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Relation between Cauchy Data for the Scattering by a Wedge

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Abstract. We justify an extension of the method of complex characteristics [6] for the Helmholtz equation in nonconvex angles. For convex angles, the method was introduced in [1] and developed in [6, 11].

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1. INTRODUCTION

This paper concerns a justification of the method of complex characteristics [1, 2, 6, 10, 11] for elliptic equations in nonconvex angles. This method finds applications to diverse problems of mathematical physics both for convex angles [5, 12] and for nonconvex ones [7, 8, 13]. The crucial part of the method is played by the *connection equation* on the Riemann surface of complex characteristics of the given elliptic operator. The connection equation generalizes well-known relations on the real characteristics of hyperbolic equations. A preliminary version of the paper was published in [4] in a sketched form. Here we present a complete exposition.

The principal object of our investigation is the Helmholtz equation

$$(\Delta+\omega^2)u(y)=0, \qquad y\in Q, \tag{1.1}$$

where Q is a nonconvex angle of magnitude $\alpha\in(\pi,2\pi)$ and

$$\omega:=\omega_1+i\omega_2\in\mathbb{C}^+:=\{z\in\mathbb{C}:\operatorname{Im} z>0\}.$$

In the polar coordinates ($y_1=\rho\cos\theta$, $y_2=\rho\sin\theta$), the angle Q is represented in the form

$$Q=\{(\rho,\theta):\beta<\theta<2\pi\}, \qquad \beta=2\pi-\alpha \qquad \text{(see Fig. 1)}.$$

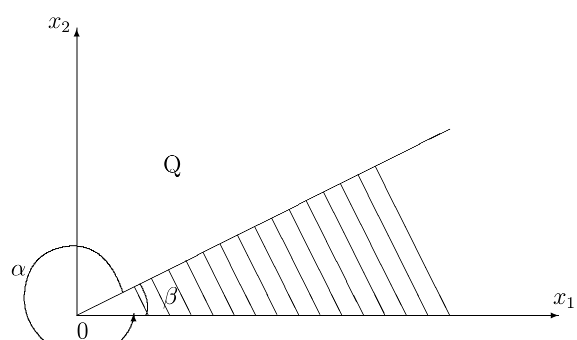

Fig. 1. Nonconvex angle.

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Equation (1.1) with $\omega \in \mathbb{C}^+$ arises when applying the Fourier–Laplace transform to nonstationary scattering problems on a wedge. Analysis of the stationary equation (1.1) enabled us to prove in [7, 8] the existence and uniqueness results and the principle of limiting amplitude for the corresponding nonstationary scattering problem. The central role in the proofs is played by the connection equation involving the Cauchy data of a given solution to (1.1). Note that the connection equation in the papers [7, 8] was applied to specific boundary-value conditions of Dirichlet and Neumann. On the other hand, this connection equation enables one to extend the method of complex characteristics [6] to equation (1.1) in nonconvex angles with general boundary-value conditions.

The connection equation first was found in [1, 2, 6] for convex angles, where this equation occurred as a direct consequence of the Paley–Wiener theorem. However, the extension of the equation to nonconvex angles is not straightforward. For nonconvex angles, the connection equation was stated for the first time in [4] for $\omega = i$ and $\alpha = \pi i/2$. In the present paper, we state and justify the connection equation for general values of $\omega \in \mathbb{C}^+$ and $\alpha \in (\pi, 2\pi)$. We prove that the connection equation is a necessary condition for the existence of solutions in the class of tempered distributions.

The connection equation is an algebraic equation on the Riemann surface of complex characteristics of equation (1.1). The justification of the connection equation is the main goal of the present paper. Namely, let $u_0(y)$ be the extension by zero of a solution $u(y)$ from $y \in Q$ to $y \in \mathbb{R}^2$. Then we have (in the sense of distributions)

$$(\Delta + \omega^2)u_0(y) = v_1(y) + v_2(y), \qquad y \in \mathbb{R}^2, \tag{1.2}$$

where v_1 and v_2 stand for tempered distributions supported by different sides of the angle Q, which can be expressed in the Cauchy data of $u(y)$. The Fourier transform of (1.2) gives (1.3),

$$(-z^2 + \omega^2)\hat{u}_0(z) = \hat{v}_1(z) + \hat{v}_2(z), \qquad z \in \mathbb{R}^2. \tag{1.3}$$

First let us consider the case of a convex angle Q corresponding to $\alpha < \pi$. Then, by the Paley–Wiener theorem, identity (1.2) can be extended to complex values of

$$z \in \mathbb{C}Q^* := \{z \in \mathbb{C}^2 : \ \operatorname{Im} z \cdot y > 0, \ y \in Q\},$$

and $\hat{u}_0(z)$, $\hat{v}_1(z)$, $\hat{v}_2(z)$ are analytic on $\mathbb{C}Q^*$. Write $V = \{z \in \mathbb{C}Q^* : \ -z^2 + \omega^2 = 0\}$, which is the Riemann surface of complex characteristics of the Helmholtz equation $\Delta + \omega^2$. Then (1.3) implies the "connection equation" (1.4),

$$\hat{v}_1(z) + \hat{v}_2(z) = 0, \qquad z \in V^*, \tag{1.4}$$

where $V^* := V \cap \mathbb{C}Q^*$.

On the other hand, in diffraction problems with $\alpha > \pi$, the angle Q is not convex. Hence, the set $\mathbb{C}Q^*$ is empty, and therefore (1.4) is meaningless. Now $W := \mathbb{R}^2 \setminus Q$ and $W_- := -W$ are convex angles. Therefore, the Paley–Wiener theorem implies that the functions $\hat{v}_1(z)$ and $\hat{v}_2(z)$ are analytic on $\mathbb{C}W^*$ but does not imply their analyticity on $\mathbb{C}W_-^*$. The main result of the present paper states that, for a nonconvex angle Q, the connection equation (1.4) holds with $V^* = V \cap \mathbb{C}W_-^*$ for the *analytic continuations* of $\hat{v}_1$ and $\hat{v}_2$ *along the Riemann surface* V.

The boundary value problems in convex angles were treated by Sobolev [16] and Shilov [15]. In the last paper, some necessary relations between the Cauchy data were found. The relations coincide with our connection equation on a curve lying on the Riemann surface. In Sobolev's paper [16], some necessary conditions on the Cauchy data were also suggested.

The connection equation for convex angles was applied in [6, 12] to solve the Ursell problem on the completeness of trapping modes on a sloping beach. The connection equation for nonconvex angles was applied in [7, 8, 13] to justify and generalize the Sommerfeld–Malyuzhinets-type representation for solutions to diffraction problems on wedges. The representation plays an important role in [7, 8] when proving the uniqueness and existence results and the limiting amplitude principle for diffraction problems.

The paper is organized as follows. In Sections 1–6, we present the basis of the general method of complex characteristics [1, 6]. In Section 7, we obtain an integral equation for the complex Fourier transform of the Cauchy data. In Section 8, we reduce the integral equation to the Riemann surface of the complex characteristics. In Section 9, we construct a special class of test functions. In Section 10, we construct a Cauchy kernel on a Riemann surface. In Sections 11–14, we reduce the integral equation to the connection equation, which is algebraic, by using analytic continuation on the Riemann surface. In the appendix, we construct a class of Schwartz functions with a special property (see [14]).

2. REDUCTION TO THE FIRST QUADRANT

The method of complex characteristics requires several steps. Our first goal is to extend equation (1.1) to $\mathbb{R}^2$.

In this section, we reduce equation (1.1) to the equation in the complement of the first quadrant. Make a linear change of variables

$$x_1 = x + y\cot\alpha, \qquad x_2 = -\frac{y}{\sin\alpha} \qquad \text{(see Fig. 2)}$$

which transforms Q to $K_- := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0 \text{ or } x_2 < 0\}$.

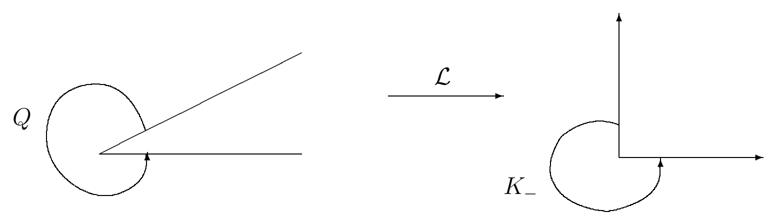

Fig. 2.

Represent the operator $\Delta + \omega^2$ in the variables (x_1, x_2),

$$(\Delta + \omega^2) = \frac{1}{\sin^2\alpha}\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - 2\frac{\partial^2}{\partial x_1 \partial x_2}\cos\alpha\Big) + \omega^2.$$

Equation (1.1) is now equivalent to

$$\Big(\Delta - 2\frac{\partial^2}{\partial x_1 \partial x_2}\cos\alpha + \omega^2\sin^2\alpha\Big)u(x) = 0, \qquad x \in K_-. \tag{2.1}$$

The solution is sought in the class $S'(K_-)$ of tempered distributions on K_-, which is the space of restrictions of the tempered distributions in $\mathbb{R}^2$ to K_-.

Everywhere below, denote the operator (2.1) as

$$\mathcal{H} = \Delta - 2\frac{\partial^2}{\partial x_1 \partial x_2}\cos\alpha + \omega^2\sin^2\alpha. \tag{2.2}$$

3. EXTENSION TO THE PLANE

We state two lemmas concerning nonconvex angles Q. The proof is similar to the case of convex angles Q, which was treated in [6].

Lemma 3.1. *Let $u(x) \in S'(K_-)$ be a solution to* (2.1). *Then the following assertions hold.*
i) *There is a distribution $u_0(x) \in S'(\mathbb{R})$ such that*

$$u_0(x) := \begin{cases} u(x), & x \in \overline{K}_-, \\ 0, & x \notin \overline{K}_-. \end{cases} \tag{3.1}$$

ii) *The traces*

$$\begin{aligned} u_1^0(x_1) &:= u(x_1, -0), \quad x_1 > 0; \qquad u_2^0(x_2) := u(0, -x_2), \quad x_2 > 0; \\ u_1^1(x_1) &:= \partial_2 u(x_1, -0), \quad x_1 > 0; \qquad u_2^1(x_2) := \partial_1 u(-0, x_2), \quad x_2 > 0, \end{aligned}$$

exist in the sense of distributions with respect to $x_1 > 0$, $x_2 > 0$.

Now we apply the operator $\mathcal{H}$ to the function $u_0(x)$ in the sense of $S'(\mathbb{R}^2)$.

Lemma 3.2. *Let $u \in S'(K_-)$ be a solution to* (2.1). *Then the following assertions hold.*
i) *There is an $u_0(x) \in S'(\mathbb{R}^2)$ satisfying* (3.1), *and*

$$\mathcal{H}u_0(x) = d_0(x), \qquad x \in \mathbb{R}^2, \tag{3.2}$$

in the sense of distributions, where

$$\begin{aligned} d_0(x_1, x_2) = -\delta(x_1) v_2^1(x_2) - \delta(x_2) v_1^1(x_1) - \delta'(x_1) v_2^0(x_2) - \delta'(x_2) v_1^1(x_1) \\ + 2\cos\alpha \cdot \delta(x_1)\partial_2 v_2^0(x_2) + 2\cos\alpha \cdot \delta(x_2)\partial_1 v_1^0(x_1). \end{aligned} \tag{3.3}$$

ii) *The distributions $v_l^\beta(x_l)$ with respect to $x_l \in \mathbb{R}$ are tempered, and*

$$\left.\begin{aligned} v_l^\beta(x_l) &= u_l^\beta(x_l), \quad x_l > 0 \\ v_l^\beta(x_l) &= 0, \qquad\quad x_l < 0 \end{aligned}\right| \quad l = 1, 2;\ \beta = 0, 1.$$

4. COMPLEX FOURIER TRANSFORM

4.1. *Fourier Transform*

Now let us apply the Fourier transform to (3.2),

$$\hat{\mathcal{H}}(\xi) \cdot \hat{u}_0(\xi) = \hat{d}_0(\xi), \qquad \xi \in \mathbb{R}^2. \tag{4.1}$$

Here $\hat{\mathcal{H}}(\xi)$ is the symbol,

$$\hat{\mathcal{H}}(\xi_1, \xi_2) = -\xi_1^2 - \xi_2^2 + 2\xi_1\xi_2\cos\alpha + \omega^2\sin^2\alpha, \qquad (\xi_1, \xi_2) \in \mathbb{R}^2, \tag{4.2}$$

of the differential operator $\mathcal{H}$ given by (2.2), and

$$\hat{d}_0(\xi_1, \xi_2) = \hat{v}_1^0(\xi_1)(i\xi_2 - 2i\xi_1\cos\alpha) - \hat{v}_1^1(\xi_1) + \hat{v}_2^0(\xi_2)(i\xi_1 - 2i\xi_2\cos\alpha) - \hat{v}_2^1(\xi_2), \qquad \xi \in \mathbb{R}^2,$$

by (3.3). For any $\omega \in \mathbb{C}^+$, there exists a $\delta_* = \delta_*(\omega) > 0$ such that

$$|\hat{\mathcal{H}}(z)| \geqslant C, \quad z \in \mathbb{C}^2, \qquad |\operatorname{Im} z_k| < \delta_*, \quad k = 1, 2, \tag{4.3}$$

for some $C = C(\omega) > 0$. Without loss of generality, we can assume that

$$\delta_* < \omega_2 := \operatorname{Im}\omega. \tag{4.4}$$

4.2. *Paley–Wiener Argument*

As is well known, the real characteristics of a differential operator provide relations between the Cauchy data of solutions. In our case, the operator $\mathcal{H}$ has no real characteristics, according to (4.3). On the other hand, any polynomial has some complex zeros and, in particular, the symbol $\hat{\mathcal{H}}$ vanishes on a Riemann surface in $\mathbb{C}^2$. We claim that the complex characteristics of the operator $\mathcal{H}$ provide the "connection equation" for analytic continuations of the Cauchy data of solutions.

