SIAM J. MATH. ANAL.
Vol. 31, No. 2, pp. 346–374

ON TRANSITIONS TO STATIONARY STATES IN A MAXWELL–LANDAU–LIFSCHITZ–GILBERT SYSTEM*

PATRICK JOLY†, ALEXANDER KOMECH‡, AND OLIVIER VACUS§

Abstract. In this paper we consider Maxwell's equations together with a dissipative nonlinear magnetic law, the Landau–Lifschitz–Gilbert equation, and we study long-time asymptotics of solutions in the 1D case in an infinite domain of propagation. We prove long-time convergence to zero of the electromagnetic field in a Fréchet topology defined by local energy seminorms: this corresponds to the local energy decay. We then introduce the set of stationary states for the Landau–Lifschitz–Gilbert equation and prove that it corresponds to the attractor set for the distribution of magnetization whose presence is one of the characteristics of ferromagnetic media.

Key words. Maxwell's equations, Landau–Lifchitz–Gilbert law, local energy decay, Liapunov theory, long-time asymptotics

AMS subject classifications. 15A15, 15A09, 15A23

PII. S0036141097329949

1. Introduction.

Ferromagnetic materials possess a spontaneous magnetization whose interaction with the magnetic field provides to this type of medium interesting absorbing properties with respect to electromagnetic waves. That is why the use of such materials as absorbing coatings for scatterers is of real importance for stealth technology. The present paper is a contribution to the mathematical theory of electromagnetic scattering by such objects. One of the main characteristics of ferromagnetic materials lies in the fact that their constitutive law, namely, the relationship between the magnetic field H and the magnetization M, is nonlinear and nonlocal with respect to time. This equation is the Landau–Lifschitz–Gilbert (LLG) equation that can be written pointwise in the form

$$\dot{M} \;=\; \gamma H_T \times M + \frac{\alpha}{|M|} M \times \dot{M}, \tag{1.1}$$

where H_T is the total magnetic field defined as

$$H_T \;=\; H + H_s + H_a(M), \tag{1.2}$$

with each of these contributions being defined as

$$\begin{cases} H & \text{is the magnetic field;} \\ H_s \;=\; H_s(\mathbf{x}) & \text{is an exterior static field (given);} \\ H_a(M) \;=\; -K\; P(M) & \text{is a field of anisotropy.} \end{cases} \tag{1.3}$$

In (1.3), $\mathbf{x} = (x, y, z)$ denotes the space variable, t denoting time; K denotes a positive coefficient, constant in time but that may depend on $\mathbf{x}$; and $P(M)$ is the orthogonal

*Received by the editors November 12, 1997; accepted for publication January 23, 1998; published electronically December 21, 1999. This research was partially supported by INRIA (Project "Ondes"), by French-Russian A.M. Liapunov Center of Moscow State University, and by RFBR research grant 96-01-00527.

http://www.siam.org/journals/sima/31-2/32994.html

†INRIA Rocquencourt, 78 153 Le Chesnay Cédex, France (patrick.joly@inria.fr).

‡Department of Mechanics and Mathematics, Moscow State University, Moscow 119899, Russia (komech@facet.inria.msu.ru).

§Dassault Aviation, DPR-DESA, 78 quai Marcel Dassault, 92214 St. Cloud Cedex, France (olivier.vacus@dassault-aviation.fr).

projection in $\mathbb{R}^3$ on the plane orthogonal to some unit vector $\mathbf{p}$, called the easy axis (this direction, which is linked to the crystallic structure of the material, may also depend on $\mathbf{x}$):

$$P(M) = M - (p \cdot M)p\ . \tag{1.4}$$

Let us mention that in (1.1), γ, the gyromagnetic factor, is a universal constant while α, the damping factor, is a phenomenologic coefficient which depends on $\mathbf{x}$. Therefore, for our applications, a propagation medium will be determined by H_s, α, K, as function of $\mathbf{x}$ (and by the initial distribution of magnetization M_0; see (1.7)).

Here we are interested in the coupling of (1.1) with Maxwell's equations in some domain Ω (typically an exterior domain, if one thinks of applications to scattering problems) with boundary $\Gamma = \partial\Omega$. We assume that space and time variables are scaled in such a way that the constants γ, ε_0 (the electric permittivity) and μ_0 (the magnetic permeability) can be taken equal to 1. One then has to solve

$$\begin{cases} \dot{E}(\mathbf{x},t) - \text{curl } H(\mathbf{x},t) = 0, \\ \dot{H}(\mathbf{x},t) + \text{curl } E(\mathbf{x},t) + \dot{M}(\mathbf{x},t) = 0, \\ \dot{M}(\mathbf{x},t) = H_T(\mathbf{x},t) \times M(\mathbf{x},t) + \dfrac{\alpha}{|M(\mathbf{x},t)|} M(\mathbf{x},t) \times \dot{M}(\mathbf{x},t), \end{cases} \quad \mathbf{x} \in \Omega,\ t > 0, \tag{1.5}$$

with a perfectly conducting boundary condition on Γ with unit normal vector n

$$E \times n\ |_\Gamma =\ 0, \tag{1.6}$$

and initial conditions on $\mathbb{R}_-$:

$$E(\mathbf{x}, t = 0)\ =\ E_0(\mathbf{x}),\ H(\mathbf{x}, t = 0)\ =\ H_0(\mathbf{x}),\ M(\mathbf{x}, t = 0)\ =\ M_0(\mathbf{x}). \tag{1.7}$$

From a mathematical point of view, even the existence and uniqueness result for system (1.5) appears to be a very difficult question (see [4], [5]). Another natural question that we wish to address here is the following: Is it possible to describe the asymptotic behavior of the solution of system (1.9) for large time? In the case of linear materials, the answer to this question has been known for a long time (see [9], [10], [11]): the electric and magnetic fields tend locally to 0. This result is known as the local energy decay. A more subtle question is an estimate of the rate of decay; this question is closely related to the geometry of the obstacle [18].

In the case of nonlinear media, there are much fewer results in that direction (see [16], [17]). Our goal in this paper is to establish a result analogous to the local energy decay in a simplified $1D$ *model problem.* More precisely we assume that all the unknowns are functions of only one space variable x (i.e., we consider the propagation of plane waves). The curl operator is then defined by

$$\text{curl } H(x,t) = \left(0, -\frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x}\right),\ \text{curl } E(x,t) = \left(0, -\frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x}\right). \tag{1.8}$$

We assume that the propagation medium is the half-space $x < 0$ and apply the perfectly conducting boundary condition at $x = 0$ ($e_x = (1, 0, 0)$):

$$E \times e_x\ =\ 0.$$

We also assume that the support of the initial magnetization M_0, which defines the ferromagnetic layer (see 2.6), is compact:

$$\text{supp } M_0\ \subset\ ] - a, 0].$$

Then, by a principle of reflection (or image principle), the analysis of (1.5), (1.6), and (1.7) can be reduced to the analysis of the pure Cauchy problem on the whole line:

$$\text{(1.9)} \quad \begin{cases} \dot{E}(x,t) - \text{curl } H(x,t) = 0, \\ \dot{H}(x,t) + \text{curl } E(x,t) + \dot{M}(x,t) = 0, \\ \dot{M}(x,t) = H_T(x,t) \times M(x,t) + \dfrac{\alpha}{|M(x,t)|} M(x,t) \times \dot{M}(x,t), \end{cases} \qquad x \in \mathbb{R},\ t > 0,$$

provided that the new initial data M_0, E_0, H_0 are appropriate extensions of the original ones:

$$\text{(1.10)} \qquad \begin{cases} M_0(-x) &=& M_0(x), \\ E_0(-x) &=& \Pi_\perp\, E_0(x), \\ H_0(-x) &=& \Pi_\parallel\, H_0(x), \end{cases}$$

where the operators $\Pi_\perp$ and $\Pi_\parallel$ are defined for any field $A(A_x, A_y, A_z)$ by

$$\text{(1.11)} \qquad \left| \begin{array}{lcl} \Pi_\perp(A_x, A_y, A_z) &=& (A_x, -A_y, -A_z), \\ \Pi_\parallel(A_x, A_y, A_z) &=& (-A_x, A_y, A_z). \end{array} \right.$$

Remark 1.1. Concerning the longitudinal components, equations (1.9) imply

$$\text{(1.12)} \qquad \begin{aligned} E_x(x,t) &= E_x^0(x), \\ H_x(x,t) + M_x(x,t) &= H_x^0(x) + M_x^0(x). \end{aligned}$$

We see here that the component E_x is constant in time, while convergence results on M_x yield results on H_x.

The outline of this article is as follows. In section 2, after having recalled some known results about weak and strong solutions of the 1D scattering problem, we state the main results of the paper, namely, Theorems A and A′ for the local energy decay (respectively, for weak and strong solutions) and Theorem B on long-time asymptotics of the magnetization M. Sections 3 and 4 are, respectively, devoted to the proof of Theorems A and A′. In section 5, we give some intermediate results about the LLG equation seen as an ordinary differential equation; these results are preparatory for section 6, in which we prove Theorem B. Finally, section 7 is devoted to some additional remarks and comments.

2. Statement of the main results.

2.1. Overview of known results in the 1D case. As we said above, existence and uniqueness results for system (1.9) are very difficult in dimension 3. In the case of the 1D model, the problem is easier and can be handled via a fixed point theorem. Let us summarize the main existence and uniqueness results from [7].

We introduce the phase space V for the system (1.9). Let $L^p = [L^p(\mathbb{R})]^3$ for $1 \le p \le \infty$, $L^{2,\infty}$ be the Banach space $L^2 \cap L^\infty$ with the norm

$$\text{(2.1)} \qquad \|M\|_{L^{2,\infty}} = \|M\|_{L^2} + \|M\|_{L^\infty},$$

and let $H(\text{curl})$ denote the Hilbert space $\{u \in L^2 \,/\, \text{curl } u \in L^2\}$ with the norm

$$\text{(2.2)} \qquad \|u\|^2_{\text{curl}} = \|\text{curl } u\|^2_{L^2} + \|u\|^2_{L^2}.$$

DEFINITION 2.1. *Let V be the Banach space*

$$V \;=\; \Big\{ (E,H,M) \in H(\text{curl}) \times H(\text{curl}) \times L^{2,\infty};\ H_x \in L^\infty(\mathbb{R}) \Big\}$$

equipped with the norm

$$\|(E,H,M)\|_V = \|E\|_{\mathrm{curl}} + \|H\|_{\mathrm{curl}} + \|M\|_{L^{2,\infty}} + \|H_x\|_{L^\infty(\mathbb{R})}. \tag{2.3}$$

Remark 2.2. The x component H_x of the magnetic field plays a particular role because of the particular form of the curl operator in the 1D case.

DEFINITION 2.3. *A function* $Y(t) = (E(x,t), H(x,t), M(x,t)) \in C^0(0,\infty;V)$ *is a global strong solution to the system* (1.9) *if*

$$(E,H) \in C^1(0,\infty;L^2)\cap C^0(0,\infty;H^1) \text{ and } M \in C^1(0,\infty;L^{2,\infty})\cap C^2(0,\infty;L^2) \tag{2.4}$$

and all equations in (1.9) *hold in the sense of distributions.*

One then shows the following theorem.

THEOREM 2.4. *Let the following assumptions hold:*

$$\begin{cases} \bullet\ \alpha(x), K(x) \in L^\infty(\mathbb{R}),\ p(x) \in L^\infty; \\ \bullet\ H_s(x) \in L^{2,\infty}, \\ \bullet\ (E^0(x), H^0(x), M^0(x)) \in V. \end{cases} \tag{2.5}$$

Then the Cauchy problem (1.9) *admits a unique global strong solution* (E,H,M), *which, moreover, satisfies*

$$|M(x,t)| \,=\, |M_0(x)| \quad a.e.\ x\in\mathbb{R}\ \forall t\geq 0, \tag{2.6}$$

$$\frac{d}{dt}\mathcal{E}(E,H,M) + \int_{\mathbb{R}} \frac{\alpha}{|M|}\left|\dot{M}\right|^2 dx \,=\, 0, \tag{2.7}$$

where $\mathcal{E}(E,H,M)$ *denotes the electromagnetic energy defined by*

$$\mathcal{E}(E,H,M) \,=\, \frac{1}{2}\int_{\mathbb{R}} \left[|E|^2 + |H|^2 + K|P(M)|^2 + |H_s - M|^2\right] dx\,. \tag{2.8}$$

Remark 2.5. For any strong solution, the electromagnetic energy is a function of class C^1 with respect to time.

