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ON A TWO-TEMPERATURE PROBLEM FOR THE KLEIN–GORDON EQUATION*

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(*Translated by the authors*)

Abstract. We consider the Klein–Gordon equation in $\mathbf{R}^n$, $n \geqq 2$, with constant or variable coefficients. The initial datum is a random function with a finite mean density of the energy and satisfies a Rosenblatt- or Ibragimov–Linnik-type mixing condition. We also assume that the random function is close to different space-homogeneous processes as $x_n \to \pm\infty$, with the distributions $\mu_\pm$. We study the distribution μ_t of the random solution at time $t \in \mathbf{R}$. The main result is the convergence of μ_t to a Gaussian translation-invariant measure as $t \to \infty$ that means the central limit theorem for the Klein–Gordon equation. The proof is based on the Bernstein "room-corridor" method and oscillatory integral estimates. The application to the case of the Gibbs measures $\mu_\pm = g_\pm$ with two different temperatures $T_\pm$ is given. It is proved that limit mean energy current density *formally* is $-\infty \cdot (0, \ldots, 0, T_+ - T_-)$ for the Gibbs measures, and it is finite and equals $-C(0, \ldots, 0, T_+ - T_-)$ with some positive constant $C > 0$ for the smoothed solution. This corresponds to the second law of thermodynamics.

Key words. Klein–Gordon equation, Cauchy problem, random initial data, mixing condition, Fourier transform, weak convergence of measures, Gaussian measures, covariance functions and matrices, characteristic functional

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1. Introduction. This paper concerns a mathematical problem of the foundation of statistical physics. The second law of thermodynamics states that the energy current is directed from a higher temperature to a lower one and is directly proportional to the difference of temperatures. We derive the law for the Klein–Gordon equation in $\mathbf{R}^n$. The key role is played by the mixing condition of Rosenblatt or Ibragimov–Linnik type for an initial measure. The mixing condition is introduced initially by Dobrushin and Suhov in their approach to the problem of the foundation of statistical physics for infinite-particle systems (see [6], [7]). The convergence to statistical equilibrium for two-temperature initial measure has been analyzed previously for (i) 1D chains of harmonic oscillators (see [2], [30]), (ii) 1D chains of anharmonic oscillators (see [17], [18], [22]), and (iii) nD harmonic crystals (see [16]). A similar result for the wave equation in $\mathbf{R}^n$ with odd $n \geqq 3$ is established in [14]. The Klein–Gordon equation shares some common features with the wave equation that is formally obtained by setting $m = 0$ in (1.1). On the other hand, the Klein–Gordon and wave equations have serious differences; see what follows. For translation-invariant initial measures the convergence to statistical equilibrium has been proved for the wave equation in [13], [24], for the Klein–Gordon equation in [12], [25], and for harmonic crystals in [15].

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We now pass to a detailed description of our results. Formal definitions and statements are given in section 2. We consider the Klein–Gordon equation in $\mathbf{R}^n$, $n \geqq 2$,

$$\begin{cases} \ddot{u}(x,\, t) = \displaystyle\sum_{j=1}^{n} \big(\partial_j - iA_j(x)\big)^2 u(x,t) - m^2\, u(x,t), \qquad x \in \mathbf{R}^n, \\ u\,|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x). \end{cases} \tag{1.1}$$

Here $\partial_j \equiv \partial/\partial x_j$, $m > 0$, and $(A_1(x), \dots, A_n(x))$ is a vector potential of a magnetic field. We assume that functions $A_j(x)$ vanish outside a bounded domain. The solution $u(x,t)$ is considered as a complex-valued function.

Denote $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot,t),\, \dot{u}(\cdot,t))$, $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0(\cdot),\, v_0(\cdot))$. Then (1.1) becomes

$$\dot{Y}(t) = \mathcal{A}Y(t), \quad t \in \mathbf{R}; \qquad Y(0) = Y_0. \tag{1.2}$$

Here by $\mathcal{A}$ we denote an operator-valued matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}, \tag{1.3}$$

where $A = \sum_{j=1}^{n}(\partial_j - iA_j(x))^2 - m^2$. We assume that the initial datum Y_0 is a random element of a functional space $\mathcal{H}$ of states with a finite local energy; see Definition 2.1 in what follows. The distribution of Y_0 is denoted by μ_0. Denote by $\mu_t(dY)$, $t \in \mathbf{R}$, the measure on $\mathcal{H}$ giving the distribution of the random solution $Y(t)$ to problem (1.2).

We identify $\mathbf{C} \equiv \mathbf{R}^2$ and denote by $\otimes$ the tensor product of real vectors. We assume that the initial correlation matrices

$$Q_0^{ij}(x,y) := \mathbf{E}\big(Y_0^i(x) \otimes Y_0^j(y)\big), \qquad x, y \in \mathbf{R}^n, \quad i,j = 0,1, \tag{1.4}$$

have the form

$$Q_0^{ij}(x,y) = \begin{cases} q_+^{ij}(x-y), & x_n, y_n > a, \\ q_-^{ij}(x-y), & x_n, y_n < -a. \end{cases} \tag{1.5}$$

Here $q_\pm^{ij}(x-y)$ are the correlation matrices of some translation-invariant measures $\mu_\pm$ with zero mean value in $\mathcal{H}$, $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $y = (y_1, \dots, y_n) \in \mathbf{R}^n$, $a > 0$. The measure μ_0 is not translation-invariant if $q_-^{ij} \neq q_+^{ij}$.

Next, we assume that the initial mean "energy" density is uniformly bounded:

$$\begin{aligned} e_0(x) &:= \mathbf{E}\Big[\big|u_0(x)\big|^2 + \big|\nabla u_0(x)\big|^2 + \big|v_0(x)\big|^2\Big] \\ &= \mathrm{tr}\,\Big(Q_0^{00}(x,x) + \big[\nabla_x \cdot \nabla_y Q_0^{00}(x,y)\big]\,\big|_{y=x} + Q_0^{11}(x,x)\Big) \leqq e_0 < \infty \end{aligned} \tag{1.6}$$

for a.a. $x \in \mathbf{R}^n$. Finally, it is assumed that the measure μ_0 satisfies a mixing condition of Rosenblatt or Ibragimov–Linnik type, which means that

(1.7) $\quad Y_0(x) \quad$ and $\quad Y_0(y) \quad$ are asymptotically independent as $|x-y| \to \infty$.

Our main result states the (weak) convergence

$$\mu_t \rightharpoondown \mu_\infty, \qquad t \to \infty \tag{1.8}$$

to an equilibrium measure μ_∞, which is a translation-invariant Gaussian measure on $\mathcal{H}$. A similar convergence holds as $t \to -\infty$ since our system is time-reversible. We construct generic examples of the initial measures μ_0 satisfying all assumptions imposed. The explicit formulas (2.13)–(2.15) for the limiting correlation matrices are given.

We apply our results to the case of the Gibbs measures $\mu_\pm = g_\pm$. Formally,

$$g_\pm(du_0, dv_0) = \frac{1}{Z_\pm} \exp\Big\{ -\frac{\beta_\pm}{2} \int \Big(\big|v_0(x)\big|^2 + \big|\nabla u_0(x)\big|^2 + m^2 \big|u_0(x)\big|^2 \Big)\, dx \Big\} \times \prod_x du_0(x)\, dv_0(x), \tag{1.9}$$

where $\beta_\pm = T_\pm^{-1}$, $T_\pm \geqq 0$ are the corresponding absolute temperatures. We adjust the definition of the Gibbs measures $g_\pm$ in section 4. The Gibbs measures $g_\pm$ have singular correlation functions and do not satisfy our assumptions (1.6). Therefore we consider Gaussian processes $u_\pm$ corresponding to the measures $g_\pm$ and define the "smoothed" measures $g_\pm^\theta$ as the distributions of the convolutions $u_\pm * \theta$, where $\theta \in D \equiv C_0^\infty(\mathbf{R}^n)$. The measures $g_\pm^\theta$ satisfy all our assumptions, and the convergence $g_t^\theta \rightharpoonup g_\infty^\theta$ follows from (1.8). This implies the weak convergence of the measures $g_t \rightharpoonup g_\infty$ since θ is arbitrary. We show that the limit energy current for g_∞ formally is

$$\bar{\mathbf{j}}_\infty = -\infty \cdot (0, \ldots, 0, T_+ - T_-).$$

Infinity denotes the "ultraviolet divergence." This relation has a finite value in the case of smoothed measures g_∞^θ,

$$\bar{\mathbf{j}}_\infty^\theta = -C_\theta \cdot (0, \ldots, 0, T_+ - T_-),$$

if $\theta(x)$ is axially symmetric with respect to Ox_n; $C_\theta > 0$ if $\theta(x) \not\equiv 0$. This corresponds to the second law of thermodynamics.

We prove convergence (1.8) first for the case of constant coefficients. We decompose the proof into three steps using the general strategy of [12], [13], [14], [15], and [16]:

I. The family of measures μ_t, $t \geqq 0$, is weakly compact in an appropriate Fréchet space.

II. The correlation matrices converge to a limit: For $i, j = 0, 1$,

$$Q_t^{ij}(x,y) = \int \big(Y^i(x) \otimes Y^j(y) \big)\, \mu_t(dY) \longrightarrow Q_\infty^{ij}(x,y), \qquad t \to \infty. \tag{1.10}$$

III. The characteristic functionals converge to the Gaussian:

$$\widehat{\mu}_t(\Psi) := \int \exp\{ i\langle Y, \Psi\rangle \}\, \mu_t(dY) \longrightarrow \exp\Big\{ -\frac{1}{2}\, \mathcal{Q}_\infty(\Psi, \Psi) \Big\}, \qquad t \to \infty, \tag{1.11}$$

where Ψ is an arbitrary element of the dual space, and $\mathcal{Q}_\infty$ is the quadratic form with the integral kernel $(Q_\infty^{ij}(x,y))_{i,j=0,1}$; $\langle Y, \Psi\rangle$ denotes the scalar product in a real Hilbert space $L^2(\mathbf{R}^n) \otimes \mathbf{R}^N$.

Property I follows from the Prokhorov compactness theorem by using methods of Vishik and Fursikov developed by them for problems of statistical hydromechanics in [5]. First, one proves a uniform bound for the mean local energy with respect to the measure μ_t. We deduce the bound from the explicit expression for the correlation matrices $Q_t^{ij}(x,y)$. The conditions of the Prokhorov theorem then follow from

Sobolev's embedding theorem. Next, we deduce property II from an analysis of oscillatory integrals arising in the Fourier transform. An important role is attributed to Proposition 5.1, reflecting the properties of the correlation functions in the Fourier transform deduced from the mixing condition.

Similarly, properties I and II have been established previously in [16] for a harmonic crystal which is a discrete model of the continuous Klein–Gordon equation. We extend here the methods of [16] to the continuous case. The main difficulty in comparison with [16] is the noncompactness of the Fourier space $\mathbf{R}^n$ (for the harmonic crystal the Fourier space is a torus T^n which is a compact space). Namely, the proofs of properties I and II rely on the uniform bounds of singular oscillatory integrals in the sense of the Cauchy principal value. The proof of uniform bounds for such integrals in [16] uses essentially the compactness of set parameters T^n which in particular provides the uniform nondegeneracy of the phase functions. In the case of the Klein–Gordon equation the corresponding phase function is nondegenerate in any finite region of the Fourier space, but is degenerate at infinity, which makes it difficult to deduce uniform bounds. In the case of translation-invariant measures, similar bounds are established in [12] for the Klein–Gordon equation; however, corresponding oscillatory integrals are less singular since they do not contain the Cauchy principal value. Therefore in the present paper the uniform bounds of oscillatory integrals demand new tools: we remove this difficulty by using Proposition 6.2, which is a modification of Proposition A.4 of [2, p. 152] to our case. Let us note that this proposition is an extension of results of Fedoryuk (see Theorems 1.8 and 1.10 in [32]). However, these results are not applied immediately to our problem because of the degeneracy of the phase function at infinity.

Let us note that we choose the initial correlation matrices in the particular form (2.9) which corresponds to the initial function (2.22). This allows us to avoid some technical assumptions on the initial correlation function (cf. [14, condition S2]).

Finally, property III follows by using a variant of the Bernstein "room-corridor" method from [12], [15]. In conclusion, we extend convergence (1.8) to the equations with variable coefficients that are constant outside a finite region. The extension follows from our result for constant coefficients, using the scattering theory for infinite energy solutions from [12].

The paper is organized as follows. The main result is stated in section 2. We apply it to Gibbs measures in section 4. Sections 3–8 deal with the case of constant coefficients. The compactness (property I) and the convergence (1.10) are proved in sections 5–7. In section 8 we prove the convergence (1.11) using the "room-corridor" method. In section 9 we establish the convergence (1.8) for variable coefficients. Appendix A is concerned with a dynamics in Fourier space; in Appendix B we prove a bound of some singular oscillatory integrals.

2. Main results.

2.1. Notation. We assume that functions $A_j(x)$ in (1.1) satisfy the following conditions:

E1. $A_j(x)$ are real C^∞-functions;

E2. $A_j(x) = 0$ for $|x| > R_0$, where $R_0 < \infty$;

E3. $\frac{\partial A_1}{\partial x_2} \not\equiv \frac{\partial A_2}{\partial x_1}$ if $n = 2$.

We assume that the initial datum Y_0 belongs to the complex phase space $\mathcal{H}$ defined in what follows.

DEFINITION 2.1. $\mathcal{H} \equiv H^1_{\rm loc}(\mathbf{R}^n) \oplus H^0_{\rm loc}(\mathbf{R}^n)$ *is the Fréchet space of pairs* $Y \equiv$

$(u(x), v(x))$ *of complex functions* $u(x), v(x)$ *endowed with local energy seminorms*

$$\|Y\|_R^2 = \int_{|x|<R} \Big(\big|u(x)\big|^2 + \big|\nabla u(x)\big|^2 + \big|v(x)\big|^2\Big)\, dx < \infty \qquad \forall\, R > 0. \tag{2.1}$$

Proposition 2.1 follows from Theorems V.3.1 and V.3.2 of [26] since the speed of propagation for (1.1) is finite.

PROPOSITION 2.1. (i) *For any* $Y_0 \in \mathcal{H}$, *there exists a unique solution* $Y(t) \in C(\mathbf{R}, \mathcal{H})$ *of the Cauchy problem* (1.2).

(ii) *For any* $t \in \mathbf{R}$, *the operator* $U(t)\colon\ Y_0 \mapsto Y(t)$ *is continuous on* $\mathcal{H}$.

Let us choose a function $\xi(x) \in C_0^\infty(\mathbf{R}^n)$ with $\xi(0) \neq 0$. Denote by $H^s_{\rm loc}(\mathbf{R}^n)$, $s \in \mathbf{R}$, the local Sobolev spaces, i.e., the Fréchet spaces of distributions $u \in D'(\mathbf{R}^n)$ with the finite seminorms

$$\|u\|_{s,R} := \big\|\Lambda^s\big(\xi(xR^{-1})\, u\big)\big\|_{L^2(\mathbf{R}^n)}.$$

Here $\Lambda^s u := F^{-1}_{k\to x}(\langle k\rangle^s \widehat{u}(k))$, $\langle k\rangle := \sqrt{|k|^2+1}$, and $\widehat{u} := Fu$, where F is the Fourier transform. For $\psi \in S \equiv S(\mathbf{R}^n)$, write $F\psi(k) = \int \exp(ikx)\, \psi(x)\, dx$.

DEFINITION 2.2. *For* $s \in \mathbf{R}$, *write* $\mathcal{H}^s \equiv H^{1+s}_{\rm loc}(\mathbf{R}^n) \oplus H^s_{\rm loc}(\mathbf{R}^n)$.

Using the standard techniques of pseudodifferential operators and Sobolev's embedding theorem (see, e.g., [20]), one can prove that $\mathcal{H}^0 = \mathcal{H} \subset \mathcal{H}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding is compact.