Proposition 4.1. i) *The function* $\tilde{d}_0(z) := \langle d_0(x), e^{ixz}\rangle$ *is analytic for* $z \in \mathbb{C}K^*_+$.

ii) *The following bound holds*:

$$|\tilde{d}_0(z)| \leqslant C(1+|z|)^{\mu}\rho^{-\nu}, \qquad z \in \mathbb{C}K^*_+, \tag{4.5}$$

where $\rho := \min(\tau_1, \tau_2)$ *and* $\tau = \operatorname{Im} z \in K_+$.

iii) *The distribution* $\hat{d}_0(\xi)$ *is a trace of an analytic function, i.e.,*

$$\tilde{d}_0(\xi + i\tau) \longrightarrow \hat{d}_0(\xi), \qquad \tau \longrightarrow 0, \tag{4.6}$$

where $\tau \in K_+$, *and the convergence holds in the sense of tempered distributions with respect to* $\xi \in \mathbb{R}^2$.

Proof. Let us prove assertions i) and iii). By Lemma 3.2 and the Paley–Wiener theorem, the distributions $\hat{v}^{\beta}_l(\xi_l) \in S'(\overline{\mathbb{R}^+})$ are the traces of the functions $\tilde{v}^{\beta}_l(z_l)$ analytic on $z_l \in \mathbb{C}^+$, and

$$|\tilde{v}^{\beta}_l(z_l)| \leqslant C(1+|z_l|)^{\mu}\tau_l^{-\nu}, \qquad \tau_l = \operatorname{Im} z_l > 0, \tag{4.7}$$

where $\mu, \nu \geqslant 0$. Therefore, the function $\hat{d}_0(\xi_1, \xi_2)$ is the trace of the following analytic function on the domain $\mathbb{C}K^*_+ = \mathbb{R}^2 \oplus iK_+$:

$$\tilde{d}_0(z_1, z_2) = \tilde{v}^0_1(z_1)(iz_2 - 2iz_1\cos\alpha) - \tilde{v}^1_1(z_1) + \tilde{v}^0_2(z_2)(iz_1 - 2iz_2\cos\alpha) - \tilde{v}^1_2(z_2), \qquad (z_1, z_2) \in \mathbb{C}K^*_+. \tag{4.8}$$

ii) The bounds (4.7) imply the corresponding bound (4.5) for $\tilde{d}_0(z_1, z_2)$.

Note that

$$\tilde{d}_0(z) = \tilde{v}_1(z) + \tilde{v}_2(z), \qquad z \in \mathbb{C}K^*_+, \tag{4.9}$$

where

$$\tilde{v}_1(z) = \tilde{v}^0_1(z_1)(iz_2 - 2iz_1\cos\alpha) - \tilde{v}^1_1(z_1), \qquad z \in \mathbb{C}^+ \times \mathbb{C}, \tag{4.10}$$

$$\tilde{v}_2(z) = \tilde{v}^0_2(z_2)(iz_1 - 2iz_2\cos\alpha) - \tilde{v}^1_2(z_2), \qquad z \in \mathbb{C} \times \mathbb{C}^+. \tag{4.11}$$

The functions $\tilde{v}_1(z)$ and $\tilde{v}_2(z)$ satisfy the bounds

$$|\tilde{v}_1(z)| \leqslant C(1+|z|)^{\mu}|\tau_1|^{-\nu}, \qquad z \in \mathbb{C}^+ \times \mathbb{C}, \tag{4.12}$$

$$|\tilde{v}_2(z)| \leqslant C(1+|z|)^{\mu}|\tau_2|^{-\nu}, \qquad z \in \mathbb{C} \times \mathbb{C}^+, \tag{4.13}$$

where $\mu, \nu \geqslant 0$, $l = 1, 2$.

The main goal of present paper is to prove that the function $\tilde{v}_1(z)$ is analytic continuation of $-\tilde{v}_2(z)$ along the Riemann surface of complex characteristics of the operator $\mathcal{H}$.

5. RIEMANN SURFACE OF COMPLEX CHARACTERISTICS

Denote by $V = V(\omega)$ the set of the complex zeros of symbol $\hat{\mathcal{H}}$ (see (4.2))

$$V(\omega) = \{(z_1, z_2) \in \mathbb{C}^2 : \hat{\mathcal{H}}(z) = -z_1^2 - z_2^2 + 2z_1 z_2 \cos\alpha + \omega^2 \sin^2\alpha = 0\}. \tag{5.1}$$

The Riemann surface V is isomorphic to a cylinder [6]. Therefore, the universal covering $\check{V}$ of the surface V is isomorphic to $\mathbb{C}$. Note that

$$(z_1 \sin\alpha)^2 + (z_2 - z_1\cos\alpha)^2 = z_1^2 + z_2^2 - 2\cos\alpha z_1 z_2 = \omega^2 \sin^2\alpha, \qquad (z_1, z_2) \in V,$$

for $(z_1, z_2) \in V$. This suggests the idea to introduce the parametrization of the universal covering surface $\check{V}$,

$$z_1 := \omega \sin\varphi, \qquad z_2 - z_1 \cos\alpha := \omega \sin\alpha \cos\varphi, \qquad \varphi \in \mathbb{C}.$$

Make the change of the variable $\varphi \to w = i\varphi$. Then

$$z_1(w) = -i\omega \sinh w, \qquad z_2(w) = -i\omega \sinh(w + i\alpha). \tag{5.2}$$

Thus, we have the following parametrization of $\check{V}$:

$$z_1(w) := -i\omega \sinh w, \qquad z_2(w) := -i\omega \sinh(w + i\alpha), \qquad w \in \mathbb{C}. \tag{5.3}$$

Denote by p the projection $p\colon \check{V} \longrightarrow V$ defined by

$$p(w) := (z_1(w), z_2(w)). \tag{5.4}$$

Moreover, consider the projections $p_l\colon \check{V} \longrightarrow \mathbb{C}$ defined by

$$p_l(w) = z_l(w), \qquad l = 1, 2.$$

Let us "lift" the functions $\tilde{v}_l^\beta(z_l)$ to $\check{V}$ by the projections (5.4). Since the domain of $\tilde{v}_l^\beta(z_l)$ is $\mathbb{C}^+$, we introduce the corresponding domains on $\check{V}$,

$$\check{\mathcal{V}}_l^+ := p_l^{-1}(\mathbb{C}^+) = \{w \in \mathbb{C} : \ \operatorname{Im} z_l > 0\}, \qquad l = 1, 2.$$

Let $\check{V}_1^+$ be the connected component of $\check{\mathcal{V}}_1^+$ which contains the point $w = i\pi/2$ and let $\check{V}_2^+$ be the connected component of $\check{\mathcal{V}}_2^+$ which contains the point $w = i(\pi/2 - \alpha)$. We can readily see that

$$\begin{aligned} \check{V}_1^+ &= \{w \in \mathbb{C} : \qquad -\pi/2 < \operatorname{Im} w < 3\pi/2, \qquad \operatorname{Im} z_1(w) > 0\}, \\ \check{V}_2^+ &= \{w \in \mathbb{C} : -\pi/2 - \alpha < \operatorname{Im} w < 3\pi/2 - \alpha, \ \operatorname{Im} z_2(w) > 0\}, \end{aligned} \tag{5.5}$$

and $\partial \check{V}_l^+ = \check{\Gamma}_l^+ \cup \check{\Gamma}_l^-$ for $l = 1, 2$, where

$$\begin{aligned} \check{\Gamma}_1^+ &= \{w \in \mathbb{C} : \operatorname{Im} z_1(w) = 0, \qquad \pi/2 < \operatorname{Im} w < 3\pi/2\}, \\ \check{\Gamma}_1^- &= \{w \in \mathbb{C} : \operatorname{Im} z_1(w) = 0, \qquad -\pi/2 < \operatorname{Im} w < \pi/2\}, \\ \check{\Gamma}_2^+ &= \{w \in \mathbb{C} : \operatorname{Im} z_2(w) = 0, \quad -\pi/2 - \alpha < \operatorname{Im} w < \pi/2 - \alpha\}, \\ \check{\Gamma}_2^- &= \{w \in \mathbb{C} : \operatorname{Im} z_2(w) = 0, \qquad \pi/2 - \alpha < \operatorname{Im} w < 3\pi i/2 - \alpha\}. \end{aligned}$$

We also introduce the domains

$$\check{\mathcal{V}}_l^- := p_l^{-1}(\mathbb{C}^-) = \{w \in \mathbb{C} : \ \operatorname{Im} z_l < 0\}, \qquad l = 1, 2.$$

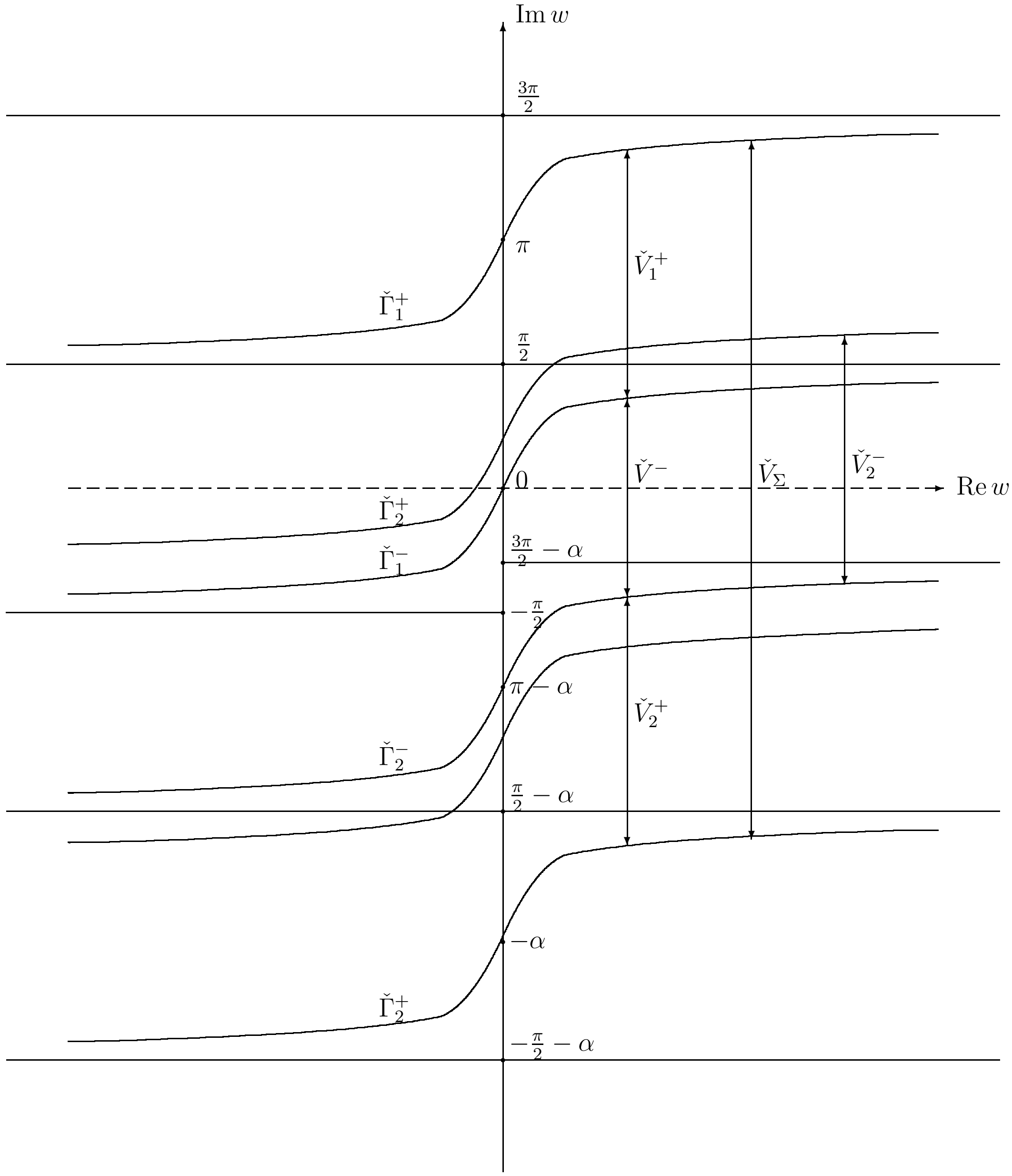

Fig. 3. Domains on the Riemann surface.

