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Scattering of solitons of the Klein–Gordon equation coupled to a classical particle

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Long-time asymptotics are established for finite energy solutions of the scalar Klein–Gordon equation coupled to a relativistic classical particle: any ‘‘scattering’’ solution is asymptotically a sum of a soliton and of a dispersive free wave packet as $t\rightarrow\pm\infty$. These asymptotics mean the nonlinear scattering of free wave packets by the soliton. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1539900]

I. INTRODUCTION: KLEIN–GORDON FIELD COUPLED TO A CLASSICAL PARTICLE

In this paper we consider the classical scalar Klein–Gordon equation coupled (noncovariantly) to a relativistic classical particle subjected to an external potential V of compact support. The system is a finite-range perturbation of the corresponding translation-invariant system without V that admits soliton-type solutions describing a particle traveling with constant velocity that is dressed by a comoving wave field. The set of all such solutions forms a finite-dimensional manifold, called the *soliton manifold,* in the phase space (a Hilbert space) of the unperturbed system. We are interested in *scattering* solutions of the perturbed system in which the particle travels to infinity as $t\rightarrow\pm\infty$.

The reason for the name *soliton manifold* resides the fact, proven in this paper, that it is an attracting set for the scattering solutions of the perturbed dynamical system. The attraction holds in the Fréchet topology defined by the local energy seminorms.

Our main result is the long-time asymptotics in the global energy norm: each scattering solution is asymptotically the sum of a soliton and a dispersive free wave as $t\rightarrow\pm\infty$. This means that the solution is scattering of the free wave by a soliton. This representation of the solutions gives a mathematical description of the wave-particle duality: for $t=-\infty$ such solution is a union of a ‘‘particle’’ = soliton and ‘‘photon’’ = free wave, for finite t the solution in general does not admit such a representation, and for $t=\infty$ the representation again appears.

Previously similar results have been proved for relativistic charged particles coupled, respectively, to the wave equation corresponding to $m=0$ (Refs. 11 and 13) and to Maxwell's equations.[2,10,16] The proof of these results is based on the nonautonomous integral inequality method[19] that uses essentially the strong Huygen's principle. In the case of the Klein–Gordon equation with $m>0$ the strong Huygen's principle fails. That is why we develop a new more general version of the integral inequality method that does not use the strong Huygen's principle. An important role play the known time decay of the Green function for the Klein–Gordon equation and a sufficiently fast spatial decay of the solitons in the case $m>0$.

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We consider a scalar wave field $\psi(x)\in\mathbb{R}$, $x\in\mathbb{R}^3$, coupled to a relativistic particle with position q and momentum p, governed by

$$\dot{\psi}(x,t)=\pi(x,t), \quad \dot{\pi}(x,t)=\Delta\psi(x,t)-m^2\psi(x,t)-\rho(x-q(t)),$$
$$\dot{q}(t)=p(t)/(1+p^2(t))^{1/2}, \quad \dot{p}(t)=-\nabla V(q(t))+\int d^3x\,\psi(x,t)\,\nabla\rho(x-q(t)) \tag{1}$$

subject to appropriate initial conditions determining the dynamics. This is a Hamiltonian system with the Hamiltonian functional

$$\mathcal{H}(\psi,\pi,q,p)=(1+p^2)^{1/2}+V(q)+\frac{1}{2}\int d^3x(|\nabla\psi(x)|^2+m^2|\psi(x)|^2+|\pi(x)|^2)$$
$$+\int d^3x\,\psi(x)\rho(x-q). \tag{2}$$

We have set the mechanical mass of the particle and the speed of wave propagation equal to one. The case of the point particle corresponds to $\rho(x)=\delta(x)$ and then the interaction term in the Hamiltonian is simply $\psi(q)$. This would result however in an energy which is not bounded from below implying for the scattering theory the well-known ultraviolet divergence. Therefore we smooth the coupling by the function $\rho(x)$ following the strategy proposed by Abraham[1] for the Maxwell field. Respectively, the system (1) is not relativistic covariant. In analogy to the Maxwell–Lorentz equations we call ρ the ''charge distribution.'' We assume the real-valued function ρ to be in the Sobolev space H^1 and of compact support, i.e.,

$$\rho,\nabla\rho\in L^2(\mathbb{R}^3), \quad \rho(x)=0 \text{ for } |x|\geqslant R_\rho. \tag{C}$$

An important assumption is that the norm of ρ in L^2 is sufficiently small,

$$\gamma_\rho:=\|\rho\|_{L^2}\ll 1 \tag{3}$$

meaning *weak* field-particle interaction.

For the potential V we introduce two sets of assumptions: smooth and bounded from below,

$$V\in C^2(\mathbb{R}^3), \quad V_0:=\inf_{q\in\mathbb{R}^3} V(q)>-\infty; \tag{$P_{\min}$}$$

and of a compact support,

$$V(x)\equiv 0 \quad \text{for } |x|>R_V>0. \tag{K}$$

Consider the corresponding nonperturbed system with $V\equiv 0$:

$$\dot{\psi}(x,t)=\pi(x,t), \quad \dot{\pi}(x,t)=\Delta\psi(x,t)-m^2\psi(x,t)-\rho(x-q(t)),$$
$$\dot{q}(t)=p(t)/(1+p^2(t))^{1/2}, \quad \dot{p}(t)=\int d^3x\,\psi(x,t)\,\nabla\rho(x-q(t)). \tag{4}$$

The system (4) has solutions traveling with constant velocity $v,|v|<1$. Up to spatial translations they are given by

$$S_v(t)=(\psi_v(x-vt),\pi_v(x-vt),vt,p_v) \tag{5}$$

with

$$\psi_v(x) = -\frac{1}{4\pi}\int \frac{e^{-m|(y-x)_{\|}+\lambda(y-x)_{\perp}|}\rho(y)d^3y}{|(y-x)_{\|}+\lambda(y-x)_{\perp}|},$$
$$\pi_v(x) = -v\cdot\nabla\psi_v(x), \quad p_v = v/\lambda. \tag{6}$$

Here we set $\lambda=\sqrt{1-v^2}$ and $x=x_{\|}+x_{\perp}$, where $v\|x_{\|}\in\mathbb{R}^3$ and $v\perp x_{\perp}\in\mathbb{R}^3$ for $x\in\mathbb{R}^3$. We call $S_v(t)$ the *soliton* with velocity v centered at $q(t)=vt$.

Let us discuss and summarize now our main results, the precise theorems to be stated in the following sections. Consider the set of scattering solutions to (1) for which $|q(t)|\to\infty$ as $t\to\infty$. Below we discuss the properties of these solutions. Since only a finite amount of energy can be dissipated to infinity, we shall show the relaxation of acceleration,

$$\ddot{q}(t)\to 0, \quad t\to\pm\infty. \tag{7}$$

More precisely, we shall establish the rate of convergence $|\ddot{q}(t)|\sim t^{-1-\sigma}$ with a $\sigma>0$. This is a crucial point of our asymptotic analysis. It implies that

$$\dot{q}(t)\to v_{\pm}, \quad t\to\pm\infty. \tag{8}$$

Also we show that the fields are asymptotically traveling waves in the sense

$$(\psi(x,t),\pi(x,t))\sim(\psi_{v_{\pm}}(x-q(t)),\pi_{v_{\pm}}(x-q(t))), \quad t\to\pm\infty. \tag{9}$$

Since the energy is conserved, the convergence here is in the sense of local energy seminorms, cf. Sec. II. Further, we shall establish the corresponding asymptotics in the *global* energy norm,

$$(\psi(x,t),\pi(x,t))\sim(\psi_{v_{\pm}}(x-q(t)),\pi_{v_{\pm}}(x-q(t)))+U(t)\Psi_{\pm}, \quad t\to\pm\infty, \tag{10}$$

where $U(t)$ is the unitary group generated by the free Klein–Gordon equation, and $\Psi_{\pm}$ are the scattering states. At last we suggest simple sufficient conditions for solutions to be scattering. Note that all finite energy solutions are scattering if $V(x)\equiv 0$. We prove (8), (9), and (10) with the assumption (3), however we suggest the same asymptotics hold in more general framework.