2.2. Random solution. Convergence to an equilibrium. Let $(\Omega, \Sigma, \mathbf{P})$ be a probability space with expectation $\mathbf{E}$, and let $\mathcal{B}(\mathcal{H})$ denote the Borel σ-algebra in $\mathcal{H}$. We assume that $Y_0 = Y_0(\omega, x)$ in (1.2) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. In other words, $(\omega, x) \mapsto Y_0(\omega, x)$ is a measurable mapping $\Omega \times \mathbf{R}^n \to \mathbf{R}^2$ with respect to the (completed) σ-algebras $\Sigma \times \mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^2)$. Then, by virtue of Proposition 2.1, $Y(t) = U(t)\, Y_0$ is again a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. Denote by $\mu_0(dY_0)$ the Borel probability measure on $\mathcal{H}$ giving the distribution of Y_0. Without loss of generality, we can assume that $(\Omega, \Sigma, \mathbf{P}) = (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_0)$ and $Y_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-a.a. $(\omega, x) \in \mathcal{H} \times \mathbf{R}^n$.

DEFINITION 2.3. *Let* μ_t *be the Borel probability measure on* $\mathcal{H}$ *which gives the distribution of* $Y(t)$:

$$\mu_t(B) = \mu_0\big(U(-t)\, B\big) \qquad \forall\, B \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbf{R}. \tag{2.2}$$

Our main objective is to derive the weak convergence of the measures μ_t in the Fréchet spaces $\mathcal{H}^{-\varepsilon}$ for each $\varepsilon > 0$,

$$\mu_t \ \overset{\mathcal{H}^{-\varepsilon}}{\longrightarrow}\ \mu_\infty \qquad \text{as} \quad t \to \infty, \tag{2.3}$$

where μ_∞ is the Borel probability measure on the space $\mathcal{H}$. By definition, this means the convergence

$$\int f(Y)\, \mu_t(dY) \longrightarrow \int f(Y)\, \mu_\infty(dY) \qquad \text{as} \quad t \to \infty \tag{2.4}$$

for any bounded continuous functional $f(Y)$ on the space $\mathcal{H}^{-\varepsilon}$. Recall that we identify $\mathbf{C} \equiv \mathbf{R}^2$, and $\otimes$ stands for the tensor product of real vectors. Denote $M^2 = \mathbf{R}^2 \otimes \mathbf{R}^2$.

We introduce the space of test functions $\mathcal{S} = S \oplus S$ and denote $\langle Y, \Psi\rangle = \langle Y^0, \Psi^0\rangle + \langle Y^1, \Psi^1\rangle$, for $Y = (Y^0, Y^1) \in \mathcal{H}$ and $\Psi = (\Psi^0, \Psi^1) \in \mathcal{S}$.

Definition 2.4. *The correlation functions of the measure μ_t are M^2-valued generalized functions defined by*

$$(2.5)\qquad Q_t^{ij}(x,y) := \mathbf{E}\big(Y^i(x,t) \otimes Y^j(y,t)\big), \qquad i,j = 0,1, \quad x,y \in \mathbf{R}^n \times \mathbf{R}^n,$$

where $\mathbf{E}$ stands for the integral with respect to the measure $\mu_0(dY)$, and the convergence of the integral is understood in the sense of distributions, i.e.,

$$(2.6)\qquad \big\langle Q_t^{ij}(x,y),\, \Psi(x,y)\big\rangle := \mathbf{E}\big\langle Y^i(x,t) \otimes Y^j(y,t),\, \Psi(x,y)\big\rangle, \qquad \Psi \in S(\mathbf{R}^{2n}).$$

For a Borel probability measure μ on the space $\mathcal{H}$, denote by $\widehat{\mu}$ the characteristic functional (the Fourier transform)

$$\widehat{\mu}(\Psi) \equiv \int \exp\big(i\langle Y,\, \Psi\rangle\big)\, \mu(dY), \qquad \Psi \in \mathcal{S}.$$

A probability measure μ is called Gaussian (with zero expectation) if its characteristic functional has the form

$$\widehat{\mu}(\Psi) = \exp\bigg(-\frac{1}{2}\, \mathcal{Q}(\Psi,\, \Psi)\bigg), \qquad \Psi \in \mathcal{S},$$

where $\mathcal{Q}$ is a real nonnegative quadratic form on $\mathcal{S}$. A measure μ is called translation-invariant if

$$\mu(T_h B) = \mu(B) \qquad \forall\, B \in \mathcal{B}(\mathcal{H}), \qquad h \in \mathbf{R}^n,$$

where $T_h Y(x) = Y(x-h)$, $x \in \mathbf{R}^n$.

2.3. Mixing condition. Let $\mathcal{O}(r)$ be a set of all pairs of open bounded subsets $\mathcal{A}, \mathcal{B} \subset \mathbf{R}^n$ at the distance $\mathrm{dist}(\mathcal{A},\, \mathcal{B}) \geqq r$, and let $\sigma(\mathcal{A})$ be a σ-algebra in $\mathcal{H}$ generated by the linear functionals $Y \mapsto \langle Y, \Psi\rangle$, where $\Psi \in \mathcal{D} = D \otimes D$ with $\mathrm{supp}\Psi \subset \mathcal{A}$. Define the Ibragimov–Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{H}$ by (cf. Definition 17.2.2 of [21, p. 391])

$$(2.7)\qquad \varphi(r) \equiv \sup_{(\mathcal{A},\mathcal{B})\in\mathcal{O}(r)} \ \sup_{\substack{A\in\sigma(\mathcal{A}),\, B\in\sigma(\mathcal{B})\\ \mu_0(B)>0}} \frac{|\mu_0(A\cap B) - \mu_0(A)\,\mu_0(B)|}{\mu_0(B)}.$$

Definition 2.5. *The measure μ_0 satisfies the* strong uniform Ibragimov–Linnik mixing condition *if*

$$(2.8)\qquad \varphi(r) \to 0, \qquad r \to \infty.$$

We specify the rate of decay of φ in what follows (see condition **S3**).

2.4. Statistical conditions and main result. We assume that the initial measure μ_0 satisfies the following conditions.

S0. μ_0 has zero expectation value, i.e., $\mathbf{E}Y_0(x) = 0$, $x \in \mathbf{R}^n$.

S1. μ_0 has correlation matrices of the form (cf. (1.5))

$$(2.9)\qquad Q_0^{ij}(x,y) = q_-^{ij}(x-y)\,\zeta_-(x_n)\,\zeta_-(y_n) + q_+^{ij}(x-y)\,\zeta_+(x_n)\,\zeta_+(y_n).$$

Here the functions $\zeta_\pm \in C^\infty(\mathbf{R})$ such that

$$\zeta_\pm(s) = \begin{cases} 1 & \text{for} \quad \pm\, s > a, \\ 0 & \text{for} \quad \pm\, s < -a. \end{cases} \tag{2.10}$$

S2. μ_0 has a finite mean energy density, i.e., (1.6) holds.

S3. μ_0 satisfies the strong uniform Ibragimov–Linnik mixing condition with

$$\overline{\varphi} \equiv \int_0^\infty r^{n-1}\varphi^{1/2}(r)\,dr < \infty. \tag{2.11}$$

Define the correlation matrix of the limit measure μ_∞. Denote by $\mathcal{E}(z)$ the fundamental solution for the operator $-\Delta + m^2$, i.e., $(-\Delta+m^2)\,\mathcal{E} = \delta(x)$ for $x \in \mathbf{R}^n$, and $\mathcal{P}(x) = -iF^{-1}[\mathrm{sign}(k_n)/\sqrt{|k|^2+m^2}]$, where F^{-1} is the inverse Fourier transform. Define the matrix-valued function

$$Q_\infty(x,y) = \big(Q_\infty^{ij}(x,y)\big)_{i,j=0,1} = \big(q_\infty^{ij}(x-y)\big)_{i,j=0,1}, \qquad x,y \in \mathbf{R}^n, \tag{2.12}$$

where

$$q_\infty^{00} = \frac{1}{2}\Big[(\mathbf{q}^+)^{00} + \mathcal{E} * (\mathbf{q}^+)^{11} + \mathcal{P} * \big((\mathbf{q}^-)^{01} - (\mathbf{q}^-)^{10}\big)\Big], \tag{2.13}$$

$$q_\infty^{10} = -q_\infty^{01} = \frac{1}{2}\Big[(\mathbf{q}^+)^{10} - (\mathbf{q}^+)^{01} + \mathcal{P} * \big((\mathbf{q}^-)^{11} + (-\Delta+m^2)(\mathbf{q}^-)^{00}\big)\Big], \tag{2.14}$$

$$\begin{aligned} q_\infty^{11} &= (-\Delta+m^2)\,q_\infty^{00} \\ &= \frac{1}{2}\Big[(\mathbf{q}^+)^{11} + (-\Delta+m^2)\big((\mathbf{q}^+)^{00} + \mathcal{P} * \big((\mathbf{q}^-)^{01} - (\mathbf{q}^-)^{10}\big)\big)\Big]. \end{aligned} \tag{2.15}$$

Here $\mathbf{q}^+ := \frac{1}{2}(q_+ + q_-)$, $\mathbf{q}^- := \frac{1}{2}(q_+ - q_-)$, and $*$ stands for the convolution of generalized functions. We show below that $D^\gamma q_\pm^{ij} \in L^2(\mathbf{R}^n)$, where $\gamma \in \mathbf{Z}^n$ with $|\gamma| \leqq 2-i-j$, $i,j=0,1$ (see (5.4)). Then the convolutions in (2.13)–(2.15) also belong to the space $L^2(\mathbf{R}^n)$. Moreover, the explicit formulas for $\mathcal{P}(x)$ and (2.13)–(2.15) imply that $q_\infty^{ij} \in L^2(\mathbf{R}^n)$, $i,j=0,1$. Applying the Fourier transform, we obtain

$$\widehat{q}_\infty(k) := \widehat{q}_\infty^{\,+}(k) + \widehat{q}_\infty^{\,-}(k), \tag{2.16}$$

where

$$\widehat{q}_\infty^{\,+}(k) := \frac{1}{2}\big(\widehat{\mathbf{q}}^+(k) + \widehat{C}(k)\,\widehat{\mathbf{q}}^+(k)\,\widehat{C}^T(k)\big), \tag{2.17}$$

$$\widehat{q}_\infty^{\,-}(k) := i\,\mathrm{sign}(k_n)\,\frac{1}{2}\big(\widehat{C}(k)\,\widehat{\mathbf{q}}^-(k) - \widehat{\mathbf{q}}^-(k)\,\widehat{C}^T(k)\big) \tag{2.18}$$

with matrix $\widehat{C}(k)$ defined by (A.3).

Let $H = L^2(\mathbf{R}^n) \oplus H^1(\mathbf{R}^n)$ denote the space of complex-valued functions $\Psi = (\Psi^0, \Psi^1)$ with a finite norm

$$\|\Psi\|_H^2 = \int_{\mathbf{R}^n} \Big(\big|\Psi^0(x)\big|^2 + \big|\nabla\Psi^1(x)\big|^2 + \big|\Psi^1(x)\big|^2\Big)\,dx < \infty. \tag{2.19}$$

Denote by $\mathcal{Q}_\infty(\Psi,\Psi)$ a real quadratic form on H defined by

$$\mathcal{Q}_\infty(\Psi,\Psi) = \sum_{i,j=0,1} \int_{\mathbf{R}^n\times\mathbf{R}^n} \big(Q_\infty^{ij}(x,y),\, \Psi^i(x)\otimes\Psi^j(y)\big)\,dx\,dy, \tag{2.20}$$

where $Q^{ij}_\infty(x,y)$ is as defined in (2.12)–(2.15), and $(\cdot\,,\cdot)$ stands for the real scalar product in $\mathbf{R}^2\times\mathbf{R}^2\cong\mathbf{R}^4$. The form $\mathcal{Q}_\infty$ is continuous on H by Corollary 5.2.

Write $D:=C_0^\infty(\mathbf{R}^n)$ and $\mathcal{D}=D\oplus D$.

THEOREM A. *Let $n\geqq 2$, $m>0$, and assume that conditions* **E1**–**E3**, **S0**–**S3** *hold. Then*

(i) *convergence* (2.3) *holds for any $\varepsilon>0$;*

(ii) *the limiting measure μ_∞ is a Gaussian equilibrium measure on $\mathcal{H}$;*

(iii) *the limiting characteristic functional has the form*

$$\widehat{\mu}_\infty(\Psi)=\exp\Big\{-\frac{1}{2}\,\mathcal{Q}_\infty(W\Psi,\,W\Psi)\Big\},\qquad \Psi\in\mathcal{D}.$$

Here $W\colon\mathcal{D}\to H$ is a linear continuous operator and $W=I$ if $A_j(x)\equiv 0$.

2.5. Examples.

2.5.1. Gaussian measures. We construct the Gaussian initial measures μ_0 satisfying conditions **S0**–**S3**. Let us take some Gaussian measures $\mu_\pm$ on $\mathcal{H}$ with correlation functions $q^{ij}_\pm(x-y)$ which are zero for $i\neq j$ and $q^{ii}_\pm\in C^2(\mathbf{R}^n)\otimes M^2$ and have a compact support

$$q^{ii}_\pm(x)=0,\qquad |x|\geqq r_0. \tag{2.21}$$

For instance, we can take $q^{ii}_\pm(x)=F^{-1}_{k\to x}[f(k_1)\cdots f(k_n)]$ with

$$f(z)=\left(\frac{1-\cos(r_0z/\sqrt{n})}{z^2}\right)^2,\qquad z\in\mathbf{R}.$$

Note that by the Minlos theorem (see [8, Chap. V]) there exist Borel probability measures $\mu_\pm$ on the space $\mathcal{H}$ because *formally* we have

$$\int\|Y\|^2_R\,\mu_\pm(dY)=|B_R|\,\mathrm{tr}\left(q^{00}_\pm(0)-\Delta q^{00}_\pm(0)+q^{11}_\pm(0)\right)<\infty,\qquad R>0.$$

Moreover, the measures $\mu_\pm$ satisfy conditions **S0**–**S2** and mixing condition (2.8), since $\varphi(r)=0$ for $r\geqq r_0$ by (2.21). Hence condition **S3** also follows.

Let us introduce (Y_-,Y_+) as a unit random function on the probability space $(\mathcal{H}\times\mathcal{H},\mu_-\times\mu_+)$. Then $Y_\pm\in\mathcal{H}$ are Gaussian independent vectors with zero mean value. Define μ_0 as the distribution of the random function

$$Y_0(x)=\zeta_-(x_n)\,Y_-(x)+\zeta_+(x_n)\,Y_+(x), \tag{2.22}$$

where the functions $\zeta_\pm$ are introduced in (2.10). Then the correlation matrices of μ_0 have the form (2.9). Hence, conditions **S0**–**S3** hold for μ_0 with the same functions $\varphi(r)$ as for $\mu_\pm$.

2.5.2. Non-Gaussian measures. Let us choose some odd nonconstant functions f^0, $f^1\in C^2(\mathbf{R}^n\times\mathbf{R}^n)$ with bounded derivatives. Define μ_0^* as the distribution of the random function $(f^0(Y_0(x)),\,f^1(Y_0(x)))$, where $Y_0(x)$ is a random function (2.22) with the Gaussian distribution μ_0. Then **S0**–**S3** hold for μ_0^*, since corresponding mixing coefficients $\varphi^*(r)=0$ for $r\geqq r_0$. The measure μ_0^* is not Gaussian since the functions f^0,f^1 are bounded and nonconstant.

