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# *On Transitions to Stationary States in One-Dimensional Nonlinear Wave Equations*

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**Abstract**

We consider the long-time asymptotics of solutions to one-dimensional nonlinear wave equations, which are infinite-dimensional Hamiltonian systems. We assume that the nonlinear term is concentrated at a finite segment of the line. We prove long-time convergence to stationary states for all finite-energy solutions in the Fréchet topology defined by local energy seminorms. This means that the set of stationary states is a point attractor for the systems in the Fréchet topology. The investigation is inspired by N. BOHR's postulate on the transitions between stationary states in quantum systems.

## 1. Introduction

We consider the long-time asymptotics of the solutions to the Cauchy problem

$$\ddot{u}(x,t) = u''(x,t) + f(x, u(x,t)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{1.1}$$

$$u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x). \tag{1.2}$$

The solutions $u(x,t)$ take values in $\mathbb{R}^d$ with $d \geqq 1$, and all the derivatives in (1.1) and everywhere below are understood in the sense of distribution. Physically, the equation (1.1) describes small crosswise oscillations of a string interacting with an elastic nonlinear medium. We assume that $f(x,u) = 0$ for $|x| \geqq a$ with some $a > 0$, and

$$f(x,u) = \chi(x)F(u), \quad F \in C^1(\mathbb{R}^d, \mathbb{R}^d), \quad \chi(x) \in C(\mathbb{R}), \tag{1.3}$$

$$F(u) = -\nabla V(u), \quad V(u) \to +\infty \quad \text{as } |u| \to \infty, \tag{1.4}$$

$$\chi(x) \geqq 0, \chi(x) \not\equiv 0, \quad \chi(x) = 0 \quad \text{for } |x| \geqq a. \tag{1.5}$$

We introduce the "configuration space" $\mathcal{Q}$ and the phase space $\mathcal{E}$ of finite-energy states for the system (1.1). We denote by $L^2$ the Hilbert space $L^2(\mathbb{R}, \mathbb{R}^d)$ with the norm $|\cdot|$, and we denote by $|\cdot|_R$ the norm in $L^2(-R, R; \mathbb{R}^d)$ for $R > 0$.

**Definition 1.1.** i) $\mathscr{Q}$ is the Hilbert space $\{u(x) \in C(\mathbb{R}, \mathbb{R}^d) : u'(x) \in L^2\}$ with the norm

$$\|u\|_{\mathscr{Q}} = |u'| + |u(0)|. \tag{1.6}$$

ii) $\mathscr{E} = \mathscr{Q} \oplus L^2$ is the Hilbert space of the pairs $(u(x), v(x))$, with the norm

$$\|(u, v)\|_{\mathscr{E}} = \|u\|_{\mathscr{Q}} + |v|. \tag{1.7}$$

iii) $\mathscr{E}_F$ is the space $\mathscr{E}$ endowed with the Fréchet topology defined by the seminorms

$$\|(u, v)\|_R \equiv |u'|_R + |u(0)| + |v|_R, \quad R > 0. \tag{1.8}$$

Note that both spaces $\mathscr{E}$ and $\mathscr{E}_F$ are metrisable and that $\mathscr{E}_F$ is not a complete space.

We denote by $V(x, u) = \chi(x)V(u)$ the potential of the nonlinear force. With the assumptions (1.3)–(1.5), the equation (1.1) is formally a Hamiltonian system with the phase space $\mathscr{E}$ and the Hamiltonian functional

$$\mathscr{H}(u, v) = \int_{\mathbb{R}} \left[\tfrac{1}{2}|v(x)|^2 + \tfrac{1}{2}|u'(x)|^2 + V(x, u(x))\right] dx \tag{1.9}$$

for $(u, v) \in \mathscr{E}$. We consider the solutions $u(x, t)$ such that $Y(t) = (u(\cdot, t), \dot{u}(\cdot, t)) \in C(\mathbb{R}, \mathscr{E})$ and we write the Cauchy problem (1.1), (1.2) in the form

$$\dot{Y}(t) = \mathscr{V}(Y(t)) \text{ for } t \in \mathbb{R}, \quad Y(0) = Y_0, \tag{1.10}$$

where $Y_0 = (u_0, v_0)$.

**Proposition 1.2.** *Let $d \geqq 1$ and let the assumptions* (1.3)–(1.5) *be fulfilled. Then*

i) *For every $Y_0 \in \mathscr{E}$ the Cauchy problem* (1.10) *has a unique solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$.*
ii) *The mapping $W_t : Y_0 \mapsto Y(t)$ is continuous in $\mathscr{E}$ and $\mathscr{E}_F$ for all $t \in \mathbb{R}$.*
ii) *The energy is conserved:*

$$\mathscr{H}(Y(t)) = \mathscr{H}(Y_0) \quad \textit{for } t \in \mathbb{R}. \tag{1.11}$$

We denote by $\mathscr{S}$ the set of all stationary states $S = (s(x), 0) \in \mathscr{E}$ for the system (1.10). We establish the long-time convergence in the Fréchet topology

$$Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{S} \quad \text{as} \quad t \to \pm\infty \tag{1.12}$$

for finite-energy solutions $Y(t)$. By definition the convergence means that for every neighborhood $\mathscr{O}(\mathscr{S})$ of $\mathscr{S}$ in $\mathscr{E}_{\mathscr{F}}$ there exist $T > 0$ such that $Y(t) \in \mathscr{O}(\mathscr{S})$ for $|t| > T$. Thus, the set $\mathscr{S}$ is the point attractor of the system (1.10) in the Fréchet topology of the space $\mathscr{E}_F$. Let us denote $\mathscr{S}^h = \{S \in \mathscr{S} : \mathscr{H}(S) \leqq h\}$ for $h \in \mathbb{R}$. Then (1.3)–(1.5) and (1.9) imply that $\mathscr{S}^h$ is a closed bounded set in $\mathscr{E}$:

$$\sup_{S \in \mathscr{S}^h} \|S\|_{\mathscr{E}} < \infty \quad \forall h \in \mathbb{R}. \tag{1.13}$$

**Proposition 1.3.** *Let assumptions* (1.3)–(1.5) *be fulfilled and, moreover, let* $d = 1$ *and let the function* $F(u)$ *be real-analytic on* $\mathbb{R}$. *Then* $\mathscr{S}^h$ *is a finite set for every* $h \in \mathbb{R}$.