We mention now some previous results which reflect the gradual progress in investigating the long-time asymptotics for coupled field-particle equations.

The results of Ref. 14 for the wave equation, $m=0$, imply the long-time convergence to the set of solitons of type (5) in the sense of local energy seminorms, as in (9).

Soliton-type asymptotics were proved for certain translation invariant completely integrable 1D equations.[18] Soliton-type asymptotics in *local* energy seminorms was proved for a translation invariant 3D system of a scalar field coupled to a particle[15] and for translation invariant 1D kinetic-reaction systems.[8]

Soliton-type asymptotics of type (10) in *global* energy norm were proved initially for *small perturbations* of soliton-type solutions of 1D nonlinear Schrödinger equations.[3,4]

Soliton-type asymptotics of type (10) in the energy norm for all finite energy scattering solutions is proved here for the first time for coupled particle-field equations (1). The asymptotics is provided by radiation of the energy to infinity which leads to the relaxation (7). The relaxation in classical electrodynamics is known as ‘‘radiative damping’’ studied by Lorentz,[17] Dirac,[5] Feynman[7] and others.[9]

Note that a lot of numerical experiments[12] confirm the long-time convergence of an arbitrary finite energy solution of a general relativistic equation to a finite sum of solitons with velocities less than the light speed and of ‘‘photons’’ propagating at the light speed. Nevertheless the proof remains an absolutely open problem.

II. EXISTENCE OF DYNAMICS, *A PRIORI* ESTIMATES

To formulate our results precisely, we need some definitions. We introduce the phase space suitable for the Cauchy problem corresponding to (1) and (2). Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with scalar product $\langle \cdot , \cdot \rangle$ and norm $|\cdot|$, and let H^1 be the Sobolev space $H^1=\{\psi \in L^2 : |\nabla \psi| \in L^2\}$ with the norm $\|\psi\|=|\psi|+|\nabla\psi|$. Let $|\psi|_R$ denote the norm in $L^2(B_R)$ for $R>0$, where $B_R=\{x\in\mathbb{R}^3:|x|\leqslant R\}$. Then the seminorms $\|\psi\|_R=|\psi|_R+|\nabla\psi|_R$ are continuous on H^1.

Definition 1: (i) The phase space $\mathcal{E}$ is the Hilbert space $H^1\times L^2\times\mathbb{R}^3\times\mathbb{R}^3$ of states $Y=(\psi,\pi,q,p)$ with finite norm

$$\|Y\|_{\mathcal{E}}=\|\psi\|+|\pi|+|q|+|p| .$$

(ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the local energy seminorms

$$\|Y\|_R=\|\psi\|_R+|\pi|_R+|q|+|p|, \quad \forall R>0 .$$

(iii) $\mathcal{F}$ is the Hilbert space $H^1\times L^2$ of fields $\Psi=(\psi,\pi)$ with finite norm

$$\|\Psi\|_{\mathcal{F}}=\|\psi\|+|\pi| .$$

(iv) $\mathcal{F}_F$ is the space $\mathcal{F}$ endowed with the Fréchet topology defined by the local energy seminorms

$$\|\Psi\|_R=\|\psi\|_R+|\pi|_R , \quad \forall R>0 .$$

Note that both spaces $\mathcal{E}$ and $\mathcal{E}_F$ are metrisable. For $\psi\in L^2$ we have

$$-\frac{1}{2m^2}|\rho|^2\leqslant\frac{m^2}{2}|\psi|^2+\langle\psi,\rho(\cdot-q)\rangle\leqslant\frac{m^2+1}{2}|\psi|^2+\frac{1}{2}|\rho|^2. \tag{11}$$

Therefore $\mathcal{E}$ is the space of finite energy states. The Hamiltonian functional $\mathcal{H}$ is continuous on the space $\mathcal{E}$ and the lower bound in (11) implies that the energy functional (2) is bounded from below, namely,

$$\inf_{Y\in\mathcal{E}}\mathcal{H}(Y)\geqslant 1+V_0-\frac{1}{2m^2}|\rho|^2. \tag{12}$$

We consider the Cauchy problem for the Hamiltonian system (1), which we write as

$$\dot{Y}(t)=\mathcal{V}_0(Y(t))+\mathcal{V}_1(Y(t)), \quad t\in\mathbb{R}, \quad Y(0)=Y^0. \tag{13}$$

All derivatives are understood in the sense of distributions. Here $Y(t)=(\psi(t),\pi(t),q(t),p(t))$, $Y^0=(\psi^0,\pi^0,q^0,p^0)\in\mathcal{E}$, and $\mathcal{V}_0:Y\mapsto(\pi,\Delta\psi-m^2\psi,0,0)$. Recall that we are interested in situations where the particle is allowed to travel to infinity, e.g., when the external potential $V(q)$ vanishes identically. The existence of dynamics is true under such conditions $(P_{\min})$.

Theorem 2: *Let (C) and $(P_{\min})$ hold. Then (i) for every $Y^0\in\mathcal{E}$ the Cauchy problem (13) has a unique solution $Y(t)\in C(\mathbb{R},\mathcal{E})$. (ii) For every $t\in\mathbb{R}$ the map $W_t:Y^0\mapsto Y(t)$ is continuous both on $\mathcal{E}$ and on $\mathcal{E}_F$. (iii) The energy is conserved, i.e.,*

$$\mathcal{H}(Y(t))=\mathcal{H}(Y^0) \quad \text{for } t\in\mathbb{R}. \tag{14}$$

(iv) The speed is bounded,

$$|\dot{q}(t)|\leqslant\overline{v}<1 \quad \text{for } t\in\mathbb{R}. \tag{15}$$

Proof: We follow Ref. 14, where the case $m=0$ is considered. Let us fix an arbitrary $b>0$ and prove (i)–(iii) for $\|Y^0\|_{\mathcal{E}}\leqslant b$ and $|t|\leqslant\varepsilon=\varepsilon(b)$ for some sufficiently small $\varepsilon(b)>0$.

ad (i) Fourier transform provides the existence and uniqueness of solution $Y_0(t)\in C(\mathbb{R},\mathcal{E})$ to the linear problem (13) with $\mathcal{V}_1=0$. Let $W_t^0:Y^0\mapsto Y_0(t)$ be the corresponding strongly continuous group of bounded linear operators on $\mathcal{E}$. Then uniqueness of solution to the (inhomogeneous) linear problem implies that (13) for $Y(t)\in C(\mathbb{R},\mathcal{E})$ is equivalent to

$$Y(t)=W_t^0Y^0+\int_0^t ds\, W_{t-s}^0\mathcal{V}_1(Y(s)), \tag{16}$$

because $\mathcal{V}_1(Y(\cdot))\in C(\mathbb{R},\mathcal{E})$ in this case. The latter follows from a local Lipschitz continuity of the map $\mathcal{V}_1$ in $\mathcal{E}$: for each $b>0$ there exist a $\kappa=\kappa(b)>0$ such that for all $Y,Z\in\mathcal{E}$ with $\|Y\|_{\mathcal{E}},\|Z\|_{\mathcal{E}}\leqslant b$,

$$\|\mathcal{V}_1(Y)-\mathcal{V}_1(Z)\|_{\mathcal{E}}\leqslant\kappa\|Y-Z\|_{\mathcal{E}}. \tag{17}$$