3. Equations with constant coefficients. In sections 3–8 we assume that coefficients $A_j(x) \equiv 0$. The problem (1.1) then becomes

$$\begin{cases} \ddot{u}(x,t) = \Delta u(x,t) - m^2 u(x,t), & t \in \mathbf{R}, \\ u\,|_{t=0} = u_0(x), \qquad \dot{u}|_{t=0} = v_0(x). \end{cases} \tag{3.1}$$

As in (1.2), we rewrite (3.1) in the form

$$\dot{Y}(t) = \mathcal{A}_0 Y(t), \quad t \in \mathbf{R}; \qquad Y(0) = Y_0. \tag{3.2}$$

Here

$$\mathcal{A}_0 = \begin{pmatrix} 0 & 1 \\ A_0 & 0 \end{pmatrix}, \tag{3.3}$$

where $A_0 = \Delta - m^2$. Denote by $U_0(t)$, $t \in \mathbf{R}$, the dynamical group for problem (3.2); then $Y(t) = U_0(t)\, Y_0$. The following proposition is well known and is proved by a standard integration by parts.

PROPOSITION 3.1. *Let $Y_0 = (u_0, v_0) \in \mathcal{H}$ and $Y(\cdot,t) = (u(\cdot,t), \dot{u}(\cdot,t)) \in C(\mathbf{R},\mathcal{H})$ be the solution to problem* (3.2). *Then the following energy bound holds*: *For $R > 0$ and $t \in \mathbf{R}$,*

$$\begin{aligned} &\int_{|x|<R} \Big(\big|\dot{u}(x,t)\big|^2 + \big|\nabla u(x,t)\big|^2 + m^2\big|u(x,t)\big|^2\Big)\,dx \\ &\qquad \leqq \int_{|x|<R+|t|} \Big(\big|v_0(x)\big|^2 + \big|\nabla u_0(x)\big|^2 + m^2\big|u_0(x)\big|^2\Big)\,dx. \end{aligned} \tag{3.4}$$

Set $\mu_t(B) = \mu_0(U_0(-t)B)$, $B \in \mathcal{B}(\mathcal{H})$, $t \in \mathbf{R}$. We formulate the main result for problem (3.2).

THEOREM B. *Let $n \geqq 1$, $m > 0$, and conditions* **S0–S3** *hold. Then the conclusions of Theorem* A *hold with $W = I$, and the limiting measure μ_∞ is translation invariant.*

Theorem B can be derived from Propositions 3.2 and 3.3 by using the methods of [5].

PROPOSITION 3.2. *The family of measures $\{\mu_t, t \in \mathbf{R}\}$ is weakly compact in $\mathcal{H}^{-\varepsilon}$ with any $\varepsilon > 0$, and the bounds $\sup_{t\geqq 0} \mathbf{E}\|U_0(t)\,Y_0\|_R^2 < \infty$, $R > 0$, hold.*

PROPOSITION 3.3. *For any $\Psi \in \mathcal{D}$,*

$$\widehat{\mu}_t(\Psi) \equiv \int \exp\big(i\langle Y, \Psi\rangle\big)\,\mu_t(dY) \longrightarrow \exp\Big\{-\frac{1}{2}\,\mathcal{Q}_\infty(\Psi,\,\Psi)\Big\}, \qquad t \to \infty. \tag{3.5}$$

Propositions 3.2 and 3.3 are proved in sections 7 and 8, respectively. The proofs essentially use the explicit formulas (A.2)–(A.6) from Appendix A.

4. Application to Gibbs measures. We apply Theorem B to the case when $\mu_\pm = g_\pm$ are the Gibbs measures (1.9) corresponding to different positive temperatures $T_- \neq T_+$.

4.1. Gibbs measures. We will define the Gibbs measures $g_\pm$ as the Gaussian measures with the correlation functions (cf. (1.9))

$$\begin{gathered} q_\pm^{00}(x-y) = T_\pm \mathcal{E}(x-y), \qquad q_\pm^{11}(x-y) = T_\pm \delta(x-y), \\ q_\pm^{01}(x-y) = q_\pm^{10}(x-y) = 0, \end{gathered} \tag{4.1}$$

where $x, y \in \mathbf{R}^n$. The correlation functions $q_\pm^{ij}$ do not satisfy condition **S2** because of singularity at $x = y$. The singularity means that the measures $g_\pm$ are not concentrated on the space $\mathcal{H}$. Let us introduce appropriate functional spaces for measures $g_\pm$. First, let us define the weighted Sobolev space with any $s, \alpha \in \mathbf{R}$.

DEFINITION 4.1. *$H_{s,\alpha}(\mathbf{R}^n)$ is the Hilbert space of the distributions $u \in S'(\mathbf{R}^n)$ with the finite norm*

$$\|u\|_{s,\alpha} \equiv \big\|\langle x\rangle^\alpha \Lambda^s u\big\|_{L_2(\mathbf{R}^n)} < \infty, \qquad \Lambda^s u \equiv F^{-1}\big[\langle k\rangle^s \widehat{u}(k)\big]. \tag{4.2}$$

Let us fix arbitrary $s, \alpha < -n/2$.

DEFINITION 4.2. *Let $G_{s,\alpha}$ be the Hilbert space $H_{s+1,\alpha}(\mathbf{R}^n) \oplus H_{s,\alpha}(\mathbf{R}^n)$ with the norm*

$$|\!|\!| Y |\!|\!|_{s,\alpha} \equiv \|u\|_{s+1,\alpha} + \|v\|_{s,\alpha} < \infty, \qquad Y = (u, v).$$

Introduce the Gaussian Borel probability measures $g_\pm^0(du)$, $g_\pm^1(dv)$ on the spaces $H_{s+1,\alpha}(\mathbf{R}^n)$ and $H_{s,\alpha}(\mathbf{R}^n)$, respectively, with characteristic functionals

$$\widehat{g}_\pm^0(\psi) = \int \exp\big\{i\langle u, \psi\rangle\big\}\, g_\pm^0(du) = \exp\bigg\{-\frac{\langle(-\Delta + m^2)^{-1}\psi,\, \psi\rangle}{2\beta_\pm}\bigg\},$$
$$\widehat{g}_\pm^1(\psi) = \int \exp\big\{i\langle v,\, \psi\rangle\big\}\, g_\pm^1(dv) = \exp\bigg\{-\frac{\langle \psi,\, \psi\rangle}{2\beta_\pm}\bigg\},$$

$\psi \in D$. By the Minlos theorem, the Borel probability measures $g_\pm^0$, $g_\pm^1$ exist on the spaces $H_{s+1,\alpha}(\mathbf{R}^n)$, $H_{s,\alpha}(\mathbf{R}^n)$, respectively, because *formally* (see Appendix B in [12, p. 31])

$$\int \|u\|^2_{s+1,\alpha}\, g_\pm^0(du) < \infty, \qquad \int \|v\|^2_{s,\alpha}\, g_\pm^1(dv) < \infty, \qquad s, \alpha < -\frac{n}{2}. \tag{4.3}$$

Finally, we define the Gibbs measures $g_\pm(dY)$ as the Borel probability measures $g_\pm^0(du) \times g_\pm^1(dv)$ on $G_{s,\alpha}$. Let $g_0(dY)$ be the Borel probability measure on $G_{s,\alpha}$ that is constructed as in section 2.5.1 with $\mu_\pm(dY) = g_\pm(dY)$. It satisfies conditions **S0** and **S1** with $q_\pm^{ij}$ from (4.1). However, g_0 does not satisfy condition **S2**. Therefore, Theorem B cannot be applied directly to $\mu_0 = g_0$. The embedding $G_{s,\alpha} \subset \mathcal{H}^s$ is continuous by the standard arguments of pseudodifferential equations [20]. The following lemma is proved easily by using the Fourier transform from the finite speed of propagation for the Klein–Gordon equation.

LEMMA 4.1. *The operators $U_0(t)\colon Y_0 \mapsto Y(t)$ allow a continuous extension $\mathcal{H}^s \mapsto \mathcal{H}^s$.*

Let Y_0 be the random function with the distribution g_0; hence $Y_0 \in G_{s,\alpha}$ a.s. Denote by g_t the distribution of the function $U_0(t)\, Y_0$.

Remark 4.1. Let s be a sufficiently large negative number. Then there exists a Gaussian Borel probability measure g_∞ on $\mathcal{H}^s$ such that

$$g_t \overset{\mathcal{H}^s}{\longrightarrow} g_\infty, \qquad t \to \infty. \tag{4.4}$$

This can be proved similarly to Theorem A. The limiting measure g_∞ is Gaussian with the correlation matrix $Q_\infty = (Q_\infty^{ij}(x,y))_{i,j=0,1}$, where

$$Q_\infty^{00}(x,y) \equiv q_\infty^{00}(x-y) = \frac{1}{2}\,(T_+ + T_-)\,\mathcal{E}(x-y), \tag{4.5}$$

$$Q_\infty^{10}(x,y) = -Q_\infty^{01}(x,y) \equiv q_\infty^{10}(x-y) = \frac{1}{2}\,(T_+ - T_-)\,\mathcal{P}(x-y), \tag{4.6}$$

$$Q_\infty^{11}(x,y) \equiv q_\infty^{11}(x-y) = \frac{1}{2}\,(T_+ + T_-)\,\delta(x-y). \tag{4.7}$$

The identities (4.5)–(4.7) follow formally from (4.1) and from (2.13)–(2.15). We will consider them as defining the functions $Q_\infty^{ij}(x,y)$.

4.2. Limit energy current for smoothed fields. Let $u(x,t)$ be the random solution to problem (3.1) with the initial measure μ_0 satisfying conditions **S0**–**S3**. The mean energy current density is $\mathbf{E}\mathbf{j}(x,t) = -\mathbf{E}\dot u(x,t)\,\nabla u(x,t)$. Therefore, in the limit $t\to\infty$,

$$\mathbf{E}\mathbf{j}(x,t) \to \bar{\mathbf{j}}_\infty = \nabla q_\infty^{10}(0)$$

by (6.7). In the case of the "Gibbs" initial measure g_0, expression (4.6) for the limiting correlation function implies *formally* that

$$\bar{\mathbf{j}}_\infty = \frac{T_+ - T_-}{2}\,\nabla\mathcal{P}(0),$$

where $[\nabla\mathcal{P}](z) = -F^{-1}[k\,\mathrm{sign}(k_n)/\sqrt{|k|^2+m^2}\,](z)$. Hence, formally we have the "ultraviolet divergence" for the limit mean of energy current density:

$$\bar{\mathbf{j}}_\infty = -\frac{T_+ - T_-}{2(2\pi)^n}\int_{\mathbf{R}^n}\frac{k\,\mathrm{sign}\,k_n}{\sqrt{|k|^2+m^2}}\,dk = -\infty\cdot(0,\dots,0,\,T_+ - T_-).$$

This is since the Gibbs measures $g_\pm$ have singular correlation functions and do not satisfy assumptions (1.6). Respectively, our results cannot be applied directly to $g_\pm$. We consider Gaussian processes $u_\pm$ corresponding to the measures $g_\pm$ and define the "smoothed" measures $g_\pm^\theta$ as the distributions of the convolutions $u_\pm * \theta$, where $\theta\in D\equiv C_0^\infty(\mathbf{R}^n)$. The measures $g_\pm^\theta$ satisfy all our assumptions, and the convergence $g_t^\theta \rightharpoonup g_\infty^\theta$ follows from Theorem B. For the convolution $U_0(t)(Y_0 * \theta)$ the corresponding limiting mean current density is finite and equals

$$\bar{\mathbf{j}}_\infty^\theta = -\frac{T_+ - T_-}{2(2\pi)^n}\int_{\mathbf{R}^n}\bigl|\hat\theta(k)\bigr|^2\frac{k\,\mathrm{sign}\,k_n}{\sqrt{|k|^2+m^2}}\,dk = -C_\theta\cdot(0,\dots,0,T_+ - T_-)$$

if $\theta(x)$ is axially symmetric with respect to Ox_n; here $C_\theta > 0$ if $\theta(x)\not\equiv 0$.

5. Bounds for initial covariance.

5.1. Mixing in terms of spectral density. The next proposition reflects the mixing property in the Fourier transforms $\widehat{q}_\pm^{ij}$ of the initial correlation functions $q_\pm^{ij}$. Condition **S2** implies that $q_\pm^{ij}(z)$ are continuous bounded functions. Therefore, $Q_0^{ij}(x,y)$ in (2.9) are also continuous bounded functions.

PROPOSITION 5.1. *Let the conditions of Theorem* B *hold. Then*

(i) *for* $i,j=0,1$, *the following bounds hold*:

$$\int_{\mathbf{R}^n} \big|Q_0^{ij}(x,y)\big|\,dy \leqq C<\infty, \qquad x\in\mathbf{R}^n,$$
$$\int_{\mathbf{R}^n} \big|Q_0^{ij}(x,y)\big|\,dx \leqq C<\infty, \qquad y\in\mathbf{R}^n,$$

where the constant C *does not depend on* $x,y\in\mathbf{R}^n$;

(ii) $\widehat{q}_{\pm}^{ij}\in L^1(\mathbf{R}^n)\otimes M^2$, $i,j=0,1$.

Proof. (i) Conditions **S0**, **S2**, and **S3** imply, by Theorem 17.2.3 of [21], that for $\alpha,\beta\in\mathbf{Z}^n$, $|\alpha|\leqq 1-i$, and $|\beta|\leqq 1-j$ with $i,j=0,1$, the following bounds hold:

$$\big|D^{\alpha,\beta}_{x,y}Q_0^{ij}(x,y)\big|\leqq Ce_0\varphi^{1/2}\big(|x-y|\big), \qquad x,y\in\mathbf{R}^n. \tag{5.1}$$

The mixing coefficient φ is bounded, and hence (5.1) and (2.11) imply

$$\begin{aligned}\int_{\mathbf{R}^n}\big|D^{\alpha,\beta}_{x,y}Q_0^{ij}(x,y)\big|^p\,dy &\leqq Ce_0^p\int_{\mathbf{R}^n}\varphi^{p/2}\big(|x-y|\big)\,dy\\ &\leqq C_1e_0^p\int_0^\infty r^{n-1}\varphi^{1/2}(r)\,dr<\infty, \qquad p\geqq 1.\end{aligned}\tag{5.2}$$

(ii) Similarly to (5.1), for $\gamma\in\mathbf{Z}^n$ with $|\gamma|\leqq 2-i-j$, $i,j=0,1$, we have

$$\big|D^\gamma_z q_{\pm}^{ij}(z)\big|\leqq Ce_0\varphi^{1/2}(|z|), \qquad z\in\mathbf{R}^n. \tag{5.3}$$

Hence, by (2.11) we obtain that for $p\geqq 1$ (cf. (5.2))

$$D^\gamma q_{\pm}^{ij}(z)\in L^p(\mathbf{R}^n)\otimes M^2. \tag{5.4}$$

Further, by Bohner's theorem, a distribution $\widehat{q}_{\pm}\equiv(\widehat{q}_{\pm}^{ij}(k))\,dk$ is a positive-definite matrix-valued measure on $\mathbf{R}^n$, and **S2** implies that the total measure $\widehat{q}_{\pm}(\mathbf{R}^n)$ is finite. Finally, (5.4) with $p=2$ implies that $\widehat{q}_{\pm}^{ij}\in L^2(\mathbf{R}^n)\otimes M^2$.

COROLLARY 5.1. (i) *Proposition* 5.1(i) *implies, by the Shur lemma, that the quadratic form* $\mathcal{Q}_0(\Psi,\Psi)=\langle Q_0(x,y),\Psi(x)\otimes\Psi(y)\rangle$ *is continuous in* $L^2(\mathbf{R}^n)\otimes\mathbf{C}^2$.

(ii) *Similarly to Proposition* 5.1(ii), *estimate* (5.4) *and Bohner's theorem imply that* $\omega^{2-i-j}(k)\,\widehat{q}_{\pm}^{ij}(k)\in L^1(\mathbf{R}^n)\otimes M^2$, $i,j=0,1$. *Hence, for the matrix* $\widehat{C}(k)$, *defined by* (A.3), *we have*

$$\widehat{C}(k)\,\widehat{q}_{\pm}(k)\,\widehat{C}^T(k),\quad \widehat{C}(k)\,\widehat{q}_{\pm}(k),\quad \widehat{q}_{\pm}(k)\,\widehat{C}^T(k)\in L^1(\mathbf{R}^n)\otimes M^4. \tag{5.5}$$

Therefore, together with (2.16)–(2.18), *it implies that* $\widehat{q}_{\infty}^{ij}\in L^1(\mathbf{R}^n)\otimes M^2$ *for all* i,j.