For a function $Y(t) \in C(\mathbb{R}, \mathscr{E})$ we denote by $O(Y)$ the orbit $\{Y(t) : t \in \mathbb{R}\} \subset \mathscr{E}$.

**Theorem 1.4.** *Let all the assumptions of Proposition 1.2 hold and let an initial state* $Y_0 \in \mathscr{E}$. *Then*

i) *For the solution* $Y(t) \in C(\mathbb{R}, \mathscr{E})$ *to the Cauchy problem* (1.10) *the orbit* $O(Y)$ *is precompact in* $\mathscr{E}_F$ *and* (1.12) *holds.*
ii) *Moreover, let* $d = 1$ *and the function* $F(u)$ *be real-analytic on* $\mathbb{R}$. *Then there exist some stationary states* $S_\pm \in \mathscr{S}$ *depending on the solution* $Y(t)$ *such that*

$$Y(t) \xrightarrow{\mathscr{E}_F} S_\pm \ \ as \ \ t \to \pm\infty. \tag{1.14}$$

Below we consider only the cases $d = 1$, because all the results for $d \geqq 1$ follow without any modifications.

*Remarks.* i) The convergence (1.14) means a "transition"

$$S_- \mapsto S_+ \tag{1.15}$$

when the time varies from $-\infty$ to $+\infty$. So the convergence gives a mathematical model of N. BOHR's transitions between stationary states in quantum systems [2].

ii) The convergence (1.14) and (1.3)–(1.9) imply by Fatou's theorem that

$$\mathscr{H}(S_\pm) \leqq \mathscr{H}(Y(t)) \equiv \mathscr{H}(Y_0), \quad t \in \mathbb{R}, \tag{1.16}$$

which is similar to a well-known property of the weak convergence in Hilbert and Banach spaces.

iii) For $d = 1$ the analyticity of $F(u)$ provides that $\mathscr{S}$ is a discrete subset of the phase space $\mathscr{E}_F$. The convergence (1.12) implies (1.14) if the attractor $\mathscr{S}$ is a discrete subset of the phase space $\mathscr{E}_F$, since the orbit $O(Y)$ is precompact in $\mathscr{E}_F$. On the other hand, (1.14) can fail if the attractor $\mathscr{S}$ is not discrete. For example, (1.14) fails for the solution

$$u(x, t) = \sin \log[(x - t)^2 + 1]$$

of the equation (1.1) with $f(x, u) \equiv 0$ for $|u| \leqq 1$. However, (1.12) holds for the solution.

iv) We assume that $f(x, u) = \chi(x) F(u)$ for the simplicity of exposition. All results of this paper can be extended easily to $f(x, u)$ without this assumption and with suitable generalization of the conditions (1.3)–(1.5) (see [17]).

As a trivial example we can consider $f(x, u) \equiv 0$. Then the equation (1.1) becomes the d'Alembert equation and the assumptions (1.4), (1.5) fail. Accordingly, for the solutions $Y(t) \in C(\mathbb{R}, \mathscr{E})$ the orbit $O(Y)$ generally is not precompact in $\mathscr{E}_F$ and the convergence (1.14) generally does not hold. In this case the convergence (1.12) holds for every solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$ and the convergence (1.14) holds if $u_0(x) = C_\pm$ and $v_0(x) = 0$ for $|x| \geqq$const. This follows evidently from the d'Alembert formula for the solution to the Cauchy problem.

Before entering into the proofs it may be useful to put our results in the context of related works. We establish here that all finite-energy solutions of a Hamiltonian system converge to an attractor, possibly consisting of an infinite number of points, in the long time limit. Such behavior is familiar from dissipative systems, however, as $t \to +\infty$ only [1, 9, 24, 37]. Moreover, the mechanism is completely different. For a dissipative system there is a local loss of "energy", whereas here energy is propagated to infinity. This scattering of energy to infinity plays the role of a dissipation and provides the convergence (1.12) and (1.14). The convergence in general fails for Hamiltonian wave equations in finite regions due to the reflections of the waves from the boundary.

For the dissipative systems the convergences (1.12) and (1.14) hold in the "global" energy metric of corresponding phase space $\mathscr{E}$ for all finite-energy solutions. On the other hand, for the Hamiltonian equation (1.1) the convergence (1.14) in general is impossible in the "global" energy metric of the space $\mathscr{E}$, because of energy conservation. Indeed, if $\|Y(t) - S_\pm\|_{\mathscr{E}} \to 0$ as $t \to \pm\infty$, then (1.11) implies that $\mathscr{H}(S_\pm) = \mathscr{H}(Y(t))$, because the Hamiltonian functional $\mathscr{H}$ is continuous on $\mathscr{E}$. Therefore, the convergence in $\mathscr{E}$ of all finite-energy solutions would imply that $\mathscr{H}(\mathscr{E}) \subset \mathscr{H}(\mathscr{S})$. However, this is impossible for any nontrivial Hamiltonian system, if the set $\mathscr{S}$ is discrete. Similarly, the convergence (1.14) of all solutions is impossible for any nontrivial finite-dimensional Hamiltonian system.

Propagation of energy to infinity is also the essence of scattering theory for Hamiltonian linear wave equations [26, 27, 30, 38–41] and for Hamiltonian relativistic-invariant nonlinear wave equations either with a unique "zero" stationary solution [3, 5–7, 31, 34, 35] (see also the surveys [32, 36]) or with small initial data [10, 12]. Note that the attractor consists then only of the zero state in contrast to the case considered here. Long-time asymptotics of solutions to nonlinear wave equations with a set of stationary solutions different from a point were not considered previously. However, the absence of local energy decay for solutions to some equations was observed in [33] and in [4].