Let $\check{V}_l^-$ be the connected component of $\check{\mathcal{V}}_l^-$ which contains the point $w = -im$ with $m := \min(\pi/2, \pi-\alpha)$ for $l = 1, 2$. Similarly to (5.5), one can see that

$$\check{V}_1^- = \{w \in \mathbb{C} : \quad -3\pi/2 < \operatorname{Im} w < \pi/2, \quad \operatorname{Im} z_1(w) < 0\},$$
$$\check{V}_2^- = \{w \in \mathbb{C} : \quad \pi/2-\alpha < \operatorname{Im} w < 5\pi/2-\alpha, \quad \operatorname{Im} z_2(w) < 0\}.$$

Write

$$\check{V}^- := \check{V}_1^- \cap \check{V}_2^-.$$

For $w = w_1 + iw_2$ and $\omega = \omega_1 + i\omega_2$, we have

$$\operatorname{Im}[z_1(w)] = \omega_2 \cosh w_1 \sin w_2 - \omega_1 \sinh w_1 \cos w_2$$

by (5.2). Hence, $\operatorname{Im}[z_1(w_1 + iw_2)] = 0$ if and only if $\tan w_2 = \frac{\omega_1}{\omega_2} \tanh w_1$, and thus

$$\check{\Gamma}_1^- = \left\{ w = w_1 + iw_2 \;\middle|\; w_1, w_2 \in \mathbb{R},\ w_2 = \arctan\left(\frac{\omega_1}{\omega_2} \tanh w_1\right) \right\}.$$

The contours can be obtained from one another by translations,

$$\check\Gamma_1^+ = \check\Gamma_1^- + \pi i, \qquad \check\Gamma_2^+ = \check\Gamma_1^- - i\alpha, \qquad \check\Gamma_2^- = \check\Gamma_2^+ + \pi i. \tag{5.6}$$

For $\nu \in \mathbb{R}$, we define the contour

$$\gamma(\nu) := \check\Gamma_1^- + i\nu.$$

In this case, the contours (5.6) can be represented as follows:

$$\check\Gamma_1^+ = \gamma(\pi), \quad \check\Gamma_1^- = \gamma(0), \quad \check\Gamma_2^+ = \gamma(-\alpha), \quad \check\Gamma_2^- = \gamma(\pi - \alpha).$$

Write $\check V_\Sigma := \check V_1^+ \cup \overline{\check V^-} \cup \check V_2^+$ (see Fig. 3, which corresponds to the case of $\operatorname{Re}\omega > 0$).

Using the definitions of $\check V_l^-$, $\check V_l^+$, $\check V^-$, and $\check V_\Sigma$, we represent the boundaries of the domains as follows:

$$\begin{aligned} \partial\check V_1^+ &= \gamma(0)\cup\gamma(\pi), & \partial\check V_2^+ &= \gamma(-\alpha)\cup\gamma(\pi-\alpha)),\\ \partial\check V_1^- &= \gamma(\pi-\alpha)\cup\gamma(0), & \partial\check V_2^- &= \gamma(\pi-\alpha)\cup\gamma(2\pi-\alpha),\\ \partial\check V^- &= \gamma(\pi-\alpha)\cup\gamma(0), & \partial\check V_\Sigma &= \gamma(-\alpha)\cup\gamma(\pi). \end{aligned}$$

6. CONNECTION EQUATION

In this section, we write out the connection equation, which is a relation between the functions $\check v_l^\beta$ on the universal covering $\check V$.

Denote by $H(\Omega)$ the set of analytic functions on an open set $\Omega \subset \mathbb{C}^n$. Let $v(z_l) \in H(\mathbb{C}^+)$, $l = 1, 2$. Define a lifting $\check v(w)$ of the function $\check v(z_l)$ as the composition:

$$\check v(w) = v(z_l(w)),$$

where the functions $z_l(w)$ are defined by (5.3). The analyticity of the functions $\tilde v_l^\beta$ on $\mathbb{C}^+$ implies the analyticity of the functions $\check v_l^\beta$ on $\check V_l^+$,

$$\check v_l^\beta(w) \in H(\check V_l^+), \qquad l = 1, 2, \quad \beta = 0, 1.$$

Using (5.2), we obtain the expressions

$$\begin{aligned} \check v_1(w) &= -\check v_1^1(w) - \omega\sinh(w - i\alpha)\ \check v_1^0(w), \qquad & w \in \check V_1^+,\\ \check v_2(w) &= -\check v_2^1(w) - \omega\sinh(w + 2i\alpha)\ \check v_2^0(w), \qquad & w \in \check V_2^+, \end{aligned} \tag{6.1}$$

from (4.10) and (4.11). Obviously,

$$\check v_l(w) \in H(\check V_l^+), \qquad l = 1, 2. \tag{6.2}$$

6.1. *Convex Angles*

Recall the connection equation in the case of a convex angle (with $\alpha < \pi$). In this case, the connection equation can be obtained by using the Paley–Wiener theorem [1, 2, 6, 10]. Namely, consider the Helmholtz equation

$$\mathcal{H}(D)u(x) = 0, \qquad x \in K_+,$$

for the solutions $u(x) \in C^\infty(\overline{K}_+)$. Then, denoting by $u_0(x)$ the extension of u by zero outside K_+, we obtain

$$\mathcal{H}(D)u_0(x) = -d_0(x), \qquad x \in \mathbb{R}^2,$$

where $d_0(x)$ is expressed in terms of the Cauchy data of $u(x)$ by formula (3.3). Paley–Wiener theorem yields now that

$$\hat{\mathcal{H}}(z)\tilde u_0(z) = -\tilde d_0(z), \qquad z \in \mathbb{C}K_+^*,$$

which is an identity for analytic functions. Since $\hat{\mathcal{H}}(z) = 0$ for $z \in V \cap \mathbb{C}K_+^*$, we obtain the connection equation

$$\tilde d_0(z) = 0, \qquad z \in V \cap \mathbb{C}K_+^*,$$

which is a relation between the Cauchy data (by formula (3.3) for $d_0(x)$).

6.2. *Nonconvex Angles*

In the case of $\alpha > \pi$, the situation is more complicated since the Paley–Wiener theorem is no longer applicable. Namely, denote by $\check{v}_l^\Sigma(w)$ the analytic extension of $\check{v}_l(w)$ to the complex domain $\check{V}_\Sigma$ if such an extension exists (see Fig. 3). The main goal of the paper is to establish the following connection equation.

Let $u(x,y) \in S'(Q)$ be a solution to equation (1.1) with $\omega \in \mathbb{C}^+$. We also assume that the functions $\check{v}_1$ and $\check{v}_2$ are defined by (6.1).

Theorem 6.1 (on the connection equation). i) *The function $\check{v}_1(w)$ admits the analytic continuation $\check{v}_1^\Sigma$ from $\check{V}_1^+$ to $\check{V}_\Sigma$, and the function $\check{v}_2(w)$ admits the analytic continuation $\check{v}_2^\Sigma$ from $\check{V}_2^+$ to $\check{V}_\Sigma$.*

ii) *The following identity holds for these analytic continuations*:

$$\check{v}_1^\Sigma(w) + \check{v}_2^\Sigma(w) = 0, \qquad w \in \check{V}_\Sigma.$$

We prove the theorem in the remaining part of the paper. It suffices to prove the theorem for $\omega_2 = 1$ (since the proof is similar for an arbitrary $\omega_2 > 0$). In this case, (4.4) reads

$$\delta_* < 1. \tag{6.3}$$

7. INTEGRAL CONNECTION EQUATION

In this section, we obtain the connection equation in an integral form. Dividing (4.1) by $\hat{\mathcal{H}}(\xi)$, we see that

$$\hat{u}_0(\xi) = \frac{\hat{d}_0(\xi)}{\hat{\mathcal{H}}(\xi)}, \qquad \xi \in \mathbb{R}^2, \tag{7.1}$$

where the division is well defined in the sense of tempered distributions by (4.3). We are going to obtain the connection equation for $d_0(x)$ by using the fact that

$$u_0(x) = 0 \quad \text{for} \quad x \in K_+.$$

Let us introduce an appropriate class of test functions supported by K_+.

Definition 7.1. Let

$$S(K_+) := \{\psi \in S(\mathbb{R}^2) : \operatorname{supp}\psi \subset \overline{K_+}\}.$$

Let us introduce a subspace of the space $S(K_+)$. For a $\delta > 0$, write

$$\mathbb{C}_\delta^+ := \{z \in \mathbb{C} : \operatorname{Im} z > -\delta\}, \qquad \mathbb{C}_\delta^- := -\mathbb{C}_\delta^+.$$

Definition 7.2. For $\delta > 0$, denote by $S_\delta(K_+)$ the set of functions $\psi \in S(K_+)$ such that the Fourier transform $\hat{\psi}(\xi)$ admits an analytic continuation $\tilde{\psi}(z)$ to the domain $\mathbb{C}_\delta^+ \times \mathbb{C}_\delta^+$ and satisfies the estimate

$$|\tilde{\psi}(z)| \leqslant C_N(1+|z|)^{-N}, \qquad \operatorname{Im} z_l > -\delta, \quad l = 1,2, \tag{7.2}$$

for any $N \in \mathbb{N}$.

Equivalently, $\psi(x) \in S_\delta(K_+)$ if $\psi \in S(K_+)$ and

$$|\psi^{(\alpha)}(x)| \leqslant C_\alpha e^{-\delta' x_1 - \delta' x_2} \quad \forall \alpha = (\alpha_1, \alpha_2)$$

for every $\delta' < \delta$.

For $\varepsilon, \delta > 0$, write

$$\mathbf{\Gamma}_\varepsilon := \{(z_1,z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 = \operatorname{Im} z_2 = \varepsilon\} \quad \text{and} \quad Z_\delta := \{z \in \mathbb{C}^2 : \operatorname{Im} z_l \in (0,\delta),\ l = 1,2\}.$$

Lemma 7.3. *Let* $\hat{u}(\xi)$ *be the Fourier transform of a distribution* $u(x) \in S'(\mathbb{R}^2)$ *with the following properties.*

i) *The function* $\hat{u}(\xi)$ *is the trace (in the sense of* (4.6)*) of an analytic function* $\tilde{u}(z)$ *in the domain* Z_δ *with some* $\delta > 0$.

ii) *The function admits the following bound in* Z_δ *for some* $\mu, \nu \geqslant 0$:

$$|\tilde{u}(z)| \leqslant C(1+|z|)^\mu \rho^{-\nu}, \qquad z \in Z_\delta,$$

where $\rho := \min(\operatorname{Im} z_1, \operatorname{Im} z_2)$.

iii) *Let us additionally assume that* $\psi \in S_\delta(K_+)$. *In this case,*

$$\langle u(x), \psi(x) \rangle = \langle \hat{u}(\xi), \hat{\psi}(-\xi) \rangle = \int_{\mathbf{\Gamma}_\varepsilon} \tilde{u}(z)\tilde{\psi}(-z)dz_1 dz_2 \tag{7.3}$$

for any $\varepsilon \in (0, \delta)$.

Proof. The lemma follows from the Parseval identity.

Let us return to the Helmholtz equation (3.2) and consider a test function $\psi \in S_\delta(K_+)$. Since $\operatorname{supp} u_0 \subset \overline{K_-}$ and $\operatorname{supp} \psi \subset \overline{K_+}$, we have

$$\langle u_0(x), \psi(x) \rangle = 0.$$

By the Parseval identity and by (7.1), this implies that

$$\langle \hat{u}_0(\xi), \hat{\psi}(-\xi) \rangle = \Big\langle \frac{\hat{d}_0(\xi)}{\hat{\mathcal{H}}(\xi)}, \hat{\psi}(-\xi) \Big\rangle = 0. \tag{7.4}$$

Recall that the Fourier transform $\hat{d}_0(\xi)$ is the trace of the analytic function defined by (4.8) in the domain $\mathbb{C}K_+^*$. Apply identity (7.3) to equation (7.4).

Proposition 7.4. *Let* $\psi \in S_\delta(K_+)$ *with some* $\delta \in (0, \delta_*)$, *where* δ_* *is as in* (4.3). *Then we have*

$$\int_{\mathbf{\Gamma}_\varepsilon} \frac{\tilde{d}_0(z)\tilde{\psi}(-z)}{\hat{\mathcal{H}}(z_1, z_2)} dz_1 dz_2 = 0 \tag{7.5}$$

for any ε, $0 < \varepsilon < \delta$.

Proof. By (4.3) and Proposition 4.1, the function $\hat{d}_0(\xi)/\hat{\mathcal{H}}(\xi)$, $\xi \in \mathbb{R}^2$, is the trace of the analytic function $\tilde{d}_0(z)/\hat{\mathcal{H}}(z)$ in the domain Z_{δ_*}, and this function admits the bound

$$\left| \frac{\tilde{d}_0(z)}{\hat{\mathcal{H}}(z)} \right| \leqslant C(1+|z|)^\mu \rho^{-\nu}, \qquad z \in Z_{\delta_*},$$

for some $\mu, \nu \geqslant 0$. Let us now apply Lemma 7.3. All the assumptions of this lemma are satisfied for any $\hat{u}(\xi) = \hat{d}_0(\xi)/\hat{\mathcal{H}}(\xi)$ and $\psi \in S_\delta(K_+)$. Therefore, (7.5) follows from (7.4) and (7.3).

Remark 7.5. We refer to (7.5) as the integral connection equation, since $\hat{d}_0(\xi)$ can be expressed by means of the Cauchy data (see (4.8)).