For example, we have

$$\left|\int d^3x\,(\psi_1(x)-\psi_2(x))\nabla\rho(x-q)\right|\leqslant|\nabla(\psi_1-\psi_2)||\rho|.$$

Moreover, by the contraction mapping principle, Eq. (16) has a unique local solution $Y(\cdot)\in C([-\varepsilon,\varepsilon],\mathcal{E})$ with $\varepsilon>0$ depending only on b. Then the existence of the global dynamics will follow from the *a priori* estimate, see in $ad\ (iii)$ below.

ad (ii) The map $W_t:Y^0\mapsto Y(t)$ is continuous in the norm $\|\cdot\|_{\mathcal{E}}$ for $|t|\leqslant\varepsilon$ and $\|Y^0\|\leqslant b$. To prove continuity of W_t in $\mathcal{E}_F$, let us consider Picard's successive approximation scheme

$$Y^N(t)=W_t^0Y^0+\int_0^t ds\, W_{t-s}^0\mathcal{V}_1(Y^{N-1}(s)),\quad N=1,2,\ldots.$$

The equation for q^N in this system implies $|\dot{q}^N(t)|<1$ and therefore $|q(t)|<|q^0|+|t|$. Now we fix $t\in\mathbb{R}$ and choose $R>|q^0|+|t|+R_\rho$ with R_ρ from (C). From the explicit solution of the free Klein–Gordon equation $W_t^0Y^0$ (see Sec. III) we conclude that every Picard's approximation $Y^N(t)$ and hence the solution $Y(t)=(\psi(x,t),\pi(x,t),q(t),p(t))$ for $|x|<R$ depends only on the initial data $(\psi^0(x),\pi^0(x),q^0,p^0)$ with $|x|<R+|t|$. Thus the continuity of W_t in $\mathcal{E}_F$ follows from the continuity in $\mathcal{E}$.

ad (iii) For $k=0,1,\ldots$ denote by $C_0^k(\mathbb{R}^3)$ the space of functions $\psi(x)\in C^k(\mathbb{R}^3)$ with compact support. For initial data $(\psi^0,\pi^0)\in C^3(\mathbb{R}^3)\times C^2(\mathbb{R}^3)$ the solution $\psi=\psi(x,t)$ satisfies $\psi\in C^2(\mathbb{R}^3\times\mathbb{R})$. Indeed, this is well known for the solution $W_t^0Y^0$ of the linear Klein–Gordon equation. The integral representation (16) then implies the same property for ψ. In addition, let Y^0 have compact support, i.e.,

$$\psi^0(x)=\pi^0(x)=0\quad\text{for}\quad|x|>R^0 \tag{18}$$

with some $R^0>0$. Since $|q(t)|<|q^0|+|t|$, (16) implies

$$\psi(x,t)=0\quad\text{for }|x|\geqslant|t|+\max\{R^0,R_\rho+|q^0|+|t|\}.$$

Thus, for such initial data energy conservation can be shown by integration by parts. Hence (iii) follows from the continuity of W_t and the fact that $C_0^3(\mathbb{R}^3)\oplus C_0^2(\mathbb{R}^3)\oplus\mathbb{R}^3\oplus\mathbb{R}^3$ is dense in $\mathcal{E}$.

We use now energy conservation to ensure the existence of a global solution and its continuity. Similar to (11) we have

$$\mathcal{H}(Y)\geqslant\frac{1}{2}|\pi|^2+\frac{1}{2}|\nabla\psi|^2+\frac{m^2}{4}|\psi|^2+\sqrt{1+p^2}+V(q)-\frac{1}{m^2}|\rho|^2,$$

and by energy conservation, for $|t|\leqslant\varepsilon$,

$$\frac{1}{2}|\pi(t)|^2+\frac{1}{2}|\nabla\psi(t)|^2+\frac{m^2}{4}|\psi|^2+\sqrt{1+p^2(t)}+V(q(t))-\frac{1}{m^2}|\rho|^2\leq\mathcal{H}(Y(t))=\mathcal{H}(Y^0). \quad (19)$$

Therefore ($P_{\min}$) implies the *a priori* estimate

$$\|\psi(t)\|+|\pi(t)|+|p(t)|\leq B \text{ for } t\in\mathbb{R} \quad (20)$$

with B depending only on the norm $\|Y^0\|_{\mathcal{E}}$ of the initial data and on $|\rho|$. Properties (i)–(iii) for arbitrary $t\in\mathbb{R}$ now follow from the same properties for small $|t|$ and from the *a priori* bound (20).

ad (iv) Note first that (20) implies $|p(t)|\leq p_0<\infty$. Hence

$$|\dot{q}(t)|/(1-\dot{q}^2(t))^{1/2}=|p(t)|\leq p_0<\infty,$$

which yields $|\dot{q}(t)|\leq q_1=:\overline{v}<1$. □

III. INTEGRAL INEQUALITY ARGUMENT

Definition 3: Let $0<\sigma<1/2$ *and let* $\alpha=(1-2\sigma)/3$. *The set* $\mathcal{E}^\sigma$ *is the set of the states* $(\psi,\pi,q,p)\in\mathcal{E}$ *such that*

$$\int_{R\leq|x|}d^3x(R^2|\psi(x)|^2+|\nabla\psi(x)|^2)=\mathcal{O}(R^{-(4+2\sigma)}) \quad (21)$$

and

$$\int_{R^\alpha\leq|x|}d^3x\ (|\psi(x)|^2+|\pi(x)|^2)=\mathcal{O}(R^{-(4+2\sigma)}) \quad (22)$$

as $R\to+\infty$.

If the soliton-type asymptotics is approximately valid, then the field should be close to the soliton centered at $q(t)$ with velocity $v(t)=\dot{q}(t)$. We therefore consider the difference

$$Z(x,t)=\Psi(x,t)-\Psi_{v(t)}(x-q(t)), \quad (23)$$

where

$$\Psi(x,t)=(\psi(x,t),\pi(x,t))$$

and $\Psi_v(x)=(\psi_v(x),\pi_v(x))$ is the field part of the soliton. Defining $\overline{\rho}(x)=(0,\rho(x))$ and $A(\psi,\pi)=(\pi,\Delta\psi-m^2\psi)$, it follows that Ψ obeys the equations of motion:

$$\dot{\Psi}(x,t)=A\Psi(x,t)-\overline{\rho}(x-q(t)). \quad (24)$$

On the other hand, for the soliton field Ψ_v with a fixed v, the equation

$$-\frac{\partial\Psi_v}{\partial x}(x-q(t))v=A\Psi_v(x-q(t))-\overline{\rho}(x-q(t)) \quad (25)$$

holds. Then (24) and (25) imply the following equation for Z:

$$\dot{Z}(x,t)=AZ(x,t)-\frac{\partial\Psi_{v(t)}}{\partial p}(x-q(t))\dot{p}(t). \quad (26)$$

Here, according to the chain rule,

$$\frac{\partial \Psi_v}{\partial p} = \frac{\partial \Psi_v}{\partial v}\frac{\partial v}{\partial p}, \tag{27}$$

where $\partial v/\partial p$ is the Jacobi matrix of the map $p \mapsto v(p) = p/\sqrt{1+p^2}$.

