Soliton-Type Asymptotics and Scattering for a Charge Coupled to the Maxwell Field

V. Imaikin*, A. Komech, and H. Spohn*****

*, **Institute of Mathematics, University of Vienna,
Boltzmanngasse 9, 1090 Vienna, Austria
E-mail: vimaikin@mat.univie.ac.at, komech@mathematik.tu-muenchen.de
***Zentrum Mathematik, TU München, 80290 München, Germany
E-mail:spohn@mathematik.tu-muenchen.de

Received July 29, 2002

Abstract. We establish soliton-type asymptotic relations for finite-energy solutions of the Maxwell–Lorentz equations describing a charge coupled to an electromagnetic field. Any solution converges to a sum of a travelling wave and an outgoing free wave. The convergence holds with respect to the global energy norm. The proof uses the method of nonautonomous integral inequalities.

1. INTRODUCTION

Consider a single charge coupled to the Maxwell field. If $q(t) \in \mathbb{R}^3$ is the position of charge at time t, then the coupled Maxwell–Lorentz equations are

$$\begin{aligned} &\operatorname{div} E(x,t) = \rho(x-q(t)), && \operatorname{rot} E(x,t) = -\dot B(x,t), \\ &\operatorname{div} B(x,t) = 0, && \operatorname{rot} B(x,t) = \dot E(x,t) + \rho(x-q(t))\dot q(t), \\ &\quad \dot q(t) = \frac{p(t)}{\sqrt{1+p^2(t)}}, \\ &\quad \dot p(t) = -\nabla V(q(t)) + \dot q(t)\wedge \operatorname{rot} A(q(t)) + \int d^3x\,[E(x,t)+\dot q(t)\wedge B(x,t)]\rho(x-q(t)). \end{aligned} \tag{1.1}$$

Here and below, all derivatives are understood in the sense of distributions. The last row is the Lorentz force equation, and the first two rows give the inhomogeneous Maxwell equations. The function ρ is the distribution of the charge, which is commented below. We use units for which the velocity of light is $c = 1$, the mechanical mass of the charge is $m = 1$, and $\varepsilon_0 = 1$.

We consider all finite-energy solutions of equations (1.1). The appropriate phase space is introduced below. First we note that the energy integral

$$\mathcal{H}(E,B,q,p) = (1+p^2)^{1/2} + V(q) + \frac{1}{2}\int d^3x \left(|E(x)|^2 + |B(x)|^2\right) \tag{1.2}$$

is conserved along the sufficiently smooth solution trajectories of (1.1). It is then natural to choose the set of all finite-energy states as the phase space. As to the external potentials, we assume that they are smooth and compactly supported,

$$V, A \in C^2(\mathbb{R}^3), \quad V(x) = 0 \quad \text{and} \quad A(x) = 0 \quad \text{for} \quad |x| > R_{\text{ex}}, \quad 0 < R_{\text{ex}} < +\infty. \tag{P}$$

*Partially supported by the Wittgenstein 2000 Award of Peter Markowich, funded by the Austria Research Fund, and by research grants of DFG (436 RUS 113/615/0-1(R)) and RFBR (01-01-04002).

**On leave Department of Mechanics and Mathematics, Moscow State University, Moscow 119992, Russia. Partially supported by Max Planck Institute for Mathematics in the Natural Sciences (Leipzig) and by the START Award of Norbert Mauser.

The real-valued charge distribution ρ is assumed to be rather smooth and compactly supported,

$$\nabla\rho,\, \rho \in L^2(\mathbb{R}^3), \qquad \rho(x) = 0 \quad \text{for} \quad |x| \geqslant R_\rho > 0. \tag{C}$$

Another important assumption is that the norm of ρ in L^2 is sufficiently small,

$$\gamma_\rho \equiv \|\rho\|_{L^2} \ll 1, \tag{1.3}$$

which means that the field-charge interaction is *weak*. We believe that this is an artifact of the mathematical technique in use. Consider the corresponding nonperturbed system with $V \equiv 0$ and $A \equiv 0$,

$$\begin{aligned} &\operatorname{div} E(x,t) = \rho(x-q(t)), && \operatorname{rot} E(x,t) = -\dot{B}(x,t),\\ &\operatorname{div} B(x,t) = 0, && \operatorname{rot} B(x,t) = \dot{E}(x,t) + \rho(x-q(t))\dot{q}(t),\\ &\dot{q}(t) = \frac{p(t)}{\sqrt{1+p^2(t)}}, && \dot{p}(t) = \int d^3x\,[E(x,t) + \dot{q}(t)\wedge B(x,t)]\rho(x-q(t)). \end{aligned} \tag{1.4}$$

System (1.4) has the set of solutions corresponding to the charge travelling with uniform velocity, v. Up to translation, they are of the form

$$S_v(t) = (E_v(x-vt), B_v(x-vt), vt, p_v) \tag{1.5}$$

with an arbitrary velocity $v \in \mathcal{V} = \{v \in \mathbb{R}^3 : |v| < 1\}$. Below, the term "soliton" means a travelling solution of (1.4).