One has to make precise the sense of the integral of $\frac{\alpha}{|M|}|\dot{M}|^2$. In fact, from the LLG equation, we have

$$\dot{M}(t) \,-\, \alpha\frac{M}{|M|}\times\dot{M}(t) \,=\, \gamma H_T(M(t))\times M(t);$$

hence we deduce, via Pythagoras's theorem, that

$$(1+\alpha^2)\,|\dot{M}(t)|^2 \,=\, \gamma^2\,|H_T(M)\times M|^2\,. \tag{2.9}$$

This observation leads to the following definition.

DEFINITION 2.6. *For the strong solution* (E,H,M) *to system* (1.9), *we set*

$$\frac{\alpha}{|M|}\left|\dot{M}\right|^2 \,=\, \gamma^2\frac{\alpha}{1+\alpha^2}\frac{|H_T\times M|^2}{|M|}, \tag{2.10}$$

which is finite since the function $M \mapsto \frac{|H_T\times M|^2}{|M|}$ *can be continuously extended by* 0 *for* $M=0$. *We have the estimate*

$$\int_{\mathbb{R}} \frac{\alpha}{|M|}\left|\dot{M}\right|^2 dx \,\leq\, \gamma^2\int_{\mathbb{R}}\frac{\alpha}{1+\alpha^2}|M||H_T|^2\,dx \tag{2.11}$$

which makes sense since one easily checks that $M \in (L^\infty)^3$ *and* $H_T \in (L^2)^3$ *for any time.*

Proof. See [7] for a complete proof. We just explain below how to obtain the two estimates of Theorem 2.4. Concerning (2.6), the product of the LLG equation with M shows that $M \cdot \dot{M} = 0$; we deduce that

$$|M(x,t)| = |M_0(x)|, \quad \text{a.e. } x \in \mathbb{R} \ \forall t \geq 0.$$

For (2.7), from Maxwell's equations we get

$$\left\{ \begin{array}{rcl} \dot{E} \cdot E - \mathrm{curl} H \cdot E &=& 0, \\ \dot{H} \cdot H + \mathrm{curl} E \cdot H &=& -\dot{M} \cdot H. \end{array} \right. \tag{2.12}$$

Summing these two equalities and integrating over $\mathbb{R}$ leads, after integration by parts, to the following identity:

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbb{R}} \left(|E|^2 + |H|^2 \right) \, dx \right\} = -\int_{\mathbb{R}} \dot{M} \cdot H \, dx. \tag{2.13}$$

We now use the LLG equation:

$$\dot{M} = \gamma H_T \times M + \frac{\alpha}{|M|} M \times \dot{M}. \tag{2.14}$$

Taking the scalar product with H_T, and using the notation $(\cdot,\cdot,\cdot)$ for the mixed product in $\mathbb{R}^3$, we get

$$\dot{M} \cdot H_T = \frac{\alpha}{|M|} \left(M, \dot{M}, H_T \right).$$

Taking the scalar product of the LLG equation by $\dot{M}$ gives

$$\left| \dot{M} \right|^2 = \gamma \left(\dot{M}, H_T, M \right).$$

Therefore, eliminating the mixed product gives

$$\dot{M} \cdot H_T = \frac{\alpha}{\gamma |M|} \left| \dot{M} \right|^2. \tag{2.15}$$

(The meaning of the right-hand side of this expression has to be understood in the sense we made precise in Definition 2.6.) Now, by definition of H_T, we have

$$\dot{M} \cdot H_T = \dot{M} \cdot H - K P(M) \cdot P\left(\dot{M} \right) + H_s \cdot \dot{M}.$$

Using the fact that $|M(x,t)| = |M_0(x)|$, we note that

$$\left\{ \begin{array}{l} H_s \cdot \dot{M} = -\dfrac{1}{2} \dfrac{\partial}{\partial t} |H_s - M|^2, \\ -K P(M) \cdot P\left(\dot{M} \right) = -\dfrac{1}{2} \dfrac{\partial}{\partial t} \left[K |P(M)|^2 \right]. \end{array} \right. \tag{2.16}$$

Therefore, we have

$$\dot{M} \cdot H_T = \dot{M} \cdot H - \frac{1}{2} \frac{\partial}{\partial t} \left[|H_s - M|^2 + K |P(M)|^2 \right]; \tag{2.17}$$

that is to say, using (2.15),

$$-\dot{M} \cdot H = -\frac{\alpha}{\gamma |M|} \left| \dot{M} \right|^2 - \frac{1}{2} \frac{\partial}{\partial t} \left[|H_s - M|^2 + K |P(M)|^2 \right]. \tag{2.18}$$

Plugging (2.18) into (2.13) leads to the energy identity

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbb{R}} \left(|E|^2 + |H|^2 + |H_s - M|^2 + K |P(M)|^2 \right) dx \right\} + \frac{1}{\gamma} \int_{\mathbb{R}} \frac{\alpha}{|M|} \left| \dot{M} \right|^2 dx = 0. \qquad \square \tag{2.19}$$

Thanks to a priori estimates (2.6) and (2.7), one is also able to obtain a theorem for weak solutions (i.e., less regular solutions) under weaker assumptions on the initial data; see the following definition.

DEFINITION 2.7. *A function* $Y(t) = (E(x,t), H(x,t), M(x,t)) \in C^0(0, \infty; V)$ *is a global weak solution to the system* (1.9) *if and only if*

(i) $M \in C^1(0, \infty; L^{2,\infty})$;

(ii) *for any* $(\varphi, \psi) \in V \times V$ *where* V *is the space of test fields*

$$V = \left\{ \varphi \in C^1(H(\text{curl})) \; / \; \text{supp } (\varphi) \textit{ is compact} \right\},$$

$$\begin{cases} \displaystyle\iint (\dot{\varphi} \cdot E + \text{curl } \varphi \cdot H) \, dxdt = -\int E_0 \cdot \varphi(\cdot, 0) dx, \\ \displaystyle\iint (\dot{\psi} \cdot H - \text{curl } \psi \cdot E) \, dxdt = -\int H_0 \cdot \psi(\cdot, 0) dx - \iint \dot{M} \cdot \psi \, dxdt; \end{cases}$$

(iii) *for almost every* $(x,t) \in \mathbb{R} \times \mathbb{R}^+$,

$$\begin{cases} \dot{M}(x,t) = H_T(x,t) \times M(x,t) + \dfrac{\alpha}{|M(x,t)|} M(x,t) \times \dot{M}(x,t), \\ M(x,0) = M_0(x). \end{cases}$$

THEOREM 2.8. *Assume that*

$$(E_0, H_0, M_0) \in (L^2)^3 \times (L^2)^3 \times (L^{2,\infty})^3;$$

then system (1.9) *admits a unique global weak solution which satisfies*

$$\int_0^\infty \int_{\mathbb{R}} \frac{\alpha}{|M|} |\dot{M}|^2 \, dx \, dt < +\infty. \tag{2.20}$$

Moreover, a.e. $x \in \mathbb{R}$, $t \mapsto M(x,t)$ *belongs to* $C^0(\mathbb{R})$.

Remark 2.9. From (2.20) and Fubini's theorem, we deduce that

$$\text{a.e. } x \in \mathbb{R}, \quad \frac{|\dot{M}(x,t)|^2}{|M_0(x)|} \in L^1(\mathbb{R}).$$

Therefore, function $t \mapsto M(x,t)$ is in $H^1(\mathbb{R}_+) \subset C^0(\mathbb{R}_+)$.

2.2. Main results of the paper. Our first main theorem concerns the convergence to 0 in Fréchet topology of the transverse components of the electromagnetic fields. In the following, we denote the transverse components $E_{\|}(x,t) = (E_y(x,t), E_z(x,t))$, $H_{\|}(x,t) = (H_y(x,t), H_z(x,t))$, and so on.

THEOREM A. *Let all assumptions* (2.5) *hold. Assume, moreover, that*

$$M_0(x) \;=\; 0 \quad \textit{for } |x| > a \tag{2.21}$$

and

$$\exists\, \alpha_* > 0 \textit{ such that } \; \alpha(x) \geq \alpha^*, \;\; \textit{a.e.} \;\; x \in [-a; a]. \tag{2.22}$$

Then, for the global solution to the Cauchy problem (1.9):

(i) *For almost every* $x \in \mathbb{R}$

$$\int_0^\infty (|E_{\|}(x,t)|^2 + |H_{\|}(x,t)|^2)dt < \infty. \tag{2.23}$$

(ii) *For every* $R > 0$

$$\int_{|x|<R} (|E_{\|}(x,t)|^2 + |H_{\|}(x,t)|^2)dx \to 0 \textit{ as } t \to \infty. \tag{2.24}$$

Our second result is a variant of Theorem A. We prove that, provided additional regularity assumptions on the initial data, we have convergence to zero of the transverse electromagnetic field, not only in the local energy norm, but also uniformly in any compact set.

THEOREM A$'$. *Let the assumptions of Theorem* A *hold (in particular* $(E_0, H_0) \in (H(\mathrm{curl}))^2$. *Then*

$$\lim_{t\to+\infty} \int_{-a}^{a} |\dot{M}(x,t)|^2 dx = 0, \tag{2.25}$$

and, for every $R > 0$,

$$\lim_{t\to+\infty} \int_{-R}^{R} \left(\left| \frac{\partial E_{\|}}{\partial x}(x,t) \right|^2 + \left| \frac{\partial H_{\|}}{\partial x}(x,t) \right|^2 \right) dx = 0, \tag{2.26}$$

$$\lim_{t\to+\infty} \sup_{|x|\leq R} \Big\{ |E_{\|}(x,t)| + |H_{\|}(x,t)| \Big\} = 0. \tag{2.27}$$

Our other results concern the asymptotic behavior of the magnetization $M(x,t)$ (and thus of the longitudinal magnetic field). We first need to introduce a nondegeneracy assumption. Let us define

$$\tilde{H}_T(x, M) \;=\; H_s + K(p \cdot M)p - (e_x \cdot M)e_x, \tag{2.28}$$

where the dependence of $\tilde{H}_T(x, M)$ with respect to x appears in the dependence of H_s, K, and p. We introduce the assumption, for any x in $\mathbb{R}$,

$$(\mathcal{H}_x) \,:\, (\forall M \in \mathbb{R}^3,\ \exists\, \lambda(x,M) \in \mathbb{R},\ \tilde{H}_T(M) = \lambda(x,M)e_x\,) \;\Rightarrow\; (\forall M \in \mathbb{R}^3,\ \lambda(x,M) \neq 0\,).$$

Remark 2.10. Assumption $(\mathcal{H}_x)$ is not satisfied if and only if the following two properties are satisfied:

(i) H_s and p are collinear to e_x ;
(ii) there exists $M \in \mathbb{R}^3$ such that $(H_s + (K-1)M) \cdot e_x \;=\; 0$.
For any x, we introduce the set

$$\mathcal{Z}(x, M_0) \;=\; \left\{ M \in \mathbb{R}^3 \text{ such that } \tilde{H}_T(x, M) \times M = 0 \text{ and } |M| = |M_0(x)| \right\}.$$

This set will be identified in section 5 as the set of stationary states of some unperturbed LLG equation at point x that are possible limits for $M(x,t)$ with the initial data $M_0(x)$. It will be proved to be a finite set containing between two and six elements (see Theorem 5.2).

Our first result for M concerns weak solutions.