8. REDUCTION TO THE RIEMANN SURFACE

Here we reduce (7.5) to an equation on the Riemann surface. Substituting (4.9) into (7.5) and representing the integral thus obtained as a sum of two terms, we obtain

$$\mathcal{I}_1(\psi) + \mathcal{I}_2(\psi) = 0 \quad \text{for any} \quad \psi \in S_\delta(K_+), \tag{8.1}$$

where

$$\mathcal{I}_1(\psi) = \int_{\mathbf{\Gamma}_\varepsilon} \frac{\tilde{v}_1(z)}{\hat{\mathcal{H}}(z)} \tilde{\psi}(-z_1, -z_2) dz_1 dz_2, \quad \mathcal{I}_2(\psi) = \int_{\mathbf{\Gamma}_\varepsilon} \frac{\tilde{v}_1(z)}{\hat{\mathcal{H}}(z)} \tilde{\psi}(-z_1, -z_2) dz_1 dz_2, \qquad \varepsilon \in (0, \delta). \tag{8.2}$$

At the next step, we restrict these integrals to the Riemann surface V (defined by (5.1)) by using the Cauchy residue theorem. The integrand of the first integral is meromorphic with respect to z_2 (for any fixed z_1) for $\operatorname{Im} z_2 < \varepsilon$, and the integrand of the second integral is meromorphic for $\operatorname{Im} z_1 < \varepsilon$. The point (z_1, z_2) corresponding to these poles belongs to V because the zeros of the symbol $\hat{\mathcal{H}}$ belong to the Riemann surface V. Therefore, $\mathcal{I}_1(\psi)$ and $\mathcal{I}_2(\psi)$ can be represented as integrals over the Riemann surface V.

To obtain this representation, introduce the pair of contours $L_1 := \{\omega + ir, r \geqslant 0\} \subset \mathbb{C}^+$ and $L_2 := \{-\omega - ir, r \geqslant 0\} \subset \mathbb{C}^-$. Let us factorize the symbol $\hat{\mathcal{H}}(z_1, z_2)$,

$$z_1^2 + z_2^2 - 2z_1 z_2 \cos\alpha - \omega^2 \sin^2\alpha = (z_2 - z_2^+(z_1))(z_2 - z_2^-(z_1)) = (z_1 - z_1^+(z_2))(z_1 - z_1^-(z_2)), \quad z \in \mathbb{C}^2.$$

Here

$$z_2^\pm(z_1) = z_1 \cos\alpha \mp \sin\alpha \sqrt{\omega^2 - z_1^2}, \qquad z_1^\pm(z_2) = z_2 \cos\alpha \mp \sin\alpha \sqrt{\omega^2 - z_2^2}, \tag{8.3}$$

and the branch of the root in (8.3) is such that $\sqrt{\omega^2 - z^2}$ is analytic on $\mathbb{C} \setminus (L_1 \cup L_2)$ and $\sqrt{\omega^2} = \omega$. Condition (6.3) implies that $z_2^\pm(z_1)$ are well defined for $\operatorname{Im} z_1 = \varepsilon < \delta < \delta_*$, and

$$\operatorname{Im} z_2^\pm(z_1) \to \pm\infty \quad \text{as} \quad \operatorname{Im} z_1 = \varepsilon \quad \text{and} \quad |\operatorname{Re} z_1| \to \infty.$$

Hence, (8.2) yields

$$\mathcal{I}_1(\psi) = \int_{\operatorname{Im} z_1 = \varepsilon} \left[\int_{\operatorname{Im} z_2 = \varepsilon} \frac{\tilde{v}_1(z)\tilde{\psi}(-z_1, -z_2)}{[z_2 - z_2^+(z_1)][z_2 - z_2^-(z_1)]} dz_2 \right] dz_1. \tag{8.4}$$

By (4.3), we can assume that

$$\operatorname{Im} z_2^+(z_1) > \delta \quad \text{and} \quad \operatorname{Im} z_2^-(z_1) < -\delta \quad \text{for} \quad \operatorname{Im} z_1 = \varepsilon \in (0, \delta).$$

Let us evaluate the inner integral in (8.4) for $\operatorname{Im} z_1 = \varepsilon \in (0, \delta)$ by closing the contour $\operatorname{Im} z_2 = \varepsilon$. Taking into account that $z_2^-(z_1) - z_2^+(z_1) = 2 \sin\alpha \sqrt{\omega^2 - z_1^2}$, we can see by the Cauchy residue theorem that

$$\int_{\operatorname{Im} z_2 = \varepsilon} \frac{\tilde{v}_1(z)\tilde{\psi}(-z_1, -z_2)}{[z_2 - z_2^+(z_1)][z_2 - z_2^-(z_1)]} dz_2 = -\frac{\pi i}{\sin\alpha} \frac{\tilde{v}_1(z)\tilde{\psi}(-z)}{\sqrt{\omega^2 - z_1^2}}.$$

Then, by (8.4), we have

$$\mathcal{I}_1(\psi) = -\frac{\pi i}{\sin\alpha} \int_{\Gamma_{1,\varepsilon}^-} \frac{\tilde{v}_1(z)\tilde{\psi}^-(z) dz_1}{\sqrt{\omega^2 - z_1^2}}, \tag{8.5}$$

where $\tilde{\psi}^-(z)$ stands for the restriction of $\tilde{\psi}(-z)$ to V and

$$\Gamma_{1,\varepsilon}^- := \{z \in V : \operatorname{Im} z_1 = \varepsilon, \ \operatorname{Im} z_2 < 0\}$$

is the contour on the Riemann surface oriented in the positive direction of $\operatorname{Re} z_1$. Similarly, we see that

$$\mathcal{I}_2(\psi) = -\frac{\pi i}{\sin\alpha}\int_{\Gamma^-_{2,\varepsilon}} \frac{\tilde v_2(z)\tilde\psi^-(z)dz_1}{\sqrt{\omega^2 - z_2^2}}, \tag{8.6}$$

where $\Gamma^-_{2,\varepsilon} := \{z \in V : \operatorname{Im} z_2 = \varepsilon, \operatorname{Im} z_1 < 0\}$ is oriented in the positive direction of $\operatorname{Re} z_2$.

Substituting (8.5) and (8.6) into (8.1), we obtain an equivalent integral equation on the Riemann surface,

$$\int_{\Gamma^-_{1,\varepsilon}} \frac{\tilde v_1(z)\tilde\psi^-(z)dz_1}{\sqrt{\omega^2 - z_1^2}} + \int_{\Gamma^-_{2,\varepsilon}} \frac{\tilde v_2(z)\tilde\psi^-(z)dz_2}{\sqrt{\omega^2 - z_2^2}} = 0, \qquad \psi \in S_\delta(K_+). \tag{8.7}$$

Let us lift this equation to $\check V$. Fisrt, we identify the lifting contours of integration $\Gamma^-_{1,\varepsilon}$ and $\Gamma^-_{2,\varepsilon}$ with the corresponding contours

$$\check\Gamma^-_{1,\varepsilon} := \{w \in \check V_1^+ : z(w) \in \Gamma^-_{1,\varepsilon}\}, \qquad \check\Gamma^-_{2,\varepsilon} := \{w \in \check V_2^+ : z(w) \in \Gamma^-_{2,\varepsilon}\}. \tag{8.8}$$

The directions of these contours correspond to the directions of $\Gamma_{l,\varepsilon}$ (see Fig. 4).

For $\varepsilon > 0$, denote by $\check V_\varepsilon^-$ the domain bounded by $\check\Gamma^-_{2,\varepsilon}$ and $\check\Gamma^-_{1,\varepsilon}$.

The definitions in (8.3) and (8.8) yield

$$\begin{aligned} \sqrt{\omega^2 - z_1^2(w)} &= \omega\cosh w, & w &\in \check\Gamma^-_{1,\varepsilon};\\ \sqrt{\omega^2 - z_2^2(w)} &= -\omega\cosh(w + i\alpha), & w &\in \check\Gamma^-_{2,\varepsilon}. \end{aligned}$$

Hence, changing the variables in (8.7) according to (5.2), we obtain the following equivalent integral equation:

$$\int_{\check\Gamma^-_{1,\varepsilon}} \check v_1(w)\check\psi^-(w)dw - \int_{\check\Gamma^-_{2,\varepsilon}} \check v_2(w)\check\psi^-(w)dw = 0, \qquad \psi \in S_\delta(K_+). \tag{8.9}$$

Theorem 8.1. i) *The integral equation* (8.9) *holds for any* $\psi \in S_\delta(K_+)$ *with* $\delta < \delta_*$, *where* δ_* *is defined by* (4.3).

ii) *The following bound holds*:

$$|\check v_l(w)| \leqslant Ce^{\kappa|w|}\rho^{-\nu}(w, \partial\hat V_l), \qquad w \in \check V_l^+, \tag{8.10}$$

where $\kappa, \nu \in \mathbb{R}$, *and* $\rho(w, \partial\check V_l)$ *is the distance from* w *to* $\partial\check V_l$, $l = 1, 2$.

Proof. Assertion i) was already proved above, and ii) follows from (4.12) and (4.13) and from the estimate

$$\operatorname{Im} z_l(w) \geqslant Ce^{|w|}\rho(w, \partial V_l^+), \quad l = 1, 2.$$

Let us note that the integral over $\check\Gamma^-_{l,\varepsilon}$ with $l = 1, 2$ in (8.9) is equal to the integral over $\check\Gamma^-_{l,\varepsilon} + i\varepsilon$ for sufficiently small values of $\varepsilon > 0$ by the Cauchy theorem and by the bounds (8.10) and (7.2). Hence, we can represent relation (8.9) in the form

$$\int_{\check\Gamma^-_1 + i\varepsilon} \check v_1(w)\check\psi^-(w)dw - \int_{\check\Gamma^-_2 - i\varepsilon} \check v_2(w)\check\psi^-(w)dw = 0, \qquad \psi \in S_\delta(K_+), \tag{8.11}$$

for sufficiently small $\varepsilon > 0$. Write

$$\check v(w) := \begin{cases} \check v_1(w), & w \in \check V_1^+,\\ -\check v_2(w), & w \in \check V_2^+. \end{cases}$$

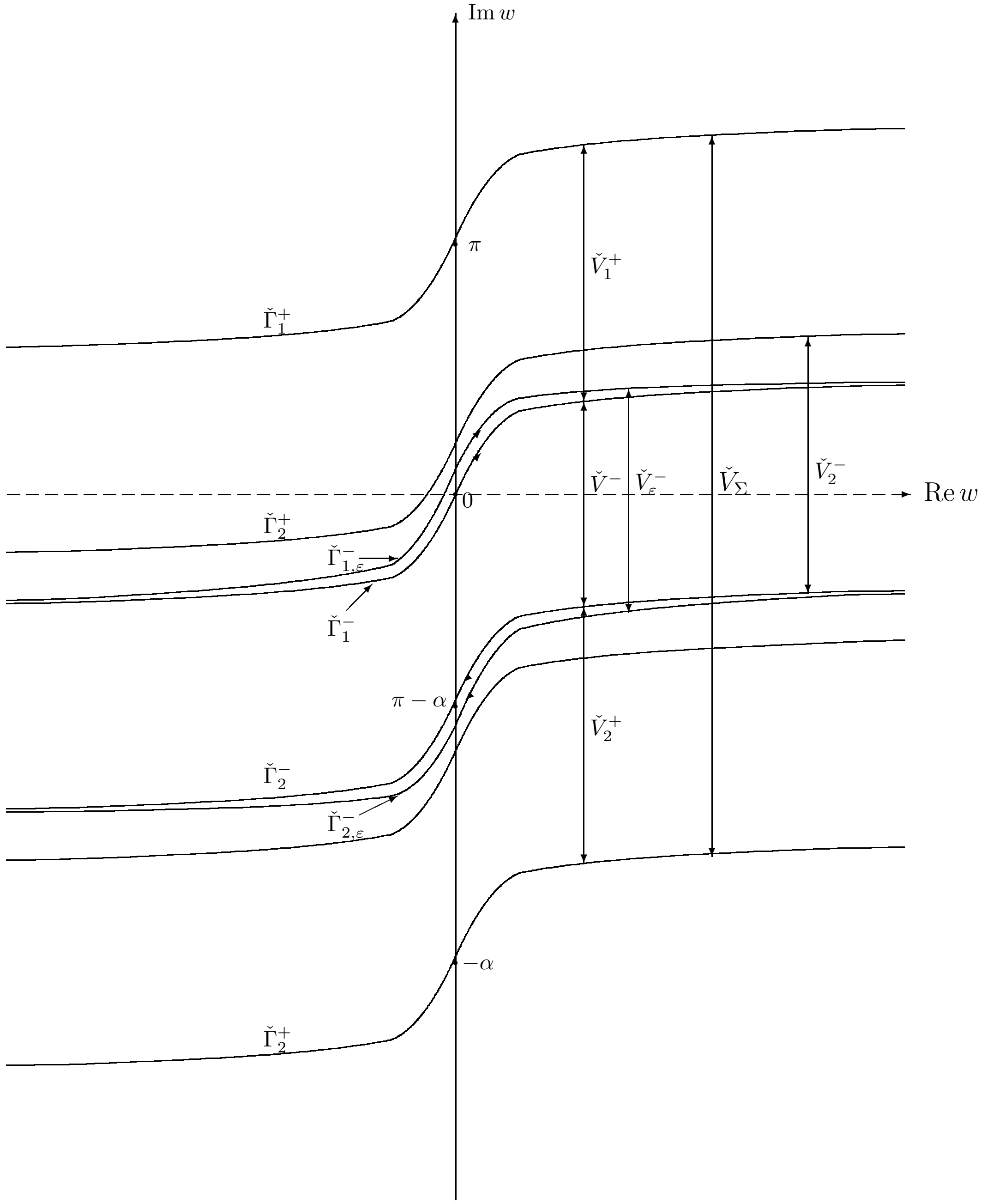

Fig. 4. Contours of integration.