Let us now discuss and summarize our main results; the precise theorems will be stated in the following section. Consider the set $\mathcal{S}$ of *scattering* solutions to (1.1) for which $|q(t)| \to \infty$ as $t \to \infty$. Below we discuss the properties of solutions of class $\mathcal{S}$, in particular, their scattering behavior will be established. Since only a finite amount of energy can be dissipated when going to infinity, we have a relaxation of the acceleration,

$$\ddot{q}(t) \to 0, \quad t \to \pm\infty, \tag{1.6}$$

an effect which is known as *radiation damping*. It differs from the usual friction for which the velocity could vanish as $t \to \infty$. Moreover, we establish the rate of convergence of the form $|\ddot{q}(t)| \sim t^{-1-\sigma}$ for some $\sigma > 0$. This is a crucial point of our asymptotic analysis. It follows that

$$\dot{q}(t) \to v_\pm, \quad t \to \pm\infty, \tag{1.7}$$

and the fields are asymptotically Coulomb travelling waves, which means that

$$(E(x,t), B(x,t)) \sim (E_{v_\pm}(x-q(t)), B_{v_\pm}(x-q(t))), \quad t \to \pm\infty. \tag{1.8}$$

Since the energy is conserved, it follows that the convergence here can be understood in the sense of local energy seminorms, cf. Section 2. Further, we establish the corresponding asymptotics in the *global* energy norm,

$$(E(x,t), B(x,t)) \sim (E_{v_\pm}(x-q(t)), B_{v_\pm}(x-q(t))) + U(t)F_\pm, \quad t \to \pm\infty, \tag{1.9}$$

where $U(t)$ is the group of the free Maxwell equation with zero charge and currents, and $F_\pm$ are the scattering states.

Note that system (1.1) can have solutions which are not scattering. For example, let $q(0)$ coincide with a local minimum point of V, and let the initial energy be sufficiently small. Then it follows from conservation of energy that $q(t)$ remains bounded for all $t \in \mathbb{R}$. We will provide simple sufficient conditions for solutions to belong to $\mathcal{S}$. Certainly, $\mathcal{S}$ coincides with the set of all finite-energy solutions in the case of $V(x) \equiv A(x) \equiv 0$.

A soliton-type asymptotics of type (1.9) for coupled Maxwell–Lorentz equations (1.1) is proved here in the global energy norm for the first time (for $V \equiv A \equiv 0$, the result was announced in [1]). In [9], the long-time convergence to the set of solitons (1.5) is established. Here we essentially use the results in [9] on the integral representation of solutions, as well as the existence of dynamics for (1.1) (see also [3]). The orbital stability of the solitons of (1.1) is proved in [2]. In [7], a general theory of orbital stability of solitons is developed.

A soliton-type asymptotics is proved for some translation-invariant completely integrable 1D equations, see [10]. The soliton-type asymptotics in *local* energy seminorms is proved for translation-invariant 3D systems of scalar fields coupled to particles, [8], and for translation-invariant 1D kinetic-reaction systems, [6]. A soliton-like asymptotics of type (1.9) in the *global* energy norm is proved initially for small perturbations of soliton-like solutions to 1D nonlinear Schrödinger translation-invariant equations, see [4] and [5].

2. MAIN RESULTS

Let us first define a suitable phase space. We refer to a point of the phase space as a *state*. Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3, \mathbb{R}^3)$ with the norm $|\cdot|$ and the inner product $\langle\cdot,\cdot\rangle$. We introduce the spaces $\mathcal{F} = L^2 \oplus L^2$ and $\mathcal{L} = \mathcal{F} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ endowed with the norms

$$\|(E(x), B(x))\|_{\mathcal{F}} = |E| + |B| \text{ and } \|Y\|_{\mathcal{L}} = |E| + |B| + |q| + |p| \text{ for } Y = (E(x), B(x), q, p) \in \mathcal{L}. \tag{2.1}$$

We regard $\mathcal{L}$ as the space of finite-energy states. The energy functional $\mathcal{H}$ is continuous on the space $\mathcal{L}$. On $\mathcal{F}$ and $\mathcal{L}$, we define the local energy seminorms by

$$\|(E(x), B(x))\|_R = |E|_R + |B|_R \text{ and } \|Y\|_R = |E|_R + |B|_R + |q| + |p| \text{ for } Y = (E(x), B(x), q, p) \tag{2.2}$$

for every $R > 0$, where $|\cdot|_R$ is the norm in $L^2(B_R)$ and B_R is the ball $\{x \in \mathbb{R}^3 : |x| < R\}$. Denote by $\mathcal{F}_F$ and $\mathcal{L}_F$ the spaces $\mathcal{F}$ and $\mathcal{L}$, respectively, equipped with the Fréchet topology induced by the above seminorms. Note that the spaces $\mathcal{L}$, $\mathcal{L}_F$, and $\mathcal{F}_F$ are metrizable, and $\mathcal{L}_F$ and $\mathcal{F}_F$ are not complete.

