by Lemma 8.2 (i), that

$$I_{22}\leqslant CN_t^2\varphi^{1/2}(\rho_t), \tag{8.10}$$

which tends to 0 as $t\to\infty$ by (8.3). Finally, it remains to check that $I_{23}\to 0$, $t\to\infty$. We have

$$\mathcal{Q}_t(\Psi_g,\Psi_g)=E\langle Y_0,\Phi_g(\cdot,t)\rangle^2=E\left(\sum_t (r_t^j+c_t^j)+l_t\right)^2,$$

according to (7.5). Therefore, by the Cauchy–Schwartz inequality,

$$\begin{aligned} I_{23}&\leqslant\left|E\left(\sum_t r_t^j\right)^2-E\left(\sum_t r_t^j+\sum_t c_t^j+l_t\right)^2\right|\\ &\leqslant CN_t\sum_t E|c_t^j|^2+C_1\left(E\left(\sum_t r_t^j\right)^2\right)^{1/2}\\ &\quad\times\left(N_t\sum_t E|c_t^j|^2+E|l_t|^2\right)^{1/2}+CE|l_t|^2. \end{aligned}\tag{8.11}$$

Then (7.6), (8.9), and (8.10) imply

$$E\left(\sum_t r_t^j\right)^2\leqslant\sum_t E|r_t^j|^2+\sum_{\substack{j\neq k\\ |j|,|k|\leqslant N_t}}|Er_t^jr_t^k|\leqslant CN_t\Delta_t/t+C_1N_t^2\varphi^{1/2}(\rho_t)\leqslant C_2<\infty.$$

Now (7.7), (7.8), (8.11), and (8.3) yield

$$I_{23}\leqslant C_1N_t^2\rho_t/t+C_2N_t(\rho_t/t)^{1/2}+C_3t^{-p}\to 0,\quad t\to\infty. \tag{8.12}$$

So, all the terms I_{21}, I_{22}, I_{23} in (8.8) tend to zero. Then (8.8) implies that

$$I_2\leqslant\frac{1}{2}\left|\sum_t E|r_t^j|^2-\mathcal{Q}_\infty(\Psi_g,\Psi_g)\right|\to 0,\quad t\to\infty. \tag{8.13}$$

Step (iii): It remains to verify that

$$I_3=\left|E\exp\left\{i\sum_t r_t^j\right\}-\exp\left\{-\frac{1}{2}E\left(\sum_t r_t^j\right)^2\right\}\right|\to 0,\quad t\to\infty. \tag{8.14}$$

Lemma 8.2, (ii) with $a=b=1$ yields

$$\begin{aligned}&\left|E\exp\left\{i\sum_t r_t^j\right\}-\prod_{-N_t}^{N_t-1}E\exp\{ir_t^j\}\right|\\&\leqslant\left|E\exp\{ir_t^{-N_t}\}\exp\left\{i\sum_{-N_t+1}^{N_t-1}r_t^j\right\}-E\exp\{ir_t^{-N_t}\}E\exp\left\{i\sum_{-N_t+1}^{N_t-1}r_t^j\right\}\right|\\&\quad+\left|E\exp\{ir_t^{-N_t}\}E\exp\left\{i\sum_{-N_t+1}^{N_t-1}r_t^j\right\}-\prod_{-N_t}^{N_t-1}E\exp\{ir_t^j\}\right|\\&\leqslant C\varphi(\rho_t)+\left|E\exp\left\{i\sum_{-N_t+1}^{N_t-1}r_t^j\right\}-\prod_{-N_t+1}^{N_t-1}E\exp\{ir_t^j\}\right|.\end{aligned}$$

Then we apply Lemma 8.2, (ii) recursively and get, according to Lemma 8.1,

$$\left|E\exp\left\{i\sum_t r_t^j\right\}-\prod_{-N_t}^{N_t-1}E\exp\{ir_t^j\}\right|\leqslant CN_t\varphi(\rho_t)\to 0,\quad t\to\infty. \tag{8.15}$$

It remains to check that

$$\left|\prod_{-N_t}^{N_t-1}E\exp\{ir_t^j\}-\exp\left\{-\frac{1}{2}\sum_t E|r_t^j|^2\right\}\right|\to 0,\quad t\to\infty. \tag{8.16}$$