THEOREM B. *Let the assumptions of Theorem* A *be satisfied. Assume that* $(\mathcal{H}_x)$ *holds almost everywhere in* x. *Then*

$$M(x,t) \to \mathcal{Z}(x, M_0) \text{ as } t \to \infty \text{ for a.e. } x \in \mathbb{R}. \tag{2.29}$$

Remark 2.11. With the set $\mathcal{Z}(x, M_0)$ being discrete, (2.29) means that for a.e. x, there exists $M_\infty \in \mathcal{Z}(x, M_0)$ such that

$$M(x,t) \to M_\infty \text{ as } t \to \infty.$$

In other words,

$$\mathcal{M} \;=\; \{\, M \in L^\infty([-a,a]) \,/\, \text{for a.e. } x \in [-a,a],\; M(x) \;=\; M_\infty \in \mathcal{Z}(x, M_0) \,\}$$

is an infinite-dimensional attractor for $M(x,t)$.

Our last result is a variant of Theorem B for strong solutions.

COROLLARY 2.12. *Let the assumptions of Theorem* A$'$ *be satisfied. Assume that* $(\mathcal{H}_x)$ *holds everywhere. For every* x, *there exists* $M_\infty(x) \in \mathcal{Z}(x, M_0)$ *such that*

$$\lim_{t\to\infty} |M(x,t) - M_\infty(x)| \;=\; 0. \tag{2.30}$$

Remark 2.13. We show with Theorem A$'$ that the transverse magnetic field $H_\|$ converges in time to 0 uniformly in space. It must be emphasized that this property is not strong enough to ensure that the convergence of the magnetization M is also uniform in space. Section 7 will be devoted to a counterexample and some comments about this assertion.

3. Proof of Theorem A: L^2 bounds of the transverse electromagnetic field for weak solutions. The first two equations of (1.9) read

$$\begin{cases} \dot{E}_x = 0, & \dot{E}_y + \dfrac{\partial H_z}{\partial x} = 0, & \dot{E}_z - \dfrac{\partial H_y}{\partial x} = 0, \\[2ex] \dot{H}_x + \dot{M}_x = 0, & \dot{H}_y - \dfrac{\partial E_z}{\partial x} + \dot{M}_y = 0, & \dot{H}_z + \dfrac{\partial E_y}{\partial x} + \dot{M}_z = 0. \end{cases} \tag{3.1}$$

We easily deduce that

$$\begin{cases} L_+(E_y + H_z) & = - & \dot{M}_z, \\ L_-(E_y - H_z) & = & \dot{M}_z, \\ L_+(E_z - H_y) & = & \dot{M}_y, \\ L_-(E_z + H_y) & = - & \dot{M}_y, \end{cases} \tag{3.2}$$

where we have introduced the two transport operators $L_\pm = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$. Considering $\dot{M}$ as known, we can solve explicitly (3.1) using the method of characteristics. After some algebraic manipulations, we end up with the following formulas:

$$(3.3)\quad \left\{ \begin{array}{rcl} E_\|(x,t) &=& \frac{1}{2}\{E_\|^0(x+t) + E_\|^0(x-t)\} + \frac{J}{2}\{H_\|^0(x+t) - H_\|^0(x-t)\} \\ && - \frac{J}{2}\left\{ \int_{\Gamma_+(x,t)} \dot{M}_\|(y,s)d\sigma - \int_{\Gamma_-(x,t)} \dot{M}_\|(y,s)d\sigma \right\}, \\ H_\|(x,t) &=& \frac{1}{2}\{H_\|^0(x+t) + H_\|^0(x-t)\} - \frac{J}{2}\{E_\|^0(x+t) - E_\|^0(x-t)\} \\ && - \left\{ \int_{\Gamma_+(x,t)} \dot{M}_\|(y,s)d\sigma + \int_{\Gamma_-(x,t)} \dot{M}_\|(y,s)d\sigma \right\}, \end{array} \right.$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$\Gamma_\pm(x,t)$ is the curve $\Gamma_\pm(x,t) = (x,t) - D_\pm$, with $D_\pm = \{(x,t)/x \pm t = 0\}$ (see Figure 3.1), and σ is the curvilinear abscissa along $\Gamma_\pm(x,t)$. Therefore, introducing

$$F(x,t) = \begin{pmatrix} E_\|(x,t) \\ H_\|(x,t) \end{pmatrix},$$

we get (note that $d\sigma = \sqrt{2}dy$, $\|J\| = 1$, and $|\dot{M}_\|| \le |\dot{M}|$)

$$(3.4)\qquad |F(x,t)| \le \frac{\sqrt{2}}{2} \int_{\Gamma^a(x,t)} |\dot{M}(y,s)|dy + \frac{1}{2}\{|F^0(x+t)| + |F^0(x-t)|\}$$

with $\Gamma^a(x,t) = \{(y,s) \in \Gamma_+(x,t) \cup \Gamma_-(x,t) :\ |y| < a\}$ (see Figure 3.1).

Then, the Cauchy–Schwarz inequality yields the estimate ($\int_{\Gamma^a(x,t)} dy = 2a$)

$$(3.5)\quad |F(x,t)| \le \sqrt{a} \left(\int_{\Gamma^a(x,t)} |\dot{M}(y,s)|^2 dy \right)^{\frac{1}{2}} + \frac{1}{2}\{|F^0(x+t)| + |F^0(x-t)|\}.$$

Integrating this inequality between 0 and T, and noticing that

$$K(x,T) \equiv \bigcup_{t=0}^{T} \Gamma^a(x,t)$$

is such that $K(x,T) \subset [-a,a] \times [0,T]$, we get the estimate

$$\int_0^T |F(x,t)|^2 dt \ \le\ 2a \int_0^T \left(\int_{|y|<a} |\dot{M}(y,s)|^2 dy \right) ds \ +\ \frac{1}{2} \sum_\pm \int_0^T |F^0(x \pm t)|^2 dt,$$

which yields, after integrating over $[-a,a]$,

$$\begin{array}{rcl} \int_0^T \int_{-a}^{a} |F(x,t)|^2 dxdt &\le& 4a^2 \int_0^T \left(\int_{|y|<a} |\dot{M}(y,s)|^2 dy \right) ds \\ && +\ \frac{1}{2} \sum_\pm \int_0^T \int_{-a}^{a} |F^0(x \pm t)|^2 dt. \end{array}$$

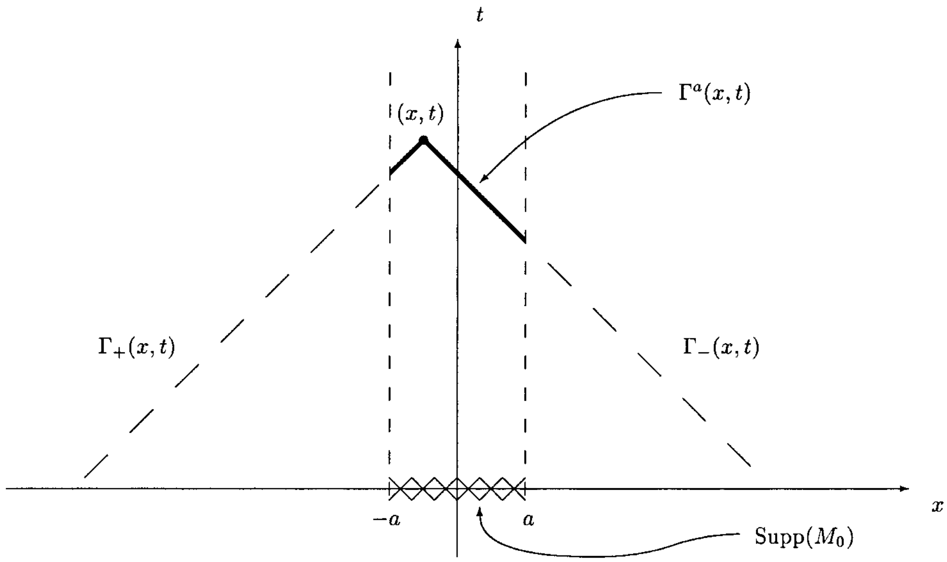

FIG. 3.1. *Characteristic lines and the 1D layer.*

Writing $|\dot{M}|^2 = \frac{|M|}{\alpha} \cdot \frac{\alpha|\dot{M}|^2}{|M|}$ and using the conservation of the norm of M and the assumption (2.22) about the damping function $\alpha(x)$, we get

$$\text{(3.6)}\quad \left|\begin{aligned} \int_0^T \int_{-a}^a |F(x,t)|^2 dxdt \;\; &\leq \;\; \frac{4a^2}{\alpha_*}\|M_0\|_\infty \int_0^T \left(\int_{|y|<a} \frac{\alpha}{|M|} |\dot{M}(y,s)|^2 dy \right) ds \\ &+ \;\; \frac{1}{2} \sum_{\pm} \int_0^T \int_{-a}^a |F^0(x \pm t)|^2 dt. \end{aligned}\right.$$

Therefore, (2.5) and (2.20) imply (2.23). Besides, (3.4) can be rewritten as

$$\left|\begin{aligned} |F(x,t)| \;\; \leq \;\; \frac{\sqrt{2}}{2} \int_{\Gamma_a^+(x,t)} |\dot{M}(y,s)|\, ds \;\; &+ \;\; \frac{\sqrt{2}}{2} \int_{\Gamma_a^-(x,t)} |\dot{M}(y,s)|\, ds \\ &+ \;\; \frac{1}{2} \left\{ |F^0(x+t)| \, + \, |F^0(x-t)| \right\}, \end{aligned}\right.$$

where $\Gamma_a^+ = \Gamma_+ \cap \Gamma^a$ and $\Gamma_a^- = \Gamma_- \cap \Gamma^a$. This yields, via the Cauchy–Schwarz inequality,

$$\left|\begin{aligned} |F(x,t)| \, \leq \, \frac{\sqrt{2a}}{2} \left(\int_{\Gamma_a^+(x,t)} |\dot{M}(y,s)|^2 ds \right)^{\frac{1}{2}} \;\; &+ \;\; \frac{\sqrt{2a}}{2} \left(\int_{\Gamma_a^-(x,t)} |\dot{M}(y,s)|^2 ds \right)^{\frac{1}{2}} \\ &+ \;\; \frac{1}{2} \left(|F^0(x+t)| + |F^0(x-t)| \right). \end{aligned}\right.$$

Therefore

$$\text{(3.7)}\quad \left|\begin{aligned} |F(x,t)|^2 \, \leq \, 2a \left(\int_{\Gamma_a^+(x,t)} |\dot{M}(y,s)|^2 ds \;\; \right. &+ \left. \;\; \int_{\Gamma_a^-(x,t)} |\dot{M}(y,s)|^2 ds \right) \\ &+ \;\; |F^0(x+t)|^2 + |F^0(x-t)|^2. \end{aligned}\right.$$

After having remarked that, for $t > R + a$,

$$\bigcup_{x=-R}^{R} \Gamma_a^{\pm}(x,t) \;\subset\; [-a,a] \,\times\, [t-(R+a),t],$$

we obtain

$$\left| \begin{array}{rcl} \displaystyle\int_{-R}^{R} |F(x,t)|^2 dx & \leq & \displaystyle 4a \int_{t-(R+a)}^{t} \int_{-a}^{a} |\dot{M}(y,s)|^2 dy ds \\ & + & \displaystyle\int_{-R}^{R} \left(|F^0(x+t)|^2 + |F^0(x-t)|^2 \right) dx \end{array} \right. \tag{3.8}$$

which leads to

$$\left| \begin{array}{rcl} \displaystyle\int_{-R}^{R} |F(x,t)|^2 dx & \leq & \displaystyle \frac{4a\|M_0\|_\infty}{\alpha_*} \int_{t-(R+a)}^{t} \int_{-a}^{a} \frac{\alpha}{|M|} |\dot{M}(y,s)|^2 dy ds \\ & + & \displaystyle\int_{-R-t}^{R-t} |F^0(x)|^2 dx \,+\, \int_{-R+t}^{R+t} |F^0(x)|^2 dx. \end{array} \right. \tag{3.9}$$

By (2.5) and (2.7),

$$\int_{-\infty}^{+\infty} |F^0(x)|^2 dx < +\infty \quad \text{and} \quad \int_0^\infty \int_{-a}^{a} \frac{\alpha}{|M|} |\dot{M}|^2 dy ds < +\infty.$$

Therefore, (3.9) implies (2.24). This concludes the proof. ∎

4. Proof of Theorem A′: Uniform bounds of the transverse electromagnetic field for strong solutions. The proof of Theorem A′ relies on a technical lemma.