Then the equation (8.9) becomes

$$\int_{\partial\check{V}_\varepsilon^-}\check{v}(w)\check{\psi}^-(w)dw=0,\qquad \psi\in S_\delta(K_+),\tag{8.12}$$

where $\check{V}_\varepsilon^-$ is the domain between the contours $\check{\Gamma}_2-i\varepsilon$ and $\check{\Gamma}_1^-+i\varepsilon$ which is placed to the right of $\partial V_\varepsilon^-=(\check{\Gamma}_1^-+i\varepsilon)\cup(\check{\Gamma}_2^--i\varepsilon)$.

Let us note that each of the functions $\check{v}_l(w)$ is analytic on one of the sides of the contour $\check{\Gamma}_l^-$, and its trace on $\check{\Gamma}_l^-$ is a distribution, due to estimates (8.10).

Definition 8.2. For $\psi \in S_\delta(K_+)$, write

$$\oint_{\check\Gamma_1^-} \check v_1(w)dw := \int_{\check\Gamma_1^- + i\varepsilon} \check v_1(w)dw, \qquad \oint_{\check\Gamma_2^-} \check v_1(w)dw = \int_{\check\Gamma_2^- - i\varepsilon} \check v_1(w)dw.$$

We can now represent (8.11) and (8.12) formally as

$$\oint_{\check\Gamma_1^-} \check v_1(w)\check\psi^-(w)dw - \oint_{\check\Gamma_2^-} \check v_2(w)\check\psi^-(w)dw = 0, \qquad \psi \in S_\delta(K_+), \tag{8.13}$$

and

$$\oint_{\partial\check V} \check v\check\psi^- = 0, \qquad \psi^- \in S_\delta, \tag{8.14}$$

where $\check V^-$ is the domain between the contours $\check\Gamma_2^-$ and $\check\Gamma_1^-$ which is placed to the right of $\partial\check V^- = \check\Gamma_1^- \cup \check\Gamma_2^-$. We systematically use Definition 8.2 in our calculations below.

In the remaining part of the paper, we derive our main theorem, Theorem 6.1, from Theorem 8.1.

Let us describe an example illustrating our main theorem (Theorem 6.1) in a model situation. Namely, let γ_1 (γ_2) be the upper (the lower) semicircle $|z| = 1$, $\operatorname{Im} z \geqslant 0$ ($|z| = 1$, $\operatorname{Im} z \leqslant 0$, respectively) which is oriented clockwise. Let $v_1(z)$ ($v_2(z)$) be a continuous function on γ_1 (γ_2), and let

$$\int_{\gamma_1} v_1(z)\psi(z)dz - \int_{\gamma_2} v_2(z)\psi(z)dz = 0 \tag{8.15}$$

for every function $\psi(z)$ which is analytic on the circle $|z| < 1$ and continuous for $|z| \leqslant 1$. Identity (8.15) is an analog of (8.9). We can also rewrite (8.15) in the form of (8.12), namely,

$$\int_{|z|=1} v(z)\psi(z) = 0, \tag{8.16}$$

where $v(z) := v_1(z)$ for $z \in \gamma_1$ and $v(z) := -v_2(z)$ for $z \in \gamma_2$. Now an analog of our main theorem, Theorem 6.1, would be the claim that the function $v(z)$ is analytic for $|z| \leqslant 1$. This fact readily follows from (8.16) if we take

$$\psi(z) := K(z', z) = \frac{1}{2\pi i}\frac{1}{z - z'} \quad \text{for} \quad |z'| > 1.$$

Indeed, the function

$$f(z') := \frac{1}{2\pi i}\int_{|z|=1} \frac{v(z)}{z - z'}dz \tag{8.17}$$

is analytic for $|z'| \neq 1$, and the Plemelj formula yields

$$f(z' + 0z') - f(z' - 0z') = v(z') \quad \text{for} \quad |z'| = 1. \tag{8.18}$$

It remains to note that $f(z')$ vanishes for $|z'| > 1$, according to the integral equation (8.16). Hence, (8.18) becomes

$$-f(z' - 0z') = v(z') \quad \text{for} \quad |z'| = 1.$$

Therefore, $v(z')$ is analytic for $|z| < 1$.

Our proof of Theorem 6.1 is technically different, though the main idea is to use the Plemelj formula for the Cauchy integrals of the type of (8.17). One of the main problems in this program is to construct an analog of the Cauchy kernel $K(z', z)$. We construct the kernels in the following two sections.

9. A SPECIAL CLASS OF TEST FUNCTIONS

Below we refer to the functions in the space $S_\delta(K_+)$ with $\delta > 0$ as the test functions. Here we introduce a special class of test functions $\psi(x) \in S_\delta(K_+)$. Define a distribution by the rule

$$R_{\lambda,\theta}(x_1, x_2) := e^{i\lambda(x_1\cos\theta+\sin\theta x_2)}\Theta(x_1\cos\theta + x_2\sin\theta)\delta(-x_1\sin\theta + x_2\cos\theta), \qquad (x_1, x_2) \in \mathbb{R}^2.$$

Obviously, $R_{\lambda,\theta}(x_1, x_2) \in S'(\mathbb{R}^2)$ for $\lambda \in \mathbb{C}^+$ and $\theta \in [0, \pi/2]$. The Fourier–Laplace transform of $R_{\lambda,\theta}$ is

$$\tilde{R}_{\lambda,\theta}(z_1, z_2) = \frac{i}{\lambda + z_1\cos\theta + z_2\sin\theta}, \qquad z \in \mathbb{C}K_+^*.$$

Write

$$\psi_{\lambda,\theta}(x_1, x_2) := (-iR_{\lambda,\theta} * \Psi)(x_1, x_2), \tag{9.1}$$

where $\Psi(x_1, x_2)$ is defined by (15.4). The Fourier–Laplace transform of $\psi_{\lambda,\theta}$ is

$$\tilde{\psi}_{\lambda,\theta}(z_1, z_2) = \frac{\tilde{\Psi}(z_1, z_2)}{\lambda + z_1\cos\theta + z_2\sin\theta}, \qquad (z_1, z_2) \in \mathbb{C}K_+^*. \tag{9.2}$$

Note that

$$\tilde{\Psi}(z_1, z_2) \in H(\mathbb{C}_1^+ \times \mathbb{C}_1^+) \tag{9.3}$$

by (6.3) and by (15.1) and (15.5).

Remark 9.1. Note that

$$\lambda + z_1\cos\theta + z_2\sin\theta \not\equiv 0, \qquad z \in V,$$

since the symbol (4.2) is an irreducible polynomial.

Proposition 9.2. *Consider a $\lambda \in \mathbb{C}^+$ and a $\theta \in [0, \pi/2]$. Then the following assertions hold.*

i) $\psi_{\lambda,\theta}(x_1, x_2) \in C^\infty(\mathbb{R}^2)$.

ii) $\operatorname{supp}\psi_{\lambda,\theta} \subset \overline{K_+}$.

iii) *The function $\tilde{\psi}_{\lambda,\theta}$ is meromorphic on $\mathbb{C}_1^+ \times \mathbb{C}_1^+$ and analytic on the subdomain $\operatorname{Im}\lambda + \operatorname{Im} z \cdot (\cos\theta, \sin\theta) > 0$ containing $\mathbb{C}K_+^*$.*

iv) *For any $N > 0$ and $\tau' > 0$, the following bound holds*:

$$\left|\tilde{\psi}_{\lambda,\theta}(z_1, z_2)\right| \leqslant C_{N,\tau'}(1 + |z|)^{-N}, \qquad \operatorname{Im}\lambda + \operatorname{Im} z \cdot (\cos\theta, \sin\theta) \geqslant \tau', \quad \operatorname{Im} z_l \geqslant -1 + \tau', \quad l = 1, 2. \tag{9.4}$$

Proof. i) The convolution (9.1) is a smooth function of x_1, x_2 because $\Psi \in S(\mathbb{R}^2)$. Assertion ii) follows from (9.1) because $\operatorname{supp} R_{\lambda,\theta} \subset \overline{K_+}$ and $\operatorname{supp}\Psi \subset \overline{K_+}$.

Assertion iii) follows from (9.2) and (15.5). The bound (9.4) follows immediately from (9.2) and (15.6).

Corollary 9.3. *For any $\theta \in [0, \pi/2]$, the function $\psi_{\lambda,\theta}(x)$ belongs to the space $S_\delta(K_+)$ of the test functions for $\delta \in (0, 1)$ and $\operatorname{Im}\lambda > \delta(\cos\theta + \sin\theta)$.*

Proof. First, note that $\operatorname{Im}(\lambda + z_1\cos\theta + z_2\sin\theta) > 0$ if $\operatorname{Im} z_l > -\delta$, $l = 1, 2$. Moreover, $\tilde{\Psi}(z_1, z_2)$ is analytic on $\mathbb{C}_\delta^+ \times \mathbb{C}_\delta^+$ by (9.3) since $\delta < 1$. Hence, the function (9.2) is analytic on $\mathbb{C}_\delta^+ \times \mathbb{C}_\delta^+$. Moreover, estimate (7.2) holds in this domain by (9.4).

Since $\delta_* < 1$ by (6.3), Theorem 8.1 and Corollary 9.3 imply the following assertion.

Corollary 9.4. *Let $\delta \in (0, \delta_*)$ and $\lambda \in \mathbb{C}^+$ be such that* $\operatorname{Im}\lambda > \delta(\cos\theta + \sin\theta)$, *and let* $\theta \in [0, \pi/2]$. *Then the integral connection equation in the forms* (8.13) *and* (8.14) *holds for* $\check{\psi}^-_{\lambda,\theta}(w)$, *namely,*

$$\oint_{\check{\Gamma}_1^-} \check{v}_1(w)\check{\psi}^-_{\lambda,\theta}(w)dw - \oint_{\check{\Gamma}_2^-} \check{v}_2(w)\check{\psi}^-_{\lambda,\theta}(w)dw = 0. \tag{9.5}$$

Proof. This follows from Corollary 9.3 and from (6.3).

Specifically, we shall prove our main theorem, Theorem 6.1, in the case of

$$\omega = i, \qquad \alpha = 3\pi/2. \tag{9.6}$$

The proof for general values of $\omega \in \mathbb{C}^+$ and $\alpha \in [\pi, 2\pi]$ is similar. In the case of (9.6), formulas (5.3) become

$$z_1(w) := \sinh w, \qquad z_2(w) := -i\cosh w, \qquad w \in \mathbb{C}. \tag{9.7}$$

Remark 9.5. In the case of (9.6), the curves $\check{\Gamma}_l^{\pm}$ are straight lines and the curves $\check{\Gamma}_1^-$ and $\check{\Gamma}_2^-$ are symmetric with respect to the point $-\pi i/4$ (see Fig. 5).

10. CAUCHY KERNEL ON THE RIEMANN SURFACE

In the previous section, we have introduced a special class of test functions. In the present section, we construct the Cauchy kernels by using the function $\check{\psi}^-_{\lambda,\theta}(w)$. By Proposition 9.2 iii), the function $\tilde{\psi}_{\lambda,\theta}(-z)$ is meromorphic on $\mathbb{C}_1^- \times \mathbb{C}_1^-$ and analytic for $\operatorname{Im} z \cdot (\cos\theta, \sin\theta) < \operatorname{Im}\lambda$. Consider the restriction of this function to the Riemann surface V and the lifting $\check{\psi}^-_{\lambda,\theta}(w)$ of this restriction to the universal covering $\check{V}$. Denote by $\Pi := \Pi(-\pi, \pi/2)$ the strip $-\pi < \operatorname{Im} w < \pi/2$.

Lemma 10.1. *The function $\check{\psi}^-_{\lambda,\theta}(w)$ is meromorphic on $w \in \Pi$, analytic at the points with* $\operatorname{Im}\sin(w - i\theta) < \operatorname{Im}\lambda$, *and continuous at the points $w \in \partial\Pi$ with* $\lambda \neq \sinh(w - i\theta)$.