According to the standard statement of the Lindeberg Central Limit Theorem (see, e.g., Ref. 16, Theorem 4.7) it suffices to verify the Lindeberg condition: $\forall\,\delta>0$,

$$\frac{1}{\sigma_t}\sum_t E_{\delta\sqrt{\sigma_t}}|r_t^j|^2\to 0,\quad t\to\infty.$$

Here $\sigma_t\equiv\Sigma_t E|r_t^j|^2$, and $E_a f:=E(X_a f)$, where X_a is the indicator of the event $|f|>a^2$. Note that (8.13) and (8.1) imply that $\sigma_t\to\mathcal{Q}_\infty(\Psi_g,\Psi_g)\neq 0$, $t\to\infty$. Hence it remains to verify that

$$\sum_t E_a|r_t^j|^2\to 0,\quad t\to\infty,\quad \text{for any } a>0.$$

This follows from the bounds for the fourth order moments as in Ref. 6, Sec. IX. This completes the proof of Proposition 2.12. □

IX. ERGODICITY AND MIXING FOR THE LIMIT MEASURES

The limit measure μ_∞ is invariant by Theorem A (iv). Let E_∞ denote the integral over μ_∞.

Theorem 9.1: *Let all assumptions of Theorem A hold for the equation (1.1) and the initial measure μ_0. Then $U(t)$ is mixing with respect to the corresponding limit measure μ_∞, i.e., $\forall f,g\in L^2(\mathcal{H}_\alpha,\mu_\infty)$,*

$$\lim_{t\to\infty}E_\infty f(U(t)Y)g(Y)=E_\infty f(Y)E_\infty g(Y). \tag{9.1}$$

In particular, the group $U(t)$ is ergodic with respect to the measure μ_∞:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(U(t)Y)dt=E_\infty f(Y)\quad (\mathrm{mod}\ \mu_\infty). \tag{9.2}$$

Proof: Since μ_∞ is Gaussian, the proof of (9.1) reduces to the proof of the following convergence: $\forall \Psi_1, \Psi_2 \in \mathcal{D}$,

$$\lim_{t\to\infty} E_\infty \langle U(t)Y, \Psi_1\rangle\langle Y, \Psi_2\rangle = 0. \tag{9.3}$$

Using the Parseval identity and (4.8) we obtain similarly to (6.4) that

$$\begin{aligned} E_\infty\langle U(t)Y,\Psi_1\rangle\langle Y,\Psi_2\rangle &= (2\pi)^{-2d}\int_{\mathbf{T}^d} (\hat{\mathcal{G}}(t,\theta)\hat{q}_\infty(\theta), f(\theta)\hat{\Psi}_1(\theta)\otimes\overline{\hat{\Psi}_2(\theta)})\, d\theta \\ &\quad + (2\pi)^{-2d}\int_{\mathbf{T}^d} (\hat{\mathcal{G}}(t,\theta)\hat{q}_\infty(\theta), g(\theta)\hat{\Psi}_1(\theta)\otimes\overline{\hat{\Psi}_2(\theta)})\, d\theta \\ &= I_f(t) + I_g(t). \end{aligned} \tag{9.4}$$

Lemma 9.2: The uniform bound holds: $\|\hat{\mathcal{G}}(t,\theta)\hat{q}_\infty(\theta)\| \leqslant G(\theta)$, $t \geqslant 0$, where $G(\theta) \in L^1(\mathbf{T}^d)$.

Proof: (4.6) implies that

$$\hat{\mathcal{G}}(t,\theta)\hat{q}_\infty(\theta) = \begin{pmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t\cdot\Omega & \cos\Omega t\cdot\Omega \end{pmatrix} \begin{pmatrix} \hat{q}_\infty^{00} & \hat{q}_\infty^{01} \\ \Omega^{-1}\hat{q}_\infty^{10} & \Omega^{-1}\hat{q}_\infty^{11} \end{pmatrix}. \tag{9.5}$$

Therefore,

$$\|\hat{\mathcal{G}}(t,\theta)\hat{q}_\infty(\theta)\| \leqslant C \sum_{i,j=0,1} \|\Omega^{-i}\hat{q}_\infty^{ij}(\theta)\|. \tag{9.6}$$