LEMMA 4.1. *The solution of the system satisfies*

$$\int_0^\infty \int_{-a}^{a} \frac{\alpha}{|M_0(x)|} |\ddot{M}(x,t)|^2 dx dt < \infty. \tag{4.1}$$

Proof. We give a proof which supposes that the electromagnetic field (E,H) is slightly more regular in time than local strong solutions. However, all the forthcoming assumptions can be justified using the method of differential quotients (see [7] for details).

Nevertheless, our proof remains rather long and will be divided into four steps.

Step 1: *Estimates using the characteristics.* In what follows, we keep the notation of the proof of Theorem A. In particular, we set

$$F(x,t) = \begin{bmatrix} E_\parallel(x,t) \\ H_\parallel(x,t) \end{bmatrix} \in \mathbb{R}^4. \tag{4.2}$$

First, we note that if we differentiate the system with respect to time, thanks to the linearity of Maxwell's equations, the relationship between $\ddot{M}$ and $\dot{E}$, $\dot{H}$ is exactly the same as the one between $\dot{M}$ and E, H. Therefore, reproducing the computations in the proof of Theorem A leads to the estimate

$$|\dot{F}(x,t)| \leq \sqrt{a} \left(\int_{\Gamma^a(x,t)} |\ddot{M}(y,s)| dy \right)^{\frac{1}{2}} + \frac{1}{2} \Big\{ |\dot{F}^0(x+t)| + |\dot{F}^0(x-t)| \Big\}. \tag{4.3}$$

Then, proceeding as in the proof of Theorem A, we easily get

$$\left| \begin{array}{rcl} \displaystyle\int_0^T \int_{-a}^a |\dot F(x,t)|^2 dx dt & \leq & \displaystyle\frac{4a^2}{\alpha_*}\|M_0\|_\infty \int_0^T \left(\int_{|y|<a} \frac{\alpha}{|M|} |\ddot M(y,s)|^2 dy \right) ds \\ & + & \displaystyle\frac{1}{2}\sum_{\pm} \int_0^T \int_{-a}^a |\dot F^0(x \pm t)|^2 dt. \end{array} \right. \tag{4.4}$$

Step 2: *Energy-like estimates.* We start from the linear Maxwell's equations after time derivation:

$$\begin{cases} \ddot E - \operatorname{curl} \dot H = 0, \\ \ddot H + \operatorname{curl} \dot E = -\ddot M. \end{cases} \tag{4.5}$$

We multiply the first equation of (4.5) by $\dot E$, the second by $\dot H$, and add the two resulting equations. After integration over x, we easily get, using Green's formula and the fact that M is supported in $[-a,a] \times \mathbb{R}^+$,

$$\frac{d}{dt}\left(\frac{1}{2} \int (|\dot E|^2 + |\dot H|^2) dx \right) + \int_{-a}^a \dot H \ddot M dx = 0. \tag{4.6}$$

Now, we differentiate in time the LLG equation which leads to

$$\ddot M = \dot H \times M + K(p \cdot \dot M) p \times M + H_T \times \dot M + \frac{\alpha}{|M|} M \times \ddot M, \tag{4.7}$$

using the fact that $|M|$ is constant in time. Multiplying (4.7) successively by $\dot H$ and $\ddot M$ gives

$$\left| \begin{array}{rcl} \ddot M \cdot \dot H & = & K(p\cdot \dot M)(p, M\dot M) + (H_T, \dot M, \dot H) + \dfrac{\alpha}{|M|}(M, \ddot M, \dot H), \\ |\ddot M|^2 & = & (\dot H, M, \ddot M) + K(p \cdot \dot M)(p, M, \ddot M) + (H_T, \dot M, \ddot M). \end{array} \right. \tag{4.8}$$

By an adequate linear combination of these two equalities, we eliminate the mixed product $(\dot H, M, \ddot M)$. This leads to

$$\ddot M \cdot \dot H = \frac{\alpha}{|M|}|\ddot M|^2 + (H_T, \dot M, \dot H) - \frac{\alpha}{|M|}(H_T, \dot M, \ddot M) \tag{4.9}$$

that we plug into (4.6) to obtain

$$\frac{d}{dt}\left[\frac{1}{2} \int (|\dot H|^2 + |\dot E|^2) dx \right] + \int_{-a}^a \frac{\alpha}{|M|} |\ddot M|^2 dx = \int_{-a}^a \left\{ \frac{\alpha}{|M|}(H_T, \dot M, \ddot M) - (H_T, \dot M, \dot H) \right\} dx. \tag{4.10}$$

Let us use the bounds

$$\left| \begin{array}{l} |(H_T, \dot M, \ddot M)| \leq \dfrac{1}{2}|\ddot M|^2 + \dfrac{1}{2}|H_T|^2 |\dot M|^2, \\ |(H_T, \dot M, \dot H)| \leq \varepsilon |\dot H|^2 + \dfrac{1}{4\varepsilon}|H_T|^2 |\dot M|^2, \end{array} \right.$$

where ε is an arbitrary strictly positive real number. We then obtain

$$\left| \begin{array}{l} \displaystyle \frac{d}{dt}\left(\frac{1}{2}\int(|\dot H|^2+|\dot E|^2)dx\right)+\frac{1}{2}\int_{-a}^{a}\frac{\alpha}{|M|}|\ddot M|^2dx \\ \displaystyle \qquad\qquad \leq \varepsilon\int_{-a}^{a}|\dot H|^2dx+\int_{-a}^{a}\left(\frac{\alpha}{2|M|}+\frac{1}{4\varepsilon}\right)|H_T|^2|\dot M|^2dx, \end{array}\right.$$

which we can integrate between 0 and T, to obtain

$$\text{(4.11)}\quad \left| \begin{array}{l} \displaystyle \frac{1}{2}\int(|\dot H(x,T)|^2+|\dot E(x,T)|^2)dx+\frac{1}{2}\int_{-a}^{a}\int_0^T\frac{\alpha}{|M|}|\ddot M|^2dxdt \\ \displaystyle \quad \leq \frac{1}{2}\int(|\dot H(x,0)|^2+|\dot E(x,0)|^2)dx+\varepsilon\int_{-a}^{a}\int_0^T|\dot H|^2dxdt \\ \displaystyle \qquad +\int_{-a}^{a}\int_0^T\left(\frac{\alpha}{2|M|}+\frac{1}{4\varepsilon}\right)|H_T|^2|\dot M|^2dxdt. \end{array}\right.$$

Step 3: *Combination of the two estimates.* We use (1.12) in Remark 1.1 to write

$$\int_{-a}^{a}\int_0^T|\dot H|^2dxdt \ \leq\ \int_{-a}^{a}\int_0^T|\dot F|^2dxdt\ +\ \int_{-a}^{a}\int_0^T|\dot M_x|^2dxdt.$$

We thus obtain, using (4.4) in (4.11),

$$\left| \begin{array}{l} \displaystyle \frac{1}{2}\int(|\dot H(x,T)|^2+|\dot E(x,T)|^2)dx+\frac{1}{2}\int_{-a}^{a}\int_0^T\left(1-\frac{8\varepsilon a^2}{\alpha_*}\|M_0\|_\infty\right)\frac{\alpha}{|M|}|\ddot M|^2dxdt \\ \displaystyle \quad \leq \frac{1}{2}\int(|\dot H(x,0)|^2+|\dot E(x,0)|^2)dx+\frac{\varepsilon}{2}\sum_{\pm}\int_{-a}^{a}\int_0^T|\dot F^0(x\pm t)|^2dxdt \\ \displaystyle \qquad +\int_{-a}^{a}\int_0^T\left(\frac{\alpha}{2|M|}+\frac{1}{4\varepsilon}\right)|H_T|^2|\dot M|^2dxdt+\varepsilon\int_{-a}^{a}\int_0^T|\dot M_x|^2dxdt. \end{array}\right.$$

We observe that

$$\int_{-a}^{a}\int_0^T|\dot F^0(x\pm t)|^2dxdt\leq 2a\int(|\dot H(x,0)|^2+|\dot E(x,0)|^2)dx$$

and we choose ε such that

$$1-\frac{8\varepsilon a^2}{\alpha_*}\|M_0\|_\infty=\frac{1}{2}. \tag{4.12}$$

Finally, if we set

$$C_0=\frac{1}{2}+2a\varepsilon,\quad C_2=\frac{1}{2}+\frac{\|M_0\|_\infty}{4\varepsilon\alpha_*},\quad C_1=\varepsilon\frac{\|M_0\|_\infty}{\alpha_*}, \tag{4.13}$$

we obtain

$$\text{(4.14)}\quad \left| \begin{array}{l} \displaystyle \frac{1}{2}\int(|\dot H(x,T)|^2+|\dot E(x,T)|^2)dx+\frac{1}{4}\int_{-a}^{a}\int_0^T\frac{\alpha}{|M|}|\ddot M|^2dxdt \\ \displaystyle \leq C_0\int(|\dot H(x,0)|^2+|\dot E(x,0)|^2)dx+C_1\int_{-a}^{a}\int_0^T\frac{\alpha}{|M|}|\dot M|^2dxdt \\ \displaystyle \qquad\qquad +C_2\int_{-a}^{a}\int_0^T\frac{\alpha}{|M|}|H_T|^2|\dot M|^2dxdt. \end{array}\right.$$

From (2.20), we already know that

$$\int_{-a}^{a}\int_{0}^{T}\frac{\alpha}{|M|}|\dot M|^2 dxdt < +\infty$$

while, because of the regularity assumptions on the initial data, one can see that

$$\int |\dot E(x,0)|^2 dx \ = \ \int |\mathrm{curl}\ H_0|^2 dx < +\infty. \tag{4.15}$$

Conversely,

$$\int |\dot H(x,0)|^2 dx \ \le \ 2\int |\mathrm{curl}\ E_0|^2 dx \ + \ 2\int |\dot M(x,0)|^2 dx, \tag{4.16}$$

and from the LLG equation one deduces that

$$|\dot M(x,0)|^2 = \frac{1}{1+\alpha^2}|H_T(x,0)|^2 |M(x,0)|^2.$$

Therefore, setting $H_T^0(x) = H_T(x,0)$, we have

$$\int |\dot M(x,0)|^2 dx \le \|M_0\|_{L^\infty}^2 \|H_T^0\|_{L^2}^2 \tag{4.17}$$

which is finite since, under the conditions on the data of the problem

$$H_T^0 = H_0 - KP(M_0) + H_s \ \in \ L^2, \tag{4.18}$$

we can conclude that

$$\int |\dot H(x,0)|^2 dx \le 2\|\mathrm{curl}\ E_0\|_{L^2}^2 + 2\|M_0\|_{L^\infty}^2 \|H_T^0\|_{L^2}^2. \tag{4.19}$$

Finally, if C denotes a constant which depends only on M_0, E_0, H_0, α_*, K, H_s, and a, we can write

$$\left|\ \begin{aligned} &\frac{1}{2}\int (|\dot H(x,T)|^2 + |\dot E(x,T)|^2)dx + \frac{1}{4}\int_0^T\int_{-a}^{a}\frac{\alpha}{|M|}|\ddot M|^2 dxdt \\ &\qquad\qquad \le C + C_2\int_0^T\int_{-a}^{a}\frac{\alpha}{|M|}|\dot M|^2 |H_T|^2 dxdt. \end{aligned}\right. \tag{4.20}$$

Step 4: *A Gronwall-type estimate.* From the definition of H_T, we easily get the following bound for $|H_T|^2$:

$$|H_T|^2 \le 3(1+\|K\|_\infty^2)\left[|H_s|^2 + |M|^2 + |H|^2\right]$$

that implies, since $H_x = -M_x$,

$$|H_T|^2 \le 3(1+\|K\|_\infty^2)(|H_s|^2 + 2|M|^2) + 3(1+\|K\|_\infty^2)|H_\parallel|^2. \tag{4.21}$$