Proposition 4: Let (C), $(P_{\min})$, (K) *hold, let the solution* $Y(t)$ *to the system (1) be scattering and* $Y(0)\in\mathcal{E}^\sigma$ *for a certain* $\sigma\in(0;1/2)$. *Then for any* $R>0$ *and sufficiently small* $\gamma_\rho := |\rho|$,

$$\|Z(\cdot + q(t),t)\|_R \leqslant C_R(Z(0),q^0,\overline{v},R_\rho)(1+|t|)^{-1-\sigma}. \tag{28}$$

Proof: First, we prove the estimate with $R=R_\rho$. Definition (23) implies $Z(\cdot,t)\in\mathcal{F}$. Solving the equations (26) we get the mild solution representation,

$$Z(t) = U(t)Z(0) - \int_0^t U(t-s)\left[\frac{\partial \Psi_{v(s)}}{\partial p}(\cdot - q(s))\dot{p}(s)\right] ds, \tag{29}$$

with $U(t)$ the group generated by the free Klein–Gordon equation in $H^1\oplus L^2$, see the explicit formulas in Sec. III A below.

Thus, the proof consists of two essential parts: (1) estimating, in local seminorms, the action of the free Klein–Gordon group $U(t)Z(0)$ and (2) estimating, in local seminorms, the free Klein–Gordon group applied to the Jacobian of the soliton field, $U(t-s)[\partial\Psi_{v(s)}/\partial p(\cdot - q(s))]$.

A. Local decay for the free Klein–Gordon group

Let us denote $S_t(x)=\{y: |y-x|=t\}$, $B_t(x)=\{y: |y-x|\leqslant t\}$. For sufficiently smooth initial data, say $u_0,v_0\in C_0^\infty(\mathbb{R}^3)$, the action of the free Klein–Gordon group in $\mathbb{R}^3$ reads [Ref. 6, Chap. 5, formulas (6.4), (6.11), (6.12)],

$$U(t)(u_0(x),v_0(x)) = (u(x,t),v(x,t)) = (u_w(x,t)-u_m(x,t), v_w(x,t)-v_m(x,t))$$

with

$$u_w(x,t) = \frac{1}{4\pi t^2}\int_{S_t(x)} d^2y\, u_0(y) + \frac{1}{4\pi t}\int_{S_t(x)} d^2y\,\frac{\partial u_0(y)}{\partial n} + \frac{1}{4\pi t}\int_{S_t(x)} d^2y\, v_0(y), \tag{30}$$

$$u_m(x,t) = \frac{m^2}{8\pi}\int_{S_t(x)} d^2y\, u_0(y) + \frac{m}{4\pi}\int_{B_t(x)} d^3y\, \dot{F}(t,x-y)u_0(y) + \frac{m}{4\pi}\int_{B_t(x)} d^3y\, F(t,x-y)v_0(y), \tag{31}$$

and

$$v_w(x,t) = \dot{u}_w(x,t) = \frac{1}{2\pi t^2}\int_{S_t(x)} d^2y\,\frac{\partial u_0(y)}{\partial n} + \frac{1}{4\pi t}\int_{S_t(x)} d^2y\,\frac{\partial^2 u_0(y)}{\partial n^2} + \frac{1}{4\pi t^2}\int_{S_t(x)} d^2y\, v_0(y) + \frac{1}{4\pi t}\int_{S_t(x)} d^2y\,\frac{\partial v_0(y)}{\partial n}, \tag{32}$$

$$\begin{aligned} v_m(x,t) = \dot{u}_m(x,t) &= \frac{m^2}{4\pi t}\int_{S_t(x)} d^2y\, u_0(y) - \frac{m^4 t}{32\pi}\int_{S_t(x)} d^2y\, u_0(y) + \frac{m^2}{8\pi}\int_{S_t(x)} d^2y\,\frac{\partial u_0(y)}{\partial n} \\ &+ \frac{m}{4\pi}\int_{B_t(x)} d^3y\, \ddot{F}(t,x-y)u_0(y) + \frac{m^2}{8\pi}\int_{S_t(x)} d^2y\, v_0(y) \\ &+ \frac{m}{4\pi}\int_{B_t(x)} d^3y\, \dot{F}(t,x-y)v_0(y). \end{aligned} \tag{33}$$

Here $n=(y-x)/|y-x|$ is the exterior unit normal vector of the sphere $S_t(x)$ at a point y,

$$F(t,z)=\frac{J_1(m\sqrt{t^2-z^2})}{\sqrt{t^2-z^2}},$$

J_1 being the Bessel function of order 1. Note that $(u_w(x,t),v_w(x,t))$ is the solution to the free wave equation corresponding to $m=0$, with the same initial conditions (u_0,v_0). From the well-known asymptotics

$$|J_1(s)|+|J_1'(s)|+|J_1''(s)|=\mathcal{O}(|s|^{-1/2})\quad \text{as } s\to\infty$$

of the Bessel function, see Ref. 20, Chap. XVII, it follows that

$$|F(t,z)|+|\dot F(t,z)|+|\ddot F(t,z)|+|\nabla_z F(t,z)|=\mathcal{O}(|t|^{-3/2})\quad \text{as } t\to\infty, \tag{34}$$

if $|z|\leqslant\nu|t|$ with $0<\nu<1$. However, near the boundary of the cone $|z|=|t|$ only some weaker decay is valid, namely for $|z|\leqslant|t|-1$,

$$|F(t,z)|+|\dot F(t,z)|+|\ddot F(t,z)|+|\nabla_z F(t,z)|=\mathcal{O}(|t|^{-3/4})\quad \text{as } t\to\infty, \tag{35}$$

Definition 5: The set $\mathcal{F}^\sigma$ for $0<\sigma<1/2$ is the set of the fields $(\psi,\pi)\in\mathcal{F}$ satisfying the conditions (21),(22).

Lemma 6: Let $(u_0,v_0)\in\mathcal{F}^\sigma$ with some $\sigma\in(0;1/2)$. Then $\forall R>0$,

$$\|U(t)(u_0,v_0)\|_R\leqslant C(u_0,v_0,R)(1+|t|)^{-1-\sigma}. \tag{36}$$

Proof: Note that for any fixed t the map $U(t):(u_0,v_0)\to(u(t),v(t))$ is continuous in $\mathcal{F}$. For initial data $(u_0,v_0)\in\mathcal{F}^\sigma$ we can approximate them with $u_0^n,\ v_0^n\in C_0^\infty$ such that the bounds (21), (22) hold for u_0, v_0 uniformly in n. Hence, it is sufficient to obtain the estimate (36) for $u_0,v_0\in C_0^\infty$, with $C(u_0,v_0,R)$ depending only on the constant of (21), (22) and on the norm of u_0,v_0 in $\mathcal{F}$. Thus, we may use the integral representation (30) to (33).