COROLLARY 5.2. *The quadratic form* $\mathcal{Q}_\infty(\Psi,\Psi)$ *is continuous in* $L^2(\mathbf{R}^n)\otimes\mathbf{C}^2$.

Proof. The proof follows from the explicit formulas (2.12)–(2.15). Indeed, first, $\mathcal{E}(z)\in L^1(\mathbf{R}^n)$. Second, for any $\psi\in L^2$ we have

$$\big\langle(\mathcal{P}*q_{\pm}^{ij})(x-y),\,\psi(x)\otimes\psi(y)\big\rangle=\big\langle q_{\pm}^{ij}(x-y),\,(\check{\mathcal{P}}*\psi)(x)\otimes\psi(y)\big\rangle,$$

where $\check{\mathcal{P}}(x):=\mathcal{P}(-x)$. Note that $\|\mathcal{P}*\psi\|_{L^2}\leqq C\|\psi\|_{L^2}$. Hence, the continuity of $Q_\infty(\Psi,\Psi)$ follows from the Shur lemma by (5.4) with $p=1$.

5.2. Splitting of the initial covariance.

LEMMA 5.1. *The Fourier transforms of the functions $\zeta_\pm \in C^\infty(\mathbf{R})$ admit the following representations*:

$$\widehat{\zeta}_\pm(k) = \pi\delta(k) \pm i\,\mathrm{PV}\Big(\frac{1}{k}\Big)\,\widehat{\alpha}_\pm(k), \tag{5.6}$$

where $\alpha_\pm \in C_0^\infty(\mathbf{R})$.

Proof. Denote $\alpha_\pm(x) := \pm\zeta'_\pm(x)$, $x \in \mathbf{R}$. Then $\zeta_+(x) = \int_{-\infty}^{x} \alpha_+(y)\,dy$, $\zeta_-(x) = \int_x^{+\infty} \alpha_-(y)\,dy$. By virtue of (2.10) the functions $\alpha_\pm$ satisfy the following properties: (i) $\alpha_\pm \in C_0^\infty(\mathbf{R})$, (ii) $\alpha_\pm(x) = 0$ for $|x| > a$, (iii) $\int_{-a}^{a} \alpha_\pm(y)\,dy = 1$; hence, $\widehat{\alpha}_\pm(0) = 1$. Therefore,

$$\zeta_\pm(x) = \int_{-\infty}^{+\infty} \theta(\pm y)\,\alpha_\pm(x-y)\,dy, \tag{5.7}$$

where $\theta(x)$ is a Heaviside function. Denote by PV the Cauchy principal part. Since $\hat{\theta}(k) = \pi\delta(k) + i\mathrm{PV}(1/k)$, $k \in \mathbf{R}$, due to (5.7) we have

$$\widehat{\zeta}_\pm(k) = \Big[\pi\delta(k) \pm i\,\mathrm{PV}\Big(\frac{1}{k}\Big)\Big]\,\widehat{\alpha}_\pm(k) = \pi\delta(k) \pm i\,\mathrm{PV}\Big(\frac{1}{k}\Big)\,\widehat{\alpha}_\pm(k).$$

Conditions **S1** and **S2** imply that $Q_0(x, y)$ is a continuous bounded function. Hence, it belongs to the Schwarz space of tempered distributions as well as its Fourier transform. Let us apply the Fourier transform to the function $Q_0(x, y)$:

$$\widehat{Q}_0(k, k') := F_{\substack{x\to k\\ y\to -k'}}\, Q_0(x, y), \qquad k, k' \in \mathbf{R}^n. \tag{5.8}$$

Then the following proposition holds.

PROPOSITION 5.2. *Let conditions* **S0**–**S3** *hold. Then*

$$\widehat{Q}_0(k, k') = \widehat{Q}_0^1(k, k') + \widehat{Q}_0^2(k, k') + \widehat{Q}_0^3(k, k'), \tag{5.9}$$

where the summands admit the following representations:

$$\widehat{Q}_0^1(k, k') = \delta(k-k')(2\pi)^n\,\frac{1}{4}\,\big(\widehat{q}_+(k) + \widehat{q}_-(k)\big), \tag{5.10}$$

$$\begin{aligned}\widehat{Q}_0^2(k, k') &= \delta(\mathbf{k}-\mathbf{k}')(2\pi)^{n-2}\\ &\quad\times \sum_\pm \mathrm{PV}\int_{-\infty}^{+\infty} \frac{\widehat{\alpha}_\pm(k_n-\xi)}{k_n-\xi}\,\frac{\overline{\widehat{\alpha}_\pm}(k'_n-\xi)}{k'_n-\xi}\,\widehat{q}_\pm(\mathbf{k}, \xi)\,d\xi,\end{aligned} \tag{5.11}$$

$$\begin{aligned}\widehat{Q}_0^3(k, k') &= \delta(\mathbf{k}-\mathbf{k}')(2\pi)^{n-2}\pi i\,\mathrm{PV}\Big(\frac{1}{k_n-k'_n}\Big)\\ &\quad\times \Big[\widehat{q}_+(k)\,\overline{\widehat{\alpha}_+}(k'_n-k_n) + \widehat{q}_+(k')\,\widehat{\alpha}_+(k_n-k'_n)\\ &\qquad - \widehat{q}_-(k)\,\overline{\widehat{\alpha}_-}(k'_n-k_n) - \widehat{q}_-(k')\,\widehat{\alpha}_-(k_n-k'_n)\Big].\end{aligned} \tag{5.12}$$

Here and in what follows we set $k = (\mathbf{k}, k_n)$, $\mathbf{k} = (k_1, \ldots, k_{n-1})$.

Proof. Using the equality $\widehat{fg} = (2\pi)^{-2n} \hat{f} * \widehat{g}$ for the tempered distributions in $\mathbf{R}^{2n}$, we get by (2.9), formally,

$$
\begin{aligned}
\widehat{Q}_0(k,k') &:= F_{\substack{x\to k\\ y\to -k'}} \Big[\sum_{\pm} \zeta_\pm(x_n)\,\zeta_\pm(y_n)\, q_\pm(x-y) \Big] \\
&= (2\pi)^{-2n} \sum_{\pm} \Big(F_{x\to k}\big(\zeta_\pm(x_n)\big)\, \overline{F_{y\to k'}\big(\zeta_\pm(y_n)\big)} \Big) * (2\pi)^n\, \widehat{q}_\pm(k)\,\delta(k-k') \\
&= (2\pi)^{n-2} \delta(\mathbf{k}-\mathbf{k}') \sum_{\pm} \int_{\mathbf{R}^1} \Big[\widehat{\zeta}_\pm(k_n-\xi)\, \overline{\widehat{\zeta}_\pm}(k_n'-\xi)\, \widehat{q}_\pm(\mathbf{k},\xi) \Big]\, d\xi, \qquad (5.13)
\end{aligned}
$$

where $*$ stands for the convolution in k and k'. The convolution exists in the sense of tempered distributions because the distribution $\widehat{\zeta}_\pm(\xi)$ is a smooth function at $\xi \neq 0$ which decreases rapidly as $|\xi| \to \infty$, and $\widehat{q}_\pm$ are bounded continuous functions. The last integral exists by the same reasoning as the limit of Riemann integral sums over ξ with the values in tempered distributions of (k,k'). We substitute (5.6) in (5.13) and obtain

$$
\begin{aligned}
\widehat{Q}_0(k,k') = (2\pi)^{n-2}\delta(\mathbf{k}-\mathbf{k}') \sum_{\pm} \mathrm{PV}\!\!\int_{\mathbf{R}^1} \widehat{q}_\pm(\mathbf{k},\xi) \Big[\pi\delta(k_n-\xi) \pm i\, \frac{\widehat{\alpha}_\pm(k_n-\xi)}{k_n-\xi} \Big] \\
\times \Big[\pi\delta(k_n'-\xi) \mp i\, \frac{\overline{\widehat{\alpha}_\pm}(k_n'-\xi)}{k_n'-\xi} \Big]\, d\xi. \qquad (5.14)
\end{aligned}
$$

Finally, (5.14) implies formulas (5.9)–(5.12).

6. Uniform bounds and convergence of covariance. In this section we prove a uniform bound and convergence (1.10) for the covariance $Q_t(x,y)$ of the measure μ_t introduced in Definition 2.4. Denote

$$
\mathcal{Q}_t(\Psi,\Psi) := \big\langle Q_t(x,y),\, \Psi(x)\otimes\Psi(y) \big\rangle, \qquad \Psi \in \mathcal{S}, \qquad (6.1)
$$

where $\mathcal{S} = S \oplus S$, and $S \equiv S(\mathbf{R}^n)$ denotes Schwarz space. Introduce a subspace of test functions $\mathcal{S}_0 \subset \mathcal{S}$:

$$
\mathcal{S}_0 = \textstyle\bigcup_N \mathcal{S}_N, \quad \mathcal{S}_N := \{\Psi \in \mathcal{S}\colon\ \widehat{\Psi}(k) = 0 \text{ for } |k| \geqq N \text{ or } |k_n| \leqq N^{-1}\}. \qquad (6.2)
$$

Lemma 6.1. *Let $\lim_{t\to\infty} \mathcal{Q}_t(\Psi,\Psi) = \mathcal{Q}_\infty(\Psi,\Psi)$ for any $\Psi \in \mathcal{S}_0$. Then the convergence holds for all $\Psi \in \mathcal{S}$.*

Proof. At first, from (A.1) it follows that

$$
\big\langle Y(x,t),\, \Psi(x) \big\rangle = \big\langle Y_0(x),\, \Phi(x,t) \big\rangle,
$$

where $\Phi(\cdot,t) = F^{-1}[\widehat{\mathcal{G}}_t^*(k)\, \widehat{\Psi}(k)]$. Therefore, $\mathcal{Q}_t(\Psi,\Psi) = \mathcal{Q}_0(\Phi(\cdot,t),\Phi(\cdot,t))$. Hence,

$$
\sup_{t\in\mathbf{R}} \big|\mathcal{Q}_t(\Psi,\Psi)\big| \leqq C \sup_{t\in\mathbf{R}} \big\|\Phi(\cdot,t)\big\|_{L^2}^2 \qquad (6.3)
$$

by Corollary 5.1(i). By the Parseval identity and (A.3), (A.4), we obtain

$$
\big\|\Phi(\cdot,t)\big\|_{L^2}^2 = (2\pi)^{-n} \int \big|\widehat{\mathcal{G}}_t^*(k)\, \widehat{\Psi}(k)\big|^2\, dk \leqq C \|\Psi\|_{H^1(\mathbf{R}^n)}^2. \qquad (6.4)
$$

For any $\Psi \in \mathcal{S}$ we can choose $\Psi_N \in \mathcal{S}_N$ such that

$$\widehat{\Psi}_N(k) = \begin{cases} \widehat{\Psi}(k) & \text{if} \quad |k| \leqq \dfrac{N}{2} \text{ and } |k_n| \geqq \dfrac{2}{N}, \\ 0 & \text{if} \quad |k| \geqq N \text{ or } |k_n| \leqq \dfrac{1}{N}, \end{cases}$$

and, moreover,

$$\|\Psi_N - \Psi\|^2_{H^1} = \int \left(|k|^2 + 1\right) \left|\widehat{\Psi}_N(k) - \widehat{\Psi}(k)\right|^2 dk \longrightarrow 0, \qquad N \to \infty. \tag{6.5}$$

Hence, Lemma 6.1 follows from (6.3)–(6.5) and Corollary 5.2.

PROPOSITION 6.1. *Let conditions* **S0**–**S3** *hold. Then*

(i) *the function* $Q_t(x,y)$ *is continuous and*

$$\sup_{t \geqq 1} \sup_{x \in B_R} \left|Q_t(x,x)\right| < \infty, \qquad R > 0; \tag{6.6}$$

(ii) *the correlation functions converge in the sense of distributions, i.e.,*

$$\mathcal{Q}_t(\Psi, \Psi) \longrightarrow \mathcal{Q}_\infty(\Psi, \Psi), \qquad t \to \infty, \quad \Psi \in \mathcal{S}. \tag{6.7}$$

Proof. Since the solution $Y(t)$ of problem (3.1) has the form $Y(t) = (\mathcal{G}_t(\cdot) * Y_0)(x)$, the correlation $Q_t(x,y)$ admits representation in the form of convolution

$$Q_t(x,y) = \int_{\mathbf{R}^{2n}} \left(\mathcal{G}_t(x - x')\, Q_0(x', y')\, \mathcal{G}^T_t(y - y')\right) dx'\, dy',$$

the existence of which is proved by the Fourier transform. Namely, let us apply the Fourier transform to the matrix $Q_t(x,y)$:

$$\widehat{Q}_t(k,k') := F_{\substack{x \to k \\ y \to -k'}} Q_t(x,y) = \widehat{\mathcal{G}}_t(k)\, \widehat{Q}_0(k,k')\, \widehat{\mathcal{G}}^T_t(-k'), \qquad k, k' \in \mathbf{R}^n,$$

where the matrix $\widehat{\mathcal{G}}_t(k)$ is defined by (A.4), and $\widehat{Q}_0(k,k')$ by (5.8). Using the equality $\widehat{\mathcal{G}}^T_t(-k') = \widehat{\mathcal{G}}^T_t(k')$ and decomposition (5.9), we split $Q_t(x,y)$ into three terms: $Q_t(x,y) = Q^1_t(x,y) + Q^2_t(x,y) + Q^3_t(x,y)$, where

$$Q^j_t(x,y) := (2\pi)^{-2n} \int_{\mathbf{R}^{2n}} e^{-ikx + ik'y} \widehat{\mathcal{G}}_t(k)\, \widehat{Q}^j_0(k,k')\, \widehat{\mathcal{G}}^T_t(k')\, dk\, dk', \tag{6.8}$$

$x, y \in \mathbf{R}^n$, $t > 0$, $j = 1, 2, 3$. Then to prove Proposition 6.1 it suffices to verify bound (6.6) and convergence (6.7) to a limit for each term $Q^j_t(x,y)$ with $j = 1, 2, 3$. We do it in Lemmas 6.2, 6.3, and 6.5 as demonstrated below.

LEMMA 6.2. (i) *The function* $Q^1_t(x,y)$ *is continuous and*

$$\sup_{t \geqq 0} \sup_{x \in \mathbf{R}^n} Q^1_t(x,x) \leqq C < \infty.$$

(ii) $Q^1_t(x,y) \to q^+_\infty(x-y)/2$ *as* $t \to \infty$ *for all* $x, y \in \mathbf{R}^n$, *where the matrix* $\widehat{q}^+_\infty$ *is defined in* (2.17).

Proof. (i) Substitute (5.10) in (6.8) and obtain

$$\widehat{Q}^1_t(k,k') = (2\pi)^n \delta(k - k')\, \widehat{\mathcal{G}}_t(k)\, \frac{1}{2}\, \widehat{\mathbf{q}}^+(k)\, \widehat{\mathcal{G}}^T_t(k), \tag{6.9}$$

where $\widehat{\mathbf{q}}^+(k) := (\widehat{q}_+(k) + \widehat{q}_-(k))/2$. Hence,

$$Q_t^1(x,y) \equiv q_t^1(x-y) = (2\pi)^{-n}\,\frac{1}{2}\int_{\mathbf{R}^n} e^{-i(x-y)k}\,\widehat{\mathcal{G}}_t(k)\,\widehat{\mathbf{q}}^+(k)\,\widehat{\mathcal{G}}_t^T(k)\,dk, \tag{6.10}$$

where $x, y \in \mathbf{R}^n$. Hence, (5.5) and (A.4) imply Lemma 6.2(i).