Using the interpolation inequality

$$\|H_\parallel\|_{L^\infty}^2 \ \le \ \|H_\parallel\|_{L^2}\left\|\frac{\partial H_\parallel}{\partial x}\right\|_{L^2},$$

we can write

$$\|H_T\|_{L^\infty}^2 \;\le\; 3(1+\|K\|_\infty^2)(\|H_s\|_{L^\infty}^2+2\|M_0\|_{L^\infty}^2) + 3(1+\|K\|_\infty^2)\|H_\parallel\|_{L^2}\left\|\frac{\partial H_\parallel}{\partial x}\right\|_{L^2}. \tag{4.22}$$

As we already know from Theorem 2.4 that $\|H_\parallel\|_{L^2}$ remains bounded, from (4.20) and (4.22), we can write

$$\left|\begin{array}{l} \dfrac{1}{2}\displaystyle\int(|\dot{H}(x,T)|^2+|\dot{E}(x,T)|^2)dx + \frac{1}{4}\int_0^T\int_{-a}^a \frac{\alpha}{|M|}|\ddot{M}|^2dxdt \\ \qquad\qquad \le C\left(1+\displaystyle\int_0^T\left(\int_{-a}^a\frac{\alpha}{|M|}|\dot{M}|^2dx\right)\left\|\frac{\partial H_\parallel}{\partial x}\right\|_{L^2}dt\right), \end{array}\right. \tag{4.23}$$

where C denotes a new positive constant depending only on the data of the problem. Finally, using Maxwell's equations, we observe that

$$\left\|\frac{\partial H_\parallel}{\partial x}\right\|_{L^2} = \left\|\dot{E}_\parallel\right\|_{L^2}. \tag{4.24}$$

Therefore, if we set

$$\left\{\begin{array}{lcl} G(t) &=& \displaystyle\int(|\dot{H}(x,t)|^2+|\dot{E}(x,t)|^2)dx, \\ m(t) &=& \displaystyle\int_{-a}^a\frac{\alpha}{|M|}|\dot{M}|^2dx, \end{array}\right. \tag{4.25}$$

we deduce from (4.23), using that $\int_0^\infty\int_{-a}^a\frac{\alpha}{|M|}|\dot{M}|^2dxdt < +\infty$, the inequality ($C$ denoting another constant)

$$G(t)\;\le\;C\left(1+\int_0^t m(s)G(s)^{\frac{1}{2}}ds\right)$$

with $m\in L^1(0,+\infty)$. Therefore, using an appropriate generalization of Gronwall's lemma, we get

$$G(t)\le\left(C^{\frac{1}{2}}+C\int_0^t m(s)ds\right)^2. \tag{4.26}$$

As $m\in L^1(0,+\infty)$, this shows that $G(t)$ remains bounded in time. Moreover, from (4.23) we also deduce

$$\int_0^T\int_{-a}^a\frac{\alpha}{|M|}|\ddot{M}|^2dxdt\le C\left(1+\int_0^T G(s)^{\frac{1}{2}}m(s)ds\right). \tag{4.27}$$

Setting $G^*=\sup_{t\ge0}G(t)$, we thus obtain

$$\int_0^T\int_{-a}^a\frac{\alpha}{|M|}|\ddot{M}|^2dxdt\le C\left(1+(G^*)^{\frac{1}{2}}\|m\|_{L^1}\right) \tag{4.28}$$

which concludes the proof of the lemma. □

Proof of Theorem A′. First, note that the inequalities

$$\left| \begin{aligned} &\int_{-a}^{a} |\dot M(x,t)|^2 dx \le \frac{\|M_0\|_{L^\infty}}{\alpha_*}\int_{-a}^{a}\frac{\alpha}{|M|}|\dot M|^2 dx,\\ &\int_{-a}^{a} |\ddot M(x,t)|^2 dx \le \frac{\|M_0\|_{L^\infty}}{\alpha_*}\int_{-a}^{a}\frac{\alpha}{|M|}|\ddot M|^2 dx, \end{aligned}\right.$$

joined to the results of Theorem A and Lemma 4.1, imply

$$\int_0^\infty \int_{-a}^{a}\left(|\dot M|^2 + |\ddot M|^2\right) dxdt < +\infty;$$

that is to say,

$$\dot M \in H^1\left(\mathbb{R}^+, L^2(-a,a)\right), \tag{4.29}$$

which implies

$$\lim_{t\to+\infty}\int_{-a}^{a} |\dot M(x,t)|^2 dx = 0. \tag{4.30}$$

Now, let us return to the formula

$$|\dot F(x,t)| \le \frac{\sqrt 2}{2}\int_{\Gamma_a(x,t)} |\dot M(y,s)|dy + \frac12\left\{|\dot F^0(x+t)| + |\dot F^0(x-t)|\right\}.$$

By the same manipulations as in the proof of Theorem A (cf. the method we used to obtain (3.9)), we get

$$\left| \begin{aligned} \int_{-R}^{R}|\dot F(x,t)|^2 dx \le \frac{4a\|M_0\|_\infty}{\alpha_*}&\int_{t-(R+a)}^{t}\int_{-a}^{a}\frac{\alpha}{|M|}|\ddot M(y,s)|^2 dyds\\ &+\int_{-R-t}^{R-t}|\dot F^0(x)|dx + \int_{-R+t}^{R+t}|\dot F^0(x)|dx. \end{aligned}\right. \tag{4.31}$$

Since $\int_{-\infty}^{+\infty}|\dot F^0(x)|^2dx < +\infty$ and $\int_0^\infty\int_{-a}^{a}\frac{\alpha}{|M|}|\ddot M|^2dyds < +\infty$ (cf. Lemma 2.1), (4.31) implies that

$$\lim_{R\to+\infty}\int_{-R}^{R}\left(\left|\dot E_\parallel\right|^2 + \left|\dot H_\parallel\right|^2\right)dx = 0. \tag{4.32}$$

Now, using Maxwell's equations we have

$$\left| \begin{aligned} &\int_{-R}^{R}\left|\frac{\partial E_\parallel}{\partial x}\right|^2 dx = \int_{-R}^{R}\left|\dot H_\parallel\right|^2 dx,\\ &\int_{-R}^{R}\left|\frac{\partial H_\parallel}{\partial x}\right|^2 dx \le 2\int_{-R}^{R}\left|\dot E_\parallel\right|^2 dx + 2\int_{-a}^{a}|\dot M|^2 dx \end{aligned}\right. \tag{4.33}$$

which, taking into account (4.30) and (4.32), lead to (2.26). Thanks to Theorem A, (2.27) is then a consequence of (2.26), since $H^1(-R,R)\subset L^\infty(-R,R)$. This concludes the proof of the theorem. □

5. Transitions to stationary states in the LLG equation. This section must be seen as an introduction to the second part of the paper, which is devoted to long-time asymptotics of the magnetization M.

5.1. Stationary states in the LLG equation. We consider the solutions to the following nonlinear evolution equation that we shall call the unperturbed LLG equation at point x:

$$\left\{ \begin{array}{l} \dot{M}(x,t) = \tilde{H}_T(x, M(x,t)) \times M(x,t) + \dfrac{\alpha}{|M(x,t)|} M(x,t) \times \dot{M}(x,t),, \\ M(x, t=0) = M_0(x), \end{array} \right. \tag{5.1}$$

where $\tilde{H}_T(x, M(x,t))$ has been defined by (2.28).

Remark 5.1. One can easily check that the first equation of (5.1) can be rewritten as

$$\dot{M}(x,t) = L(x, M(x,t)), \tag{5.2}$$

where we have defined

$$L(x,M) \;=\; \frac{1}{1+\alpha^2}\left[\tilde{H}_T(x,M) \times M + \frac{\alpha}{|M|} M \times \Big(\tilde{H}_T(x,M) \times M\Big)\right]. \tag{5.3}$$

We now introduce the set $\mathcal{S}(x)$ of the stationary states for (5.1) as

$$\mathcal{S}(x) \;=\; \Big\{ M_0 \in \mathbb{R}^3 / \dot{M}(x,t) \;=\; 0 \quad \forall t > 0 \Big\}. \tag{5.4}$$

The main property of this set is that its intersection with any sphere $\Sigma(R)$ is a discrete set. (See Figure 5.1).

THEOREM 5.2. *Let $R > 0$. Under assumption $(\mathcal{H}_x)$ (see section 2.2), the intersection of $\mathcal{S}(x)$ with the sphere $\Sigma(R)$ with center at the origin and of radius R is a set which contains at least two elements and at most six.*

This result is of interest because (see (2.9))

$$|\dot{M}| \;=\; 0 \quad \Leftrightarrow \quad |\tilde{H}_T(x,M) \times M| \;=\; 0.$$

This shows that, for the set $\mathcal{Z}(x, M_0)$ defined in section2.2,

$$\mathcal{Z}(x, M_0) \;=\; \mathcal{S}(x) \cap \Sigma(|M_0(x)|),$$

and we have the following corollary.

COROLLARY 5.3. *The set $\mathcal{Z}(x, M_0)$ is a discrete set which contains at least two elements and at most six.*

Proof. We give a geometrical proof of Theorem 5.2: we characterize $\mathcal{S}(x)$ in every possible case. Toward this end, let us consider

$$\tilde{H}_T(x,M) \times M \;=\; 0. \tag{5.5}$$

This equality leads to two different problems depending on whether or not H_s is equal to 0.

Case 1. $H_s = 0$. If M belongs to $\mathcal{S}(x)$, we deduce from (5.5) that

$$\exists\, \lambda \in \mathbb{R}, \quad K\,(p \cdot M)p \,-\, (e_x \cdot M)e_x \;=\; \lambda M, \tag{5.6}$$

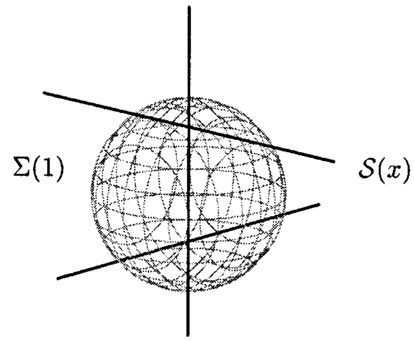

FIG. 5.1. *The unit sphere and the set $\mathcal{S}(x)$ with six intersection points ($H_s = 0.5\, e_z$, $K = 0.7$, $p = e_y$).*

which corresponds to a simple eigenvalue problem. Assumption $(\mathcal{H}_x)$ ensures that p and e_x are not collinear and that $K \neq 0$. Thus the operator defined by $M \mapsto K(p \cdot M)p - (e_x \cdot M)e_x$ is a real symmetric operator A; in the basis $(e_x, p, e_x \times p)$, we have

$$A = \begin{bmatrix} -1 & -(e_x \cdot p) & 0 \\ K(e_x \cdot p) & K & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The operator A has three distinct real eigenvalues: one is zero, and the other ones have opposite signs (as $K\left[(e_x \cdot p)^2 - 1\right] < 0$). Thus there exist three different eigendirections, (e_i) and $\mathcal{S}(x)$ is made up of three lines passing through the center of the $\Sigma(R)$. There are exactly six stationary states such that $|M_0| = R$, namely $M_0 = Re_i$, $i = 1, \ldots, 3$.

Case 2. $H_s \neq 0$. Let us denote $u = e_x$, $v = p$, and $w = H_s/|H_s|$. We shall distinguish three cases:

- *Case* 2.1 (*general case, see Figure* 5.2): $(u, v, w) \neq 0$.
  In the basis $\{u, v, w\}$, we denote M by $(x, y, z)^t$. Equality (5.5) then reads

  $$(\, |H_s|\, w \,+\, Ky\, v \,-\, x\, u\,) \times (\, x\, u \,+\, y\, v \,+\, z\, w\,) \;=\; 0,$$

  which yields

  $$|H_s|x\; w{\times}u + |H_s|y\; w{\times}v + Kxy\; v{\times}u + Kyz\; v{\times}w - xy\; u{\times}v - xz\; u{\times}w \;=\; 0.$$

  As $\{u \times v, u \times w, v \times w\}$ is also a basis of $\mathbb{R}^3$, we deduce that

  $$\left\{ \begin{array}{rcl} x\,(|H_s| + z) & = & 0, \\ y\,(|H_s| - Kz) & = & 0, \\ x\,y\,(K+1) & = & 0. \end{array} \right. \tag{5.7}$$

  This shows that $\mathcal{S}(x)$ is made up of two or three lines: $d_1 = \{x = 0, y = 0\}$, $d_2 = \{y = 0, z = -|H_s|\}$ and, if $K \neq 0$, $d_3 = \{x = 0, z = |H_s|/K\}$. We deduce that the intersection between $\mathcal{S}(x)$ and a sphere of radius R has at least two points (because the line d_1 passes through the point $(0, 0, 0)$) and at most six points. All the values between 2 and 6 are possible, depending on R and on the distance of the lines d_2 and d_3 from the point $(0, 0, 0)$.