System (1.1) is overdetermined. Therefore, its actual phase space is a nonlinear submanifold of the linear space $\mathcal{L}$.

Definition 2.1. i) Introduce the phase space $\mathcal{M}$ for the Maxwell–Lorentz equations (1.1) as the metric space of states $(E(x), B(x), q, p) \in \mathcal{L}$ satisfying the constraints

$$\operatorname{div} E(x) = \rho(x - q) \quad \text{and} \quad \operatorname{div} B(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^3. \tag{2.3}$$

The metric on $\mathcal{M}$ is induced through by the embedding $\mathcal{M} \subset \mathcal{L}$.

ii) For $0 \leqslant \sigma \leqslant 1$, let $\mathcal{M}^\sigma$ be the set of states $(E(x), B(x), q, p) \in \mathcal{M}$ such that $\nabla E(x)$ and $\nabla B(x)$ are of class L^∞_{loc} outside the ball B_R for some $R = R(Y) > 0$, and

$$|E(x)| + |B(x)| + |x|\Big(|\nabla E(x)| + |\nabla B(x)|\Big) \leqslant C|x|^{-1-\sigma} \quad \text{for} \quad |x| > R. \tag{2.4}$$

iii) Let $\mathcal{M}_F$ be the space $\mathcal{M}$ endowed with the Fréchet topology induced by the embedding $\mathcal{M} \subset \mathcal{L}_F$.

Remark. $\mathcal{M}$ is a complete metric space which is a nonlinear submanifold of $\mathcal{L}$. The space $\mathcal{M}_F$ is metrizable.

Let us rewrite system (1.1) as a dynamical equation on $\mathcal{M}$,

$$\dot{Y}(t) = F(Y(t)) \quad \text{for} \quad t \in \mathbb{R}, \tag{2.5}$$

where $Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in \mathcal{M}$.

Proposition 2.1 [9]. *Let* (C) *and* (P) *hold and let* $Y^0 = (E^0(x), B^0(x), q^0, p^0) \in \mathcal{M}$.

i) *System* (1.1) *has a unique solution* $Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{M})$ *with* $Y(0) = Y^0$.

ii) *The energy is conserved, i.e.,*

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y^0) \quad for\ t \in \mathbb{R}. \tag{2.6}$$

iii) *The inequality*

$$\sup_{t\in\mathbb{R}} |\dot{q}(t)| \leqslant \overline{v} < 1 \tag{2.7}$$

holds with $\overline{v}$ *depending only on the initial data* Y^0.

The field components of the soliton have the form

$$\begin{aligned} E_v(x) &= -\nabla\phi_v(x) + v\cdot\nabla\ A_v(x), & \phi_v(x) &= \int \frac{d^3y}{4\pi|v(y-x)_{\|} + \lambda(y-x)_{\perp}|}\rho(y), \\ B_v(x) &= \operatorname{rot} A_v(x), & A_v(x) &= v\phi_v(x), \qquad p_v = \frac{v}{\sqrt{1-v^2}}. \end{aligned} \tag{2.8}$$

Here $\lambda = \sqrt{1-v^2}$, and we set $x = vx_{\|} + x_{\perp}$, where $x_{\|} \in \mathbb{R}$ and $v \perp x_{\perp} \in \mathbb{R}^3$ for $x \in \mathbb{R}^3$.

Write $F(x,t) = (E(x,t), B(x,t))$ and $F_v(x) = (E_v(x), B_v(x))$. Set $\gamma_\rho = \|\rho\|_{L^2}$. Let

$$\mathcal{F}_s = \{Z(x) \in \mathcal{F} : \operatorname{div} Z(x) = 0\}.$$

Denote by $U(t)$ the group of the free Maxwell–Lorentz equations on $\mathcal{F}_s$. The existence of this group follows from the appendix in [9] if we set $j = 0$ and $\rho = 0$. The action of this group is isometric on $\mathcal{F}_s$ according to the corresponding energy conservation law.

Theorem 2.1. *Let* $\gamma_\rho = \|\rho\|_{L^2}$ *be sufficiently small,* $\gamma_\rho \leqslant \gamma(\overline{v}, R_\rho)$.