The transitions to stationary states (1.12) and (1.14) in some infinite-dimensional Hamiltonian systems are established in [13–21] (see the survey [22]). The results of [13–16] concern Lamb's system [25, 11], i.e., the equation (1.1) with $f(x, u) = \delta(x)F(u)$, while the results of [17] concern the equation (1.1) with $f(x, u) = \sum_1^N \delta(x - x_k)F_k(u)$. The results [18, 19, 21] concern the three-dimensional scalar wave equation coupled to a particle, and the results [20] concern the three-dimensional Maxwell-Lorentz system with a charge [29]. A Liapunov-type criterion of asymptotic stability is established in [23] for stationary solutions to general $n$-dimensional equations and systems (1.1) with space-localized nonlinear terms, i.e., with $f(x, u) = 0$ for $|x| >$ const.

## 2. Existence of Dynamics and A Priori Estimates

We prove Proposition 1.2 by the contraction-mapping principle. Let $W_t^0$ be the dynamical group corresponding to the linear equation (1.1) with $f(x,u) \equiv 0$. Then the Cauchy problem (1.10) for $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is equivalent to the integral equation

$$Y(t) = W_t^0 Y_0 + \int_0^t W_{t-\tau}^0 (0, f(\cdot, u(\cdot, \tau)))d\tau. \tag{2.1}$$

Therefore the contraction-mapping principle implies the existence and uniqueness of a local solution $Y(t) \in C(-\varepsilon, \varepsilon; \mathcal{E})$ with some $\varepsilon > 0$. The continuity of $W_t$ in $\mathcal{E}$ and $\mathcal{E}_F$ follows for small $|t|$ from this construction due to corresponding properties of $W_t^0$.

To prove the energy conservation, let us assume for a moment that $u^0(x) \in C^2(\mathbb{R})$, $v^0(x) \in C^1(\mathbb{R})$ and

$$u^0(x) = v^0(x) = 0 \quad \text{for } |x| \geqq R^0. \tag{2.2}$$

Then the integral representation (2.1) implies that $u(x,t) \in C^2(\mathbb{R} \times (-\varepsilon, \varepsilon))$ and

$$u(x,t) = 0 \text{ for } |x| \geqq \bar{R} + |t|, \quad \bar{R} = \max(R^0, a). \tag{2.3}$$

Hence we get conservation (1.11) by partial integration for small $|t|$. For arbitrary $(u^0, v^0) \in \mathcal{E}$ the energy conservation follows from density and continuity reasons.

Now the energy conservation (1.11) and the existence of the local solution imply the existence of global solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$ and all the properties claimed for every $t \in \mathbb{R}$. □

We need however a finer characterization of the properties of the solution.

**Proposition 2.1.** *Let the assumptions* (1.3)–(1.5) *be fulfilled. Then*

i) *The mapping* $W_t : Y_0 \mapsto Y(t)$ *is Lipschitz continuous in* $\mathcal{E}_F$*, i.e., for every* $R, T > 0$,

$$\|W_t Y_1 - W_t Y_2\|_R \leqq L_T \|Y_1 - Y_2\|_{R+T} \quad \textit{for} \quad |t| \leqq T, \tag{2.4}$$

*where* $L_T$ *is bounded for bounded norms* $\|Y_1\|_{R+T}$, $\|Y_2\|_{R+T}$.

ii) *The a priori estimate*

$$|u(x,t)| \leqq \alpha + \beta\sqrt{|x|} \equiv b(x) \quad \textit{for} \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{2.5}$$

*holds, where* $\alpha$ *and* $\beta$ *are bounded for bounded energy* $\mathcal{H}(Y_0)$.

iii) $u(x,\cdot) \in C(\mathbb{R}, H^1_{\text{loc}}(\mathbb{R}))$ *and* $u'(x,\cdot) \in C(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}))$.

iv) *For a.a.* $x \in \mathbb{R}$

$$\int_t^{t+1} (|\dot{u}(x,s)|^2 + |u'(x,s)|^2 + |u(x,s)|^2)ds \leqq e(x) < \infty \quad \textit{for} \quad t \in \mathbb{R}, \tag{2.6}$$

*where* $e(x)$ *may depend on* $x$ *and* $\mathcal{H}(Y_0)$*, and does not depend on* $t \in \mathbb{R}$.

**Proof.** i) The Lipschitz continuity (2.4) follows for small $T > 0$ from the construction of $W_t$ by the contraction-mapping principle due to corresponding properties of $W_t^0$. The extension to arbitrary $T > 0$ follows.

ii) The energy conservation (1.11) and (1.4), (1.5) imply that

$$D = \sup_{t\in\mathbb{R}} \int |u'(x,t)|^2 dx < \infty, \tag{2.7}$$

and $D$ is bounded for bounded energy $\mathscr{H}(Y_0)$. Therefore by the Cauchy-Schwarz inequality,

$$|u(x,t) - u(x_0,t)| = \Big| \int_{x_0}^{x} u'(y,t)dy \Big| \leqq \sqrt{D}\sqrt{|x-x_0|} \quad \text{ for } x, x_0, t \in \mathbb{R}. \tag{2.8}$$

Let us choose $x_0$ such that $\chi(x_0) > 0$. Then (1.11) and (1.4), (1.5) imply that $\sup_{t\in\mathbb{R}} |u(x_0,t)| < \infty$. Therefore (2.8) implies (2.5).

iii) Let us use the integral representation (2.1). The claimed properties hold for the first summand in the right-hand side of (2.1) and the same is true for the integral summand due to $u(x,t) \in C(\mathbb{R}^2)$.