(ii) Applying (A.6) to $\widehat{q}(k) := \widehat{\mathbf{q}}^+(k)$, we obtain

$$q_t^1(z) = (2\pi)^{-n}\,\frac{1}{2}\int_{\mathbf{R}^n} e^{-izk}\,\widehat{q}_\infty^+(k)\,dk + o(1), \qquad t\to\infty, \quad z\in\mathbf{R}^n,$$

since the remaining oscillatory integrals in (6.10) vanish as $t\to\infty$ by (5.5) and the Lebesgue–Riemann theorem. Lemma 6.2 is proved.

Lemma 6.3. (i) *The function $Q_t^2(x,y)$ is continuous and*

$$\sup_{t\geqq 1}\sup_{x\in B_R} Q_t^2(x,x) \leqq C < \infty \qquad \textit{for any} \quad R>0.$$

(ii) $\lim_{t\to\infty}\langle Q_t^2(x,y), \Psi(x)\otimes\Psi(y)\rangle = \frac{1}{2}\langle q_\infty^+(x-y), \Psi(x)\otimes\Psi(y)\rangle$, $\Psi\in\mathcal{S}$.

Proof. (i) Substitute (5.11) in (6.8) and obtain

$$\begin{aligned} Q_t^2(x,y) &\equiv (2\pi)^{-2n}\int_{\mathbf{R}^{2n}} e^{-ikx+ik'y}\,\widehat{\mathcal{G}}_t(k)\,\widehat{Q}_0^2(k,k')\,\widehat{\mathcal{G}}_t^T(k')\,dk\,dk' \\ &= (2\pi)^{-n-2}\sum_{\pm}\int_{\mathbf{R}^{n+1}} d\mathbf{k}\,dk_n\,dk_n' \\ &\qquad\times\left[e^{-ikx+ik'y}\,\widehat{\mathcal{G}}_t(k)\,\mathrm{PV}\int_{-\infty}^{+\infty}\frac{\widehat{\alpha}_\pm(k_n-\xi)}{k_n-\xi}\,\frac{\overline{\widehat{\alpha}_\pm}(k_n'-\xi)}{k_n'-\xi}\right. \\ &\qquad\qquad\left.\times\,\widehat{q}_\pm(\mathbf{k},\xi)\,d\xi\,\widehat{\mathcal{G}}_t^T(k')\right]\Big|_{k'=(\mathbf{k},k_n')}. \end{aligned} \tag{6.11}$$

We change variables and obtain the representation

$$Q_t^2(x,y) = (2\pi)^{-n-2}\sum_{\pm}\int_{\mathbf{R}^n} e^{-ik(x-y)}J_\pm(t,x_n,k)\,\widehat{q}_\pm(k)\,J_\pm^*(t,y_n,k)\,dk, \tag{6.12}$$

where by $J_\pm(t,x_n,k) = (J_\pm^{ij}(t,x_n,k))_{i,j=0,1}$ we denote the matrix-valued integral

$$J_\pm(t,x_n,k) := \mathrm{PV}\int_{-\infty}^{+\infty} e^{-i\xi x_n}\,\frac{\widehat{\alpha}_\pm(\xi)}{\xi}\,\widehat{\mathcal{G}}_t(\mathbf{k},k_n+\xi)\,d\xi, \tag{6.13}$$

and $J_\pm^*$ stands for the Hermitian conjugate.

Proposition 6.2. *For any $k\in\mathbf{R}^n$ the functions $J_\pm(t,x_n,k)$ are continuous and uniformly bounded on $t>1$ and $x_n\in[-R,R]$; moreover,*

$$\sup_{t\geqq 1,|x_n|\leqq R}\left|J_\pm^{ij}(t,x_n,k)\right| < C_1 + C_2\left|\widehat{C}^{ij}(k)\right|, \qquad i,j=0,1, \tag{6.14}$$

where $\widehat{C}^{ij}(k)$ are defined in (A.3) *and the constants C_1, C_2 do not depend on k.*

The proof of Proposition 6.2 is shown in Appendix B. Now item (i) of Lemma 6.3 follows from (6.12) and from estimate (6.14) by (5.5) and the Lebesgue dominated convergence theorem.

(ii) According to Lemma 6.1 it suffices to consider $\Psi \in \mathcal{S}_N$ with any fixed $N \in \mathbf{N}$. By (6.12), we obtain

$$\Big\langle Q_t^2(x,y),\, \Psi(x)\otimes\Psi(y)\Big\rangle = (2\pi)^{-2n}\Big\langle \widehat{\mathcal{G}}_t(k)\,\widehat{Q}_0^2(k,k')\,\widehat{\mathcal{G}}_t^T(k'),\, \widehat{\Psi}(k)\otimes\overline{\widehat{\Psi}}(k')\Big\rangle$$

$$= (2\pi)^{-n-2}\sum_{\pm}\int_{\mathbf{R}^n} j_\pm(t,k)\,\widehat{q}_\pm(k) j_\pm^*(t,k)\,dk, \tag{6.15}$$

where by $j_\pm(t,k)$ we denote an inner vector-valued integral

$$j_\pm(t,k) := \mathrm{PV}\int_{-\infty}^{+\infty}\frac{\widehat{\alpha}_\pm(\xi)}{\xi}\,\overline{\widehat{\Psi}}(\mathbf{k},k_n+\xi)\,\widehat{\mathcal{G}}_t(\mathbf{k},k_n+\xi)\,d\xi. \tag{6.16}$$

Recall $\Psi \in \mathcal{S}_N$; therefore $k \in \operatorname{supp}\widehat{\Psi} \subset B_N^0 := \{k \in B_N\colon\ |k_n| \geqq 1/N\}$, where by B_N we denote a ball of a radius N.

Lemma 6.4. *Let $\Psi \in \mathcal{S}_N$ with some $N \in \mathbf{N}$. Then for $k \in \mathbf{R}^n$,*

$$j_\pm(t,k) = -\pi\,\operatorname{sign} k_n\overline{\widehat{\Psi}}(k)\left[\sin\omega(k)\,t - \cos\omega(k)\,t\,\widehat{C}(k)\right] + o(1) \tag{6.17}$$

as $t\to+\infty$; $o(1)$ tends to zero uniformly on $k \in \mathbf{R}^n$.

Proof. From (A.4) it follows that it suffices to prove (6.17) for the integrals of the form

$$j_*(t,k) := \mathrm{PV}\int_{-\infty}^{+\infty} e^{\pm i\omega(\mathbf{k},k_n+\xi)\,t}\,\frac{\widehat{\alpha}_+(\xi)}{\xi}\,g(\mathbf{k},k_n+\xi)\,d\xi, \tag{6.18}$$

where $g(k) \in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp} g \subset B_N^0$. Since $g(\mathbf{k},k_n+\xi) = 0$ for $|k_n+\xi| \leqq 1/N$, we have

$$\big|\nabla_n\omega(\mathbf{k},k_n+\xi)\big| = \frac{|k_n+\xi|}{\omega(\mathbf{k},k_n+\xi)} \geqq C(N) > 0,$$

for $(\mathbf{k},k_n+\xi) \in \operatorname{supp} g(\mathbf{k},k_n+\xi)$ and $k \in B_N^0$. Therefore we can apply Lemma 5 from Chapter VII of [4, p. 151] to the integral $j_*(t,k)$ and, since $\widehat{\alpha}_+(0) = 1$, we conclude that $j_*(t,k) = g(k)\,e^{\pm i\omega(k)\,t}\pi i\ \operatorname{sign}(\pm\nabla_n\omega(k)) + o(1)$, $t\to+\infty$. Lemma 6.4 is proved.

Substituting (6.17) in (6.15) and applying (A.7) to $\widehat{q}(k) = \widehat{q}_+(k) + \widehat{q}_-(k)$, we obtain, by (2.17), that

$$\begin{aligned}&\big\langle Q_t^2(x,y),\, \Psi(x)\otimes\Psi(y)\big\rangle\\ &\quad= (2\pi)^{-n-2}\pi^2\int_{\mathbf{R}^n}\overline{\widehat{\Psi}}(k)\big[\sin\omega(k)\,t - \cos\omega(k)\,t\,\widehat{C}(k)\big]\big(\widehat{q}_+(k)+\widehat{q}_-(k)\big)\\ &\qquad\qquad\times\big[\sin\omega(k)\,t\,I - \cos\omega(k)\,t\,\widehat{C}^T(k)\big]\,\widehat{\Psi}(k)\,dk + o(1)\\ &\quad= \frac{1}{(2\pi)^n}\,\frac{1}{2}\,\big\langle\widehat{q}_\infty^+(k), \widehat{\Psi}(k)\otimes\overline{\widehat{\Psi}}(k)\big\rangle + o(1), \qquad t\to\infty,\end{aligned} \tag{6.19}$$

since the remaining oscillatory integrals vanish as $t\to\infty$ by the Lebesgue–Riemann theorem and Corollary 5.1. Lemma 6.3 is proved.

Lemma 6.5. (i) *The function $Q_t^3(x,y)$ is continuous and*

$$\sup_{t\geqq 1}\,\sup_{x\in B_R} Q_t^3(x,x) \leqq C < \infty \qquad \textit{for any}\quad R > 0.$$

(ii) $\lim_{t\to\infty}\langle Q_t^3(x,y),\Psi(x)\otimes\Psi(y)\rangle = \langle q_\infty^-(x-y),\Psi(x)\otimes\Psi(y)\rangle$ *for any* $\Psi\in\mathcal{S}$, *where the matrix* $\widehat{q}_\infty^-$ *is defined in* (2.18).

Proof. (i) Substituting (5.12) in (6.8) we obtain

$$Q_t^3(x,y) \equiv (2\pi)^{-2n}\int_{\mathbf{R}^{2n}} e^{-ikx+ik'y}\,\widehat{\mathcal{G}}_t(k)\,\widehat{Q}_0^3(k,k')\,\widehat{\mathcal{G}}_t^T(k')\,dk\,dk'$$

$$= (2\pi)^{-n-2}\pi i\ \mathrm{PV}\int_{\mathbf{R}^{n+1}}\Big[e^{-ikx+ik'y}\,\widehat{\mathcal{G}}_t(k)\,\widehat{q}_0^{\,3}(k,k')\,\widehat{\mathcal{G}}_t^{\,T}(k')\Big]\Big|_{k'=(\mathbf{k},k_n')}dk_n'\,dk. \tag{6.20}$$

Here we denote

$$\widehat{q}_0^3(k,k') := \frac{1}{k_n-k_n'}\Big[\overline{\widehat{\alpha}_+}(k_n'-k_n)\,\widehat{q}_+(k)+\widehat{\alpha}_+(k_n-k_n')\,\widehat{q}_+(k') \\ -\overline{\widehat{\alpha}_-}\,(k_n'-k_n)\,\widehat{q}_-(k)-\widehat{\alpha}_-(k_n-k_n')\,\widehat{q}_-(k')\Big]. \tag{6.21}$$

Let us substitute (6.21) in (6.20) and consider the first of the arising integrals,

$$I_t(x,y) := (2\pi)^{-n-2}\pi i \\ \times\mathrm{PV}\int_{\mathbf{R}^{n+1}}\Big[e^{-ikx+ik'y}\widehat{\mathcal{G}}_t(k)\,\widehat{q}_+(k)\,\frac{\overline{\widehat{\alpha}_+}(k_n'-k_n)}{k_n-k_n'}\,\widehat{\mathcal{G}}_t^T(k')\Big]\Big|_{k'=(\mathbf{k},k_n')}dk_n'\,dk. \tag{6.22}$$

Changing variables $k_n'\to k_n'-k_n=\xi$ we obtain

$$I_t(x,y) = -(2\pi)^{-n-2}\pi i\int_{\mathbf{R}^n} e^{-i(x-y)k}\,\widehat{\mathcal{G}}_t(k)\,\widehat{q}_+(k)\,J_+^*(t,y_n,k)\,dk, \tag{6.23}$$

where $J_+(t,y_n,k)$ is defined in (6.13). Equalities (6.23) and (A.4), Proposition 6.2, and (5.5) imply that for $x,y\in B_R$

$$\big|I_t(x,y)\big| \leqq C\int_{\mathbf{R}^n}\big(1+|\widehat{C}(k)|\big)\big|\widehat{q}_+(k)\big|\big(1+|\widehat{C}^T(k)|\big)\,dk \leqq C_1<\infty,$$

which proves item (i) of Lemma 6.5.

(ii) According to Lemma 6.1, it suffices to prove item (ii) of Lemma 6.5 for $\Psi\in\mathcal{S}_N$ with any fixed $N\in\mathbf{N}$. Applying (6.20) we obtain

$$\big\langle Q_t^3(x,y),\Psi(x)\otimes\Psi(y)\big\rangle = (2\pi)^{-2n}\big\langle\widehat{Q}_t^3(k,k'),\ \widehat{\Psi}(k)\otimes\overline{\widehat{\Psi}}(k')\big\rangle \\ = (2\pi)^{-n-2}\pi i\,\mathrm{PV}\int_{\mathbf{R}^{n+1}}\Big[\overline{\widehat{\Psi}}(k)\,\widehat{\mathcal{G}}_t(k)\,\widehat{q}_0^3(k,k')\,\widehat{\mathcal{G}}_t^T(k')\,\widehat{\Psi}(k')\Big]\Big|_{k'=(\mathbf{k},k_n')}dk_n'\,dk. \tag{6.24}$$

Let us substitute (6.21) in (6.24) and consider, for example, $\langle I_t(x,y),\Psi(x)\otimes\Psi(y)\rangle$, where $\Psi\in\mathcal{S}_N$ and $I_t(x,y)$ defined in (6.22). Changing variables $k_n'\to k_n'-k_n=\xi$, we obtain that

$$I_t(\Psi)\equiv\big\langle I_t(x,y),\Psi(x)\otimes\Psi(y)\big\rangle = -(2\pi)^{-n-2}\pi i\int_{\mathbf{R}^n}\overline{\widehat{\Psi}}(k)\,\widehat{\mathcal{G}}_t(k)\,\widehat{q}_+(k)\,j_+^*(t,k)\,dk, \tag{6.25}$$

where $j_+(t,k)$ is defined in (6.16). Substituting (6.17) in the integral in the right-hand side of (6.25), we get

$$I_t(\Psi) = (2\pi)^{-n-2}\pi^2 i \\ \times\int_{\mathbf{R}^n}\overline{\widehat{\Psi}}(k)\,\widehat{\mathcal{G}}_t(k)\,\widehat{q}_+(k)\,\big[\sin\omega t-\cos\omega t\,\widehat{C}^T(k)\big]\ \mathrm{sign}(k_n)\,\widehat{\Psi}(k)\,dk+o(1).$$

Further, we apply (A.8) with $\widehat{q}(k)=\widehat{q}_+(k)$ and conclude that as $t\to\infty$

(6.26)
$$I_t(\Psi)=(2\pi)^{-n}\frac{i}{8}\int_{\mathbf{R}^n}\overline{\widehat{\Psi}}(k)\Big(\widehat{C}(k)\,\widehat{q}_+(k)-\widehat{q}_+(k)\,\widehat{C}^T(k)\Big)\ \mathrm{sign}(k_n)\widehat{\Psi}(k)\,dk+o(1),$$

since the remaining oscillatory integrals tend to zero as $t\to\infty$ by the Lebesgue–Riemann theorem and (5.5). This implies the convergence of $I_t(\Psi)$ to a limit as $t\to\infty$. Similar arguments give the limits of type (6.26) for all remaining terms in (6.24). Hence, finally,

$$\begin{aligned}&\big\langle Q_t^3(x,y),\,\Psi(x)\otimes\Psi(y)\big\rangle\\&\quad=(2\pi)^{-n}\,\frac{i}{2}\int_{\mathbf{R}^n}\overline{\widehat{\Psi}}(k)\Big(\widehat{C}(k)\,\widehat{\mathbf{q}}^-(k)-\widehat{\mathbf{q}}^-(k)\,\widehat{C}^T(k)\Big)\ \mathrm{sign}(k_n)\widehat{\Psi}(k)\,dk\\&\quad=(2\pi)^{-n}\big\langle\widehat{q_\infty^-}(k),\,\widehat{\Psi}(k)\otimes\overline{\widehat{\Psi}}(k)\big\rangle+o(1),\end{aligned}$$

where $t\to+\infty$, $\widehat{\mathbf{q}}^-(k)=(\widehat{q}_+(k)-\widehat{q}_-(k))/2$. Lemma 6.5 is proved.