Let $Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{M})$ *be a solution of* (1.1) *belonging to* $\mathcal{S}$, *and let* $Y(0) \in \mathcal{M}^\sigma$ *with some* $\sigma \in (0,1]$. *Then* (1.6) *holds, and the solution* $Y(t)$ *admits the following long-time asymptotics.*

i) *There exist limits* $v_\pm = \lim_{t\to\pm\infty} \dot{q}(t) \in \mathcal{V}$ *such that*

$$|\dot{q}(t) - v_\pm|, \leqslant C(1+|t|)^{-\sigma}, \tag{2.9}$$

$$\|F(x+q(t),t) - F_{v_\pm}(x)\|_R \leqslant C_R(1+|t|)^{-\sigma}, \quad \forall R > 0. \tag{2.10}$$

ii) *There exist limits* $F_\pm \in \mathcal{F}_s$ *such that*

$$\|F(x,t) - F_{v(t)}(x-q(t)) - U(t)F_\pm\|_{\mathcal{F}} \leqslant C(1+|t|)^{-\sigma}. \tag{2.11}$$

Remark. In [9], the relaxation of acceleration (1.6) and the convergence

$$\|F(x+q(t),t) - F_{v(t)}(x)\|_R \to 0$$

were established under the Wiener condition $\hat{\rho}(k) \neq 0$. The technique developed here avoids this condition at the expense of the constraint $\|\rho\|_{L^2} \ll 1$ and even gives a bound for the rate of convergence.

Let us formulate a criterion for a solution $Y(t)$ to be scattering. Write $v(t) := \dot{q}(t)$.

Theorem 2.2. *Consider solutions* $Y(t)$ *of system* (1.1) *with the initial data* $Y(0) \in \mathcal{M}^\sigma$ *for some* $\sigma \in (0,1]$. *Then, for* $|v(0)|$ *close enough to* 1 *and sufficiently small* γ_ρ, *the solution* $Y(t)$ *belongs to* $\mathcal{S}$, *i.e.,*

$$\lim_{t\to\pm\infty} |q(t)| = \infty.$$

Consider a solution $Y(t) \in \mathcal{S}$ of system (1.1). If t is sufficiently large, then $Y(t)$ obeys the nonperturbed equations (1.4). Since system (1.1) is invariant with respect to time translations, we can assume that $Y(t)$ obeys equations (1.4) for $t \geqslant 0$.

If the soliton-like asymptotics is approximately valid, then the field should be close to the soliton centered at $q(t)$ and having the velocity $v(t) = \dot{q}(t)$. We therefore consider the difference

$$Z(x,t) = F(x,t) - F_{v(t)}(x - q(t)). \tag{3.1}$$

Setting $\overline{\rho}(x) = (\rho(x), 0)$ and $A(E,B) = (\operatorname{rot} B, -\operatorname{rot} E)$, we see that F satisfies the equations of motion

$$\dot{F}(x,t) = AF(x,t) - \overline{\rho}(x - q(t))v(t). \tag{3.2}$$

On the other hand, for the soliton field F_v with a fixed v, the following equation holds:

$$-v \cdot \nabla F_v(x - q(t)) = AF_v(x - q(t)) - \overline{\rho}(x - q(t))v.$$

Then we have the equation

$$\dot{Z}(x,t) = AZ(x,t) - \dot{p}(t) \cdot \nabla_p F_{v(t)}(x - q(t)) \tag{3.3}$$

for Z. According to the chain rule,

$$\nabla_p F_v = \nabla_v F_v \, dv(p), \tag{3.4}$$

where $dv(p)$ is the differential of the mapping $p \mapsto v(p) = p/\sqrt{1+p^2}$. In the Cartesian coordinate system, $dv(p)$ is represented by the Jacobi matrix $\partial v_i/\partial p_j$.

Lemma 3.1. *Under the assumptions of Theorem* 2.1, *the following bound holds for any* $R > 0$:

$$\|Z(\cdot + q(t), t)\|_R \leqslant C_R(1 + |t|)^{-1-\sigma}, \tag{3.5}$$

where C_R *depends also on the initial data,* $\overline{v}$, *and* R_ρ.

Proof. First, let us prove the estimate for $R = R_\rho$. Definition (3.1) and the Maxwell–Lorentz equations (1.1) for F and F_v imply that $\operatorname{div} Z = 0$. Therefore, $Z(\cdot, t) \in \mathcal{F}_s$. Solving equations (3.3), we obtain the representation for the mild solution Z,

$$Z(t) = U(t)Z(0) - \int_0^t U(t-s)[\dot{p}(s) \cdot \nabla_p F_{v(s)}(\cdot - q(s))] \, ds. \tag{3.6}$$

The action of the group on the integrand is well defined since $\operatorname{div} \nabla_p F_{v(s)} = 0$. Denote by

$$Z_1(x,t) = E(x,t) - E_{v(t)}(x - q(t)) \quad \text{and} \quad Z_2(x,t) = B(x,t) - B_{v(t)}(x - q(t))$$

the components of $Z(x,t)$ and observe that

$$\int d^3x [E_v(x) + v \wedge B_v(x)]\rho(x) = 0$$

for $|v| < 1$ because the soliton (1.5) is a solution of (1.4). Then formula (1.1) implies

$$\dot{p}(t) = \int d^3x [Z_1(x + q(t), t) + \dot{q}(t) \wedge Z_2(x + q(t), t)]\rho(x). \tag{3.7}$$