iv) The estimate (2.6) follows from (1.11), (2.5) and from integral representation of type (2.1):

$$Y(s) = W_{s-t}^0 Y(t) + \int_0^{s-t} W_\theta^0(0, f(\cdot, u(\cdot, t+\theta))d\theta, \tag{2.9}$$

which holds due to the uniqueness of the solution. Namely, the estimates of type (2.6) hold for the first summand in the right-hand side of (2.9) due to the bounds (1.11) uniform in $t$ for $Y(t)$. The same holds for the second summand due to the estimates (2.5) uniform in $t$. □

*Remark.* The conditions (1.3)–(1.5) imply that the Hamiltonian functional $\mathscr{H}$ is Fréchet differentiable in $\mathscr{E}$ and

$$\frac{\delta\mathscr{H}}{\delta v(x)} = v(x), \quad \frac{\delta\mathscr{H}}{\delta u(x)} = -u''(x) - f(x, u(x)). \tag{2.10}$$

So equation (1.1) can be written in a Hamiltonian form:

$$\dot{u} = \frac{\delta\mathscr{H}}{\delta v}, \quad \dot{v} = -\frac{\delta\mathscr{H}}{\delta u}\,. \tag{2.11}$$

## 3. Stationary States

We prove Proposition 1.3. To find all the stationary solutions, we substitute $u(x,t) = s(x)$ to (1.1). Then (1.3) and (1.5) imply that

$$\begin{aligned} s''(x) + f(x, s(x)) = 0 \quad &\text{for } x \in [-a, a], \\ s(x) = s(\pm a) \quad &\text{for } \pm x \geqq a \end{aligned} \tag{3.1}$$

since $s'(x) \in L^2(\mathbb{R})$. Therefore the continuous map $I : \mathscr{E}_F \to \mathbb{R}$ defined by $I(u(x), v(x)) = u(-a)$ is an injection on $\mathscr{S}$. Hence Proposition 1.3 follows from the next lemma.

**Lemma 3.1.** *$Z^h \equiv I\mathscr{S}^h$ is a finite set for every $h \in \mathbb{R}$.*

**Proof.** $\mathscr{S}^h$ is a compact subset in $\mathscr{E}_F$ due to (1.13) and (3.1). Hence, $Z^h$ is a closed bounded subset in $\mathbb{R}$. It remains to prove that $Z^h$ has no limit points. Let us assume to the contrary that

$$z_k \in Z^h \text{ and } z_k \to \bar{z} \in Z^h \text{ as } k \to \infty. \tag{3.2}$$

Let us denote by $s_\lambda(x)$ the solution to the problem

$$\begin{aligned} s''_\lambda(x) + f(x, s_\lambda(x)) = 0 \quad &\text{for } x \in [-a, a], \\ s'_\lambda(-a) = 0, \quad &s_\lambda(-a) = \lambda, \end{aligned} \tag{3.3}$$

if the solution exists, and let $\Lambda$ denote the set of all $\lambda \in \mathbb{R}$ such that the solution $s_\lambda(x)$ exists. We extend $s_\lambda(x)$ to $|x| > a$ by constants:

$$s_\lambda = s_\lambda(\pm a) \quad \text{for } \pm x > a. \tag{3.4}$$

Then $S_\lambda = (s_\lambda(x), 0) \in \mathscr{E}$ for every $\lambda \in \Lambda$. Let us define the map $T : \Lambda \to \mathbb{R}$ by

$$T : \lambda \mapsto s'_\lambda(a - 0). \tag{3.5}$$

Then $Z^h = \{\lambda \in \Lambda : T(\lambda) = 0, \mathscr{H}((s_\lambda(x), 0)) \leqq h\}$. $\Lambda$ is an open set, whence

$$\Lambda = \cup_1^\infty \Lambda_j, \quad \Lambda_j = (\lambda_j^-, \lambda_j^+) \neq \emptyset, \quad \lambda_j^\pm \notin \Lambda. \tag{3.6}$$

Of course, $\bar{z} \in \Lambda_l$ with some $l$. We show that

$$|\lambda_l^\pm| < \infty, \quad \lambda_l^\pm \in \Lambda. \tag{3.7}$$

This contradicts (3.6) and completes the proof of Lemma 3.1.

At first, the map $T : \Lambda \to \mathbb{R}$ is real-analytic and $T(z) = 0$ for $z \in Z$. Therefore (3.2) implies that $T(\lambda) = 0$ for all $\lambda \in \Lambda_l$, i.e.,

$$(s_\lambda(x), 0) \in \mathscr{S} \quad \forall \lambda \in \Lambda_l. \tag{3.8}$$

Denote by $\mathscr{U}$ the potential-energy functional in the configuration space $\mathscr{Q}$:

$$\mathscr{U}(u) \equiv \mathscr{H}(u, 0) = \int_{-\infty}^{\infty} \left(\tfrac{1}{2}|u'(x)|^2 + V(x, u(x))\right) dx \quad \text{for } u \in \mathscr{Q}. \tag{3.9}$$

Then (3.1) is equivalent to the identity

$$\delta \mathscr{U}(s) = 0 \tag{3.10}$$

(following also from (2.11)), where $\delta\mathscr{U}$ is the Fréchet differential of $\mathscr{U}$ in the space $\mathcal{Q}$. Therefore (3.8) implies that

$$\frac{d}{d\lambda}\mathscr{U}(s_\lambda) = \left\langle \delta\mathscr{U}(s_\lambda),\ \frac{d}{d\lambda}s_\lambda \right\rangle = 0 \quad \text{for} \ \ \lambda \in \Lambda_l; \tag{3.11}$$

hence the function $\lambda \mapsto \mathscr{U}(s_\lambda)$ is constant on $\Lambda_l$. Therefore as in (2.5) the a priori estimate

$$|s_\lambda(x)| \leqq \alpha_1 + \beta_1\sqrt{a} \quad \text{for} \quad |x| \leqq a \quad \text{and} \ \ \lambda \in \Lambda_l \tag{3.12}$$

holds with some $\alpha_1$ and $\beta_1$ not depending on $\lambda \in \Lambda_l$. For instance, the interval $\Lambda_l$ is bounded because $s_\lambda(-a) = \lambda$. On the other hand, the uniform bounds (3.12) and the equation (3.3) imply that the set of functions $\{s_\lambda(x) : \lambda \in \Lambda_l\}$ is precompact in $\mathcal{Q}$, and therefore $\lambda_l^\pm \in \Lambda_l$. □

Proposition 1.3 is proved. □

## 4. Long-Time Asymptotics

We prove the Theorem 1.4.