At first consider (u_w,v_w). For the free wave equation the following energy inequality is well known:

$$\int_{B_R} d^3x(|\nabla u_w(x,t)|^2+|v_w(x,t)|^2)\leqslant\int_{B_{t+R}} d^3x(|\nabla u_0(x)|^2+|v_0(x)|^2).$$

Further, from the strong Huygen's principle it follows that for $t>R$ the solution $(u_w(x,t),v_w(x,t))$ does not change if one replaces $u_0(x)$, $v_0(x)$ by zero inside the ball B_{t-R}. Hence,

$$\int_{B_R} d^3x(|\nabla u_w(x,t)|^2+|v_w(x,t)|^2)\leqslant\int_{B_{t-R,t+R}} d^3x(|\nabla u_0(x)|^2+|v_0(x)|^2),$$

where $B_{t-R,t+R}=\{x\in\mathbb{R}^3: t-R\leqslant|x|\leqslant t+R\}$. Then the conditions (21), (22) imply, for sufficiently large t,

$$|\nabla u_w(\cdot,t)|_R+|v_w(\cdot,t)|_R\leqslant C(R,u_0,v_0)(1+|t|)^{-1-\sigma}. \tag{37}$$

It remains to estimate $|u_w(\cdot,t)|_R$, $\|u_m(\cdot,t)\|_R$, and $|v_m(\cdot,t)|_R$. We claim that if $(u_0,v_0)\in\mathcal{F}^\sigma$, then for sufficiently large t any of the norms $|I(\cdot,t)|_R$, $\|J(\cdot,t)\|_R$, $|K(\cdot,t)|_R$ is bounded by $C(u_0,v_0,R)(1+|t|)^{-1-\sigma}$ for any integral $I(x,t)$ of (30), $J(x,t)$ of (31), $K(x,t)$ of (33). At first consider the spherical integrals. For example, let us prove these estimates for the integrals

$$I(x,t)=\frac{1}{4\pi t}\int_{S_t(x)} d^2y\,\frac{\partial u_0(y)}{\partial n},\quad J(x,t)=\frac{m^2}{8\pi}\int_{S_t(x)} d^2y\,u_0(y),\quad K(x,t)=\frac{m^4 t}{32\pi}\int_{S_t(x)} d^2y\,u_0(y).$$

For $I(x,t)$ we should estimate $|I(\cdot,t)|_R$. We have

$$|I(\cdot,t)|_R^2=\int_{B_R} d^3x(I(x,t))^2$$
$$=\frac{C}{t^2}\int_{B_R} d^3x\left(\int_{S_t(x)} d^2y\,\frac{\partial u_0(y)}{\partial n}\right)^2$$
$$\leqslant\frac{C}{t^2}\int_{B_R} d^3x\ 4\pi t^2\int_{S_t(x)} d^2y\left(\frac{\partial u_0(y)}{\partial n}\right)^2=C_1\int_{B_R} d^3x\int_{S_t(x)} d^2y\left(\frac{\partial u_0(y)}{\partial n}\right)^2.$$

For a non-negative continuous function u and $t\geqslant R$ the following bound follows by integration in polar coordinates and geometric argument:

$$\int_{B_R} d^3x\int_{S_t(x)} d^2y\ u(y)\leqslant 8\pi R^2\int_{B_{t-R,t+R}} d^3x\ u(x). \tag{38}$$

Hence

$$|I(\cdot,t)|_R^2\leqslant C_2R^2\int_{B_{t-R,t+R}} d^3x\left(\frac{\partial u_0(x)}{\partial n}\right)^2 \tag{39}$$

for $t>R$. Thus, from the condition (21) the stated bound follows.

For $J(x,t)$ we should estimate $\|J(\cdot,t)\|_R=|J(\cdot,t)|_R+|\nabla J(\cdot,t)|_R$. Consider $|J(\cdot,t)|_R$. Similar to (39) we obtain

$$|J(\cdot,t)|_R^2\leqslant CR^2t^2\int_{B_{t-R,t+R}} d^3x(u_0(x))^2.$$

Then from the condition (21) the required estimate follows. For $\nabla J(x,t)=(m^2/8\pi)\int_{S_t(x)}d^2y\ \nabla u_0(y)$ the estimate is analogous. Consider $|K(\cdot,t)|_R$. Similarly to (39) we obtain

$$|K(\cdot,t)|_R^2\leqslant CR^2t^4\int_{B_{t-R,t+R}} d^3x(u_0(x))^2.$$

Then the estimate we need follows from the condition (21).

Now estimate the integrals over the balls. For example, consider the integral

$$J(x,t)=\frac{m}{4\pi}\int_{B_t(x)} d^3y\ F(t,x-y)v_0(y).$$

We should estimate $|J(\cdot,t)|_R$ and $|\nabla J(\cdot,t)|_R$. We have

$$|J(x,t)|=C\left|\int_{B_t(x)} d^3y\ F(t,x-y)v_0(y)\right|$$
$$\leqslant C\left|\int_{B_{t^\alpha}(x)} d^3y\ F(t,x-y)v_0(y)\right|+C\left|\int_{B'_{t^\alpha}(x)} d^3y\ F(t,x-y)v_0(y)\right|,$$

where $B_{t^\alpha}(x)=\{y:|y-x|\leqslant t^\alpha\}$, $B'_{t^\alpha}(x)=\{y:t^\alpha\leqslant|y-x|\leqslant t\}$. The first integral is bounded by

$$\left(\int_{B_{t^\alpha}(x)} d^3y\ F^2(t,x-y)\right)^{1/2}\left(\int_{B_{t^\alpha}(x)} d^3y\ v_0^2(y)\right)^{1/2}\leqslant C_1t^{(3\alpha-3)/2}|v_0|\leqslant C_1t^{-1-\sigma}|v_0|$$

for $\alpha=(1-2\sigma)/3$. The second integral is bounded by

$$\left|\int_{B_{t^\alpha,t-1}(x)} d^3y\, F(t,x-y)v_0(y)\right| + \left|\int_{B_{t-1,t}(x)} d^3y\, F(t,x-y)v^0(y)\right|, \tag{40}$$

where $B_{t^\alpha,t-1}(x)=\{y:t^\alpha\leqslant|y-x|\leqslant t-1\}$, $B_{t-1,t}(x)=\{y:t-1\leqslant|y-x|\leqslant t\}$. Further, for sufficiently large t the first integral of (40) is bounded by

$$\left(\int_{B_{t^\alpha,t-1}(x)} d^3y\, F^2(t,x-y)\right)^{1/2}\left(\int_{B_{t^\alpha,t-1}(x)} d^3y\, v_0^2(y)\right)^{1/2}\leqslant C(t^{3-3/2})^{1/2}(t^{-7/2-2\sigma})^{1/2}=Ct^{-1-\sigma},$$

due to (35) and (22). The second integral of (40) is bounded by

$$C\,t\left(\int_{B_{t-1,t}(x)} d^3y\, v_0^2(y)\right)^{1/2}\leqslant Ct^{-1-\sigma}$$

by (22). Thus we have the pointwise bound $|J(x,t)|\leqslant Ct^{-1-\sigma}$ that implies the stated integral bound.

Now let us estimate $|\nabla J(\cdot,t)|_R$. Note that

$$\nabla J(x,t)=\frac{m}{4\pi}\int_{B_t(x)} d^3y\, F(t,x-y)\nabla v_0(y).$$

Since $F(t,x-y)=m/2$ for $|x-y|=t$, the partial integration gives

$$\left|\int_{B_t(x)} d^3y\, F(t,x-y)\frac{\partial}{\partial y_i}v_0(y)\right|\leqslant\frac{m}{2}\int_{S_t(x)} d^2y\,|v_0(y)|+\int_{B_t(x)} d^3y\left|\frac{\partial}{\partial x_i}F(t,x-y)v_0(y)\right|.$$

Then the estimates for both the spherical integral and the integral over the ball are made as above. Hence, the bound $|\nabla J(\cdot,t)|_R\leqslant C(v_0,R)(1+|t|)^{-1-\sigma}$ follows from the condition (22).