Thus, using the inequality $|\dot{q}(t)| < 1$ and condition (C), we obtain

$$|\dot{p}(t)| \leqslant C\gamma_\rho \|Z(\cdot + q(t),t)\|_{R_\rho}. \tag{3.8}$$

Let us write $\overline{F}_v = \nabla_p F_v$, $S_{t-s}(x) = \{y: \ |y-x| = t-s\}$, and

$$\overline{F}(\cdot,t,s) = U(t-s)[\overline{F}_{v(s)}(\cdot - q(s))]. \tag{3.9}$$

Then the formula for $U(t-s)$ in the appendix of [9] implies the representation

$$\overline{F}(x,t,s) = \sum_{|\alpha|\leqslant 1} (t-s)^{|\alpha|-2} \int_{S_{t-s}(x)} d^2y\, m_\alpha(x-y)\partial_y^\alpha \overline{F}_{v(s)}(y-q(s)) \tag{3.10}$$

because $\operatorname{div} \overline{F}_{v(s)} = 0$. The coefficients $m_\alpha(\cdot)$ are bounded 6×6 matrix functions, and the sums are taken over the multiindices $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ with the integers $\alpha_j \geqslant 0$. Therefore, $\overline{F}(x+q(t),t,s)$ can be represented by integrals of type (3.10) over the shifted sphere $S_{t-s}(x+q(t))$ in which $x+q(t)$ replaces x in $m_\alpha(x-y)$. If $|x| \leqslant R_\rho$, then on this sphere we have

$$|y-q(s)| = |(y-x-q(t)) + (x+q(t)-q(s))| \geqslant (t-s) - |x| - \overline{v}(t-s) \geqslant (1-\overline{v})(t-s) - R_\rho \tag{3.11}$$

by the bound (2.7) on $\dot{q}(t)$. On the other hand, by the Cauchy–Schwarz inequality, the integral representation (2.8) yields the bounds of type (2.4) for $\sigma = 1$,

$$|\overline{F}_v(x)| + |x||\nabla \overline{F}_v(x)| \leqslant \gamma_\rho C(\overline{v},R_\rho)|x|^{-2}, \qquad |x| \geqslant 2R_\rho, \quad |v| \leqslant \overline{v}. \tag{3.12}$$

Substituting (3.12) and (3.11) into the formula for $\overline{F}(x+q(t),t,s)$, we obtain the pointwise bound

$$|\overline{F}(x+q(t),t,s)| \leqslant \gamma_\rho \sum_{|\alpha|\leqslant 1} (t-s)^{|\alpha|-2} \frac{C_1(\overline{v},R_\rho)(t-s)^2}{(1+|t-s|)^{|\alpha|+2}} \leqslant \gamma_\rho \frac{C_2(\overline{v},R_\rho)}{1+(t-s)^2} \tag{3.13}$$

for $|x| \leqslant R_\rho$ provided that $t-s \geqslant 3R_\rho/(1-\overline{v})$. Therefore, for large $t-s$, formula (3.13) implies the integral estimate

$$\|\overline{F}(x+q(t),t,s)\|_{R_\rho} \leqslant \gamma_\rho \frac{C_3(\overline{v},R_\rho)}{1+(t-s)^2}. \tag{3.14}$$

On the other hand, for bounded $t-s$, this integral estimate follows from (3.9) because the mapping $U(t-s)$ is isometric, and $\|\nabla_p F_v\|_{\mathcal{F}} \leqslant \gamma_\rho C(\overline{v},R_\rho)$ by condition (C). Finally, (3.8) and (3.14) imply

$$\|\dot{p}(s)\cdot(\overline{F}(x+q(t),t,s)\|_{R_\rho} \leqslant \gamma_\rho^2 C_4(\overline{v},R_\rho)\, \frac{\|Z(\cdot+q(s),s)\|_{R_\rho}}{1+(t-s)^2}. \tag{3.15}$$

Now, let us bound the first term on the right-hand side of (3.6); more precisely, we should estimate $\|[U(t)Z(0)](\cdot + q(t),t)\|_{R_\rho}$. Let us note that we have derived (3.14) from the bounds (3.12), which correspond to (2.4) with $\sigma = 1$. Since $Y(0) \in \mathcal{M}^\sigma$ by assumption, it follows that the function $F(x,0) = (E(x,0),B(x,0))$ satisfies the bounds (2.4) with some $\sigma \in (0,1]$. On the other hand, $F_{v(0)}(x-q(0))$ satisfies the same bounds (2.4) with $\sigma = 1$. Hence, $Z(x,0) = F(x,0) - F_{v(0)}(x-q(0))$ satisfies (2.4) with the same σ. Therefore, following (3.12)–(3.14) in the same way, we see that

$$\|[U(t)Z(0)](\cdot + q(t),t)\|_{R_\rho} \leqslant C(\bar{v},R_\rho)(1+|t|)^{-1-\sigma}. \tag{3.16}$$

For bounded t, this estimate follows from conservation of energy for the free Maxwell–Lorentz equation.