It remains to prove that $\Omega^{-i}\hat{q}_\infty^{ij}(\theta) \in L^1(\mathbf{T}^d)$. Since $\hat{q}_\infty(\theta) \in L^1(\mathbf{T}^d)$ by Remark 2.9, it suffices to verify that $\Omega^{-1}(\theta)\hat{q}_\infty^{1j}(\theta) \in L^1(\mathbf{T}^d)$, $j = 0,1$. This also follows from Remark 2.9 if $\mathcal{C}_0 = \emptyset$. Otherwise, we will use the condition **ES**. Namely, owing to (2.22), we have

$$\Omega^{-1}(\theta)\hat{q}_\infty^{ij}(\theta) = \sum_{\sigma=1}^{s+1} \Pi_\sigma(\theta)\Omega^{-1}(\theta) M_0^{ij}(\theta)\Pi_\sigma(\theta), \tag{9.7}$$

since $\Omega^{-1}(\theta)$ commutes with its spectral projection $\Pi_\sigma(\theta)$. At last, (2.21) and **ES** imply

$$\Omega^{-1}M_0^{10} = \tfrac{1}{2}(\Omega^{-1}\hat{q}_0^{10} - \hat{q}_0^{01}\Omega^{-1}) \in L^1(\mathbf{T}^d),$$

$$\Omega^{-1}M_0^{11} = \tfrac{1}{2}(\Omega^{-1}\hat{q}_0^{11} + \hat{q}_0^{00}\Omega) \in L^1(\mathbf{T}^d).$$

□

The Lemma 9.2 together with (3.11) and Lemma 3.3 imply that $\forall\delta > 0\ \exists\varepsilon > 0$ such that

$$|I_f(t)| \leqslant \delta, \qquad t \geqslant 0. \tag{9.8}$$

It remains to study the oscillatory integral $I_g(t)$. Rewrite it using (5.1), in the form

$$I_g(t) = \sum_m \sum_{\pm,\, k\in \mathrm{I}_n} \int_{\mathbf{T}^d} g_m(\theta) e^{\pm i\omega_k(\theta)t} a_k^{\pm}(\theta)(\hat{q}_\infty(\theta), \hat{\Psi}_1(\theta)\otimes\overline{\hat{\Psi}_2(\theta)})\, d\theta. \tag{9.9}$$

Here all phase functions $\omega_k(\theta)$ and the amplitudes $a_k^{\pm}(\theta)$ are smooth functions in the $\mathrm{supp}\, g_m$. Furthermore, $\nabla\omega_k(\theta) = 0$ only on the set of the Lebesgue measure zero. This follows similarly to (2.4) since $\nabla\omega_k(\theta) \not\equiv \mathrm{const}$ by the condition **E4**. Hence,

$$I_g(t) \to 0 \quad \text{as } t \to \infty, \tag{9.10}$$

by the Lebesgue–Riemann Theorem since $\hat{q}_\infty \in L^1(\mathbf{T}^d)$. Finally, (9.4)–(9.10) imply (9.3) since $\delta > 0$ is arbitrary. □

Remark: A similar result for wave and Klein–Gordon equations has been proved in Refs. 4, 5.

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APPENDIX: CROSSING POINTS

1. Proof of Lemmas 2.2 and 2.3

Step 1: By the condition **E1** the matrix $\hat{V}(\theta)$ is an analytic function in a connected open (complex) neighborhood $\mathcal{O}_c(\mathbf{T}^d)$ of $\mathbf{T}^d$ in $\mathbf{T}_c^d := \mathbf{T}^d \oplus i\mathbf{R}^d$. Consider the analytic function $d(\theta,\omega) := \det(\hat{V}(\theta) - \omega^2)$ in $\mathcal{O}_c(\mathbf{T}^d) \times \mathbf{C}$ and the analytic subset defined by the equation $d(\theta,\omega) = 0$ in $\mathcal{O}_c(\mathbf{T}^d) \times \mathbf{C}$. The subset consists of the points $(\theta, \pm\omega_k(\theta))$, $k \in \mathrm{I}_n$. It is important that $d(\theta,\omega) \not\equiv 0$ for any fixed $\theta \in \mathcal{O}_c(\mathbf{T}^d)$, hence the function d satisfies the *Weierstrass condition* of Ref. 15, Section 2.1.1. Therefore, by the Weierstrass Preparation Theorem in Ref. 15, Thm 2.1, there exists a proper analytic *discriminant subset* $\Delta \subset \mathcal{O}_c(\mathbf{T}^d)$ s.t.: for $\Theta \in \mathcal{O}_c(\mathbf{T}^d) \setminus \Delta$ there exists a (complex) neighborhood $\mathcal{O}_c(\Theta)$ of Θ in $\mathcal{O}_c(\mathbf{T}^d)$ where each of $\omega_k(\theta)$ can be chosen as a holomorphic function. More precisely, this is established in the proof of Ref. 15, Proposition 2.1 which is the main step to the proof of the Weierstrass Theorem. We set $\mathcal{C}_* := \Delta \cap \mathbf{T}^d$ and $\mathcal{O}(\Theta) = \mathcal{O}_c(\Theta) \cap \mathbf{T}^d$ for $\Theta \in \mathbf{T}^d \setminus \mathcal{C}_*$. Then Lemma 2.2 (ii) follows for $\omega_k(\theta)$.