Proof. By (9.2),

$$\tilde{\psi}_{\lambda,\theta}(-z_1, -z_2) = \frac{\tilde{\Psi}(-z_1, -z_2)}{\lambda - z_1\cos\theta - z_2\sin\theta}, \qquad (z_1, z_2) \in (\mathbb{C}_1^- \times \mathbb{C}_1^-). \tag{10.1}$$

First, let us consider the lifting of the function $\tilde{\Psi}(-z_1, -z_2)$. For $\omega = i$, we have

$$\tilde{\Psi}(-z_1, -z_2) = \tilde{\Lambda}(-z_1)\tilde{\Lambda}(-z_2) = e^{-\sqrt[4]{-z_1+i} - \sqrt[4]{-z_2+i}}, \qquad (z_1, z_2) \in (\mathbb{C}_1^- \times \mathbb{C}_1^-),$$

by (15.5). By Remark 15.1, the function $\tilde{\Lambda}_1(z_1)$ admits an analytic continuation to $\mathbb{C} \setminus \mathcal{L}$ and is continuous on each of the sides of $\mathcal{L}$. The function $-z_1(w)$ is a two-sheeted mapping of the strip $\Pi_1 := \Pi(-3\pi/2, \pi/2))$ onto $\mathbb{C} \setminus \mathcal{L}^-$, where $\mathcal{L}^- := -\mathcal{L}$, and is continuous up to $\partial\Pi_1$. Therefore, the function $\Lambda_1^-(w) := \tilde{\Lambda}(-z_1(w))$ is analytic on Π_1 and continuous on $\overline{\Pi_1}$. Similarly, the function $\Lambda_2^-(w) = \tilde{\Lambda}(-z_2(w))$ is analytic on $\Pi_2 := \Pi(-\pi, \pi)$ and continuous on $\overline{\Pi_2}$. By (15.5) and (15.7),

$$\check{\Psi}^-(w) := \tilde{\Psi}(-z_1(w), -z_2(w)) = \Lambda_1^-(w)\Lambda_2^-(w).$$

Therefore, the function $\check{\Psi}^-(w)$ is analytic on $\Pi := \Pi_1 \cap \Pi_2$ and continuous in $\overline{\Pi}$. Substituting (9.7) into (10.1), we obtain

$$\check{\psi}^-_{\lambda,\theta}(w) = \frac{\check{\Psi}^-(w)}{\lambda - \sinh(w - i\theta)},$$
$$\check{\Psi}^-(w) = \tilde{\Lambda}(-z_1(w))\tilde{\Lambda}(-z_2(w)) = e^{-\sqrt[4]{-\sinh w + i} - \sqrt[4]{i\cosh w + i}}, \quad w \in \Pi.$$

This implies the assertion of the lemma.

We can now construct a special class of Cauchy kernels. Write

$$\lambda = \sinh(w' - i\theta), \qquad w' \in \mathbb{C}. \tag{10.2}$$

Note that $\operatorname{Im}\lambda > 0$ for $w' - i\theta \in \check{V}_1^+$.

Definition 10.2. For $(w', w) \in \Pi \times \Pi$ and $\theta \in [0, \pi/2]$, write

$$\mathcal{K}_\theta(w', w) := -M_\theta(w')\check{\psi}^-_{\lambda,\theta}(w) = M_\theta(w')\frac{\check{\Psi}^-(w)}{\sinh(w - i\theta) - \sinh(w' - i\theta)}, \tag{10.3}$$

where

$$M_\theta(w') := \frac{\cosh(w' - i\theta)}{\check{\Psi}^-(w')}. \tag{10.4}$$

Proposition 10.3. i) *For any $\theta \in [0, \pi/2]$, the function $\mathcal{K}_\theta(w', w)$ is meromorphic on $\Pi \times \Pi$ and continuous at every point $(w', w) \in \overline{\Pi} \times \overline{\Pi}$ with $\sinh(w' - i\theta) \neq \sinh(w - i\theta)$.*

ii) *$K_\theta(w', w)$ is analytic for $w' \neq w$ and $w' \neq S_\theta w'$, where $S_\theta w := -w - \pi i + 2i\theta$ is the symmetry with respect to $w = -\pi i/2 + i\theta$.*

iii) *The residue at $w = w'$ is equal to one,*

$$\operatorname{res}_{w=w'} \mathcal{K}_\theta(w', w) = 1.$$

iv) *For $(w', w) \in \overline{\Pi} \times \overline{\Pi}$ with $|\operatorname{Re} w| \geqslant C(w')$, the following bound holds*:

$$|\mathcal{K}_\theta(w', w)| \leqslant C(w')e^{-\sigma \exp \frac{1}{4}|\operatorname{Re} w|} \qquad \text{for} \quad \sigma > 0. \tag{10.5}$$

Proof. *Step 1.* Lemma 10.1 implies that $\check{\psi}_{\sinh(w'-i\theta),\theta}(w)$ is meromorphic in Π and continuous at every point $w \in \partial\Pi$ with $\sinh(w' - i\theta) \neq \sin(w - i\theta)$ by (10.2). Therefore, assertions i), iii) follow from (10.3) and (10.4) since

$$e^{-\sqrt[4]{-\sinh w' + i} - \sqrt[4]{i \cosh w' + i}} \neq 0 \quad \text{for} \quad w' \in \mathbb{C}.$$

Step 2. The function $M_\theta(w')$ is analytic on $w' \in \Pi$, and $\check{\Psi}^-(w)$ is analytic on Π. Hence, assertion ii) follows from the identity

$$\sinh(w - i\theta) - \sinh(w' - i\theta) = 2 \sinh \frac{w - w'}{2} \cosh \frac{w + w' - 2i\theta}{2}.$$

The bound (10.5) follows now from (10.3) and (15.11). This completes the proof of the proposition.

11. ANALYTIC CONTINUATION

We are going to construct an analytic continuation of the function $\check{v}_1(w)$ (of $\check{v}_2(w)$) to $w \in \check{V}_1$ (to $w \in \check{V}_2$, respectively).

To construct an analytic continuation of $\check{v}_1$, denote by $K_1(w', w)$ the function $\mathcal{K}_\theta(w', w)$ with $\theta = 0$. Then definitions (10.3) and (10.4) imply that

$$K_1(w', w) = \frac{\cosh w'}{\sinh w - \sinh w'} \frac{\check{\Psi}^-(w)}{\check{\Psi}^-(w')}. \tag{11.1}$$

It follows from Proposition 10.3 that $K_1(w', w)$ is meromorphic on $\Pi \times \Pi$ and analytic for $w' \neq w$ and $w' \neq S_1 w$, where $S_1 w := -w - \pi i$. Obviously, $S_1 w$ is the symmetry with center at $w = -\pi i/2$.

Introduce the contour $\gamma^1 := \gamma(\pi/2)$ with the direction from the right to the left and take the strip $\check{V}_2^- := \Pi(-\pi/2, 0)$ (see Fig. 5).

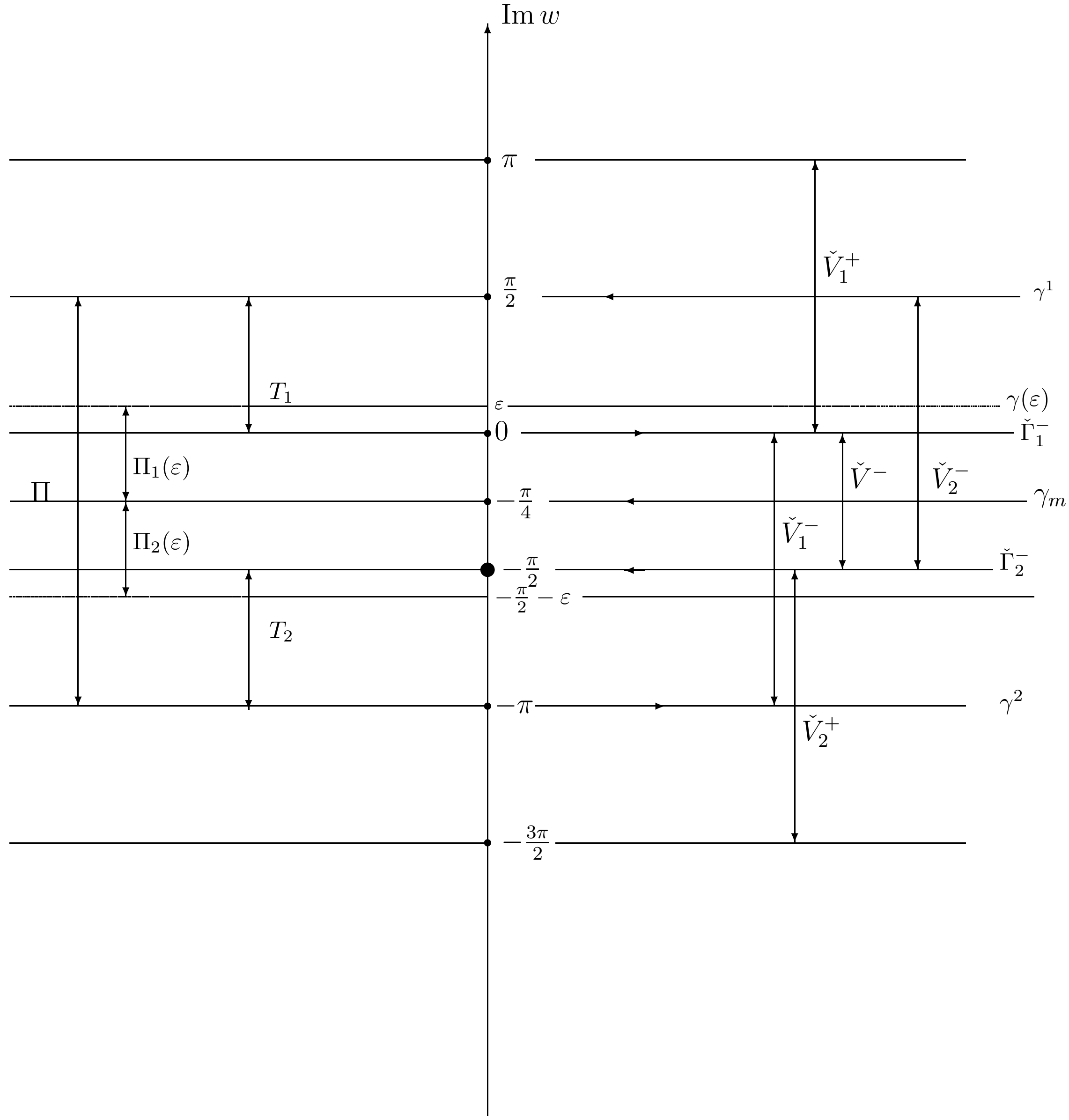

Fig. 5. Domain of analyticity for $\check{v}_1(w)$.

To construct an analytic continuation of $\check{v}_2$, we set $\theta = \pi/2$ and $K_2 := \mathcal{K}_\theta(w', w)$. Then definitions (10.3) and (10.4) yield

$$K_2(w', w) = \frac{\sinh w'}{\cosh w - \cosh w'} \frac{\check{\Psi}^-(w)}{\check{\Psi}^-(w')}. \tag{11.2}$$

Proposition 10.3 implies that K_2 is meromorphic on $\Pi \times \Pi$ and analytic for $w \neq w'$ and $w \neq -w'$. Introduce the contour $\gamma^2 := \gamma(-\pi)$ with the direction from the left to the right and take the strip $\check{V}_1^- := \Pi(-\pi, 0)$ (see Fig. 5).

Proposition 11.1. i) *The function $v_1(w')$ admits an analytic continuation to $\check{V}_2^-$, and the continuation is given by the formula*

$$\check{v}_1(w') := \frac{1}{2\pi i}\Big(\int_{\gamma^1} \check{v}_1(w) K_1(w', w)\,dw + I\!\!\!\!\int_{\check{\Gamma}_2^-} \check{v}_2(w) K_1(w', w)\,dw\Big), \qquad w' \in \check{V}_2^-. \tag{11.3}$$

ii) *The function* $v_2(w')$ *admits an analytic continuation to* $\check V_1^-$, *and the continuation is given by the formula*

$$\check v_2(w') := \frac{1}{2\pi i}\Big(\not\!\!\oint_{\check\Gamma_1^-} \check v_1(w)K_2(w',w)dw + \int_{\gamma^2} \check v_2(w)K_2(w',w)dw\Big), \qquad w' \in \check V_1^-. \tag{11.4}$$

Proof. Let us prove assertion i) (assertion ii) can be proved in a similar way).

I. First, let us prove (11.3) for $w' \in \gamma(\pi/4)$, where $\operatorname{Im} w' = \pi/4$.

Step 1. By Proposition 10.3, the function $K_1(w',w)$ with $w \in \Pi(0,\pi/2) \subset \Pi$ has a unique pole at $w = w'$ with the residue 1 and admits the bounds (10.5). On the other hand, the function $\check v_1(w)$ is analytic on $\Pi(0,\pi/2) \subset \check V_1^+$ by (6.2) and admits the bounds

$$|\check v_1(w)| \leqslant C(\varepsilon)e^{\kappa|\operatorname{Re} w|}, \quad w \in \overline{\Pi(\varepsilon,\pi/2)} \tag{11.5}$$

for any $\varepsilon > 0$ by (8.10). Hence,

$$\frac{1}{2\pi i}\Big(\int_{\gamma^1} \check v_1(w)K_1(w',w)dw + \not\!\!\oint_{\check\Gamma_1^-} \check v_1(w)K_1(w',w)dw\Big) = \check v_1(w'), \qquad w' \in \Pi(0,\pi/2), \tag{11.6}$$

by the Cauchy residue theorem and by Definition 8.2. In fact, the Cauchy theorem implies (11.6) with the contour $\check\Gamma_1^-$ replaced by $\check\Gamma_1^- + i\varepsilon$ with $\varepsilon < \operatorname{Im} w'$. Hence, (11.6) also follows from Definition 8.2 since $\varepsilon > 0$ can be taken to be arbitrarily small.