Now Proposition 6.1 follows from Lemmas 6.2, 6.3, and 6.5.

7. Compactness of measures family. Proposition 3.2 can be deduced from the bound (7.19) with the help of the Prokhorov theorem (see Lemma 3.1 in [5]). As a preliminary we prove two auxiliary lemmas.

LEMMA 7.1. *The function $\nabla_x\cdot\nabla_y Q_t^{00}(x,y)$ is continuous and*

$$\sup_{t\geqq 1}\sup_{x\in B_R}\Big(\nabla_x\cdot\nabla_y Q_t^{00}(x,y)\,\big|_{x=y}\Big)\leqq C<\infty,\qquad R>0.\tag{7.1}$$

Proof. For simplicity, let us consider the case $Y_0^0(x)\equiv 0$ a.e. (The general case $Y_0^0(x)\not\equiv 0$ is proved similarly.) Then

$$\begin{aligned}\nabla_x\cdot\nabla_y Q_t^{00}(x,y)&=(2\pi)^{-2n}\int_{\mathbf{R}^{2n}}e^{-ikx+ik'y}\,k\cdot k'\Big[\widehat{\mathcal{G}}_t(k)\,\widehat{Q}_0(k,k')\,\widehat{\mathcal{G}}_t^T(k')\Big]^{00}dk\,dk'\\&=(2\pi)^{-2n}\int_{\mathbf{R}^{2n}}e^{-ikx+ik'y}\frac{\sin\omega(k)\,t}{\omega(k)}\,k\cdot k'\widehat{Q}_0^{11}(k,k')\,\frac{\sin\omega(k')\,t}{\omega(k')}\,dk\,dk'.\end{aligned}\tag{7.2}$$

As in the proof of Proposition 6.1, we represent $\nabla_x\cdot\nabla_y Q_t^{00}(x,y)$ as a sum:

$$\nabla_x\cdot\nabla_y Q_t^{00}(x,y)=\sum_{j=1}^{3}\nabla_x\cdot\nabla_y\big[Q_t^j(x,y)\big]^{00},\tag{7.3}$$

where each term $\nabla_x\cdot\nabla_y[Q_t^j(x,y)]^{00}$ is defined similarly to (7.2) with the function $[\widehat{Q}_0^j(k,k')]^{11}$ in the integrand instead of $\widehat{Q}_0^{11}(k,k')$. Further, we estimate each term $\nabla_x\cdot\nabla_y[Q_t^j(x,y)]^{00}$ separately by the methods of Lemmas 6.2, 6.3, and 6.5.

I. From (5.10) and (7.2) it follows that

(7.4)
$$\nabla_x\cdot\nabla_y[Q_t^1(x,y)]^{00}=(2\pi)^{-n}\frac{1}{4}\int_{\mathbf{R}^n}e^{-ik(x-y)}\ \frac{|k|^2(\widehat{q}_+^{11}(k)+\widehat{q}_-^{11}(k))}{\omega^2(k)}\big(\sin\omega(k)\,t\big)^2\,dk.$$

Hence, by Proposition 5.1(ii), we obtain that the function $\nabla_x \cdot \nabla_y [Q_t^1(x,y)]^{00}$ is continuous by the Lebesgue dominated convergence theorem. Moreover,

$$\sup_{t\geqq 0} \Big| \nabla_x \cdot \nabla_y \big[Q_t^1(x,y)\big]^{00} \Big|_{y=x} \Big| \leqq C \int \Big(\big|\widehat{q}_+^{11}(k)\big| + \big|\widehat{q}_-^{11}(k)\big| \Big)\, dk < \infty. \tag{7.5}$$

II. Consider the second term in the right-hand side of (7.3) (cf. (6.11)):

$$\begin{aligned} &\nabla_x \cdot \nabla_y \big[Q_t^2(x,y)\big]^{00} \\ &\quad = (2\pi)^{-n-2} \sum_{\pm} \int_{\mathbf{R}^{n+1}} d\mathbf{k}\, dk_n\, dk_n' \Big[e^{-ikx+ik'y}\, k\cdot k' \, \frac{\sin\omega(k)\,t}{\omega(k)}\, \frac{\sin\omega(k')\,t}{\omega(k')} \\ &\qquad \times \mathrm{PV} \int_{-\infty}^{+\infty} \frac{\widehat{\alpha}_\pm(k_n-\xi)}{k_n-\xi}\, \frac{\overline{\widehat{\alpha}_\pm}(k_n'-\xi)}{k_n'-\xi}\, \widehat{q}_\pm^{11}(\mathbf{k},\xi)\, d\xi \Big] \Big|_{k'=(\mathbf{k},k_n')}. \end{aligned} \tag{7.6}$$

Changing variables, we obtain that

$$\begin{aligned} \nabla_x \cdot \nabla_y \big[Q_t^2(x,y)\big]^{00} = C \sum_{\pm} \int_{\mathbf{R}^n} e^{-ik(x-y)} \Big(& J_\pm^{01}(t,x_n,k)\, \overline{J_\pm^{01}(t,y_n,k)}\, |\mathbf{k}|^2 \\ & + \tilde{J}_\pm(t,x_n,k)\, \overline{\tilde{J}_\pm(t,y_n,k)} \Big)\, \widehat{q}_\pm^{11}(k)\, dk, \end{aligned} \tag{7.7}$$

where $J_\pm^{01}(t,x_n,k)$ is defined in (6.13), and

$$\tilde{J}_\pm(t,x_n,k) := \mathrm{PV} \int_{-\infty}^{+\infty} e^{-i\xi x_n}\, \frac{\widehat{\alpha}_\pm(\xi)}{\xi}\, \sin\omega(\mathbf{k},k_n+\xi)\,t\, \frac{k_n+\xi}{\omega(\mathbf{k},k_n+\xi)}\, d\xi. \tag{7.8}$$

By Lemma B.1, we obtain the estimate $|J_\pm^{01}(t,x_n,k)| \leqq C_1/\omega(k)$. Since

$$\left| \frac{k_n+\xi}{\omega(\mathbf{k},k_n+\xi)} - \frac{k_n}{\omega(k)} \right| \leqq C|\xi|,$$

it follows that $\sup_{t\geqq 1, |x|\leqq R} |\tilde{J}_\pm(t,x_n,k)| \leqq C_1 < \infty$ by Lemma B.1. Hence, by virtue of Proposition 5.1(ii), the function $\nabla_x \cdot \nabla_y [Q_t^2(x,y)]^{00}$ is continuous by the Lebesgue dominated convergence theorem. Moreover,

$$\Big| \nabla_x \cdot \nabla_y \big[Q_t^2(x,y)\big]^{00} \big|_{x=y} \Big| \leqq C \int_{\mathbf{R}^n} \left(\frac{C_1 |\mathbf{k}|^2}{\omega^2(k)} + C_2 \right) \Big(\big|\widehat{q}_+^{11}(k)\big| + \big|\widehat{q}_-^{11}(k)\big| \Big)\, d\mathbf{k}\, dk_n < \infty. \tag{7.9}$$

III. Applying (5.12) and (7.2), we obtain (cf. (6.20))

$$\begin{aligned} \nabla_x \cdot \nabla_y \big[Q_t^3(x,y)\big]^{00} = C_0\, \mathrm{PV} \int_{\mathbf{R}^{n+1}} & e^{-ixk+iyk'}\, \frac{\sin\omega(k)\,t}{\omega(k)}\, k\cdot k' \big[\widehat{q}_0^3(k,k')\big]^{11} \\ & \times \frac{\sin\omega(k')\,t}{\omega(k')} \Big|_{k'=(\mathbf{k},\,k_n')} dk\, dk_n', \end{aligned} \tag{7.10}$$

where $C_0 = (2\pi)^{-n-2}\pi i$ and $\widehat{q}_0^3(k,k')$ is defined in (6.21). We substitute (6.21) and estimate one of the integrals (for the remaining integrals the proof is similar):

$$\begin{aligned} I_t(x,y) := C_0\, \mathrm{PV} \int_{\mathbf{R}^{n+1}} & e^{-ixk+iyk'}\, \frac{\sin\omega(k)\,t}{\omega(k)}\, k\cdot k'\, \widehat{q}_+^{11}(k) \\ & \times \frac{\sin\omega(k')\,t}{\omega(k')}\, \frac{\overline{\widehat{\alpha}_+}(k_n'-k_n)}{k_n-k_n'} \Big|_{k'=(\mathbf{k},\,k_n')} dk\, dk_n'. \end{aligned} \tag{7.11}$$

Changing variables $k'_n \to k'_n - k_n = \xi$, we obtain that

$$I_t(x,y) = -C_0 \int_{\mathbf{R}^n} e^{-i(x-y)k} \frac{\sin\omega(k)\,t}{\omega(k)} \,\widehat{q}_+^{11}(k)\, J_2(t,y_n,k)\,dk, \tag{7.12}$$

where

$$J_2(t,y_n,k) := \mathrm{PV}\int_{-\infty}^{+\infty} e^{i\xi y_n} \frac{\overline{\widehat{\alpha_+}(\xi)}}{\xi} \sin\omega(\mathbf{k},k_n+\xi)\,t\, \frac{k^2+k_n\xi}{\omega(\mathbf{k},k_n+\xi)}\,d\xi. \tag{7.13}$$

Note that

$$\left|\frac{k^2+k_n\xi}{\omega(\mathbf{k},k_n+\xi)} - \frac{k^2}{\omega(k)}\right| \leqq |\xi|\,\frac{k^2}{\omega(k)}.$$

Hence, by Lemma B.2, $\sup_{t\geqq 1,|y_n|\leqq R} |J_2(t,y_n,k)| \leqq Ck^2/\omega(k)$. Therefore, from (7.13) and Proposition 5.1(ii) it follows that the function $\nabla_x\cdot\nabla_y[Q_t^3(x,y)]^{00}$ is continuous by the Lebesgue dominated convergence theorem. Moreover,

$$\sup_{t\geqq 1, x\in B_R} \big|I_t(x,x)\big| \leqq C\int_{\mathbf{R}^n} \left|\sin\omega(k)\,t\,\frac{|k|^2}{\omega^2(k)}\,\widehat{q}_+^{11}(k)\right| dk \leqq C\|\widehat{q}_+^{11}\|_{L^1} < \infty.$$

Lemma 7.1 is proved.

Denote

$$e_t(x,x') := Q_t^{00}(x,x') + \nabla_x\cdot\nabla_{x'}Q_t^{00}(x,x') + Q_t^{11}(x,x'). \tag{7.14}$$

Lemma 7.2. *For any $R>0$, the following equality holds*:

$$\mathbf{E}\big\|U_0(t)\,Y_0(\cdot)\big\|_R^2 = \int_{|x|<R} e_t(x,x)\,dx, \qquad t\in\mathbf{R}. \tag{7.15}$$

Proof. The estimate (3.4) for $U_0(t)\,Y_0 = Y(x,t) = (Y^0(x,t),Y^1(x,t))$ implies that

$$\mathbf{E}\big\|Y(\cdot,t)\big\|_R^2 \leqq \mathbf{E}\big\|Y_0(\cdot)\big\|_{R+t}^2 < \infty \tag{7.16}$$

by condition **S2** and the Fubini theorem. Hence, the mathematical expectation $\mathbf{E}\|Y(\cdot,t)\|_R^2$ is finite for any $R>0$, $t\geqq 0$. Therefore, by the Fubini theorem, we obtain that

$$\mathbf{E}\Big(\big|Y^0(x,t)\big|^2 + \big|\nabla Y^0(x,t)\big|^2 + \big|Y^1(x,t)\big|^2\Big) < \infty, \qquad x\in X\subset\mathbf{R}^n,$$

where $\mathrm{mes}(\mathbf{R}^n\setminus X)=0$. Hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned}&\mathbf{E}\Big(\big|Y^0(x,t)\,Y^0(x',t)\big| + \big|\nabla_x Y^0(x,t)\cdot\nabla_{x'}Y^0(x',t)\big|\\ &\quad + \big|Y^1(x,t)\,Y^1(x',t)\big|\Big) < \infty, \qquad x,x'\in X.\end{aligned} \tag{7.17}$$

We take $\theta_k(x) = k^n\theta(kx)$, where $\theta(x)\in C_0^\infty(\mathbf{R}^n)$, $\int\theta(x)\,dx = 1$, and $\theta(x)\geqq 0$. Then definition (2.6) of the correlation functions gives

$$\mathbf{E}\big\|\theta_k * Y(\cdot,t)\big\|_R^2 = \int_{|x|\leqq R} dx \int_{\mathbf{R}^{2n}} \theta_k(x-y)\,\theta_k(x-y')\,e_t(y,y')\,dy\,dy'. \tag{7.18}$$

It is obvious that $\theta_k(x) \to \delta(x)$ as $k \to \infty$. Therefore, the right-hand side of (7.18) converges to $\int_{|x| \leqq R} e_t(x,x)\,dx$, since $e_t(\cdot,\cdot) \in C(\mathbf{R}^n \times \mathbf{R}^n)$, and the left-hand side converges to $\mathbf{E}\|Y(\cdot,t)\|_R^2$ by the Lebesgue dominated convergence theorem. Indeed, $\|\theta_k * Y(\cdot,t)\|_R \to \|Y(\cdot,t)\|_R$ as $k \to \infty$, and $\|\theta_k * Y(\cdot,t)\|_R \leqq \|Y(\cdot,t)\|_{R+R(\theta)}$, where $\|Y(\cdot,t)\|^2_{R+R(\theta)}$ is a summable majorant, by virtue of (7.16). Hence, (7.18) implies (7.15) as $k \to \infty$.

Corollary 7.1. *The estimate* (7.17) *implies the convergence of the integrals in* (2.5), *for a.e.* $(x,y) \in \mathbf{R}^{2n}$.

Lemma 7.3. *Let conditions* **S0**–**S3** *hold. Then*

$$\sup_{t \geqq 0} \mathbf{E}\big\|U_0(t)\,Y_0\big\|_R^2 < \infty, \qquad R > 0. \tag{7.19}$$

Proof. From Lemma 7.2 it follows that

$$\mathbf{E}\big\|Y(\cdot,t)\big\|_R^2 = \int_{|x|<R} e_t(x,x)\,dx.$$

Proposition 6.1(i) and Lemma 7.1 imply that

$$\sup_{t \geqq 1,\, x \in B_R} e_t(x,x) \leqq \bar{e} < \infty.$$

Hence,

$$\mathbf{E}\big\|U_0(t)\,Y_0\big\|_R^2 = \int_{B_R} e_t(x,x)\,dx \leqq \bar{e}|B_R| < \infty.$$

Lemma 7.3 is proved.

Finally, Proposition 3.2 follows from (7.19), by the methods of [5].

8. Convergence of characteristic functionals. In this section we apply the Bernstein "room-corridor" method to prove Proposition 3.3. We rewrite (3.5) in the form

$$\int \exp\big(i\langle U_0(t)\,Y_0, \Psi\rangle\big)\,\mu_0(dY_0) \longrightarrow \exp\Big\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\Big\}, \qquad t \to \infty. \tag{8.1}$$

We use the standard integral representation for $U_0(t)\,Y_0$, divide the domain of integration into "rooms" and "corridors," and evaluate their contribution. As the result, the expression $\langle U_0(t)\,Y_0, \Psi\rangle$ in (8.1) is represented as the sum of weakly dependent random variables. Further, we evaluate the variances of these random variables. A similar method was used in [12, section 7, pp. 17–19]. However, the proofs are not identical since in the present paper we study nontranslation-invariant measures.