### *4.1. Compact Attracting Set*

Let us construct a compact attracting set $\mathscr{A}$ for the trajectory $Y(t)$. Let $\bar\alpha$, $\bar\beta$ denote some positive constants to be chosen later.

**Definition 4.1.** $\mathscr{A} = \mathscr{A}_{\bar\alpha\bar\beta} = \{S_\lambda = (s_\lambda(x), 0) \in \mathscr{E} : \ \lambda \in \Lambda, \quad |s_\lambda(x)| \leqq \bar\alpha + \bar\beta\sqrt{x} \quad \text{for}\ |x| \leqq a\}$.

$\mathscr{A}$ is a compact set in $\mathscr{E}_F$ due to equation (3.1). We prove the next lemma in the following section.

**Lemma 4.2.** *Let all assumptions of Theorem 1.4 hold. Then*

$$Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{A} = \mathscr{A}_{\bar\alpha\bar\beta} \quad \text{as} \ \ t \to \pm\infty, \tag{4.1}$$

*if the constants $\bar\alpha$ and $\bar\beta$ are sufficiently large.*

*4.2. Proof of Theorem 1.3*

i) Lemma 4.2 implies that the orbit $O(Y)$ is precompact in $\mathscr{E}_F$. Therefore, the next lemma implies (1.12). We denote by $\Omega(Y)$ the omega-limit set of the trajectory $Y(t)$ in the Fréchet topology of the space $\mathscr{E}_F$ : $\bar{Y} \in \Omega(Y)$ if and only if $Y(t_k) \xrightarrow{\mathscr{E}_F} \bar{Y}$ for some sequence $t_k \to \pm\infty$.

**Lemma 4.3.** *$\Omega(Y)$ is a subset of $\mathscr{S}$.*

**Proof.** $\Omega(Y) \subset \mathscr{A}$, since $\mathscr{A}$ is an attracting set. Moreover, the set $\Omega(Y)$ is invariant with respect to $W_t$, $t \in \mathbb{R}$, due to the continuity of $W_t$ in $\mathscr{E}_F$. Hence, for every $\bar{Y} \in \Omega(Y)$ there exists a $C^2$-curve $t \mapsto \lambda(t) \in \mathbb{R}$ such that $W_t\bar{Y} = S_{\lambda(t)}$. Then $S_{\lambda(t)}$ is the solution to (1.2). Therefore, $\lambda(t) \equiv \lambda$ and $\bar{Y} = S_\lambda \in \mathscr{S}$.

ii) $\Omega(Y) \subset \mathscr{S}^h$ follows with $h = \mathscr{H}(Y_0)$ as in (1.16). Hence, $Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{S}^h$ due to (1.12). However, $\mathscr{S}^h$ is a finite set by Proposition 1.3. Therefore, (1.14) follows by the continuity of $Y(t)$. □

## 5. Attraction to a Compact Set

We deduce Lemma 4.2 from the following lemma on "attraction in the mean", which we prove in the next section. For $R > 0$ let us denote

$$\rho_R(t) = \inf_{S \in \mathscr{A}} \|Y(t) - S\|_R \quad \text{for } t \in \mathbb{R}. \tag{5.1}$$

**Lemma 5.1.** *For every $R > 0$,*

$$\int_0^\infty \rho_R^2(t)\, dt < \infty. \tag{5.2}$$

Let us fix a metric $\rho(\cdot,\cdot)$ on $\mathscr{E}$, defining the topology of $\mathscr{E}_F$. We prove (4.1) ad absurdum: Let us assume that there exist $\varepsilon > 0$ and a sequence $t_k \to \infty$, such that

$$\rho(Y(t_k), \mathscr{A}) \geqq \varepsilon \quad \text{for all } k = 1, 2, \ldots. \tag{5.3}$$

We show that this is impossible and thus complete the proof of Lemma 4.2. We may assume that $t_k + 1 < t_{k+1}$ for every $k$. Then (5.2) implies by the Fatou theorem that

$$\int_0^1 \sigma_R(\theta)\, d\theta < \infty, \quad \text{where } \sigma_R(\theta) = \sum_1^\infty \rho_R^2(t_k + \theta). \tag{5.4}$$

Therefore, $\sigma_R(\theta) < \infty$ for every $\theta \in \Theta(R) \subset [0, 1]$, and $\int_{\Theta(R)} dx = 1$. Then for every $R > 0$,

$$\rho_R(t_k + \theta) \to 0 \quad \text{as } k \to \infty \quad \text{for } \theta \in \Theta = \cap_{R \in \mathbb{N}}^\infty \Theta(R). \tag{5.5}$$

Hence $Y(t_k+\theta) \xrightarrow{\mathscr{E}_F} \mathscr{A}$ as $k \to \infty$ for every $\theta \in \Theta \subset [0,1]$, and $\int_\Theta dx = 1$. Then for every $\theta \in \Theta$ the compactness of $\mathscr{A}$ in $\mathscr{E}_F$ implies that for some sequence $k(\theta) \to \infty$,

$$Y(t_{k(\theta)}+\theta) \xrightarrow{\mathscr{E}_F} \bar{Y}(\theta) \in \mathscr{A} \quad \text{as } k(\theta) \to \infty \quad \text{for } \theta \in \Theta. \tag{5.6}$$

Then the continuity of the map $W_{-\theta}$ in $\mathscr{E}_F$ also implies that

$$Y(t_{k(\theta)}) \xrightarrow{\mathscr{E}_F} W_{-\theta}\bar{Y}(\theta) \quad \text{as } k(\theta) \to \infty \quad \text{for } \theta \in \Theta. \tag{5.7}$$