Step 2. Now let us apply the integral connection equation (9.5). We have $\operatorname{Im}\lambda = \operatorname{Im}\sinh w' > \delta := 0.5.$ because $\operatorname{Im} w' = \pi/4$. Thus, $\psi_{\lambda,0}(x) \in S_\delta(K_+)$ by Corollary 9.3 because $\theta = 0$. Hence, relation (9.5) holds for $K_1(w',w)$ with $\operatorname{Im} w' = \pi/4$ because $K_1(w',w) = C(w')\check\psi_{\lambda,0}(w)$ by (11.1). Therefore, in (11.6), we can replace the contour $\check\Gamma_1^-$ by $\check\Gamma_2^-$ and the function $\check v_1$ by $\check v_2$ and obtain relation (11.3) for $w' \in \gamma(\pi/4)$.

II. We extend now relation (11.3) from $w' \in \gamma(\pi/4)$ to all $w' \in \check V_2^-$. It suffices to prove that the right-hand side of (11.3) is analytic on $\check V_2^-$ because the function $\check v_1(w)$ is analytic on $\check V_1^+ \supset \gamma(\pi/4)$ (see (6.2)).

For $w' \in \check V_2^-$, the kernel $K_1(w',w)$ is continuous for $w \in \gamma^1 \cup \check\Gamma_2^-$ by Proposition 10.3 i) since $\sinh w' \neq \sinh w$ for these w', w. Hence, all "integrals" in (11.3) are well defined by estimates (10.5) and (8.10). It remains to prove that this function is differentiable with respect to $w' \in \check V_2^-$. This follows from the expression

$$\frac{\partial}{\partial w'}K_1(w',w) = \check\Psi^-(w)\frac{M_0'(w')(\sinh w' - \sinh w) - M_0(w')\cosh w'}{(\sinh w - \sinh w')^2},$$

which admits estimates (10.5) for $w' \in \check V_2^- \subset \Pi$ and $w \in \gamma^1 \cup \check\Gamma_2^-$.

12. UNIVERSAL CAUCHY KERNEL

We have established the representations (11.3) and (11.4) for $v_l \in \check V^-$ with $l = 1, 2$, respectively. These representations contain *different* kernels K_θ with $\theta = 0$ and $\theta = \pi/2$, respectively, which gives us no possibility to identify $\check v_1$ with $-\check v_2$ on $\check V^-$. In this section, we construct representations which use a universal kernel for both the functions. These representations, together with the analyticity of $\check v_l^-$ on $\check V^-$, enables us to prove the main theorem.

Consider the kernel $K_\theta(w',w)$ defined by (10.3) and (10.4) with $\theta = \pi/4$ and $\lambda := \sinh(w' - \pi/4)$,

$$K(w',w) = \frac{\cosh(w' - \pi i/4)}{\sinh(w - \pi i/4) - \sinh(w' - \pi i/4)}\,\frac{\check\Psi^-(w)}{\check\Psi^-(w')}. \tag{12.1}$$

By Proposition 10.3, this function admits a meromorphic continuation to $\Pi\times\Pi$, and it is analytic at the points $(w',w)\in\Pi\times\Pi$ with $\sinh(w-\pi i/4)\neq\sinh(w'-\pi i/4)$ and continuous for all pairs $(w',w)\in\partial(\Pi\times\Pi)$ with $\sinh(w-\pi i/4)\neq\sinh(w'-\pi i/4)$.

In what follows, we use the fact that

$$\operatorname{Im}\lambda=\operatorname{Im}\sinh(w'-\pi i/4)>0\quad\text{for}\quad w'\in\Pi\Big(\frac{\pi}{4},\pi\Big)$$

since we have there $w'-\frac{\pi i}{4}\in\check V_1^+$. Therefore, we have $K(w',\cdot)\in S_\delta$ for any sufficiently small $\delta>0$ (the interval for δ depends on w' by Corollary 9.3).

Just as in Proposition 11.1, we obtain representations of type (11.3) and (11.4) for $\check v_1$ and $\check v_2$ with the universal kernel (12.1). Write $T_1:=\Pi(0,\pi/2)$ and $T_2:=\Pi(-\pi,-\pi/2)$.

Proposition 12.1. i) *The function $v_1(w')$ on the domain T_1 can be represented by the formula*

$$\check v_1(w'):=\frac{1}{2\pi i}\Big(\int_{\gamma^1}\check v_1(w)K(w',w)dw+\oint_{\check\Gamma_2^-}\check v_2(w)K(w',w)dw\Big),\qquad w'\in T_1.\tag{12.2}$$

ii) *The function $v_2(w')$ on the domain T_2 can be represented by the formula*

$$\check v_2(w'):=\frac{1}{2\pi i}\Big(\oint_{\check\Gamma_1^-}\check v_1(w)K(w',w)dw+\int_{\gamma^2}\check v_2(w)K(w',w)dw\Big),\qquad w'\in T_2.\tag{12.3}$$

Proof. Let us prove assertion i) (assertion ii) can be proved in a similar way).

I. First, let us prove (12.2) for $w'\in\gamma(3\pi/8)$, where $\operatorname{Im}w'=3\pi/8$.

Step 1. Repeating the arguments of *Step 1* in the proof of Proposition 11.1, we obtain the representation (11.6) with K instead of K_1.

Step 2. Let us apply now the integral connection equation (9.5). For $\delta<\sin(\pi/8)/\sqrt2$, we have

$$\operatorname{Im}\lambda=\operatorname{Im}\sinh(w'-\pi i/4)=\sin(\pi/8)>\delta(\cos(\pi/4)+\sin(\pi/4))$$

since $\operatorname{Im}w'=3\pi/8$. Thus,

$$\psi_{\lambda,\pi/4}(x)\in S_\delta(K_+)$$

by Corollary 9.3 for such δ. Hence, (12.2) follows for $w'\in\gamma(3\pi/8)$ similarly to *Step 2* in Proposition 11.1.

II. Now we extend (12.2) from $w'\in\gamma(3\pi/4)$ to all $w'\in T_1$. It suffices to prove that the right-hand side of (11.3) is analytic on $\check T_1$ since the function $\check v_1(w)$ is analytic on $\check V_1^+\supset\gamma(3\pi/4)$ (see (6.2)).

For $w'\in T_1$, the kernel $K(w',w)$ is continuous for $w\in\gamma^1\cup\check\Gamma_2^-$ by Proposition 10.3 i) since $\sinh(w'-\pi i/4)\neq\sinh(w-\pi i/4)$ for these (w',w). Thus, repeating the arguments of Part II in the proof of Proposition 11.1, we obtain (12.2).

13. BOUNDS FOR THE ANALYTIC CONTINUATION

Let us choose a sufficiently small $\varepsilon>0$. In this section, we obtain important estimates for the function $\check v_1$ on the set $\Pi_1(\varepsilon)$ (which is the closure of $\Pi(-\pi/4,\varepsilon)$) and for the function $\check v_2$ on the set $\Pi_2(\varepsilon)$ (which is the closure of $\Pi(-\pi/2-\varepsilon,-\pi/4)$).

Lemma 13.1. *For any sufficiently small $\varepsilon>0$, there exists a $C(\varepsilon)>0$ such that*

$$|\sinh w'-\sinh w|\geqslant Ce^{|\operatorname{Re}w'|},\qquad w\in\gamma^1\cup\check\Gamma_2^-,\qquad w'\in\Pi_1(\varepsilon),\tag{13.1}$$

$$|\cosh w'-\cosh w|\geqslant Ce^{|\operatorname{Re}w'|},\qquad w\in\gamma^2\cup\check\Gamma_1^-,\qquad w'\in\Pi_2(\varepsilon).\tag{13.2}$$

Proof. Let us prove inequality (13.1) (inequality (13.2) can be proved in a similar way).

Step I. Let us first consider a point $w \in \gamma^1$, in which case we have $w_2 := \operatorname{Im} w = \pi/2$. Then

$$-\frac{3\pi}{8} \leqslant \frac{w_2' - w_2}{2} \leqslant \frac{\varepsilon}{2} - \frac{\pi}{4}$$

for $w' \in \Pi_1(\varepsilon)$. Hence,

$$\Big|\sinh \frac{w' - w}{2}\Big| \geqslant Ce^{|\frac{w'-w}{2}|}, \qquad w' \in \Pi_1(\varepsilon), \quad w \in \gamma^1. \tag{13.3}$$

Second, let us consider $w \in \check{\Gamma}_2^-$, in which case we have $w_2 = -\pi/2$. Then

$$\frac{\pi}{8} \leqslant \frac{w_2' - w_2}{2} \leqslant \frac{a}{2} + \frac{\pi}{4}$$

for $w' \in \Pi_1(\varepsilon)$. Hence, (13.3) also holds in this case. Therefore,

$$\Big|\sinh \frac{w' - w}{2}\Big| \geqslant Ce^{|\frac{w'-w}{2}|}, \qquad w' \in \Pi_1(\varepsilon), \quad w \in \gamma^1 \cup \check{\Gamma}_2^-, \quad C > 0. \tag{13.4}$$

Step II. Let us first consider points $w \in \gamma^1$ and $w' \in \Pi_1(\varepsilon)$. Then

$$\frac{\pi}{8} \leqslant \frac{w_2 + w_2'}{2} \leqslant \frac{\pi}{4} + \frac{\varepsilon}{2}.$$

Second, let us consider points $w \in \check{\Gamma}_2^-$ and $w' \in \Pi_1(\varepsilon)$. Then

$$-\frac{3\pi}{8} \leqslant \frac{w_2 + w_2'}{2} \leqslant -\frac{\pi}{4} + \frac{\varepsilon}{2}.$$

Hence,

$$\Big|\cosh \frac{w' + w}{2}\Big| \geqslant Ce^{|\frac{w+w'}{2}|}, \qquad w' \in \Pi_1(\varepsilon), \quad w \in \gamma^1 \cup \check{\Gamma}_2^-, \quad C > 0. \tag{13.5}$$

Step III. Therefore, it follows from (13.4) and (13.5) that

$$|\sinh w' - \sinh w| = 2\Big|\sinh\Big(\frac{w' - w}{2}\Big)\cosh\Big(\frac{w' + w}{2}\Big)\Big| \geqslant Ce^{|\frac{w'-w}{2}|+|\frac{w'+w}{2}|} \geqslant Ce^{|w_1'|},$$
$$w \in \gamma^1 \cup \check{\Gamma}_2^-, \quad C > 0.$$

Corollary 13.2. *By* (13.1) *and* (13.2), *the functions* $K_l(w', w)$ *defined by* (11.1) *and* (11.2) *admit the estimates*

$$|K_1(w', w)| \leqslant C(\varepsilon)\Big|\frac{\check{\Psi}^-(w)}{\check{\Psi}^-(w')}\Big|, \qquad w \in \gamma^1 \cup \check{\Gamma}_2^-, \quad w' \in \Pi_1(\varepsilon), \tag{13.6}$$
$$|K_2(w', w)| \leqslant C(\varepsilon)\left|\frac{\check{\Psi}^-(w)}{\check{\Psi}^-(w')}\right|, \qquad w \in \gamma^2 \cup \check{\Gamma}_1^-, \quad w' \in \Pi_2(\varepsilon),$$

for all sufficiently small $\varepsilon < 0$ *and for some* $C(\varepsilon) < \infty$.

Proof. The bound (13.6) follows from (11.1) because

$$\Big|\frac{\cosh w'}{\sinh w - \sinh w'}\Big| \leqslant C, \qquad w \in \gamma^1 \cup \check{\Gamma}_2^-, \quad w' \in \Pi_1(\varepsilon),$$

by Lemma 13.1. For $l = 2$, the bound follows similarly from the representation (11.2).

Corollary 13.3. *For all sufficiently small* $\varepsilon > 0$, *the functions* $\check{v}_l(w)$ *admit the bounds*

$$|\check{v}_l(w)| \leqslant C(\varepsilon)|\check{\Psi}^-(w)|^{-1}, \qquad w \in \Pi_l(\varepsilon), \tag{13.7}$$

for some $C(\varepsilon) < \infty$, $l = 1, 2$.

Proof. The assertion follows from (11.3), (11.4), (11.5), and (13.6).

14. SINGULAR INTEGRAL EQUATION

In the previous section, we have obtained the representations (12.2) and (12.3) which make use of the universal kernel. However, the contours of integration are different, and the closures of the domains of validity, T_1 and T_2, are disjoint because the strip $\check{V}^-$ lies between T_1 and T_2.

Therefore, a straightforward application of (12.2) and (12.3) gives no possibility to immediately identify the functions $\check{v}_1$ and $-\check{v}_2$ to prove the main theorem. Hence, it is natural to replace all contours of integration in (12.2) and (12.3) by the contour γ_m.

Proposition 14.1. *The following identities hold*:

$$\int_{\gamma_m} [\check{v}_1(w) + \check{v}_2(w)]K(w', w)dw = 0, \qquad w' \in \Pi\Big(-\frac{\pi}{4}, \frac{\pi}{2}\Big); \tag{14.1}$$

$$\int_{\gamma_m} [\check{v}_1(w) + \check{v}_2(w)]K(w', w)dw = 0, \qquad w' \in \Pi\Big(-\pi, -\frac{\pi}{4}\Big). \tag{14.2}$$

Remark 14.2. i) This proposition implies our main theorem, Theorem 6.1, by the Plemelj formula.

ii) We shall derive the proposition from the representations (12.2) and (12.3) with the Cauchy kernel K. However, the proof heavily depends on the analyticity of $\check{v}_1$ and $\check{v}_2$ on the domain $\check{V}^-$.

iii) The desired analyticity on $\check{V}^-$ follows by Proposition 11.1 from the representations (11.3) and (11.4) with the Cauchy kernels K_1 and K_2. On the other hand, the analyticity cannot be derived from the representations (12.2) and (12.3) by the method of Proposition 11.1. This is related to the fact that the function $K(w', w)$ for $w \in \check{\Gamma}_2^-$ has a pole at $w' = -w - \pi i \in \check{\Gamma}_1^-$.

iv) For this reason, we need all three representations for $\check{v}_1$ and $\check{v}_2$ (with the Cauchy kernels K, K_1, and K_2).