Step 2: The identity (2.4) will follow from the next general Proposition.

Proposition 10.1: Let $\mathcal{M}$ be a proper analytic subset of $\mathcal{O}_c(\mathbf{T}^d)$. Then the Lebesgue measure of the intersection $M = \mathcal{M} \cap \mathbf{T}^d$ is zero.

Proof: Let us use the analytic stratification of the analytic sets which is constructed in Ref. 12, Thm 19 of Chapter II.E and Thm 10 of Chapter III.A. Namely, for each $\Theta \in M$ there exists a complex neighborhood $\mathcal{O}_c(\Theta)$ s.t. $\mathcal{M} \cap \mathcal{O}_c(\Theta) = \cup_{0 \leq \delta \leq d-1} \mathcal{M}_\delta$, where each $\mathcal{M}_\delta$ is an analytic submanifold of the complex dimension $\delta \leq d-1$: here we use that $\mathcal{M}$ is the proper analytic subset in $\mathcal{O}_c(\Theta)$. Now

$$M \cap \mathcal{O}_c(\Theta) = \cup_{0 \leq \delta \leq d-1} (\mathcal{M}_\delta \cap \mathbf{T}^d).$$

Lemma 10.2: Let $\Theta \in M$ and $\delta = 0, \ldots, d-1$. Then there exists a (real) neighborhood $\mathcal{O}(\Theta)$ of Θ in $\mathbf{T}^d$ such that the intersection $\mathcal{M}_\delta \cap \mathcal{O}(\Theta)$ is contained in a smooth submanifold of $\mathbf{T}^d$ of the real dimension $d-1$.

Proof: We may assume that (i) $\mathcal{M}_\delta$ is defined by the equations $h_j(\theta) = 0$, $j = 1, \ldots, d-\delta$, with the holomorphic functions h_j in $\mathcal{O}_c(\Theta)$; and (ii) $\nabla_c h_j(\theta) \neq 0$, $\theta \in \mathcal{O}_c(\Theta)$, where ∇_c stands for the complex gradient. It is important that $d - \delta \geq 1$ so we have at least one function $h_1(\theta)$. Then $h_1(\theta) = f_1(\theta) + i g_1(\theta)$ with the real smooth functions f_1, g_1, and $f_1(\theta) = g_1(\theta) = 0$, $\theta \in M_\delta \cap \mathcal{O}_c(\Theta)$. However, $\nabla_c h_j(\theta) = \nabla_r f_1(\theta) + i \nabla_r g_1(\theta) \neq 0$, where ∇_r stands for the real gradient. Therefore, either $\nabla_r f_1(\Theta) \neq 0$ or $\nabla_r g_1(\Theta) \neq 0$. □

Now Proposition 10.1 obviously follows. □

This proposition implies (2.4) since Δ is a proper analytic subset of $\mathcal{O}_c(\mathbf{T}^d)$. Lemma 2.3 also follows from Proposition 10.1 since **E4** implies that $\det \hat{V}(\theta) \not\equiv 0$ in $\mathbf{T}^d$ and $D_k(\theta) \not\equiv 0$ in $\mathbf{T}^d \setminus \mathcal{C}_*$.

Step 3: Lemma 2.2 (iii) follows from the construction in Ref. 15, Sec. 2.1. Lemma 2.2 (iv) follows from (2.6) since the projection $\Pi_\sigma(\theta)$ can be expressed by the Cauchy integral over the contour surrounding the isolated eigenvalue $\omega_{r_\sigma}(\theta)$.