On the other hand, the compactness of $\mathscr{A}$ in $\mathscr{E}_F$ implies that there exists a sequence $\theta_j \in \Theta$ such that $\theta_j \to 0$ as $j \to \infty$ and

$$\bar{Y}(\theta_j) \xrightarrow{\mathscr{E}_F} Y^* \in \mathscr{A} \quad \text{as } j \to \infty. \tag{5.8}$$

Now the uniform Lipschitz continuity (2.4) of $W_{-\theta}$ with $\theta \in [0,1]$ and the convergence $W_{-\theta_j}Y^* \xrightarrow{\mathscr{E}_F} Y^*$ as $j \to \infty$ imply

$$W_{-\theta_j}\bar{Y}(\theta_j) \xrightarrow{\mathscr{E}_F} Y^* \text{ as } j \to \infty. \tag{5.9}$$

However this convergence together with (5.7) for $\theta = \theta_j$ contradict (5.3). □

## 6. Attraction in the Mean

We prove Lemma 5.1. It suffices to construct for sufficiently large $\bar{\alpha}, \bar{\beta}, \bar{T} > 0$ a function $S_{\lambda(t)} \in \mathscr{A}$ defined for $t \geqq \bar{T}$ such that for every $R > 0$

$$\int_{\bar{T}}^{\infty} \|Y(t) - S_{\lambda(t)}\|_R^2 \, dt < \infty. \tag{6.1}$$

We establish this inequality with $\lambda(t) = u(-a, n)$ for $n \leqq t < n+1, n = 0, 1, \ldots$. We may replace the seminorm $\|\cdot\|_R$ from (1.8) with an equivalent seminorm with $|u(-a)|$ instead of $|u(0)|$. Then (6.1) means for $R > a$ that

$$\int_{\bar{T}}^{\infty} \left( \int_{|x|<a} (|u'(x,t) - s'_{\lambda(t)}(x)|^2 + |\dot{u}(x,t)|^2) \, dx + |u(-a,t) - \lambda(t)|^2 \right.$$
$$\left. + \int_{a<|x|<R} (|u'(x,t)|^2 + |\dot{u}(x,t)|^2) \, dx \right) dt < \infty. \tag{6.2}$$

*6.1. Energy Scattering to Infinity*

**Lemma 6.1.** *For the functions $y_\pm(t) = u(\pm a, t)$ and $z_\pm(t) = u'(\pm a, t)$ the following bound holds:*

$$\int_0^\infty (|\dot y_-(t)|^2 + |z_-(t)|^2 + |\dot y_+(t)|^2 + |z_+(t)|^2)\, dt < \infty. \tag{6.3}$$

**Proof.** This follows from the d'Alembert representation

$$u(x,t) = f_\pm(t-x) + g_\pm(t+x), \quad \pm x > a, \quad t \in \mathbb{R}, \tag{6.4}$$

and from the finiteness of the energy flow to infinity. Namely, the d'Alembert representations (6.4) imply that (6.3) is equivalent to

$$\int_0^\infty (|f_-'(t+a)|^2 + |g_-'(t-a)|^2 + |f_+'(t-a)|^2 + |g_+'(t+a)|^2)\, dt < \infty. \tag{6.5}$$

The integrals for $f_-'$, $g_+'$ are finite due to the d'Alembert formulas

$$f_-(-x) = \tfrac12 u_0(x) - \tfrac12 \int_{-a}^x v_0(s)\, ds \quad \text{for} \quad -x < -a,$$
$$g_+(x) = \tfrac12 u_0(x) + \tfrac12 \int_a^x v_0(s)\, ds \quad \text{for} \quad x > a$$

and due to the fact that $(u_0, v_0) \in \mathscr{E}$. To derive (6.5) for $g_-'$, $f_+'$ we introduce the energy functional on the segment $\Delta = [-a, a]$ for $Y = (u(x), v(x)) \in \mathscr{E}$,

$$\mathscr{H}_\Delta(Y) = \int_\Delta \left[\tfrac12 |v(x)|^2 + \tfrac12 |u'(x)|^2 + V(x, u(x))\right] dx. \tag{6.6}$$

Then we consider the energy flow from $\Delta$ at first for smooth initial data $(u_0, v_0)$. Then (1.1) and (6.4) imply, that

$$\begin{aligned} \frac{d}{dt}\mathscr{H}_\Delta(Y(t)) &= \dot u u'\Big|_{x=-a-0}^{x=a+0} \\ &= |f_-'(t+a)|^2 - |g_-'(t-a)|^2 + |g_+'(t+a)|^2 - |f_+'(t-a)|^2 \\ &\quad \text{for a.a. } t \in \mathbb{R}. \end{aligned} \tag{6.7}$$

Integrating this equation, we get the energy identity

$$\begin{aligned} &\mathscr{H}_\Delta(Y(t)) + \int_0^t (|g_-'(s-a)|^2 + |f_+'(s-a)|^2)\, dt \\ &= \mathscr{H}_\Delta(Y(0)) + \int_0^t (|f_-'(s+a)|^2 + |g_+'(t+a)|^2)\, dt \quad \text{for} \quad t \in \mathbb{R}. \end{aligned} \tag{6.8}$$

For general initial data $(u_0, v_0) \in \mathscr{E}$ the same identity follows by density and continuity reasons. Finally, (6.5) for $g_-'$, $f_+'$ follows from (6.8) and from (6.5) for $f_-'$, $g_+'$, because $\mathscr{H}_\Delta(Y(t)) \geqq \text{const}$ due to (1.3)–(1.5). □

### *6.2. Nonlinear Goursat Problem*

We consider the Goursat problem for the wave equation (1.1) with Cauchy data on the lines $x = \text{const}$:

$$\begin{aligned} \ddot{u}(x,t) &= u''(x,t) + f(x, u(x,t)), \\ u|_{x=r} = y(t),\ \ u'|_{x=r} &= z(t),\ \ t \in \mathbb{R}. \end{aligned} \tag{6.9}$$

We establish the continuity of the map $G_{r,x} : (y(\cdot), z(\cdot)) \mapsto (u(x,\cdot), u'(x,\cdot))$ and then we deduce (6.2) from (6.3) by this continuity in the next subsection.