Proof of Proposition 14.1. *Step 1.* Let us first prove (14.1) for $w' \in T_1 = \Pi(0, \pi/2)$. We shall replace the contours in (12.2) by using the analyticity of the function $\check{v}_2(w)$ on $\check{V}_1^-$ (see Proposition 11.1) and estimates (13.7).

I. Let us lift the contour of integration $\check{\Gamma}_2^-$ up to γ_m in (12.2).

1. We replace first the formal integral over the contour $\check{\Gamma}_2^-$ in (13.5) to the integral over $\check{\Gamma}_2^- - i\varepsilon$ by using Definition 8.2 (for a small $\varepsilon > 0$).

2. Second, we lift the contour $\check{\Gamma}_2^- - i\varepsilon$ up to γ_m and obtain

$$\check{v}_1(w') = \frac{1}{2\pi i}\Big(\int_{\gamma^1} \check{v}_1(w)K(w', w)dw + \int_{\gamma_m} \check{v}_2(w)K(w', w)dw\Big), \qquad w' \in \Pi\Big(0, \frac{\pi}{2}\Big). \tag{14.3}$$

This follows from the Cauchy theorem. In fact, *a)* the contours $\check{\Gamma}_2^- - i\varepsilon$ and γ_m bound the strip $\Pi_2(\varepsilon)$, *b)* the function $\check{v}_2(w)$ is analytic on $w \in \check{V}_1^- \supset \Pi_2(\varepsilon)$ by Proposition 11.1, and *c)* the function $K(w', w)$ is analytic with respect to $w \in \Pi(-\pi/2, -\pi/4)$ for $w' \in \Pi(0, \pi/2)$ by Proposition 10.3 ii) with $\theta = \pi/4$. Finally, the integrand $\check{v}_2(w)K(w', w)$ admits the bound

$$|\check{v}_2(w)K(w', w)| \leqslant C(w')e^{-|w|}, \qquad w \in \Pi_2(\varepsilon), \tag{14.4}$$

which follows from estimates (13.7) and from the representation (12.1).

II. We move down the contour γ^1 in formula (14.3). Let us make this movement in two steps. Recall that $w' \in T_1$.

First, let us deform the contour γ^1 to the contour $\gamma(\varepsilon)$ for a small $0 < \varepsilon < \operatorname{Im} w'$ by using the Cauchy residue theorem and the "standard" bound (8.10) for $l = 1$. Then we replace the contour $\gamma(\varepsilon)$ by γ_m (using the bound (10.5)).

Let us equip the contour $\gamma(\varepsilon)$ with the direction of the contour γ^1, i.e., from the right to the left. Since $w' \in \check{V}_1^+$, the bound (8.10) for $l = 1$ holds on the strip $\overline{\Pi(\varepsilon, \pi/2)}$. Hence,

$$\int_{\gamma^1} \check{v}_1(w)K(w', w)dw = \int_{\gamma(\varepsilon)} \check{v}_1(w)K(w', w)dw + \check{v}_1(w'). \tag{14.5}$$

Replace now the contour $\gamma(\varepsilon)$ by γ_m in the second integral in (14.5). Namely, we again apply the bound (13.7) for $l = 1$ to the function $\check{v}_1(w)K(w', w)$. The representation (12.1) implies that $\check{v}_1(w)K(w', w)$ admits a bound similar to the bound in (14.4) of the strip $\Pi_1(\varepsilon)$. Therefore,

$$\int_{\gamma(\varepsilon)} \check{v}_1(w)K(w', w)dw = \int_{\gamma_m} \check{v}_1(w)K(w', w)dw \tag{14.6}$$

because the function $\check{v}_1$ is analytic on $\check{V}_2^-$ and the function $K(w', w)$ is analytic on the strip $\Pi_1(\varepsilon)$ by the Cauchy theorem (taking into account the directions of the contours $\gamma(\varepsilon)$ and γ_m). Hence, substituting first the expression in (14.6) into (14.5) and then the expression in (14.5) into (14.3), we obtain

$$v_1(w') = v_1(w') + \frac{1}{2\pi i}\Big(\int_{\gamma_m} \check{v}_1(w)K(w', w)dw + \int_{\gamma_m} \check{v}_2(w)K(w', w)dw\Big), \qquad w' \in T_1. \tag{14.7}$$

Further, relation (14.7) yields

$$\int_{\gamma_m} [\check{v}_1(w) + \check{v}_2(w)]K(w', w)dw = 0, \qquad w' \in T_1.$$

Step 2. It remains to extend this identity to any point $w' \in \Pi(-\pi/4, \pi/2)$. This follows from the analyticity of the left-hand side of (14.1) on the strip $\Pi(-\pi/4, \pi/2)$.

Identity (14.2) can be proved in a similar way.

Proof of the main theorem (Theorem 6.1). It suffices to show that

$$\check{v}_1(w) + \check{v}_2(w) \equiv 0, \qquad w \in \gamma_m.$$

Write

$$\mathcal{I}(w') := \int_{\gamma_m} [\check{v}_1(w) + \check{v}_2(w)]K(w', w)dw, \qquad w' \in \check{V}^- \setminus \gamma_m.$$

By (14.1) and (14.2), we have

$$\mathcal{I}(w') \equiv 0, \qquad w' \in \check{V}^- \setminus \gamma_m.$$

Hence, by the Plemelj theorem,

$$[\check{v}_1(w) + \check{v}_2(w)] = \mathcal{I}(w' + i0) - \mathcal{I}(w' - i0) = 0, \qquad w' \in \gamma_m.$$

15. APPENDIX. ON A CLASS OF SCHWARTZ FUNCTIONS

For $\omega_2 > 0$, denote by $\Lambda(x)$, $x \in \mathbb{R}$, the tempered distribution with respect to $x \in \mathbb{R}$ with the Fourier transform

$$\tilde{\Lambda}(z) = e^{-\sqrt[4]{z + i\omega_2}}, \qquad z \in \mathbb{R}, \tag{15.1}$$

where $\sqrt[4]{z}$ is analytic outside the cut $\mathcal{L} := [0, -i\infty)$ and we have $\operatorname{Re}\sqrt[4]{z} > 0$ and $\operatorname{Im}\sqrt[4]{z} > 0$ for $z \in \mathbb{C}^+$.

Remark 15.1. The function $\tilde{\Lambda}(z)$ admits an analytic continuation from the points $z \in \mathbb{R}$ to all points $z \in \mathbb{C} \setminus \mathcal{L}(\omega_2)$, and this function is continuous on each side of the ray $\mathcal{L}(\omega_2)$, where $\mathcal{L}(\omega_2) = \{-i\omega_2 - ir : r \geqslant 0\}$.

Lemma 15.2. i) *The uniform bounds*

$$\left|\tilde{\Lambda}^{(k)}(\xi+i\tau)\right| \leqslant Ce^{-|\xi|/4}, \qquad \xi\in\mathbb{R}, \quad \tau\geqslant\tau', \tag{15.2}$$

hold for any $k=0,1,2,\dots$ *and* $\tau'>-\omega_2$.

ii) *The function* $\tilde{\Lambda}(z)$, $z\in\mathbb{C}^+$, *is the Fourier–Laplace transform of the function* $\Lambda(x)\in S(\mathbb{R})$ *with* $\operatorname{supp}\Lambda\subset\overline{\mathbb{R}^+}$.

Proof. i) We have $\tilde{\Lambda}\in C^\infty(\mathbb{R})$ because $\tilde{\Lambda}(z)$ is analytic on $\overline{\mathbb{C}}^+$ by (15.1). The bound (15.2) follows from the inequality

$$\operatorname{Re}\sqrt[4]{z+i\omega_2}\geqslant C_{\omega_2}|z|^{1/4}-1, \qquad z\in\mathbb{C}. \tag{15.3}$$

ii) The bound (15.2) implies that $\tilde{\Lambda}(\xi)\in S(\mathbb{R})$. Hence,

$$\Lambda := F^{-1}\tilde{\Lambda}\in S(\mathbb{R}).$$

Finally, by the Paley–Wiener theorem, we have $\operatorname{supp}\Lambda(x)\subset\overline{\mathbb{R}^+}$ (because the function $\tilde{\Lambda}$ is analytic on $\mathbb{C}^+$ and the bound (15.2) holds). This completes the proof of the lemma.

Definition 15.3. Write

$$\Psi(x):=\Psi(x_1,x_2):=\Lambda(x_1)\Lambda(x_2) \quad \text{for} \quad x=(x_1,x_2)\in\mathbb{R}^2. \tag{15.4}$$

By Lemma 15.2,

$$\begin{gathered}\Psi(x)\in S(\mathbb{R}^2), \qquad \operatorname{supp}\Psi\subset\overline{K_+}, \qquad \tilde{\Psi}(z_1,z_2)\in H(\mathbb{C}^+_{\omega_2}\times\mathbb{C}^+_{\omega_2});\\ \tilde{\Psi}(z_1,z_2)=\tilde{\Lambda}(z_1)\tilde{\Lambda}(z_2)=e^{-\sqrt[4]{z_1+\omega_2 i}-\sqrt[4]{z_2+\omega_2 i}}, \qquad (z_1,z_2)\in(\mathbb{C}^+_{\omega_2}\times\mathbb{C}^+_{\omega_2}).\end{gathered} \tag{15.5}$$

It follows from the bound (15.2) that the function $\tilde{\Psi}(z)$ satisfies the following inequality:

$$\left|\frac{\partial^{k_1+k_2}}{\partial z_1^{k_1}\partial z_2^{k_2}}\tilde{\Psi}(\xi+i\tau)\right|\leqslant C_{k,\tau',N}(1+|\xi_1|)^{-N}(1+|\xi_2|)^{-N}, \qquad \tau_{1,2}\geqslant\tau', \tag{15.6}$$

for every $\tau'>-\omega_2$.

Let us consider the restriction of the function $\tilde{\Psi}(-z_1,-z_2)$ to the domain V and the lifting of the restriction to the universal covering $\check{V}$. Take $\omega_2=1$. By (9.7), the liftings of the restrictions of $\tilde{\Lambda}(-z_1)$ and $\tilde{\Lambda}(-z_2)$ are of the form

$$\check{\Lambda}_1^-(w):=e^{-\sqrt[4]{-\sinh w+i}}, \quad w\in\check{V}_1^+; \qquad \check{\Lambda}_2^-(w):=e^{-\sqrt[4]{i\cosh w+i}}, \quad w\in\check{V}_2^+ . \tag{15.7}$$

Lemma 15.4. *For* $w\in\overline{\Pi(-3\pi/2,\pi/2)}$, *the following bounds hold*:

$$K_1e^{-C_1\exp\frac{|w_1|}{4}}\leqslant|\Lambda_1^-(w)|\leqslant K_2e^{-C_2\exp\frac{|w_1|}{4}}. \tag{15.8}$$

For $w\in\overline{\Pi(-\pi,\pi)}$, *the following bounds hold*:

$$K_1e^{-C_1\exp\frac{|w_1|}{4}}\leqslant|\Lambda_2^-(w)|\leqslant K_2e^{-C_2\exp\frac{|w_1|}{4}}. \tag{15.9}$$

Here $C_{1,2}$ *and* $K_{1,2}$ *stand for some positive constants.*

Proof. For $w \in \overline{\Pi(-3\pi/2, \pi/2)}$, consider the function

$$\check{\Lambda}_1^-(w) := \tilde{\Lambda}(-z_1(w)) = \tilde{\Lambda}(-\sinh w) = e^{-\sqrt[4]{-\sinh w + i}}. \tag{15.10}$$

Since $w \in \overline{\Pi(-3\pi/2, \pi/2)}$ only if $z_1 \in \mathbb{C}$, it follows that

$$\operatorname{Re} \sqrt[4]{-z_1(w) + i} \geqslant C|z_1(w)|^{1/4} - 1$$

by (15.3). Hence, the second inequality in (15.8) follows by virtue of (15.10). The first inequality is obvious. Inequalities (15.9) can be proved in a similar way.

Finally, let us consider the lifting $\check{\Psi}^-(w)$ of the function $\tilde{\Psi}(-z_1, -z_2)$ to the universal covering $\check{V}$. By (15.5) and (15.7), we have

$$\check{\Psi}^-(w) = \tilde{\Lambda}(-z_1(w))\tilde{\Lambda}(-z_2(w)) = e^{-\sqrt[4]{-\sinh w + i} - \sqrt[4]{i\cosh w + i}}, \qquad w \in \Pi.$$

Corollary 15.5. *The function $\check{\Psi}^-(w)$ admits the bound*

$$K_1 e^{-C_1 \exp \frac{|w_1|}{4}} \leqslant |\check{\Psi}^-(w)| \leqslant K_2 e^{-C_2 \exp \frac{|w_1|}{4}}, \qquad w \in \Pi, \tag{15.11}$$

with some $C_{1,2} > 0$ and $K_{1,2} > 0$.

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