Step 4: It remains to prove Lemma 2.2 (ii). Let $\mathcal{O}(\Theta)$ denote a small real neighborhood of a point $\Theta \in \mathbf{T}^d \setminus \mathcal{C}_*$ and $E_\sigma(\theta) = \Pi_\sigma(\theta) \mathbf{R}^n$. It suffices to construct an orthonormal basis $\{e_k(\theta) : k \in (r_{\sigma-1}, r_\sigma]\}$ in $E_\sigma(\theta)$ which depends real-analytically on $\theta \in \mathcal{O}(\Theta)$.

Let us choose an arbitrary basis $\{b_k(\Theta):k\in(r_{\sigma-1},r_\sigma]\}$ in $E_\sigma(\Theta)$. Then $\Pi_\sigma(\theta)b_k(\Theta)$ depend real-analytically on $\theta\in\mathcal{O}(\Theta)$, and $\{\Pi_\sigma(\theta)b_k(\Theta):k\in(r_{\sigma-1},r_\sigma]\}$ is a basis of $E_\sigma(\theta)$ for θ from a reduced neighborhood $\mathcal{O}'(\Theta)$. Finally, construct the orthonormal basis $\{e_k(\theta):k\in(r_{\sigma-1},r_\sigma]\}$ by the standard Hilbert–Schmidt orthogonalization process applied to $\{\Pi_\sigma(\theta)b_k(\Theta):k\in(r_{\sigma-1},r_\sigma]\}$ for each $\theta\in\mathcal{O}'(\Theta)$. □

Remark 10.3: Lemma 2.2 (iii) also follows from Ref. 15, Sec. 2.1 since the enumeration (2.5), (2.6) corresponds to the factorization of the type in Ref. 15, Eq. (2.5) for the function $d(\theta,\omega)$, into the product of the irreducible factors, with the multiplicities $r_\sigma-r_{\sigma-1}$, which is constructed in Ref. 15, Thm 2.1.

2. Proof of Lemma 2.4

Step 1: Let us fix arbitrary $k,l\in\mathrm{I}_n$ and consider $\omega_k(\theta)$ as the functions of $V\in\mathcal{R}_N$ and of $\theta\in\mathbf{T}^d$. It suffices to prove that $D_k(\theta)$ and $\nabla(\omega_k(\theta)\pm\omega_l(\theta))$ are analytic and are not zero in an open dense subset in $\mathcal{R}_N\times\mathbf{T}^d$.

Let us consider $V_{k'l'}(x)$, $k',l'\in\mathrm{I}_n$, $|x_i|\leqslant N$, as the coordinates of the matrix-function V in the region $\mathcal{R}_N$. Condition **E2** allows us to consider $V_{k'l'}(x)$ as independent real variables for any $k',l'\in\mathrm{I}_n$ and the points x with either $x_1>0$, or $x_1=0$ and $x_2>0$, or $x_1=x_2=0$ and $x_3>0$, etc. Let us identify $\mathcal{R}_N$ with corresponding range $\mathbf{R}^M$ of the independent real variables $V_{k'l'}(x)$.

Step 2: Consider $\omega_k(\theta)$ as the functions of $\{V_{k'l'}(x)\}$ and θ in $\mathbf{C}^M\times\mathbf{T}_c^d$. As above, each $\omega_k(\theta)$ can be chosen as a holomorphic function outside a proper analytic discriminant subset $\Delta\subset\mathbf{C}^M\times\mathbf{T}_c^d$. Lemma 10.2 implies that the region $O:=(\mathbf{R}^M\times\mathbf{T}^d)\setminus\Delta$ is an open dense subset in $\mathbf{R}^M\times\mathbf{T}^d$. Therefore, it suffices to prove that the functions D_k and $\nabla(\omega_k\pm\omega_l)$ are not identically zero in each connected open component of O. However, the region of analyticity $\mathcal{O}:=(\mathbf{C}^M\times\mathbf{T}_c^d)\setminus\Delta$ is connected. Hence, it remains to construct a point of $\mathbf{C}^M\times\mathbf{T}_c^d$ such that the functions D_k and $\nabla(\omega_k\pm\omega_l)$ are holomorphic and nonidentically zero in a neighborhood of the point. It is easy to construct such point for any $n\geqslant1$: we can choose an arbitrary $\theta\in\mathbf{T}^d$ and the nearest neighbor crystal (1.12) repeated n times with distinct masses m_k, $k\in\mathrm{I}_n$. □

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