*Remark.* Our assumptions (1.4), (1.5) ensure that the Cauchy problem (1.1), (1.2) is well posed globally in $t$. The Goursat problem (6.9) generally is not well posed globally in $x \in \mathbb{R}$. However, the Goursat problem is well posed locally in $x$ and this is sufficient for our purposes. To deduce (6.1) from (6.3) we need to prove the continuity of the map $G_{b,x}$ for $b = -a$ and for bounded $x \in [-R, R]$ only. The continuity holds "for large $t$" and "along" the global solution $u(x,t)$ considered.

Let $\sigma$ denote an arbitrary segment in $\mathbb{R}$ of length $|\sigma|$.

**Definition 6.2.** $\mathscr{E}(\sigma)$ is the Hilbert space of functions $(y(t), z(t)) \in H^1(\sigma) \oplus L^2(\sigma)$, such that

$$\|(y,z)\|_{\mathscr{E}(\sigma)} = |\dot{y}|_\sigma + |y|_\sigma + |z|_\sigma < \infty, \tag{6.10}$$

where $|\cdot|_\sigma$ is the norm in $L^2(\sigma)$.

**Definition 6.3.** $\bar{\mathscr{E}}$ denotes the space of functions $(y(t), z(t)) \in H^2_{\text{loc}}(\mathbb{R}) \oplus L^2(\mathbb{R})$, such that

$$\|(y,z)\|_{\bar{\mathscr{E}}} = \sup_{|\sigma| \geqq 1} \frac{\|(y,z)\|_{\mathscr{E}(\sigma)}}{\sqrt{|\sigma|}} < \infty. \tag{6.11}$$

*Remark.* Propositions 2.1 iii), iv) imply that $(u, u')|_{x=r} \in \bar{\mathscr{E}}$ for every $r \in \mathbb{R}$ and, moreover, that $\|(u,u')|_{x=r}\|_{\bar{\mathscr{E}}} \leqq 2e(r)$.

We consider the solutions $u(x,t)$ to the Goursat problem (6.9) with $(y,z) \in \bar{\mathscr{E}}$ such that $(u, u') \in C(r-\varepsilon, r+\varepsilon; \bar{\mathscr{E}})$ with some $\varepsilon > 0$. For such solutions the Goursat problem is equivalent to the integral identity

$$Z(x) = W^0_{x-r} Z_r - \int_r^x W^0_{x-s}(0, f(s, u(s,\cdot)))\, ds, \tag{6.12}$$

which is similar to (2.1), where $Z(x) = (u(x,\cdot), u'(x,\cdot))$ and $Z_r = (y(\cdot), z(\cdot))$.

**Lemma 6.4.** *Let assumptions* (1.3)–(1.5) *be fulfilled, and let* $Z_r \in \bar{\mathscr{E}}$. *Then*

i) *The Goursat problem* (6.9) *has a unique solution*

$$Z(x) =: G_{r,x} Z_r \in C(r-\varepsilon, r+\varepsilon; \bar{\mathscr{E}})$$

*with some* $\varepsilon > 0$.

ii) *Here* $\varepsilon = \varepsilon(R, B) > 0$ *depends only on* $R$ *and* $B$ *for* $r \leqq R$ *and* $\|Z_r\|_{\bar{\mathscr{E}}} \leqq B$.

iii) *For every* $R, B > 0$, $|r| \leqq R$, $\|Z_r\|_{\bar{E}} \leqq B$, $|x - r| < \varepsilon(R, B)$ *and every segment* $\sigma \subset \mathbb{R}$ *the function* $Z(x, \cdot)|_\sigma$ *depends on* $Z_r|_\Sigma$ *only, where* $\Sigma$ *is a* $\delta$*-neighborhood of the segment* $\sigma$ *in* $\mathbb{R}$ *with* $\delta = |x - r|$.

iv) *The map* $G_{r,x} : Z_r|_\Sigma \mapsto Z(x, \cdot)|_\sigma$ *for* $\|Z_r\|_{\bar{\mathscr{E}}} \leqq B$ *is Lipschitz continuous from* $\mathscr{E}(\Sigma)$ *to* $\mathscr{E}(\sigma)$, *and*

$$\|G_{r,x}Z_r^1 - G_{r,x}Z_r^2\|_{\mathscr{E}(\sigma)} \leqq L(R, B)\|Z_r^1 - Z_r^2\|_{\mathscr{E}(\Sigma)} \quad \text{for } |r| \leqq R \text{ and } \delta = |x - r| \leqq \varepsilon(R, B) \tag{6.13}$$

*for every* $Z_r^j \in \bar{\mathscr{E}}$, $j = 1, 2$. *The Lipschitz constant* $L(R, B)$ *does not depend on the segment* $\sigma$.

**Proof.** The contraction-mapping principle implies the existence and uniqueness of the solution $Z(x)$ to (6.12) such that $Z(x) \in C(r - \varepsilon, r + \varepsilon; \mathscr{E}(\sigma))$ for every segment $\sigma \subset \mathbb{R}$. The crucial point is that $\varepsilon = \varepsilon(R, B) > 0$ does not depend on the segment $\sigma$ due to the uniform bounds for $\|Z_r\|_{\mathscr{E}(\sigma)}$ with bounded $|\sigma| \geqq 1$, and to the homogenuity of the problem in $t$.

The properties (iii) and (iv) follow from the same properties of the successive Picard approximations due to the corresponding properties of the operators $W^0_{x-s}$. □

*6.3. Proof of the Attraction in the Mean*

We deduce (6.2) from (6.3) and (6.13). We choose $\lambda(t) = y_-(n) \equiv u(-a, n)$ for $n \leqq t < n + 1$, $n = 0, 1, \ldots$ .