First, we evaluate $\langle U_0(t)\,Y_0, \Psi\rangle$ in (8.1) by using a dual group. For $t \in \mathbf{R}$, introduce the "formal adjoint" operators $U_0'(t)$, $U'(t)$ from space $\mathcal{D}$ to a suitable space of distributions. For example,

$$\big\langle Y, U_0'(t)\,\Psi\big\rangle = \big\langle U_0(t)\,Y, \Psi\big\rangle, \qquad \Psi \in \mathcal{D}, \quad Y \in \mathcal{H}. \tag{8.2}$$

Denote $\Phi(\cdot,t) = U_0'(t)\,\Psi$. Then (8.2) can be rewritten as

$$\big\langle Y(t),\, \Psi\big\rangle = \big\langle Y_0,\, \Phi(\cdot,t)\big\rangle, \qquad t \in \mathbf{R}. \tag{8.3}$$

The adjoint groups admit a convenient description. Lemma 8.1 in what follows displays that the action of groups $U_0'(t)$, $U'(t)$ coincides, respectively, with the action of groups $U_0(t)$, $U(t)$, up to the order of the components. In particular, $U_0'(t)$, $U'(t)$ are continuous groups of operators from $\mathcal{D}$ to $\mathcal{D}$.

LEMMA 8.1 (see Lemma 7.1 in [12, p. 17]). *For* $\Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$,

$$U_0'(t)\,\Psi = \big(\dot\phi(\cdot,t),\, \phi(\cdot,t)\big), \qquad U'(t)\,\Psi = \big(\dot\psi(\cdot,t),\, \psi(\cdot,t)\big), \tag{8.4}$$

where $\phi(x,t)$ *is the solution of* (3.1) *with the initial date* $(u_0, v_0) = (\Psi^1, \Psi^0)$ *and* $\psi(x,t)$ *is the solution of* (1.1) *with the initial date* $(u_0, v_0) = (\Psi^1, \Psi^0)$.

Next we divide $\mathbf{R}^n$ into "rooms" and "corridors." Given $t > 0$, choose $d \equiv d_t \geqq 1$ and $\rho \equiv \rho_t > 0$ as follows: take $0 < \delta < 1$ and $\rho_t \sim t^{1-\delta}$, $d_t \sim t/\log t$, $t \to \infty$. Set $h = d + \rho$ and

$$a^j = jh, \quad b^j = a^j + d, \qquad j \in \mathbf{Z}. \tag{8.5}$$

We call the slabs $R_t^j = \{x \in \mathbf{R}^n \colon a^j \leqq x_n \leqq b^j\}$ "rooms" and the slabs $C_t^j = \{x \in \mathbf{R}^n \colon b^j \leqq x_n \leqq a^{j+1}\}$ "corridors." Here $x = (x_1, \dots, x_n)$, d is the width of a room, and ρ is the width of a corridor.

Denote by χ_r the indicator of the interval $[0, d]$ and by χ_c the indicator of the interval $[d, h]$ so that $\sum_{j\in\mathbf{Z}} (\chi_r(s - jh) + \chi_c(s - jh)) = 1$ for a.a. $s \in \mathbf{R}$. Then the following decomposition holds:

$$\big\langle Y_0,\, \Phi(\cdot,t)\big\rangle = \sum_{j\in\mathbf{Z}} \Big(\big\langle Y_0,\, \chi_r^j \Phi(\cdot,t)\big\rangle + \big\langle Y_0,\, \chi_c^j \Phi(\cdot,t)\big\rangle\Big), \tag{8.6}$$

where $\chi_r^j := \chi_r(x_n - jh)$ and $\chi_c^j := \chi_c(x_n - jh)$. Consider the random variables r_t^j, c_t^j, where

$$r_t^j = \big\langle Y_0,\, \chi_r^j \Phi(\cdot,t)\big\rangle, \quad c_t^j = \big\langle Y_0,\, \chi_c^j \Phi(\cdot,t)\big\rangle, \qquad j \in \mathbf{Z}. \tag{8.7}$$

Then (8.3) and (8.6) imply that

$$\big\langle U_0(t)\, Y_0, \Psi\big\rangle = \sum_{j\in\mathbf{Z}} (r_t^j + c_t^j). \tag{8.8}$$

Note that the series in (8.8) is a finite sum. In fact, the support $\operatorname{supp}\Psi \subset B_{\overline{r}}$ with an $\overline{r} > 0$. Then, by Huygen's principle, the support of the function Φ at $t > 0$ is a subset of an "inflated future cone" (see [12, p. 18]):

$$\operatorname{supp}\Phi \subset \big\{(x,t) \in \mathbf{R}^n \times \mathbf{R}_+ \colon\, |x| \leqq t + \overline{r}\big\}. \tag{8.9}$$

Hence, (8.7) implies that

$$r_t^j = c_t^j = 0 \quad \text{for} \quad jh + t < -\overline{r} \quad \text{or} \quad jh - t > \overline{r}. \tag{8.10}$$

Therefore, series (8.8) becomes a sum

$$\big\langle U_0(t)\, Y_0, \Psi\big\rangle = \sum_{-N_t}^{N_t} \big(r_t^j + c_t^j\big), \qquad N_t \sim \frac{t}{h}. \tag{8.11}$$

LEMMA 8.2. *Let* $n \geqq 1$, $m > 0$, *and conditions* **S0–S3** *hold. Then the following bounds hold for* $t > 1$:

$$\mathbf{E}|r_t^j|^2 \leqq C(\Psi)\, \frac{d_t}{t}, \quad \mathbf{E}|c_t^j|^2 \leqq C(\Psi)\, \frac{\rho_t}{t}, \qquad j \in \mathbf{Z}. \tag{8.12}$$

Proof. We prove the first bound in (8.12) only; the second is done in a similar way. Let us express $\mathbf{E}|r_t^j|^2$ in correlation matrices. Definition (8.7) and condition **S2** imply, by the Fubini theorem, that

$$\mathbf{E}|r_t^j|^2 = \big\langle \chi_r^j(x_n)\, \chi_r^j(y_n)\, Q_0(x,y),\, \Phi(x,t) \otimes \Phi(y,t) \big\rangle. \tag{8.13}$$

For the function $\Phi(x,t)$, the following bound holds (cf. Theorem XI.17(b) of [29, p. 54]):

$$\sup_{x\in\mathbf{R}^n} \big|\Phi(x,t)\big| = O(t^{-n/2}), \qquad t \to \infty. \tag{8.14}$$

Applying (8.9) and (8.14) to equality (8.13), we obtain that

$$\mathbf{E}|r_t^j|^2 \leqq Ct^{-n} \int_{|x|\leqq t+\overline{r}} \chi_r^j(x_n) \bigg(\int_{\mathbf{R}^n} \big\|Q_0(x,y)\big\|\, dy \bigg)\, dx, \tag{8.15}$$

where $\|Q_0(x,y)\|$ stands for the norm of a matrix $(Q_0^{ij}(x,y))$. Therefore, the first bound (8.12) follows from Proposition 5.1(i). Lemma 8.2 is proved.

Now convergence (8.1) follows just as in sections 8 and 9 in [12, pp. 20–25].

9. Variable coefficients: The scattering theory for infinite energy solutions. In this section we prove Theorem A. We deduce it from Propositions 9.1 and 9.2 below by using the arguments as in sections 10 and 11 in [12, pp. 25–29].

Consider the operators $U'(t)$, $U_0'(t)$ in $H = L^2(\mathbf{R}^n) \oplus H^1(\mathbf{R}^n)$ (see (2.19)). The energy conservation for the Klein–Gordon equation implies the following corollary.

COROLLARY 9.1. *There exists a constant $C > 0$ such that for any $\Psi \in H$,*

$$\big\|U_0'(t)\,\Psi\big\|_H \leqq C\,\|\Psi\|_H, \quad \big\|U'(t)\,\Psi\big\|_H \leqq C\|\Psi\|_H, \qquad t \in \mathbf{R}. \tag{9.1}$$

Lemma 9.1 follows from the results of Vainberg (see Theorems 3–5 in [3]). Consider a family of finite seminorms in H,

$$\|\Psi\|_{(R)}^2 = \int_{|x|\leqq R} \Big(\big|\Psi^0(x)\big|^2 + \big|\Psi^1(x)\big|^2 + \big|\nabla\Psi^1(x)\big|^2 \Big)\, dx, \qquad R > 0.$$

Denote by $H_{(R)}$ the subspace of functions from H with a support in the ball B_R.

DEFINITION 9.1. *H_c denotes the space $\bigcup_{R>0} H_{(R)}$ endowed with the following convergence: A sequence Ψ_n converges to Ψ in H_c as $n \to \infty$ if and only if there exists $R > 0$ such that all $\Psi_n \in H_{(R)}$, and Ψ_n converges to Ψ in the norm $\|\cdot\|_{(R)}$ as $n \to \infty$.*

In what follows, we speak of continuity of maps from H_c in the sense of sequential continuity. Given $t \geqq 0$, denote

$$\varepsilon(t) = \begin{cases} (t+1)^{-3/2}, & n \geqq 3, \\ (t+1)^{-1} \log^{-2}(t+2), & n = 2. \end{cases} \tag{9.2}$$

LEMMA 9.1 (see [3]). *Let $n \geqq 2$ and conditions* **E1**–**E3** *hold. Then, for any $R, R_0 > 0$, there exists a constant $C = C(R,R_0)$ such that for $\Psi \in H_{(R)}$,*

$$\big\|U'(t)\,\Psi\big\|_{(R_0)} \leqq C\varepsilon(t)\, \big\|\Psi\big\|_{(R)}, \qquad t \geqq 0. \tag{9.3}$$

Given $t \geqq 0$, set

$$\varepsilon_1(t) = \begin{cases} (t+1)^{-1/2}, & n \geqq 3, \\ \log^{-1}(t+2), & n = 2. \end{cases} \tag{9.4}$$

THEOREM 9.1. *Let $n \geqq 2$ and conditions* **E1**–**E3** *and* **S0**–**S3** *hold. Then there exist linear continuous operators $W, r(t)\colon H_c \to H$ such that for $\Psi \in H_c$,*

$$U'(t)\,\Psi = U_0'(t)\,W\,\Psi + r(t)\,\Psi, \qquad t \geqq 0, \tag{9.5}$$

and the following bounds hold: For any $R > 0$ and $\Psi \in H_{(R)}$,

$$\|r(t)\,\Psi\|_H \leqq C(R)\,\varepsilon_1(t)\,\|\Psi\|_{(R)}, \qquad t \geqq 0, \tag{9.6}$$

$$\mathbf{E}\big|\langle Y_0, r(t)\,\Psi\rangle\big|^2 \leqq C(R)\,\varepsilon_1^2(t)\,\|\Psi\|_{(R)}^2, \qquad t \geqq 0. \tag{9.7}$$

Proof. Relations (9.5) and (9.6) are proved just as in section 10 in [12, p. 25–27]. It remains to prove (9.7). First, similarly to (8.13),

$$\mathbf{E}\big\langle Y_0,\, r(t)\,\Psi\big\rangle^2 = \big\langle Q_0(x,y),\, r(t)\,\Psi(x) \otimes r(t)\,\Psi(y)\big\rangle =: \mathcal{Q}_0\big(r(t)\,\Psi, r(t)\,\Psi\big). \tag{9.8}$$

Hence, Corollary 5.1(i) and (9.6) imply the following inequality for $\Psi \in H_{(R)}$:

$$\mathbf{E}\big\langle Y_0,\, r(t)\,\Psi\big\rangle^2 \leqq C\,\big\|r(t)\,\Psi\big\|_{L^2}^2 \leqq C\,\big\|r(t)\,\Psi\big\|_H^2 \leqq C(R)\,\varepsilon_1^2(t)\,\|\Psi\|_{(R)}^2. \tag{9.9}$$

Theorem 9.1 is proved.

Finally, Theorem A follows from the two propositions below.

PROPOSITION 9.1. *The family of the measures $\{\mu_t,\ t \in \mathbf{R}\}$ is weakly compact in $\mathcal{H}^{-\varepsilon}$, for any $\varepsilon > 0$.*

PROPOSITION 9.2. *For any $\Psi \in \mathcal{D}$,*

$$\widehat{\mu}_t(\Psi) \equiv \int \exp\big(i\langle Y,\, \Psi\rangle\big)\,\mu_t(dY) \longrightarrow \exp\Big\{-\frac{1}{2}\,\mathcal{Q}_\infty(W\Psi,\, W\Psi)\Big\} \tag{9.10}$$

as $t \to \infty$.

We deduce these propositions from Propositions 3.2 and 3.3, respectively, with the help of Theorem 9.1.

Proof of Proposition 9.1. Similarly to Proposition 3.2, Proposition 9.1 follows from the bounds

$$\sup_{t \geqq 0} \mathbf{E}\big\|U(t)\,Y_0\big\|_R < \infty, \qquad R > 0, \tag{9.11}$$

which follow from Theorem 9.1 and Proposition 3.2 as in [12].

Proof of Proposition 9.2. Equations (9.5) and (9.7) imply, by the Cauchy–Schwarz inequality, that

$$\begin{aligned} &\Big|\mathbf{E}\exp\big\{i\langle U(t)\,Y_0, \Psi\rangle\big\} - \mathbf{E}\exp\big\{i\big\langle Y_0, U_0'(t)\,W\Psi\big\rangle\big\}\Big| \\ &\quad \leqq \mathbf{E}\big|\langle Y_0, r(t)\,\Psi\rangle\big| \leqq \big(\mathbf{E}\big|\langle Y_0,\, r(t)\,\Psi\rangle\big|^2\big)^{1/2} \longrightarrow 0, \qquad t \to \infty. \end{aligned}$$

It remains to prove that

$$\mathbf{E}\exp\big\{i\langle Y_0,\, U_0'(t)\,W\Psi\rangle\big\} \longrightarrow \exp\Big\{-\frac{1}{2}\,\mathcal{Q}_\infty(W\Psi,\, W\Psi)\Big\}, \qquad t \to \infty. \tag{9.12}$$

This does not follow directly from Proposition 3.3, since generally, $W\Psi \notin \mathcal{D}$. We approximate $W\Psi$ by functions from $\mathcal{D}$. It is possible since $W\Psi \in H$, and $\mathcal{D}$ is dense in H. Hence, for any $\varepsilon > 0$ there exists $\Phi \in \mathcal{D}$ such that

$$\|W\Psi - \Phi\|_H \leqq \varepsilon. \tag{9.13}$$

Now we can derive (9.12) by the triangle inequality

$$\begin{aligned}
&\Big|\mathbf{E}\exp\big\{i\langle Y_0,\, U_0'(t)\, W\Psi\rangle\big\} - \exp\Big\{-\frac{1}{2}\,\mathcal{Q}_\infty(W\Psi,\, W\Psi)\Big\}\Big| \\
&\quad\leqq \Big|\mathbf{E}\exp\big\{i\langle Y_0,\, U_0'(t)\, W\Psi\rangle\big\} - \mathbf{E}\exp\big\{i\langle Y_0,\, U_0'(t)\,\Phi\rangle\big\}\Big| \\
&\qquad + \mathbf{E}\Big|\exp\big\{i\langle U_0(t)\, Y_0, \Phi\rangle\big\} - \exp\Big\{-\frac{1}{2}\,\mathcal{Q}_\infty(\Phi,\Phi)\Big\}\Big| \\
&\qquad + \Big|\exp\Big\{-\frac{1}{2}\,\mathcal{Q}_\infty(\Phi,\,\Phi)\Big\} - \exp\Big\{-\frac{1}{2}\,\mathcal{Q}_\infty(W\Psi,\, W\Psi)\Big\}\Big|.
\end{aligned} \tag{9.14}$$

Applying the Cauchy–Schwarz inequality, we get, similarly to (9.8) and (9.9), that

$$\mathbf{E}\big|\big\langle Y_0,\, U_0'(t)(W\Psi - \Phi)\big\rangle\big| \leqq \Big(\mathbf{E}\big|\big\langle Y_0,\, U_0'(t)(W\Psi-\Phi)\big\rangle\big|^2\Big)^{1/2} \leqq C\big\|U_0'(t)(W\Psi - \Phi)\big\|_H.$$

Hence, (9.1) and (9.13) imply that

$$\mathbf{E}\big|\big\langle Y_0, U_0'(t)(W\Psi - \Phi)\big\rangle\big| \leqq C\varepsilon, \qquad t \geqq 0. \tag{9.15}$$

Now we can estimate each term in the right-hand side of (9.14). The first term is $O(\varepsilon)$ uniformly in $t > 0$ by (9.15). The second term converges to zero as $t \to \infty$ by Proposition 3.3, since $\Phi \in \mathcal{D}$. Finally, the third term is $O(\varepsilon)$ owing to (9.13) and the continuity of the quadratic form $\mathcal{Q}_\infty(\Psi,\Psi)$ in $L^2(\mathbf{R}^n) \otimes \mathbf{C}^2$ (Corollary 5.2). Now the convergence (9.12) follows since $\varepsilon > 0$ is arbitrary.