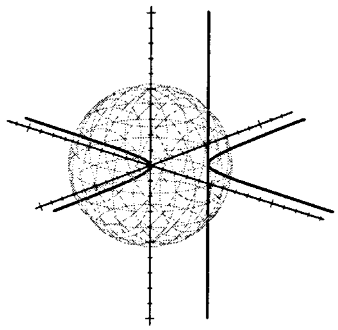

FIG. 5.2. *The unit sphere and the set $\mathcal{S}(x)$ with six intersection points ($H_s = 0.5\,(e_x + p)$, $K = 0.5$, $p = e_y$).*

- *Case* 2.2: $(u, v, w) = 0$, $(u, v, u \times v) \neq 0$.
  In the basis $\{u, v, u \times v\}$, we denote M by $(x, y, z)^t$ and H_s by $(h_u, h_v, 0)^t$. The computations now lead to

$$\left\{\begin{array}{rcl} x\,(h_v + Ky) \,-\, y\,(h_u + x) & = & 0, \\ z\,(h_v + Ky) & = & 0, \\ z\,(h_u - x) & = & 0. \end{array}\right. \tag{5.8}$$

  In this case, the set $\mathcal{S}(x)$ is made up of a hyperbola ($\{z = 0, (K+1)xy + h_v x - h_u y = 0\}$) and, if $K \neq 0$, a line ($\{x = h_u, y = -h_v/K\}$). As one of the branches of the hyperbola passes through the point $(0, 0, 0)$, the conclusions are exactly the same as in the previous case.
- *Case* 2.3: u, v, and w are collinear.
  In the canonical basis $\{e_x, e_y, e_z\}$, we have $p = e_x$, $H_s = |H_s| e_x$, and $M = (x, y, z)^t$. The computations now lead to

$$\left\{\begin{array}{rcl} y\,(|H_s| + Ky - x) & = & 0, \\ z\,(|H_s| + Ky - x) & = & 0. \end{array}\right. \tag{5.9}$$

  As

$$(\mathcal{H}_x) \quad \Leftrightarrow \quad (|H_s| + Ky - x) \,\neq\, 0,$$

  the set $\mathcal{S}(x)$ reduces in this case to the line $\{y = 0, z = 0\}$, and there are always exactly two intersection points. ∎

5.2. Free transitions to stationary states. In this section we still consider the unperturbed case (i.e., $H_\parallel = 0$). It is indeed easy to show in this case the convergence of $M(x,t)$ to some element of $\mathcal{Z}(x, M_0)$. It is also possible to see which of these positions are stable and which are not (see Remark 5.6).

THEOREM 5.4. *Let $\mathcal{Z}(x, M_0)$ be defined as in section* 2.2. *The solution to system* (5.1) *satisfies*

$$\exists\, M_\infty(x) \in \mathcal{Z}(x, M_0) \quad \textit{such that} \quad \lim_{t\to\infty} M(x,t) \,=\, M_\infty(x).$$

Proof. Note that

$$\tilde{H}_T(x, M(x,t)) \,=\, H_s(x) + K(p \cdot M(x,t))p - (e_x \cdot M(x,t))e_x \tag{5.10}$$

can also be defined as

$$\tilde{H}_T(x, M(x,t)) \;=\; (\nabla_M V)\,(x, M(x,t)), \tag{5.11}$$

where we define

$$V(x,M) \;=\; H_s(x)\cdot M(x,t) - \frac{1}{2}(e_x \cdot M(x,t))^2 - \frac{1}{2}|P(M)|^2 \quad \forall x\in\mathbb{R},\ \forall M \in \mathbb{R}^3.$$

For this reason, we have

$$\tilde{H}_T(x, M(x,t))\cdot \dot{M}(x,t) \;=\; \frac{d}{dt}V(x, M(x,t)). \tag{5.12}$$

Additionally, we deduce from (5.2) that

$$\tilde{H}_T(x, M(x,t))\cdot \dot{M}(x,t) \;=\; \frac{\alpha}{1+\alpha^2}\,\frac{1}{|M|}\,\left|\tilde{H}_T(x, M(x,t)) \times M(x,t)\right|^2. \tag{5.13}$$

Thus

$$\frac{d}{dt}V(x, M(x,t)) \;=\; g(x, M(x,t)), \tag{5.14}$$

where

$$g(x,M) \;=\; \frac{\alpha}{1+\alpha^2}\,\frac{1}{|M|}\,\left|\tilde{H}_T(x,M)\times M\right|^2 \quad \forall x\in\mathbb{R},\ \forall M\in\mathbb{R}^3. \tag{5.15}$$

The main properties of this function g are, respectively,

(i) $g(x,M) \;\geq\; 0$;

(ii) $(\, g(x,M) = 0$ and $|M| = |M_0(x)|\,) \;\Leftrightarrow\; (M \in \mathcal{Z}(x, M_0))$;

(iii) $g(x,M) \;=\; \frac{\alpha}{1+\alpha^2}\,\frac{1}{|M|}\,|\nabla_M V(x,M)\times M|^2$.

Therefore, (5.14) means that the function $M \to V(x,M)$ is a strict anti-Liapunov function for the system (5.1). Classical results on dynamic systems then ensure the convergence to these stationary states (see, for instance, [6]). □

Remark 5.5. Property (iii) of function g means, in particular, that the set of extremal points of $V(x,M)$ on the sphere is included in $\mathcal{Z}(x,M_0)$.

Moreover, in the general case ($(e_x, p, H_s) \neq 0$), it can be shown that, if $M \in \mathcal{Z}(x, M_0)$ is neither a maximum nor a minimum for V, then it is a saddlepoint for V (it cannot be a local extremum).

As an illustration of these results and Theorem 5.2, we represent below two trajectories of the vector $M(t)$ on the sphere of radius $|M_0| = 1$ (see Figure 5.3). These trajectories have been computed numerically and correspond to the following data: ($H_s = 2\,e_z$, $K = 0$) and ($H_s = 0.5\,e_z$, $K = 0.7$, $p = e_y$), respectively. One can check that the first case corresponds to two stationary states on the sphere while the second one corresponds to six states (all are indicated by bold arrows). The initial position, indicated by a bold dot, is the same in both cases.

Remark 5.6. Although it is not the main objective of this paper, one could complete the analysis by results on the stability of the stationary points. Let us first recall the following definition.

DEFINITION 5.7. *A stationary state M_∞ is stable for the trajectory $M(x,t)$ if and only if*

$$\exists\,\mathcal{V}(M_\infty)\ \forall M_0 \in \mathcal{V}(M_\infty), \quad \lim_{t\to\infty} M(x,t) \;=\; M_\infty, \tag{5.16}$$

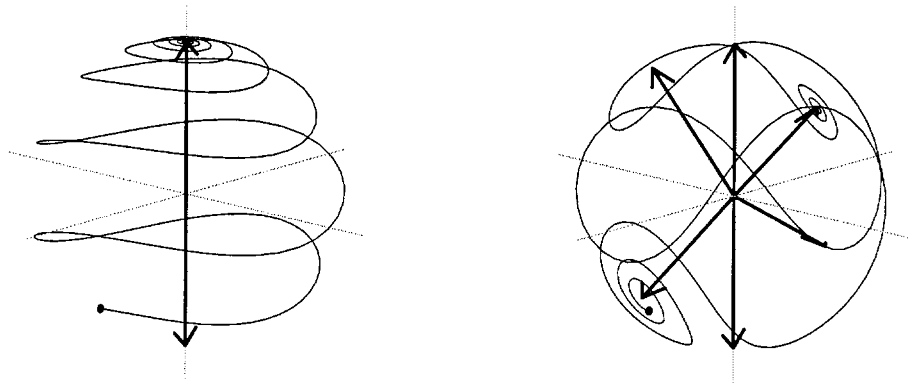

FIG. 5.3. *Trajectories of $M(t)$ with two and six stationary states.*

where $\mathcal{V}(M_\infty)$ *is a neighborhood of* M_∞ *and* $M(x,t)$ *is the solution to* (5.1) *associated with the initial data* M_0. *(Otherwise,* M_∞ *is said to be unstable.)*

Two results can now be pointed out:

1. In the general case $((e_x, p, H_s) \neq 0)$, one can show that only the points $M_\infty \in \mathcal{Z}(x, M_0)$, such that
$$V(M_\infty) \;=\; \max_{M\in S} V,$$
are stable stationary states, while the others are unstable.
2. This is not necessarily true when $(e_x, p, H_s) = 0$.

6. Proof of Theorem B: Attraction of M for weak solutions. Using the notation of section 5, one can check that the LLG equation of (1.5) can be rewritten (see Remark 5.1) in the form

$$\dot{M}(x,t) \;=\; L(x, M(x,t)) + R(x,t), \tag{6.1}$$

where

$$R(x,t) = \frac{1}{1+\alpha^2}\left[H_\parallel(x,t)\times M(x,t) + \frac{\alpha}{|M|}M(x,t)\times\big(H_\parallel(x,t)\times M(x,t)\big)\right]. \tag{6.2}$$

By the definition of V and g (see section 5, proof of Theorem 5.4), one easily shows that

$$\frac{d}{dt}V(x, M(x,t)) = g(x, M(x,t)) + r(x,t),\;\; t>0, \tag{6.3}$$

where

$$r(x,t) \;=\; -\tilde{H}_T(x, M(x,t))\cdot R(x,t) + \frac{\alpha}{1+\alpha^2}\,\frac{1}{|M|}\,\left|H_\parallel(x,t)\times M(x,t)\right|^2. \tag{6.4}$$

As a consequence of Theorem A and (2.6), we see that $R(x,M)$ satisfies

$$\int_0^\infty |R(x, M(x,t), H_\parallel(x,t))|^2 dt \;<\; \infty \text{ for a.e. } x\in\mathbb{R}. \tag{6.5}$$

and thus

$$\int_0^\infty |r(x, M(x,t), H_\|(x,t))|dt \; < \; \infty \text{ for a.e. } x \in \mathbb{R}. \tag{6.6}$$

Let us fix an $x \in \mathbb{R}$ such that (6.1)–(6.6) hold. For this reason we shall write $M(t)$, $R(t)$, and $r(t)$ instead of $M(x,t)$, $R(x,t)$, and $r(x,t)$, respectively, and similarly $g(M)$ and $V(M)$ instead of $g(x,M)$ and $V(x,M)$. By Theorem 2.8, we know that all these functions are, almost everywhere in x, continuous functions of time. In the following, we shall consider such an x. We shall also write $\mathcal{Z}$ instead of $\mathcal{Z}(x, M_0)$ and denote by S the sphere of radius $|M_0(x)|$.