Altogether, we obtain that for sufficiently large t the estimate (36) is true. For bounded t this estimate follows from the energy conservation for the free Klein–Gordon equation.

□

Remark: The statement of the lemma is true under some weaker conditions on initial data, than (21), (22). Namely, it suffices to assume that

$$\int_{R\leqslant|x|\leqslant R+1} d^3x(R^2|u_0(x)|^2+|\nabla v_0(x)|^2)=\mathcal{O}(R^{-(4+2\sigma)}),$$

$$\int_{R^\alpha\leqslant|x|\leqslant R} d^3x\ (|u_0(x)|^2+|v_0(x)|^2)=\mathcal{O}(R^{-(4+2\sigma)})$$

as $R\to+\infty$; $\alpha=(1-2\sigma)/3$.

Thus, for the first term on the right-hand side of (29) we have

$$\|U(t)Z(0)\|_{R_\rho}\leqslant\frac{C(Z(0),\overline{v},R_\rho)}{(1+|t|)^{1+\sigma}}.$$

Then from (15) the estimate

$$\|U(t)Z(0)(\cdot+q(t))\|_{R_\rho}\leqslant\frac{C(Z(0),q^0,\overline{v},R_\rho)}{(1+|t|)^{1+\sigma}} \tag{41}$$

follows.

B. Decay of the soliton field subject to free Klein–Gordon group

Denote by $Z_1(x,t)=\psi(x,t)-\psi_{v(t)}(x-q(t))$ the first component of $Z(x,t)$ and observe that $\langle\psi_v,\nabla\rho\rangle=0$ for $|v|<1$ because the soliton (5) is a solution to (4). Note that for scattering solutions, for sufficiently large $|t|$, the fourth equation of the system (1) transforms to the fourth nonperturbed equation of the system (4) Then for these $|t|$,

$$\dot{p}(t)=\langle Z_1(x+q(t),t),\nabla\rho(x)\rangle\,. \tag{42}$$

Thus we obtain,

$$|\dot{p}(t)|\leqslant C\|Z(\cdot+q(t),t)\|_{R_\rho}\gamma_\rho\,. \tag{43}$$

Denote $S_{t-s}(x)=\{y\colon\ |y-x|=t-s\}$, $B_{t-s}(x)=\{y\colon\ |y-x|\leqslant t-s\}$, $\psi_v^p=\partial\psi_v/\partial p$, $\pi_v^p=\partial\pi_v/\partial p$, and

$$(\psi^U(\cdot,t,s),\pi^U(\cdot,t,s))=U(t-s)\left[\frac{\partial\Psi_{v(s)}}{\partial p}(\cdot-q(s))\right]. \tag{44}$$

Then the formulas (30), (31) for $U(t-s)$ imply

$$\begin{aligned}\psi^U(x,t,s)={}&\frac{1}{4\pi(t-s)}\int_{S_{t-s}(x)}d^2y\ \pi^p_{v(s)}(y-q(s))-\frac{m}{4\pi}\int_{B_{t-s}(x)}d^3y\,F(t-s,x-y)\,\pi^p_{v(s)}(y-q(s))\\&+\frac{1}{4\pi(t-s)^2}\int_{S_{t-s}(x)}d^2y\ \psi^p_{v(s)}(y-q(s))+\frac{1}{4\pi(t-s)}\int_{S_{t-s}(x)}d^2y\,\frac{\partial}{\partial n}\psi^p_{v(s)}(y-q(s))\\&-\frac{m^2}{8\pi}\int_{S_{t-s}(x)}d^2y\ \psi^p_{v(s)}(y-q(s))-\frac{m}{4\pi}\int_{B_{t-s}(x)}d^3y\,\dot{F}(t-s,x-y)\,\psi^p_{v(s)}(y-q(s)).\end{aligned} \tag{45}$$

From this one derives the explicit formula for $\nabla\psi^U(x,t,s)$; (32) and (33) give the formula for $\pi^U(x,t,s)$.

Now $\psi^U(x+q(t),t,s)$ can be represented as the sum of type (45) of integrals over the shifted sphere $S_{t-s}(x+q(t))$ and ball $B_{t-s}(x+q(t))$ and with $x+q(t)$ replacing x in F and $\dot{F}$. Denote $\psi_S^U(x+q(t),t,s)$ the sum of the integrals over the sphere $S_{t-s}(x+q(t))$ and $\psi_B^U(x+q(t),t,s)$ the sum of the integrals over the ball $B_{t-s}(x+q(t))$. Let us estimate $\psi_S^U(x+q(t),t,s)$. If $|x|\leqslant R_\rho$, we have on the sphere $S_{t-s}(x+q(t))$,

$$|y-q(s)|=|(y-x-q(t))+(x+q(t)-q(s))|\geqslant(t-s)-|x|-\overline{v}(t-s)\geqslant(1-\overline{v})(t-s)-R_\rho \tag{46}$$

by the bound (15) on $\dot{q}(t)$. On the other hand, the integral representation (6) yields by Cauchy–Schwartz inequality,

$$\sup_{|v|\leqslant\overline{v}}\sup_{|x|\geqslant 2R_\rho}e^{m|x|}(|\partial^\alpha\psi_v^p(x)|+|\partial^\beta\pi_v^p(x)|)\leqslant C(\overline{v},R_\rho)\,\gamma_\rho<\infty \tag{47}$$

for all multi-indices α,β with $|\alpha|\leqslant 2$, $|\beta|\leqslant 1$, recall that $\gamma_\rho:=|\rho|$. Then (47) and (46) imply the following pointwise bound for $\psi_S^U(x+q(t),t,s)$:

$$|\psi_S^U(x+q(t),t,s)|\leqslant C_1(\overline{v},R_\rho)\,\gamma_\rho e^{-m(t-s)} \tag{48}$$

for $|x|\leqslant R_\rho$ and provided $t-s\geqslant 3R_\rho/(1-\overline{v})$.

Now let us estimate $\psi_B^U(x+q(t),t,s)$. Set $\mu=(1-2\sigma)/6$ and consider two regions $B_\mu=\{y:|y-q(s)|\leqslant(t-s)^\mu\}$ and

$$B'_\mu = B_{t-s}(x+q(t)) - B_\mu = \{y : |y-(x+q(t))| \leqslant (t-s) \ \& \ |y-q(s)| > (t-s)^\mu\}.$$

Represent every integral over the ball $B_{t-s}(x+q(t))$ as the sum of the integrals over B_μ and B'_μ. Note that the volume of B_μ is of order $(t-s)^{3\mu}$, the volume of B'_μ is of order $(t-s)^3-(t-s)^{3\mu}$. Furthermore, for sufficiently large $t-s$ and $y\in B_\mu$ we have $|y-x|\leqslant \nu|t|$ with some positive $\nu<1$. Hence, by (34) the following estimate is true:

$$|F(t-s,x-y)| + |\dot F(t-s,x-y)| + |\ddot F(t-s,x-y)| \leqslant C|t-s|^{-3/2}.$$

Then for the integrals over B_μ we have

$$\left|\int_{B_\mu} \cdots \right| \leqslant C_1(t-s)^{3\mu}(t-s)^{-3/2}C_2\gamma_\rho = C_3\gamma_\rho(t-s)^{3\mu-3/2} = C_3\gamma_\rho(t-s)^{-1-\sigma}$$

for $\mu=(1-2\sigma)/6$.