Step 1. The bounds (6.3) and the d'Alembert representation (6.4) imply the convergence of the integral $\int_{\bar{T}}^\infty \int_{a<|x|<R} \cdots$ in (6.2).

Step 2. The integral $\int_0^\infty |u(-a, t) - \lambda(t)|^2 \, dt$ also converges, because it is equal to

$$\sum_0^\infty \int_n^{n+1} |y_-(t) - y_-(n)|^2 \, dt \leqq \int_0^\infty |\dot{y}_-(t)|^2 \, dt < \infty.$$

Step 3. Let us verify the bound

$$\sum_{n=\bar{N}}^\infty \int_n^{n+1} \Big( \int_{-a}^a (|u'(x, t) - s'_{\lambda(n)}(x)|^2 + |\dot{u}(x, t)|^2) \, dx \Big) \, dt < \infty \tag{6.14}$$

for sufficiently large $\bar{N}$. Proposition 2.1 iv) means that the solution $Z(x) = (u(x, \cdot), u'(x, \cdot)) = G_{-a,x}(y_-(\cdot), z_-(\cdot))$ to the equation (6.12) satisfies

$$\|Z(r)\|_{\bar{\mathscr{E}}} \leqq \bar{B} = 2e(a) \quad \text{for } r \in [-a, a]. \tag{6.15}$$

On the other hand, the function $S_n(x) = (s_{\lambda(n)}(x), 0) = G_{-a,x}(y_-(n), 0)$ is also a solution to the equation (6.12) for every $n = 0, 1, \ldots$. Therefore, we can apply the Lipschitz continuity from Lemma 6.4 (iv) to estimate the difference between these two solutions.

**Lemma 6.5.** *For sufficiently large $n \geqq \bar{N}$ there exists the solution $S_n(x) = G_{-a,x}(y_-(n), 0)$ to the equation* (6.12) *and for every $x = -a + \delta \in [-a, a]$,*

$$\|Z(x) - S_n(x)\|^2_{\mathscr{E}([n,n+1])} \leqq \bar{L} \int_{n-\delta}^{n+1+\delta} (|z_-(t)|^2 + |\dot{y}_-(t)|^2)\, dt \quad \textit{for} \quad n \geqq \bar{N}. \tag{6.16}$$

We prove this lemma below. Summing (6.16) over $n \geqq \bar{N}$ and integrating the sum over $x \in [-a, a]$, we get (6.14) due to (6.3).

Step 4. $S_n(x) \in \mathscr{A}_{\bar{\alpha}\bar{\beta}}$ for sufficiently large $\bar{\alpha}, \bar{\beta} > 0$. Indeed, (6.14) and bounds (2.7) imply for sufficiently large $\bar{N}$, that

$$\bar{D} = \sup_{n \geqq \bar{N}} \int |s'_{\lambda(n)}(x)|^2 dx < \infty. \tag{6.17}$$

Moreover, (2.5) with $x = -a$ implies

$$\bar{d} = \sup_{n \geqq 0} |s_{\lambda(n)}(-a)| < \infty. \tag{6.18}$$

Hence, (as with (2.8)) (6.17) implies that

$$\sup_{n \geqq 0} |s_{\lambda(n)}(x)| \leqq \bar{\alpha} + \bar{\beta}\sqrt{x} \quad \text{for } |x| \leqq a, \tag{6.19}$$

for sufficiently large $\bar{\alpha}$ and $\bar{\beta}$. □

**Proof of Lemma 6.5.** Let us denote $\bar{\varepsilon} = \varepsilon(a, \bar{B})$ and prove the existence of the solution $S_n(x) = G_{-a,x}(y_-(n), 0)$ and the bounds (6.16) for $-a + (k-1)\bar{\varepsilon} \leqq x \leqq -a + k\bar{\varepsilon}$ by induction in $k = 1, 2, \ldots$ with $k \leqq 2a/\bar{\varepsilon} + 1$.

$k = 1$. For $-a \leqq x \leqq -a + \bar{\varepsilon}$ the existence of the solution

$$S_n(x) = G_{-a,x}(y_-(n), 0)$$

and the bounds (6.16) for all $n \geqq 0$ follow directly from (6.13) with $r = -a$ for two solutions $Z(x)$ and $S_n(x)$, because $Z(-a) - S_n(-a) = (y_-(\cdot) - y_-(n), z_-(\cdot))$ and

$$\|Z(-a) - S_n(-a)\|^2_{\mathscr{E}([n-\delta,n+1+\delta])} \leqq C(\delta) \int_{n-\delta}^{n+1+\delta} (|z_-(t)|^2 + |\dot{y}_-(t)|^2)\, dt. \tag{6.20}$$

The bounds (6.13) holds for the solutions because of (6.15) with $r = -a$ and because of a similar estimate for $S_n(-a) = (y_-(n), 0)$.

$k = 2$. For $-a + \bar{\varepsilon} \leqq x \leqq -a + 2\bar{\varepsilon}$ the existence of the solution $S_n(x) = G_{-a,x}(y_-(n), 0) = G_{-a+\bar{\varepsilon},x} S_n(-a + \bar{\varepsilon})$ and the bounds (6.16) follow by double application of (6.13) for sufficiently large $n \geqq N_1$, provided that

$$\|S_n(-a + \bar{\varepsilon})\|_{\mathscr{E}} \leqq \bar{B} \quad \text{for } n \geqq N_1. \tag{6.21}$$

Such an $N_1 < \infty$ exists due to (6.15) and the bound (6.16) with $x = -a + \bar{\varepsilon}$ proved above, because $\int_{n-\delta}^{n+1+\delta} (|z_-(t)|^2 + |\dot{y}_-(t)|^2)\, dt \to 0$ as $n \to \infty$ due to (6.3).

Induction in $k$ completes the proof. □

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