The first step of the proof is the convergence of $V(M(t))$. Let $V(\mathcal{Z}) = \{V_0, V_1, \ldots, V_N\}$ with $V_0 < V_1 < \cdots < V_N$ (In particular, by Remark 5.5 V_0 is the minimum of V on S and V_N is the maximum). Let us introduce

$$V^+ = \limsup\nolimits_{t\to\infty} V(M(t)) \in]V_i, V_{i+1}] \quad \text{for some } i. \tag{6.7}$$

Lemma 6.1. *Let V_{i+1} be defined by* (6.7)*. Then we necessarily have*

$$V(M(t)) \to V_{i+1} \textit{ as } t \to \infty. \tag{6.8}$$

Proof. We prove (6.8) by contradiction. Let us assume to the contrary that

$$V^- \; = \; \liminf_{t\to\infty} \; V(M(t)) \; < \; V_{i+1}.$$

Using also (6.7), we deduce that there exists $\varepsilon > 0$ and $V^0 \in \mathbb{R}$ such that

$$\max(V^-, V_i + \varepsilon) < V^0 < \min(V^+, V_{i+1} - \varepsilon). \tag{6.9}$$

Therefore, by continuity of $V(M(t))$ there exists a sequence $t_k \to \infty$ such that

$$V(M(t_k)) = V^0. \tag{6.10}$$

If we prove that, for sufficiently large T and $t > T$,

$$V(M(t)) \; \geq \; V_{i+1} - \varepsilon, \tag{6.11}$$

we will get a contradiction to (6.10) since $t_k \to \infty$ and $V^0 < V_{i+1} - \varepsilon$. To prove (6.11), first note that the properties of g imply the existence of some $\delta > 0$ such that

$$V(M) \in \left[V_i + \frac{\varepsilon}{2}, V_{i+1} - \frac{\varepsilon}{2}\right] \; \Rightarrow \; g(M) \geq \delta. \tag{6.12}$$

Let us introduce for each k the set

$$\mathcal{E}_k = \left\{ t > t_k : \; V(M(\tau)) \in \left[V_i + \frac{\varepsilon}{2}, V_{i+1} - \frac{\varepsilon}{2}\right] \text{ for } t_k < \tau < t \right\}, \tag{6.13}$$

It is easy to see that by construction $\mathcal{E}_k$ is connected and that, taking into account (6.9) and (6.10) as well as the continuity of $V(M(t))$, it is not empty. One can also prove that this set is bounded. Indeed, let $\tau \in \mathcal{E}_k$; integrating (6.3) between t_k and τ, we get

$$V(M(\tau)) - V(M(t_k)) \; = \; \int_{t_k}^{\tau} g(M(s))ds + \int_{t_k}^{\tau} r(s)ds. \tag{6.14}$$

By definition of $\mathcal{E}_k$, we have

$$\forall\, t_k < s < \tau, \quad V(M(s)) \in \left[V_i + \frac{\varepsilon}{2}, V_{i+1} - \frac{\varepsilon}{2}\right].$$

Therefore, using (6.14) and (6.12), we get

$$V(M(\tau)) - V(M(t_k)) \;\geq\; \delta(\tau - t_k) - I_k, \tag{6.15}$$

and thus,

$$\tau \;\leq\; t_k + \frac{1}{\delta}\left(\max_{M\in S} V(M) - \min_{M\in S} V(M) + I_k\right), \tag{6.16}$$

where

$$I_k \;=\; \int_{t_k}^{\infty} |r(s)|ds. \tag{6.17}$$

Therefore $\mathcal{E}_k$ is bounded by

$$t_k + \frac{1}{\delta}\left(\max_{M\in S} V(M) - \min_{M\in S} V(M) + I_k\right). \tag{6.18}$$

Now, let us introduce

$$\tau_k = \sup \mathcal{E}_k < +\infty. \tag{6.19}$$

By continuity, $V(M(\tau_k))$ is equal to $V_i + \frac{\varepsilon}{2}$ or $V_{i+1} - \frac{\varepsilon}{2}$. Let us show that $V(M(\tau_k)) = V_{i+1} - \frac{\varepsilon}{2}$ at least for k large enough. Indeed, from (6.14), we deduce in particular that (g is positive and we take $\tau = \tau_k$)

$$V(M(\tau_k)) - V(M(t_k)) \;\geq\; -I_k. \tag{6.20}$$

I_k tends to 0 when k tends to $+\infty$ because of (6.6). Thus, there exists $\bar{k} = \bar{k}(\varepsilon)$ such that, for $k \geq \bar{k}$,

$$V(M(\tau_k)) - V(M(t_k)) \;\geq\; -\frac{\varepsilon}{2} \text{ for } t_k < \tau < \tau_k. \tag{6.21}$$

Therefore

$$V(M(\tau_k)) \geq V(M(t_k)) - \frac{\varepsilon}{2} = V^0 - \frac{\varepsilon}{2} > V_i + \frac{\varepsilon}{2}.$$

This proves that (see Figure 6.1)

$$k \geq \bar{k} \Rightarrow V(M(\tau_k)) = V_{i+1} - \frac{\varepsilon}{2}.$$

Now, we introduce the set

$$\mathcal{F}_k \;=\; \left\{t \geq \tau_k \,:\, V(M(\tau)) \geq V_{i+1} - \frac{\varepsilon}{2} \text{ for } \tau_k \leq \tau \leq t\right\}. \tag{6.22}$$

Equations (6.9) and (6.10) ensure that $\mathcal{F}_k$ is bounded for every k. Let us set

$$\theta_k = \max \mathcal{F}_k. \tag{6.23}$$

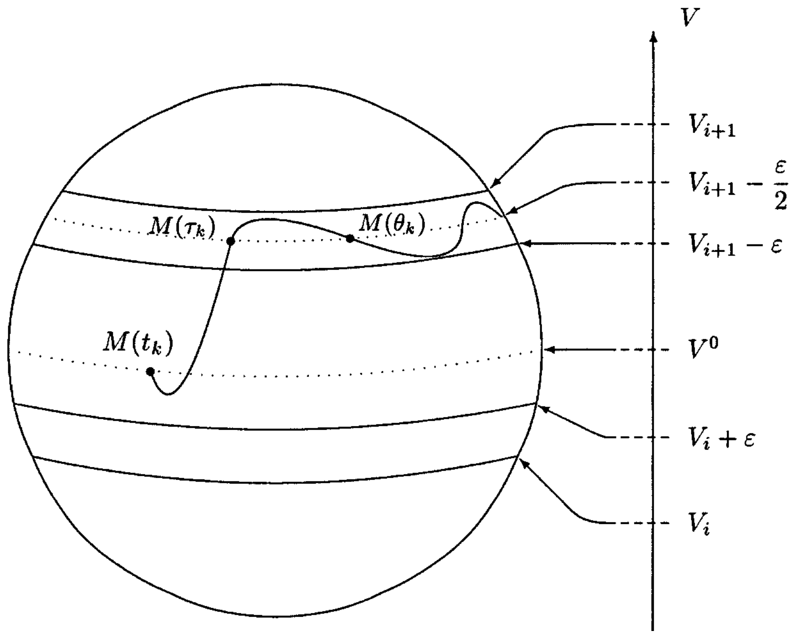

FIG. 6.1. *Behavior of a Liapunov function.*

By definition of $\mathcal{F}_k$ and continuity of $V(M(t))$, we have

$$V(M(\theta_k)) = V_{i+1} - \frac{\varepsilon}{2}. \tag{6.24}$$

Integrating (6.3) between θ_k and $t > \theta_k$ and using once again the positivity of g, we get, for $k \geq \overline{k}$,

$$\forall\, t > \theta_k \quad V(M(k)) - V_{i+1} + \frac{\varepsilon}{2} \geq -I_k \geq -\frac{\varepsilon}{2} \tag{6.25}$$

which proves (6.11) with $T = \theta_{\overline{k}}$. □

Now to prove Theorem B, let us assume by contradiction that the conclusion is not true. This means that there exists $\rho > 0$ and a strictly increasing sequence $t_k \to \infty$ such that (see Figure 6.2)

$$\text{dist}(M(t_k), \mathcal{Z}) \;\geq\; \rho. \tag{6.26}$$

We are going to prove that this contradicts Lemma 6.1.

First, using the properties of g, we know that there exists $\delta > 0$ such that

$$\text{dist}(M(t_k), \mathcal{Z}) \;\geq\; \frac{\rho}{2} \;\Rightarrow\; g(M) \geq \delta. \tag{6.27}$$

By continuity of $M(t)$ and by (6.26), we know that the set

$$\mathcal{G}_k \;=\; \left\{ t > t_k \,/\, \tau \in]t_k, t[\Rightarrow \text{dist}(M(\tau), \mathcal{Z}) \;\geq\; \frac{\rho}{2} \right\}$$

is not empty. Let us check that, for k large enough,

$$]t_k, t_k + T] \subset \mathcal{G}_k \;\text{ where }\; T \,=\, C\rho, C > 0. \tag{6.28}$$

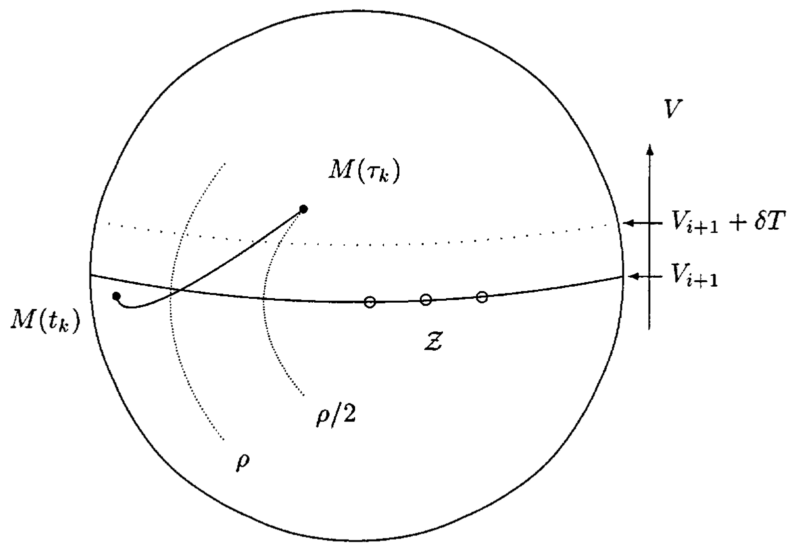

FIG. 6.2

Let $\tau_k = \sup \mathcal{G}_k$. As the case $\tau_k = +\infty$ is obvious ($]t_k, t_k + T] \subset]t_k, +\infty[$), let us assume that $\tau_k < +\infty$. By continuity, $\text{dist}(M(\tau_k), \mathcal{Z}) = \rho/2$, and using (6.26),

$$|M(t_k) - M(\tau_k)| \geq \frac{\rho}{2}. \tag{6.29}$$

Conversely, integrating (6.1) between t_k and τ_k and using (6.5) and the Cauchy–Schwarz inequality, we get

$$|M(\tau_k) - M(t_k)| = \left| \int_{t_k}^{\tau_k} L(M(s))ds + \int_{t_k}^{\tau_k} R(s)ds \right| \leq \omega(\tau - t_k) + J_k(\tau_k - t_k)^{\frac{1}{2}}, \tag{6.30}$$

where $\omega > 0$, and

$$J_k^2 = \int_{t_k}^{\infty} |R(s)|^2 ds. \tag{6.31}$$

Regrouping (6.29) and (6.30), we get, using Young's inequality,

$$\frac{\rho}{2} \leq \omega(\tau_k - t_k) + J_k(\tau_k - t_k)^{\frac{1}{2}} \leq 2\omega(\tau_k - t_k) + \frac{J_k^2}{2\omega}. \tag{6.32}$$

As J_k^2 tends to 0 when $k \to \infty$, we have, for $k \geq \tilde{k}$, $J_k^2/2\omega < \rho/4$. Therefore,

$$k \geq \tilde{k} \Rightarrow \frac{\rho}{4} \leq 2\omega(\tau_k - t_k) \Rightarrow (\tau_k - t_k) \geq CT. \tag{6.33}$$

Now, integrating (6.3) between t_k and $t_k + T$, we get

$$V(M(t_k + T)) - V(M(t_k)) = \int_{t_k}^{t_k+T} g(M(s))ds + \int_{t_k}^{t_k+T} r(s)ds. \tag{6.34}$$

By (6.26)–(6.28), $g(M(s)) \geq \delta$ for $s \in [t_k, t_k + T]$ and $k \geq \tilde{k}$. Therefore,

$$V(M(t_k + T)) - V(M(t_k)) \;\geq\; \delta T \;-\; I_k, \tag{6.35}$$

where $I_k = \int_{t_k}^{+\infty} |r(s)| ds$ tends to 0 when $k \to +\infty$. Therefore, for k large enough,

$$V(M(t_k + T)) - V(M(t_k)) \;\geq\; \frac{\delta T}{2} \tag{6.36}$$

which of course contradicts Lemma 6.1. □

7. On the attraction of $\mathcal{M}$ for strong solutions. In this last section, the assumptions are those of Theorem A′. First, note that the proof of Corollary 2.12 is obvious: indeed, the proof given in the previous section for weak solutions also applies to strong solutions and ensures the convergence of $M(x,t)$ for every $x \in \mathbb{R}$.