Combining (3.6), (3.15), and (3.16), we arrive at the inequality

$$\|Z(\cdot+q(t),t)\|_{R_\rho} \leqslant \frac{C(\overline{v},R_\rho)}{(1+|t|)^{1+\sigma}} + \gamma_\rho^2 C_4(\overline{v},R_\rho)\int_0^t \frac{\|Z(\cdot+q(s),s)\|_{R_\rho}}{1+(t-s)^2}\,ds, \quad t\geqslant 0. \tag{3.17}$$

Therefore, setting $M(t)=\max_{0\leqslant s\leqslant t}(1+|s|)^{1+\sigma}\|Z(\cdot+q(s),s)\|_{R_\rho}$, we obtain

$$M(t)\leqslant C+\gamma_\rho^2 C_4(\overline{v},\rho)I_\sigma M(t),$$

where

$$I_\sigma=\sup_{t\geqslant 0}(1+|t|)^{1+\sigma}\int_0^t \frac{(1+|s|)^{-1-\sigma}}{(1+|t-s|^2)}\,ds<\infty \quad \text{for} \quad \sigma\in(0,1].$$

It remains to choose $\gamma_\rho^2 C_4(\overline{v},\rho)I_\sigma<1$, which implies (3.5) with $R=R_\rho$. We claim that the bound (3.5) with $R=R_\rho$ implies (3.5) for any $R>0$. Indeed, relations (3.14)–(3.17) hold with the norm $\|\cdot\|_R$ instead of $\|\cdot\|_{R_\rho}$ on the *left*-hand sides and with $C_i(\overline{v},R)$ instead of $C_i(\overline{v},R_\rho)$ on the *right*-hand sides. Then (3.17) (with this generalization) and (3.5) (with $R=R_\rho$) imply (3.5) for any $R>0$.

Proof of Theorem 2.1. i) (3.5) with $R=R_\rho$ and (3.7) imply the equivalence

$$|\dot{p}(t)|\leqslant C(1+|t|)^{-1-\sigma} \Longleftrightarrow |\ddot{q}(t)|\leqslant C_1(1+|t|)^{-1-\sigma}. \tag{3.18}$$

Then the limits (1.7) exist, and (2.9) follows. Therefore, (3.5) implies (2.10).

ii) We must prove that $\|Z(x,t)-U(t)F_\pm\|_{\mathcal{F}}\leqslant C(1+|t|)^{-\sigma}$. This is equivalent to

$$\|U(-t)Z(x,t)-F_\pm\|_{\mathcal{F}}\leqslant C(1+|t|)^{-\sigma}$$

since the group $U(t)$ is isometric in $\mathcal{F}_s$. Apply $U(-t)$ to the integral equation (3.6); this gives

$$U(-t)Z(t)=Z(0)-\int_0^t U(-s)[\dot{p}(s)\overline{F}_{v(s)}(\cdot-q(s))]\,ds.$$

By condition (2.7), the norm of $F_{v(s)}(\cdot-q(s))$ in $\mathcal{F}$ is bounded uniformly with respect to s. Then (3.18) implies the convergence of the integral in $\mathcal{F}_s$ at the desired rate.

Theorem 2.1 is proved.

4. CONSTRUCTING SCATTERING SOLUTIONS

In this section we prove Theorem 2.2. Since system (1.1) is time-invertible, we consider only the case $t\to+\infty$. Consider the charge with initial data $q(0),v(0)$. Introduce $e:=v(0)/|v(0)|$. The orthogonal projections of the vectors $v(t),p(t),q(t)$ to e are $v_e(t)e$, $p_e(t)e$, $q_e(t)e$, respectively, with $v_e(t):=v(t)\cdot e$, $p_e(t):=p(t)\cdot e$, $q_e(t):=q(t)\cdot e$, where the dot means the inner product in $\mathbb{R}^3$. Note that the vectors $v(t)$ and $p(t)$, $v_e(t)$, and $p_e(t)$ are of the same direction, and $v_e(0)=|v(0)|$, $p_e(0)=|p(0)|$. Introduce the layer in $\mathbb{R}^3$ of the form $L(e,R_{\rm ex})=\{x:|x\cdot e|\leqslant R_{\rm ex}\}$. In this case, $\operatorname{supp}V\subset L(e,R_{\rm ex})$ and $\operatorname{supp}A\subset L(e,R_{\rm ex})$.

The statement of the theorem follows from the three propositions proved below. Since system (1.1) is invariant with respect to time translations, we start from $t=0$ in each of the propositions.

Proposition 4.1. *Let $|q(0)|>R_{\rm ex}$, let $|v(0)|$ be close enough to* 1, *and let e be directed towards $L(e,R_{\rm ex})$. Then the charge enters $L(e,R_{\rm ex})$ at a certain instant τ with $|v_e(\tau)|$ close to* 1.