For the integrals over B'_μ we obtain, due to (47), provided $t-s\geqslant(2R_\rho)^{1/\mu}$,

$$\left|\int_{B'_\mu} \cdots \right| \leqslant C_4((t-s)^3-(t-s)^{3\mu})C_5\exp(-m(t-s)^\mu)\gamma_\rho \leqslant \frac{C_6\gamma_\rho}{(t-s)^{1+\sigma}}$$

for sufficiently large $t-s$. Thus, the sum of the integrals over B'_μ is bounded by $(C_7(\overline{v},R_\rho)\gamma_\rho)/(1+(t-s)^{1+\sigma})$. So we come to

$$|\psi^U_B(x+q(t),t,s)| \leqslant \frac{C_8(\overline{v},R_\rho)\gamma_\rho}{1+(t-s)^{1+\sigma}} \tag{49}$$

for $|x|\leqslant R_\rho$ and sufficiently large $t-s$. Therefore (48) and (49) imply for large $t-s$, together with similar bounds for $\nabla\psi^U(x+q(t),t,s)$ and $\pi^U(x+q(t),t,s)$, the integral estimate

$$\|(\psi^U(\cdot+q(t),t,s),\pi^U(\cdot+q(t),t,s))\|_{R_\rho} \leqslant \frac{C_9(\overline{v},R_\rho)\gamma_\rho}{1+(t-s)^{1+\sigma}}. \tag{50}$$

On the other hand, for bounded $t-s$ this integral estimate follows from (44) by energy conservation for the group $U(t-s)$ since $\|\partial\Psi_v/\partial p\|_{\mathcal F}\leqslant C(\overline{v},R_\rho)\gamma_\rho$ by (C). Finally, (43) and (50) imply

$$\|\dot p(s)\cdot(\psi^U(\cdot+q(t),t,s),\pi^U(\cdot+q(t),t,s))\|_{R_\rho} \leqslant C_{10}(\overline{v},R_\rho)\gamma_\rho \frac{\|Z(\cdot+q(s),s)\|_{R_\rho}\gamma_\rho}{1+(t-s)^{1+\sigma}}. \tag{51}$$

C. Completing the proof of Proposition 4

The method was initially developed in Ref. 13 for $m=0$, see also Ref. 11. Combining (29) to (51) and (41) we arrive at

$$\|Z(\cdot+q(t),t)\|_{R_\rho} \leqslant \frac{C(Z(0),q^0,\overline{v},R_\rho)}{(1+|t|)^{1+\sigma}} + \gamma_\rho^2 C_{10}(\overline{v},R_\rho)\int_0^t \frac{\|Z(\cdot+q(s),s)\|_{R_\rho}}{1+(t-s)^{1+\sigma}}ds, \quad t\geqslant 0. \tag{52}$$

Therefore, setting $M(t)=\max_{0\leqslant s\leqslant t}(1+|s|)^{1+\sigma}\|Z(\cdot+q(s),s)\|_{R_\rho}$, we have

$$M(t)\leqslant C_0(Z(0),q^0,\overline{v},R_\rho)+\gamma_\rho^2 C(\overline{v},R_\rho)I_\sigma M(t),$$

where

$$I_\sigma = \sup_{t\geqslant 0}(1+|t|)^{1+\sigma}\int_0^t \frac{(1+|s|)^{-1-\sigma}}{(1+|t-s|^{1+\sigma})}\,ds<\infty.$$

It remains to choose $\gamma_\rho^2 C(\overline{v},R_\rho)I_\sigma<1$, then (28) with $R=R_\rho$ follows.

Remark: It is important that $\overline{v}$ is bounded for bounded γ_ρ and fixed initial data.

At last, we claim that the bound (28) with $R=R_\rho$ implies (28) for any $R>0$. Indeed, (50)–(52) hold with the norm $\|\cdot\|_R$ instead of $\|\cdot\|_{R_\rho}$ on the *left*-hand sides and with $C_i(\overline{v},\rho,R)$ instead of $C_i(\overline{v},R_\rho)$ on the *right*-hand sides. Then (52) with this generalization and (28) with $R=R_\rho$ imply (28) for any $R>0$. □

IV. SCATTERING

Theorem 7: *Under the conditions of Proposition 4, for sufficiently small γ_ρ, the convergence (7) holds, and the solution $Y(t)$ displays the following long-time asymptotics:*

(i) There exist $v_\pm=\lim_{t\to\pm\infty}\dot q(t)\in\mathcal{V}$ such that

$$|\dot q(t)-v_\pm|\leqslant C\ (1+|t|)^{-\sigma}, \tag{53}$$

$$\|\Psi(\cdot+q(t),t)-\Psi_{v_\pm}\|_R\leqslant C_R(1+|t|)^{-\sigma},\quad \forall R>0. \tag{54}$$

(ii) There exist $\Psi_\pm\in\mathcal{F}$ such that

$$\|\Psi(\cdot,t)-\Psi_{v(t)}(\cdot-q(t))-U(t)\Psi_\pm\|_{\mathcal{F}}\leqslant C(1+|t|)^{-\sigma}. \tag{55}$$

Proof: (i) Equation (28) with $R=R_\rho$ and (43) imply

$$|\dot p(t)|\leqslant C(1+|t|)^{-1-\sigma}\Leftrightarrow|\ddot q(t)|\leqslant C_1(1+|t|)^{-1-\sigma}. \tag{56}$$

Then the limits (8) exist, and (53) follows. Therefore, (28) implies (54).

(ii) We have to prove that $\|Z(x,t)-U(t)F_\pm\|_{\mathcal{F}}\leqslant C(1+|t|)^{-\sigma}$. This is equivalent to $\|U(-t)Z(x,t)-F_\pm\|_{\mathcal{F}}\leqslant C(1+|t|)^{-\sigma}$ since the group $U(t)$ is isometric in $\mathcal{F}$. Apply $U(-t)$ to the integral equation (29). We obtain

$$U(-t)Z(t)=Z(0)-\int_0^t U(-s)\left[\frac{\partial\Psi_{v(s)}}{\partial p}(\cdot-q(s))\dot p(s)\right]ds.$$

The condition (15) implies that the norm of $\Psi_{v(s)}(\cdot-q(s))$ in $\mathcal{F}$ is bounded uniformly with respect to s. Then (56) implies the convergence of the integral in $\mathcal{F}$ at the stated rate and Theorem 7 is proved. □

A. Constructing scattering solutions

Let us formulate a criterion for a solution $Y(t)$ to be scattering. Introduce the energy of the field part of a solution

$$h(t)=\frac{1}{2}\int d^3x(|\nabla\psi(x,t)|^2+m^2|\psi(x,t)|^2+|\pi(x,t)|^2).$$

Set $G=\sup_{x\in\mathbb{R}^3}|\nabla V(x)|$ and $v(t)=\dot q(t)$.