One could think that the uniform convergence to 0 of the transverse field $H_{\|}(x,t)$ would yield that the convergence of the magnetization distribution $M(x,t)$ to $M_\infty(x,t)$ is itself uniform. In fact, such a result is not obvious at all, and may not be true. More precisely, we are going to prove, with the help of a suitable counterexample, that it is not possible to prove the uniform convergence of $M(x,t)$ with the only assumption that the convergence of $H_{\|}(x,t)$ is uniform.

To construct this counterexample, we denote by S the unit sphere, set $M_\infty = e_z = (0,0,1)^t$, and $M_{-\infty} = -M_\infty$. We shall need two simple lemmas which apply to the evolution equation

$$\begin{cases} \dot{M}(t) = \tilde{H}_T(M(t)) \times M(t) + \dfrac{\alpha}{|M(t)|} M(t) \times \dot{M}(t), \\ M(t=0) = M_0 \in S, \end{cases} \tag{7.1}$$

where $\tilde{H}_T(M(t)) = -(M(t)\cdot e_x)e_x + 2e_z + H_{\|}(t)$, which corresponds to $H_s = 2e_z$ and $K = 0$. In such a case, one easily verifies that

$$\mathcal{Z} \;=\; \{\, M_\infty, M_{-\infty} \}\,.$$

Lemma 7.1. *Assuming that $H_{\|}(t) = 0$, for any solution $M(t)$ to system* (7.1)*, and for all $M_0 \in S$, we have*

$$(M_0 \neq M_{-\infty}) \Rightarrow \Big(V(M_0) \;>\; V_{\min} = V(M_{-\infty}) \text{ and } \lim_{t\to\infty} M(t) \;=\; M_\infty \Big), \tag{7.2}$$

where

$$\forall M \in S, \quad V(M) \;=\; 2M\cdot e_z - \frac{1}{2}|M\cdot e_x|^2.$$

Remark 7.2. This result means that M_∞ can be a limit state if and only if $M_0 = M_{-\infty}$, in which case the solution is stationary.

Proof. It is easy to check that

$$\max_{M\in S} V(M) \;=\; V_{\max} \;=\; V(M_\infty),$$

and

$$\min_{M\in S} V(M) \;=\; V_{\min} \;=\; (V(M_{-\infty})).$$

This is true as soon as $|H_s| > 1$ (see section 4). Moreover, for the same reason, V admits no other critical point on S. The lemma is then a direct consequence of the fact that V is a strict anti-Liapunov function for the system (7.1). ◻

LEMMA 7.3. *Let $\varepsilon \in]0,1]$, and let us assume that $H_{\|}(t) = \varepsilon e_y$ and $M_0 = M_{-\infty}$. Then, for all $T > 0$, the solution to system* (7.1) *on* $[0,T]$ *is such that*

$$V(M(T)) \;<\; V_{\min} \;=\; V(M_{-\infty})$$

where V is as defined in Lemma 7.1.

Proof. In this case, the total magnetic field is

$$\tilde{H}_T(M) = -(M(t)\cdot e_x)e_x + 2e_z + \varepsilon e_y.$$

The associated strict anti-Liapunov function is

$$\forall M \in S,\; V'(M) \;=\; -\frac{1}{2}|M\cdot e_x|^2 + 2M\cdot e_z + \varepsilon M\cdot e_y, \tag{7.3}$$

which is such that

$$\min_{M\in S} V'(M) \;=\; V'_{\min} \;=\; V'(M_{-\infty}) \;=\; V(M_{-\infty}),$$

where V has been defined in Lemma 7.1, and

$$\max_{M\in S} V'(M) \;=\; V'_{\max} \;=\; V'(M_{\infty}) \;=\; V(M_{\infty}).$$

Thus,

$$V'(M(T)) \;>\; V'_{\min} \;=\; V(M_{\infty}). \tag{7.4}$$

Let us define

$$\mathcal{C}' \;=\; \{M \in S \setminus V'(M) \;=\; V'(M(T))\} \tag{7.5}$$

and

$$V_T \;=\; \min_{M\in \mathcal{C}'} V(M). \tag{7.6}$$

As $V'_{\min} = V_{\min}$ is reached in M_∞ only, we see that

$$(\, V_T = V_{\min}\,) \implies (\, V'(M(T)) = V'_{\min}\,), \tag{7.7}$$

which is not true; whence the result, since

$$V'(M(T)) \;\geq\; V_T \;>\; V_{\min}. \tag{7.8}$$ ◻

We can now state the following theorem.

THEOREM 7.4. *Let $\Omega = [0,1]$ be a ferromagnetic layer defined by the initial distribution*

$$M_0(x) \;=\; M_{-\infty} \quad \forall x \in [0,1],$$

$H_s = 2e_z$, *and $K = 0$. Let us consider a transverse magnetic field defined by*

$$\left|\begin{array}{lcll} H_{\|}(x,t) & = & 0 & \forall x \notin [0,1]\,, \\ & = & 0 & \forall t \notin \left[\dfrac{1}{x}, \dfrac{2}{x}\right], \\ & = & x & \forall t \in \left[\dfrac{1}{x}, \dfrac{2}{x}\right]. \end{array}\right.$$

Then, $H_{\|}(x,t)$ *converges uniformly to* 0 *when* t *goes to* $+\infty$; *however,*

(i) $\lim_{t\to\infty} M(0,t) = M_{-\infty}$;

(ii) $\forall x \in]0,1]$, $\lim_{t\to\infty} M(x,t) = M_{\infty}$;

(iii) $\forall \varepsilon \in [0,2[$ $\forall T > 0$, $\exists x \in]0,1]$, $\exists t > T$ *such that*

$$\left| M(x,t) - \lim_{t\to\infty} M(x,t) \right| = |\, M(x,t) - M_{\infty} \,| = 2 > \varepsilon.$$

Proof. First of all, it is easy to see that

$$\sup_{x\in]0,1]} H_{\|}(x,t) = \frac{2}{t} \tag{7.9}$$

which ensures that the convergence of $H_{\|}$ to 0 in time is uniform in space.

Concerning the three assertions of the theorem, result (i) is clear since $H_{\|}(0,t) = 0$ for all $t \geq 0$. For (ii) and (iii), let us consider $\varepsilon \in [0,2[$ and $T > 0$. We choose $x \in]0,1]$ such that $1/x > T$. Then, on the one hand,

$$\forall t \in [0,T], \quad H_{\|}(x,t) = 0 \text{ and } M(x,t) = M_{-\infty}. \tag{7.10}$$

On the other hand, by Lemma 7.3,

$$V\left(M\left(x, t = \frac{2}{x}\right)\right) > V_{\min} \tag{7.11}$$

and thus, by Lemma 7.1,

$$\lim_{t\to\infty} M(x,t) = M_{\infty} \qquad \square \tag{7.12}$$

Remark 7.5. The previous example is only a counterexample to the fact that the uniform convergence of M could be proven by using uniform convergence of $H_{\|}$. It is not a counterexample to the uniform convergence of M: indeed, it is not a solution of the coupled Maxwell–LLG system.

Remark 7.6. One might also believe that $M(t)$ always converges to a stable stationary state. This is not so obvious, and may not be true. In fact, one can show that a "perturbation" $H_{\|}(x,t)$—that is to say, a function C^1 in time vanishing to 0 with $t \to \infty$— can lead the magnetization M from a stable position to an unstable one. Let us consider

$$\begin{cases} \dot{\mu}(t) = -\,\mu(t) \times (H_T(\mu(t)) \times \mu(t)), \\ \mu(t=0) = \mu_0 \in S, \end{cases} \tag{7.13}$$

where $H_T(\mu) = -(\mu \cdot e_x)e_x + 2e_z$. The solution to this problem is such that

$$\forall \mu_0 \neq M_{\infty}, \quad \lim_{t\to\infty} \mu(t) = M_{-\infty}, \tag{7.14}$$

because $V(\mu)$ is a strict anti-Liapunov function for (7.13). Let us now define the function $h(t)$ as

$$h(t) = \frac{1}{1+\alpha^2} H_T(\mu(t)) \times \mu(t) - \frac{1+\alpha+\alpha^2}{1+\alpha^2} \mu(t) \times (H_T(\mu(t)) \times \mu(t)). \tag{7.15}$$

This function is such that

(i) $h(t) \in C^1(\mathbb{R})$ because $\mu(t) \in C^1(\mathbb{R})$;

(ii) $h(t) \to 0$ when $t \to \infty$ because $\mu(t) \times H_T(\mu(t)) \to 0$ when $t \to \infty$;

(iii) additionally, computations lead to

$$(H_T(\mu(t)) + h(t)) \times \mu(t) + \alpha\mu(t) \times [(H_T(\mu(t)) + h(t)) \times \mu(t)] = \\ - \mu(t) \times (H_T(\mu(t)) \times \mu(t)).$$

In other words, function $\mu(t)$, the solution to problem (7.13), is also the solution to problem (7.1) with the perturbation $H_{\parallel}(t) = h(t)$.

REFERENCES

[1] W. B. Brown, *Micromagnetics*, Interscience Tracts of Physics, Wiley Intescience, New York, 1963.

[2] B. Lax and K. J. Button, *Microwave Ferrites and Ferrimagnetics*, McGraw-Hill, New York, 1962.

[3] W. L. Miranker and B. E. Willner, *Global analysis of magnetic domains*, Quart. Appl. Math., 37 (1979/80), pp. 219–238.

[4] A. Visintin, *On Landau–Lifschitz' equations for ferromagnetism*, Japan J. Appl. Math., 2 (1985), pp. 69–84.

[5] J. L. Joly, G. Métivier, and J. Rausch, *On Landau–Lifschitz' Equations for Ferromagnetism*, in preparation.

[6] A. Haraux, *Nonlinear Evolution Equations–Global Behavior of Solutions*, Springer-Verlag, Berlin, 1981.

[7] P. Joly and O. Vacus, *Maxwell's Equations in a* 1*D Ferromagnetic Medium: Existence and Uniqueness of Strong Solutions*, Rapport de Recherche INRIA 3052, France, 1997.

[8] P. Joly and O. Vacus, *Propagation d'ondes en milieu ferromagnétique* 1*D: Existence et unicité de solutions faibles*, Note soumise au Compte Rendu de l'Académie des Sciences, 1996.

[9] P. D. Lax, C. S. Morawetz, and R. S. Phillips, *Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle*, Comm. Pure Appl. Math., 16 (1963), pp. 477–486.

[10] C. S. Morawetz, *The decay of solutions of the exterior initial-boundary value problem for the wave equation*, Comm. Pure Appl. Math., 14 (1961), pp. 561–568.

[11] C. S. Morawetz and W. A. Strauss, *Decay and scattering of solutions of a nonlinear relativistic wave equation*, Comm. Pure Appl. Math., 25 (1972), pp. 1–31.

[12] A. Komech, *On transitions to stationary states in some infinite dimensional Hamiltonian systems*, Dokl. Math., 53 (1996), pp. 208–210.

[13] A. I. Komech, *On the stabilization of string-oscillators interaction*, Russian J. Math. Phys., 3 (1995), pp. 227–248.

[14] A. I. Komech and H. Spohn, *Soliton-like asymptotics for a classical particle interacting with a scalar wave field*, Nonlinear Anal., 33 (1998), pp. 13–24.

[15] A. I. Komech, H. Spohn, and M. Kunze, *Long-time asymptotics for a classical particle interacting with a scalar wave field*, Comm. Partial Differential Equations, 22 (1997), pp. 307–335.

[16] A. I. Komech, *On stabilization of string-nonlinear oscillator interaction*, J. Math. Anal. Appl., 196 (1995), pp. 384–409.

[17] A. I. Komech and B. R. Vainberg, *On asymptotic stability of stationary solutions to nonlinear wave and Klein-Gordon equations*, Arch. Rational Mech. Anal., 134 (1996), pp. 227–248.

[18] P. D. Lax and R. S. Phillips, *Scattering Theory*, Academic Press, New York, 1967.