Appendix A. Fourier transform. We consider the dynamics and correlation functions of system (3.2). Denote by $F\colon\, w \mapsto \widehat{w}$ the Fourier transform of a tempered distribution $w \in D'(\mathbf{R}^n)$ (see, e.g., [19]). We also use this notation for vector- and matrix-valued functions.

In the Fourier representation, system (3.2) becomes $\dot{\widehat{Y}}(k,t) = \widehat{\mathcal{A}}_0(k)\,\widehat{Y}(k,t)$, and hence,

$$\widehat{Y}(k,t) = \widehat{\mathcal{G}}_t(k)\,\widehat{Y}_0(k), \qquad \widehat{\mathcal{G}}_t(k) = \exp\big(\widehat{\mathcal{A}}_0(k)\,t\big). \tag{A.1}$$

Here we denote

$$\widehat{\mathcal{A}}_0(k) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \qquad \widehat{\mathcal{G}}_t(k) = \begin{pmatrix} \cos\omega t & \dfrac{\sin\omega t}{\omega} \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}, \tag{A.2}$$

where $\omega = \omega(k) = \sqrt{|k|^2 + m^2}$. Denote by I the identity matrix and

$$\widehat{C}(k) \equiv \big(\widehat{C}^{ij}(k)\big)_{i,j=0}^{1} := \begin{pmatrix} 0 & \omega^{-1}(k) \\ -\omega(k) & 0 \end{pmatrix}. \tag{A.3}$$

Then

$$\widehat{\mathcal{G}}_t(k) = \cos\omega t\, I + \sin\omega t\,\widehat{C}(k). \tag{A.4}$$

Hence,

$$\begin{aligned}\widehat{\mathcal{G}}_t(k)\,\widehat{Q}(k,k')\,\widehat{\mathcal{G}}_t^T(k') &= \cos\omega(k)\,t\,\cos\omega(k')\,t\,\widehat{Q}(k,k')\\ &\quad+\sin\omega(k)\,t\,\sin\omega(k')\,t\,\widehat{C}(k)\,\widehat{Q}(k,k')\,\widehat{C}^T(k')\\ &\quad+\cos\omega(k)\,t\,\sin\omega(k')\,t\,\widehat{Q}(k,k')\,\widehat{C}^T(k')\\ &\quad+\cos\omega(k)\,t\,\sin\omega(k')\,t\,\widehat{C}(k)\,\widehat{Q}(k,k')\\ &=\frac{1}{2}\sum_{\pm}\Big\{\cos\big(\omega(k)\pm\omega(k')\big)\,t\,\Big(\widehat{Q}(k,k')\,\mp\widehat{C}(k)\,\widehat{Q}(k,k')\,\widehat{C}^T(k')\Big)\\ &\quad+\frac{1}{2}\sin\big(\omega(k)\pm\omega(k')\big)\,t\,\Big(\widehat{C}(k)\,\widehat{Q}(k,k')\,\pm\widehat{Q}(k,k')\,\widehat{C}^T(k')\Big)\Big\}.\end{aligned}\tag{A.5}$$

In the particular case when $\widehat{Q}(k,k')=\delta(k-k')\,\widehat{q}(k)$, we obtain

$$\begin{aligned}\widehat{\mathcal{G}}_t(k)\,\widehat{q}(k)\,\widehat{\mathcal{G}}_t^T(k) &= \frac{1}{2}\Big\{\widehat{q}(k)+\widehat{C}(k)\,\widehat{q}(k)\,\widehat{C}^T(k)\Big\}\\ &\quad+\frac{1}{2}\cos 2\omega(k)\,t\,\Big\{\widehat{q}(k)-\widehat{C}(k)\,\widehat{q}(k)\,\widehat{C}^T(k)\Big\}\\ &\quad+\frac{1}{2}\sin 2\omega(k)\,t\,\Big\{\widehat{C}(k)\,\widehat{q}(k)+\widehat{q}(k)\,\widehat{C}^T(k)\Big\}.\end{aligned}\tag{A.6}$$

The following formulas are used in the proofs of Lemmas 6.3 and 6.5, respectively:

$$\begin{aligned}&\Big[\sin\omega t\,I-\cos\omega t\,\widehat{C}(k)\Big]\,\widehat{q}(k)\Big[\sin\omega t\,I-\cos\omega t\,\widehat{C}^T(k)\Big]\\ &\quad=\frac{1}{2}\Big\{\widehat{q}(k)+\widehat{C}(k)\,\widehat{q}(k)\,\widehat{C}^T(k)\Big\}-\frac{1}{2}\cos 2\omega(k)\,t\,\Big\{\widehat{q}(k)-\widehat{C}(k)\,\widehat{q}(k)\,\widehat{C}^T(k)\Big\}\\ &\quad-\frac{1}{2}\sin 2\omega(k)\,t\,\Big\{\widehat{C}(k)\,\widehat{q}(k)+\widehat{q}(k)\,\widehat{C}^T(k)\Big\},\end{aligned}\tag{A.7}$$

$$\begin{aligned}&\widehat{\mathcal{G}}_t(k)\,\widehat{q}(k)\Big[\sin\omega(k)\,t\,I-\cos\omega(k)\,t\,\widehat{C}^T(k)\Big]\\ &\quad=\frac{1}{2}\Big\{\widehat{C}(k)\,\widehat{q}(k)-\widehat{q}(k)\,\widehat{C}^T(k)\Big\}-\frac{1}{2}\cos 2\omega(k)\,t\,\Big\{\widehat{q}(k)\,\widehat{C}^T(k)+\widehat{C}(k)\,\widehat{q}(k)\Big\}\\ &\quad+\frac{1}{2}\sin 2\omega(k)\,t\,\Big\{\widehat{q}(k)-\widehat{C}(k)\,\widehat{q}(k)\,\widehat{C}^T(k)\Big\}.\end{aligned}\tag{A.8}$$

Appendix B. Singular oscillatory integrals. By virtue of (A.4), Proposition 6.2 follows from the following lemma.

LEMMA B.1. *Let $\omega(k)=\sqrt{|k|^2+m^2}$; the function $\Omega(k)$ be one of the functions $\omega(k)$, $\omega^{-1}(k)$ or 1; and $x_n\in[-R,R]$. Then the matrix-valued integral*

$$I(t,x_n,k):=\mathrm{PV}\int_{\mathbf{R}}e^{-i\xi x_n}e^{\pm i\omega(\mathbf{k},k_n+\xi)\,t}\,\frac{\widehat{\alpha}_+(\xi)}{\xi}\,\Omega(\mathbf{k},k_n+\xi)\,d\xi$$

is uniformly bounded:

$$\sup_{|x_n|\leqq R,t\geqq 1}\big|I(t,x_n,k)\big|\leqq C\,\Omega(k),\tag{B.1}$$

where a constant C does not depend on k.

The bound (B.1) follows from the inequalities

$$\left.\begin{aligned}\big|\omega(\mathbf{k},k_n+\xi)-\omega(k)\big| &\leqq C|\xi|\\ \big|\omega^{-1}(\mathbf{k},k_n+\xi)-\omega^{-1}(k)\big| &\leqq C|\xi|\,\omega^{-1}(k)\end{aligned}\right\}\quad\forall\,k\in\mathbf{R}^n,$$

$|e^{i\xi x_n}-1| \leqq \min\{|\xi||x_n|, 2\}$ and the following lemma.

LEMMA B.2. *The integral* $J_t(k) := \mathrm{PV}\int_{-\infty}^{+\infty} e^{i\omega(\mathbf{k},k_n+\xi)\,t}\widehat{\alpha}_+(\xi)/\xi\,d\xi$ *is bounded for all* $t>1$, $k=(\mathbf{k},k_n)\in\mathbf{R}^n$.

Proof. Note that a similar integral around the circle (instead of $\mathbf{R}$) has been considered in Proposition A.4 of [2] for $n=1$ and the condition $\omega^{(p)}(0)=0$ for $p=1,\dots,m-1$ and $\omega^{(m)}(0)\neq 0$ for a finite number m. Moreover, it is assumed in [2] that the inequality and the estimates for all functions are fulfilled uniformly on the parameter, which belongs to a compact set. In our case the parameter belongs to an infinite space, and these estimates are not uniform because all derivatives $\partial^p_{k_n}\omega(\mathbf{k},k_n)$ vanish as $|\mathbf{k}|\to\infty$. Then Proposition A.4(ii) from [2] is not applied directly and we have to modify it to apply it to our case. Since $\widehat{\alpha}_+\in S(\mathbf{R}^1)$, it suffices to prove that the integral

$$j_t(k) := \mathrm{PV}\int_{-\delta}^{\delta}\frac{e^{i\omega(\mathbf{k},k_n+\xi)\,t}}{\xi}\,d\xi \tag{B.2}$$

is bounded uniformly on $t>1$ and $k\in\mathbf{R}^n$ for a small enough $\delta>0$. Let us consider separately two cases $|k_n|\geqq B\delta$ and $|k_n|\leqq B\delta$, where B is a large enough number.

(i) Let $|k_n|\geqq B\delta$. Then $\partial_{k_n}\omega(k)=k_n/\omega(k)\neq 0$. We change the variables

$$\xi\to z\equiv z(k,\xi) := \frac{\omega(\mathbf{k},k_n+\xi)-\omega(k)}{\partial_{k_n}\omega(k)}.$$

Then $\omega(\mathbf{k},k_n+\xi)=\omega(k)+zk_n/\omega(k)$ and $z\,|_{\xi=0}=0$, $(\partial z/\partial\xi)\,|_{\xi=0}=1$. Therefore, denoting by $\xi=\varphi(k,z)$ the inverse function to $z=z(k,\xi)$, we obtain that for small enough δ for $|k_n|\geqq B\delta$

$$j_t(k)=e^{i\omega(k)\,t}\,\mathrm{PV}\int_{-\delta}^{\delta}\frac{e^{iztk_n/\omega(k)}\partial_z\varphi(k,z)}{\varphi(k,z)}\,dz+O(1)$$

uniformly on k. Note that $(\partial_z\varphi(k,z))/\varphi(k,z)=1/z+\chi(k,z)$, where $\chi(k,z)$ is a bounded function for $|z|\leqq\delta$ uniformly on $|k_n|\geqq B\delta$ for large enough B. Hence,

$$j_t(k)=e^{i\omega(k)\,t}\,\mathrm{PV}\int_{-\delta}^{\delta}\frac{e^{iztk_n/\omega(k)}}{z}\,dz+O(1)$$

uniformly on k for $|k_n|\geqq B\delta$. Further, denote $\lambda := tk_n/\omega(k)$. Hence, $j_t(k)=e^{i\omega(k)\,t}I(\lambda)+O(1)$, where $I(\lambda):=\mathrm{PV}\int_{-\delta}^{\delta}(e^{iz\lambda}/z)\,dz$. For $|\lambda|\leqq C<\infty$, we have $|I(\lambda)|\leqq|\lambda|2\delta\leqq C_1$. For $|\lambda|\geqq C$, using the formula $\lim_{\lambda\to\pm\infty}I(\lambda)=\pm\pi i$, we obtain the uniform boundedness of the integral $I(\lambda)$ for all λ. Hence, for the case $|k_n|\geqq B\delta$ the integral $j_t(k)$ is uniformly bounded in t and k.

(ii) Let $|k_n|\leqq B\delta$ with fixed B above. We use the following relation:

$$\partial^2_{k_n}\omega(\mathbf{k},0)=\frac{1}{\omega(\mathbf{k},0)}\neq 0,\qquad \mathbf{k}\in\mathbf{R}^{n-1}.$$

The phase function in the integral (B.2) admits a representation in the form

$$\omega(\mathbf{k},k_n+\xi)=\omega(\mathbf{k},0)+(k_n+\xi)^2\,\frac{\partial^2_{k_n}\omega(\mathbf{k},0)}{2}+\cdots=\omega(\mathbf{k},0)+C_{\mathbf{k}}\mu^2(\mathbf{k},\,k_n+\xi),$$

where $C_{\mathbf{k}} := (2\omega(\mathbf{k},0))^{-1}$. Moreover, $\mu(\mathbf{k},0) = 0$ and $\mu'_{k_n}(\mathbf{k},0) = 1$, and the integral (B.2) becomes

$$j_t(k) = e^{i\omega(\mathbf{k},0)\,t}\,\mathrm{PV}\int_{-\delta}^{\delta} \frac{e^{iC_{\mathbf{k}}\mu^2(\mathbf{k},k_n+\xi)\,t}}{\xi}\,d\xi.$$

Take a new variable ζ by the rule $\mu(\mathbf{k},k_n+\xi) = \mu(\mathbf{k},k_n)+\zeta$, and express a variable ξ by ζ. Denote by $\varphi(\mathbf{k},\mu) = k_n$ the inverse function to $\mu = \mu(\mathbf{k},k_n)$. Then $k_n+\xi = \varphi(\mathbf{k},\zeta+\mu(\mathbf{k},k_n))$. Hence,

$$\xi = \varphi\big(\mathbf{k},\zeta+\mu(\mathbf{k},k_n)\big) - k_n = \varphi\big(\mathbf{k},\zeta+\mu(\mathbf{k},k_n)\big) - \varphi\big(\mathbf{k},\mu(\mathbf{k},k_n)\big).$$

In particular, $\partial_\zeta\xi = \partial_\mu\varphi(\mathbf{k},\zeta+\mu(k))$, and then,

$$\begin{aligned} j_t(k) &= e^{i\omega(\mathbf{k},0)\,t}\,\mathrm{PV}\int_{-\delta}^{\delta} \frac{e^{iC_{\mathbf{k}}t[\zeta+\mu(k)]^2}\partial_\mu\varphi(\mathbf{k},\zeta+\mu(k))}{\varphi(\mathbf{k},\zeta+\mu(k))-\varphi(\mathbf{k},\mu(k))}\,d\zeta + O(1)\\ &= e^{i\omega(\mathbf{k},0)\,t}\,\mathrm{PV}\int_{-\delta}^{\delta} \frac{e^{iC_{\mathbf{k}}t[\zeta+\mu(k)]^2}}{\zeta}\,d\zeta + O(1) \end{aligned} \tag{B.3}$$

uniformly in k for $|k_n| \leqq B\delta$. In contrast to Proposition A.4 of [2], in our case $C_{\mathbf{k}} \neq$ const., and, moreover, $C_{\mathbf{k}} \to 0$ as $|\mathbf{k}| \to \infty$. Let us take a new parameter $\lambda := C_{\mathbf{k}}t$. Then

$$j_t(k) = e^{i\omega(\mathbf{k},0)\,t}\,\mathrm{PV}\int_{-\delta}^{\delta} \frac{e^{i\lambda[z+\mu(k)]^2}}{z}\,dz + O(1)$$

uniformly in k for $|k_n| \leqq B\delta$. Note that $|\mu(k)| \leqq |k_n| \leqq B\delta$. Hence, the uniform boundedness of the last integral $\mathrm{PV}\int_{-\delta}^{\delta}$ follows by Proposition A.4 (ii) of [2].

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