Proposition 4.2. *Let $|q(0)|\leqslant R_{\rm ex}$, and let $|v(0)|$ be close to* 1. *Then the charge leaves $L(e,R_{\rm ex})$ at a certain instant τ such that $|v_e(\tau)|>0$, and $v_e(\tau)e$ is directed outside $L(e,R_{\rm ex})$.*

Proposition 4.3. *Let $|q(0)| \geqslant R_{\text{ex}}$, let $|v(0)| > 0$, and let e be directed outside $L(e, R_{\text{ex}})$. Then the charge never enters $L(e, R_{\text{ex}})$ and $|q_e(t)| \to \infty$ as $t \to +\infty$.*

Proof of Proposition 4.1. For $v_e(t)$, we have the estimate

$$v_e(t) \geqslant v_e(0) - \int_0^t |\dot{v}(s)| ds = |v(0)| - \int_0^t |\dot{v}(s)| ds.$$

Since the free equations (1.4) are satisfied outside $L(e, R_{\text{ex}})$, the following estimate is valid (see (3.5) and (3.8)):

$$|\dot{v}(t)| \leqslant \frac{C\gamma_\rho}{(1+|t|)^{\sigma+1}}, \tag{4.1}$$

where the finite number C is determined by the initial data, $\overline{v}$, and R_ρ. Thus,

$$v_e(t) \geqslant |v(0)| - \int_0^\infty \frac{C\gamma_\rho dt}{(1+|t|)^{\sigma+1}} = |v(0)| - \frac{C\gamma_\rho}{\sigma},$$

and we obtain the desired result for any sufficiently small γ_ρ.

Proof of Proposition 4.2. Let us first show that the growth of the field energy

$$h(t) := \frac{1}{2} \int d^3x (|E(x,t)|^2 + |B(x,t)|^2)$$

is not very fast.

Lemma 4.1.

$$h(t) \leqslant (\sqrt{h(0)} + \sqrt{2}\gamma_\rho t)^2. \tag{4.2}$$

Proof. Taking system (1.1) into account, we obtain

$$\begin{aligned}\dot{h}(t) = \langle E, \dot{E} \rangle + \langle B, \dot{B} \rangle &= \langle E, \operatorname{rot} B - \rho(x - q(t))\dot{q}(t) \rangle - \langle B, \operatorname{rot} E \rangle \\ &= \langle E, \operatorname{rot} B \rangle - \langle B, \operatorname{rot} E \rangle - \langle E, \rho(x - q(t))\dot{q}(t) \rangle.\end{aligned}$$

Since $\langle E, \operatorname{rot} B\rangle - \langle B, \operatorname{rot} E \rangle = 0$ (see details in [9]) and $|\dot{q}(t)| \leqslant 1$, we have $\dot{h}(t) \leqslant \|E\| \, \|\rho\| \leqslant \sqrt{2}\gamma_\rho \sqrt{h}$. Integrating this differential inequality with respect to t, we come to $\sqrt{h(t)} \leqslant \sqrt{h(0)} + \sqrt{2}\gamma_\rho t$, which proves (4.2).

Let us now prove the proposition.

Set $G := \sup_{x \in \mathbb{R}^3}(|\nabla V(x)| + |\operatorname{rot} A(x)|)$. Recall that $v = p/\sqrt{1+p^2}$, and hence $p = v/\sqrt{1-v^2}$. Thus, $|v|$ is close to 1 if and only if $|p|$ is large. Due to (4.2), from the equation

$$\dot{p}(t) = -\nabla V(q(t)) + \dot{q}(t) \wedge \operatorname{rot} A(q(t)) + \int d^3x \, [E(x,t) + \dot{q}(t) \wedge B(x,t)] \rho(x - q(t)),$$

we see that

$$|\dot{p}| \leqslant G + (\|E(t)\| + \|B(t)\|)\gamma_\rho \leqslant G + (2h(t))^{1/2}\gamma_\rho \leqslant G + ((2h(0))^{1/2} + 2\gamma_\rho t)\gamma_\rho = G_1 + 2\gamma_\rho^2 t$$

with $G_1 = G + (2h(0))^{1/2}\gamma_\rho$. Then we obtain the lower bound

$$|p(t)| \geqslant |p(0)| - \int_0^t |\dot{p}(s)| ds \geqslant |p(0)| - G_1 t - \gamma_\rho^2 t^2 = P - f(t),$$

where $P := |p(0)|$ and $f(t) := G_1 t + \gamma_\rho^2 t^2$. For $\dot v(t) = \dot p(t)(1+p^2(t))^{-3/2}$, we obtain the estimate

$$|\dot v(t)| = \frac{|\dot p(t)|}{(1+|p(t)|^2)^{3/2}} \leqslant \frac{|\dot p(t)|}{|p(t)|^3} \leqslant \frac{|\dot p(t)|}{(P-f(t))^3} \leqslant \frac{G_1 + 2\gamma_\rho^2 t}{(P-f(t))^3},$$

and hence

$$v_e(t) \geqslant |v_0| - \int_0^t |\dot v(s)| ds \geqslant |v_0| - \int_0^t \frac{G_1 + 2\gamma_\rho^2 s}{(P-f(t))^3} ds \geqslant |v_0| - \frac{1}{2(P-f(t))^2}. \tag{4.3}$$