Theorem 8: *Let (C), $(P_{\min})$, (K) hold. Consider solutions $Y(t)$ to the system (1) with initial data $Y(0)\in\mathcal{E}^\sigma$, $0<\sigma<1/2$. Let $R_V,G,h(0),|q(0)|$ be finite. Then for $|\dot q(0)|$ close enough to 1 and sufficiently small γ_ρ, we have*

$$\lim_{t\to\pm\infty}|q(t)|=\infty. \tag{57}$$

Proof: Since the system (1) is time invertible, we consider only the case $t\to+\infty$. Consider the particle with initial data $q(0),v(0):=\dot q(0)$. Introduce $e=v(0)/|v(0)|$. The orthogonal projection of the vectors $v(t),p(t),q(t)$ onto e read $v_e(t)e$, $p_e(t)e$, $q_e(t)e$, respectively, with $v_e(t):=v(t)\cdot e$, $p_e(t):=p(t)\cdot e$, $q_e(t):=q(t)\cdot e$, here the dot means the scalar product in $\mathbb{R}^3$. Note that the vectors $v(t)$ and $p(t)$, $v_e(t)$ and $p_e(t)$ are of the same directions and $v_e(0)=|v(0)|$, $p_e(0)=|p(0)|$. Introduce the layer in $\mathbb{R}^3$, $L(e,R_V)=\{x:|x\cdot e|\le R_V\}$, then $\operatorname{supp} V\subset L(e,R_V)$.

The statement of the theorem follows from the three propositions below. Since the system (1) is invariant with respect to time translations, we start from $t=0$ in either proposition.

Proposition 9: Let $|q(0)|>R_V$, $|v(0)|$ *be close enough to 1, let* e *be directed toward* $L(e,R_V)$. *Then the particle enters* $L(e,R_V)$ *at a certain moment* τ *with* $|v_e(\tau)|$ *close to 1.*

Proposition 10: Let $|q(0)|\le R_V$, *let* $|v(0)|$ *be close to 1. Then the particle leaves* $L(e,R_V)$ *at a certain moment* τ *such that* $|v_e(\tau)|>0$ *and* $v_e(\tau)e$ *is directed outside* $L(e,R_V)$.

Proposition 11: Let $|q(0)|\ge R_V$, $|v(0)|>0$ *and* e *is directed outside* $L(e,R_V)$. *Then the particle never enters* $L(e,R_V)$ *and* $|q_e(t)|\to\infty$ *as* $t\to+\infty$.

Proof of Proposition 9: For $v_e(t)$ we have the estimate

$$v_e(t)\ge v_e(0)-\int_0^t|\dot{v}(s)|ds=|v(0)|-\int_0^t|\dot{v}(s)|ds.$$

Since outside $L(e,R_v)$ the free equations (4) are satisfied, the following estimate (see [28) and (43)] is valid:

$$|\dot{v}(t)|\le\frac{C(Z(0),q(0),\overline{v},R_\rho)\gamma_\rho}{(1+|t|)^{\sigma+1}}. \tag{58}$$

Here $C(Z(0),q(0),\overline{v},R_\rho)$ is bounded uniformly with respect to the values $q(0),\psi(0),\pi(0)$ under consideration. Thus,

$$v_e(t)\ge|v(0)|-\int_0^\infty\frac{C\gamma_\rho\,dt}{(1+|t|)^{\sigma+1}}=|v(0)|-\frac{C\gamma_\rho}{\sigma},$$

and we obtain the required result for sufficiently small γ_ρ. □

Proof of Proposition 10: First we check that the growth of the field energy is not very fast.

Lemma 12: The following bound holds:

$$h(t)\le(\sqrt{h(0)}+\sqrt{2}\gamma_\rho t)^2. \tag{59}$$

Proof: Multiply the equation $\ddot{\psi}=\Delta\psi-m^2\psi-\rho$ by $\dot{\psi}$ and integrate over $\mathbb{R}^3$. We obtain $\dot{h}(t)=-\int d^3x\,\rho\dot{\psi}$ and hence $\dot{h}(t)\le\sqrt{2}\gamma_\rho\sqrt{h}$. Integrating this differential inequality in t we come to $\sqrt{h(t)}\le\sqrt{h(0)}+\sqrt{2}\gamma_\rho t$ which proves (59). □

Let us now prove the proposition. Recall that $v=p/\sqrt{1+p^2}$ and hence, $p=v/\sqrt{1-v^2}$. Thus, $|v|$ is close to 1 if and only if $|p|$ is large. From the equation

$$\dot{p}(t)=-\nabla V(q(t))+\int d^3x\,\psi(x,t)\,\nabla\rho(x-q(t))$$

we obtain, due to (59), $|\dot{p}|\le G+\|\psi\|\gamma_\rho\le G+(2h(t))^{1/2}\gamma_\rho\le G+((2h(0))^{1/2}+2\gamma_\rho t)\gamma_\rho=G_1+2\gamma_\rho^2 t$ with $G_1=G+(2h(0))^{1/2}\gamma_\rho$. The conditions of the theorem imply that G_1 is bounded. We obtain the following lower and upper bounds:

$$p_e(t)\ge p_e(0)-\int_0^t|\dot{p}(s)|ds\ge|p(0)|-G_1t-\gamma_\rho^2t^2=P-f(t),$$

$$|p(t)|\le|p(0)|+\int_0^t|\dot{p}(s)|ds\le|p(0)|+G_1t+\gamma_\rho^2t^2=P+f(t),$$

where $P:=|p(0)|,f(t):=G_1t+\gamma_\rho^2t^2$. These estimates imply for $v_e(t)$,

$$v_e(t)=\frac{p_e(t)}{|p(t)|}\left(1+\frac{1}{|p(t)|^2}\right)^{-1/2}\geqslant\frac{P-f(t)}{P+f(t)}\left(1-\frac{1}{(P-f(t))^2}\right)$$
$$=\frac{1-a^2-2af(t)+a^2f^2(t)}{1-a^2f^2(t)}$$
$$\geqslant(1-a^2-2af(t)+a^2f^2(t))(1+a^2f^2(t))=1-a^2+g(t), \tag{60}$$

where $a:=P^{-1}$, $g(t):=-2af(t)+(2a^2-a^4)f^2(t)-2a^3f^3(t)+a^4f^4(t)$. The corresponding estimate for $q_e(t)$ is

$$q_e(t)\geqslant q_e(0)+(1-a^2)t+\int_0^t g(s)ds. \tag{61}$$

Take sufficiently large P, that is small a, then from the estimates (61), (60) the statement of the proposition follows. □

Proof of Proposition 11: We claim that there exist such small $\gamma_\rho>0,\underline{v}>0$ that $\forall\ t>0\, v_e(t)\geqslant\underline{v}$. Indeed, set $T=\sup\{t>0:\ v_e(t)>\underline{v}\}$. If $\underline{v}<v_e(0)/2$, then, by continuity, $T>0$. We claim that it is possible to choose such small $\gamma_\rho>0$, $\underline{v}>0$ that $T=+\infty$. Indeed, for $t\in[0,T]$ the free equations (4) are satisfied, hence the estimate (58) is valid. Take

$$0<\underline{v}<v_e(0)-\int_0^\infty\frac{C\gamma_\rho}{(1+|t|)^{\sigma+1}}=v_e(0)-\frac{C\gamma_\rho}{\sigma};$$

this choice is possible for sufficiently small γ_ρ. If $T<+\infty$, then $v_e(T)>\underline{v}$, hence, by continuity, $v_e(T+\varepsilon)>\underline{v}$ for some $\varepsilon>0$. This contradicts to the definition of T. Thus, $T=+\infty$. Hence, for $t>0$ one obtains $q_e(t)\geqslant q_e(0)+\underline{v}t$. □

Note that from the proof of the theorem the following statement follows.

Corollary 13: Let (*C*), ($P_{\min}$), (*K*) *hold, let* $Y(0)\in\mathcal{E}^\sigma,0<\sigma<1/2$. *Let* $R_V,h(0),|q(0)|$ *be finite. Then for* $\dot{q}(0)\neq0$ *and sufficiently small* γ_ρ, *G the solution* $Y(t)$ *is scattering.*

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