The corresponding estimate for $q_e(t)$ is

$$q_e(t) \geqslant q_e(0) + |v_0| t - \int_0^t \frac{ds}{2(P-f(s))^2}. \tag{4.4}$$

For sufficiently large P, the statement of the proposition follows from the estimates (4.4) and (4.3).

Proof of Proposition 4.3. We claim that there are small numbers $\gamma_\rho > 0$ and $\underline{v} > 0$ such that $v_e(t) \geqslant \underline{v}$ for any $t > 0$. Indeed, set $T = \sup\{t > 0 : \ v_e(t) > \underline{v}\}$. If $\underline{v} < v_e(0)/2$, then $T > 0$ by continuity. Further, it is possible to choose small numbers $\gamma_\rho > 0$ and $\underline{v} > 0$ such that $T = +\infty$. Moreover, for $t \in [0, T]$, the free equations (1.4) are satisfied, and hence the estimate (4.1) is valid. Take

$$0 < \underline{v} < v_e(0) - \int_0^\infty \frac{C\gamma_\rho}{(1+|t|)^{\sigma+1}} = v_e(0) - \frac{C\gamma_\rho}{\sigma};$$

the choice is possible for any sufficiently small γ_ρ. If $T < +\infty$, then $v_e(T) > \underline{v}$, and hence we have $v_e(T+\varepsilon) > \underline{v}$ for some $\varepsilon > 0$ by continuity. This contradicts the definition of T. Thus, $T = +\infty$. Hence, for $t > 0$, one obtains $q_e(t) \geqslant q_e(0) + \underline{v} t$.

Analyzing the proof we obtain the following statement.

Corollary 4.1. *Let $Y(0) \in \mathcal{M}^\sigma$, where $0 < \sigma \leqslant 1$. Then, for $v(0) \neq 0$ and for sufficiently small γ_ρ and G, the solution $Y(t)$ is scattering.*

REFERENCES

1. Komech, A., Spohn, H., and Imaikin, V., Soliton-Like Asymptotics for the Coupled Maxwell–Lorentz Equations, in *Mathematical and Numerical Aspects of Wave Propagation, Santiago de Compostela, Spain,* SIAM, Philadelphia, 2000, pp. 329–333.
2. Bambusi, D. and Galgani, L., Some Rigorous Results on the Pauli-Fierz Model of Classical Electrodynamics, *Ann. Henri Poincaré*, 1993, vol. 58, no. 1, pp. 155–171.
3. Bauer, G. and Dürr, D., The Maxwell–Lorentz System of a Rigid Charge, *Ann. Henri Poincaré*, 2001, vol. 2, no. 1, pp. 179–196.
4. Buslaev, V. S. and Perelman, G. S., On Nonlinear Scattering of States Which Are Close to a Soliton, in: *Méthodes semi-classiques, vol. 2. Colloque international (Nantes, 1991)*, *Asterisque*, 1992, no. 210 (6), pp. 49–63.
5. Buslaev, V. S. and Perelman, G. S., On the Stability of Solitary Waves for Nonlinear Schrödinger Equations, *Trans. Amer. Math. Soc.*, 1995, vol. 164, pp. 75–98.
6. Fleckinger, J. and Komech, A., On Soliton-Like Asymptotics for 1D Nonlinear Reaction Systems, *Russian J. Math. Phys.*, 1997, vol. 5, no. 3, pp. 295–306.
7. Grillakis, M., Shatah, J., and Strauss, W. A., Stability Theory of Solitary Waves in the Presence of Symmetry, I; II, *J. Funct. Anal.*, 1987, vol. 74, no. 1, pp. 160–197; 1990, vol. 94, no. 2, pp. 308–348.
8. Komech, A. and Spohn, H., Soliton-Like Asymptotics for a Classical Particle Interacting with a Scalar Wave Field, *Nonlinear Anal.*, 1998, vol. 33, no. 1, pp. 13–24.
9. Komech, A. and Spohn, H., Long-Time Asymptotics for the Coupled Maxwell-Lorentz Equations, *Comm. Partial Differential Equations*, 2000, vol. 25, no. 3/4, pp. 559–584.
10. Novikov, S. P., Manakov, S. V., Pitaevskii, L. P., and Zakharov, V. E., *Theory of Solitons: The Inverse Scattering Method*, Consultants Bureau, 1984.