

# On a Two-Temperature Problem for Wave Equation

T.V. Dudnikova<sup>1,\*</sup>, A.I. Komech<sup>2,†</sup> and H. Spohn<sup>3</sup>

<sup>1</sup> Mathematics Department, Elektrostal Polytechnic Institute, Elektrostal 144000, Russia.  
E-mail: misis@elsite.ru

<sup>2</sup> Mechanics and Mathematics Department, Moscow State University, Moscow 119899, Russia. E-mail: komech@mech.math.msu.su

<sup>3</sup> Zentrum Mathematik, Technische Universität, München D-80290, Germany.  
E-mail: spohn@mathematik.tu-muenchen.de

Received August 28, 2001, revised December 12, 2001

*Dedicated to M.I. Vishik on the occasion of his 80th anniversary*

**Abstract.** Consider the wave equation with constant or variable coefficients in  $\mathbf{R}^3$ . The initial datum is a random function with a finite mean density of energy that also satisfies a Rosenblatt- or Ibragimov–Linnik-type mixing condition. The random function converges to different space-homogeneous processes as  $x_3 \rightarrow \pm\infty$ , with the distributions  $\mu_{\pm}$ . We study the distribution  $\mu_t$  of the random solution at a time  $t \in \mathbf{R}$ . The main result is the convergence of  $\mu_t$  to a Gaussian translation-invariant measure as  $t \rightarrow \infty$  that means central limit theorem for the wave equation. The proof is based on the Bernstein ‘room-corridor’ argument. The application to the case of the Gibbs measures  $\mu_{\pm} = g_{\pm}$  with two different temperatures  $T_{\pm}$  is given. Limiting mean energy current density *formally* is  $-\infty \cdot (0, 0, T_+ - T_-)$  for the Gibbs measures, and it is finite and equals  $-C(0, 0, T_+ - T_-)$  with  $C > 0$  for the convolution with a nontrivial test function.

KEYWORDS: wave equation, random function, Gaussian measure, correlation function, characteristic functional, Radon transformation

AMS SUBJECT CLASSIFICATION: 60Fxx, 60Gxx, 82xx

\*Supported partly by research grants of DFG (436 RUS 113/615/0-1) and of RFBR (01-01-04002)

†Supported partly by CONACYT (Mexico), by the Institute of Physics and Mathematics of the University of Michoacan (Morelia, Mexico), Max-Planck Institute for Mathematics in the Sciences (Leipzig, Germany), research grants of DFG (436 RUS 113/615/0-1) and of RFBR (01-01-04002)

## 1. Introduction

The paper concerns a mathematical problem of foundations of statistical physics. We consider the Second Law of thermodynamics in a reversible infinite dimensional Hamiltonian equations. The Law states that the energy current is directed from higher temperature to lower temperature. We derive the Law for wave equations in  $\mathbf{R}^3$  with constant and variable coefficients. The key role plays the mixing condition of Rosenblatt- or Ibragimov–Linnik-type for an initial measure. The mixing condition is introduced initially by R.L. Dobrushin and Yu.M. Suhov in their approach to the problem of foundation of statistical physics for infinite-particle systems, [4, 5]. The mixing condition is used also in the paper [2] which concerns a discrete version of our result for a 1D chain of harmonic oscillators. Let us explain our result in the case of constant coefficients,

$$\begin{cases} \ddot{u}(x, t) = \Delta u(x, t), & x \in \mathbf{R}^3, \\ u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x). \end{cases} \quad (1.1)$$

We use the notation  $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot, t), \dot{u}(\cdot, t))$ ,  $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0, v_0)$ . Then (1.1) becomes

$$\dot{Y}(t) = \mathcal{F}(Y(t)), \quad t \in \mathbf{R}, \quad Y(0) = Y_0. \quad (1.2)$$

We assume that the initial datum  $Y_0$  is a random function with zero mean living in a functional phase space  $\mathcal{H}$  of states of finite local energy; the distribution of  $Y_0$  is denoted by  $\mu_0$ . Denote by  $\mu_t(dY)$ ,  $t \in \mathbf{R}$ , the measure on  $\mathcal{H}$  giving the distribution of the random solution  $Y(t)$  to problem (1.2). We assume that the initial correlation functions  $Q_0^{ij}(x, y) \equiv E(Y_0^i(x)Y_0^j(y))$ ,  $i, j = 0, 1$ , and some of their derivatives are continuous and decaying as  $|x - y| \rightarrow \infty$ . In particular, the initial mean energy density is bounded:

$$\begin{aligned} & E[|\nabla u_0(x)|^2 + |v_0(x)|^2] \\ &= [\nabla_x \cdot \nabla_y Q_0^{00}(x, y)]|_{y=x} + Q_0^{11}(x, x) \leq C < \infty, \quad x \in \mathbf{R}^3. \end{aligned} \quad (1.3)$$

Next, we assume that the initial correlation matrix  $(Q_0^{ij}(x, y))_{i,j=0,1}$  has the form

$$Q_0^{ij}(x, y) = \begin{cases} q_-^{ij}(x - y), & x_3, y_3 < -a, \\ q_+^{ij}(x - y), & x_3, y_3 > a. \end{cases} \quad (1.4)$$

Here  $q_{\pm}^{ij}(x - y)$  are the correlation functions of some translation-invariant measures  $\mu_{\pm}$  with zero mean value in  $\mathcal{H}$ ,  $x = (x_1, x_2, x_3)$ ,  $y = (y_1, y_2, y_3) \in \mathbf{R}^3$ ,

and  $a > 0$ . The measure  $\mu_0$  is not translation-invariant if  $q_-^{ij} \neq q_+^{ij}$ . Finally, we assume that the initial measure  $\mu_0$  satisfies a mixing condition. Roughly speaking, it means that

$$Y_0(x) \text{ and } Y_0(y) \text{ are asymptotically independent as } |x - y| \rightarrow \infty. \quad (1.5)$$

Our main result establishes the (weak) convergence

$$\mu_t \rightharpoonup \mu_\infty, \quad t \rightarrow \infty, \quad (1.6)$$

to an equilibrium measure  $\mu_\infty$ , which is a translation-invariant Gaussian measure on  $\mathcal{H}$ . The similar convergence holds for  $t \rightarrow -\infty$  since our system is time-reversible. We construct generic examples of the random initial datum satisfying all assumptions imposed. We get the explicit formulas (2.13)–(2.15) for the limiting correlation matrices.

We apply our results to the case of the Gibbs measures  $\mu_\pm = g_\pm$ . Formally

$$\begin{aligned} g_\pm(du_0, dv_0) &= \frac{1}{Z_\pm} \exp\left(-\frac{\beta_\pm}{2} \int (|\nabla u_0(x)|^2 + |v_0(x)|^2) dx\right) \\ &\times \prod_x du_0(x) dv_0(x), \quad \beta_\pm = T_\pm^{-1}, \end{aligned} \quad (1.7)$$

where  $T_\pm \geq 0$  are the corresponding absolute temperatures. We adjust the definition of the Gibbs measures  $g_\pm$  in Section 3. The Gibbs measures  $g_\pm$  have singular correlation functions and do not satisfy our assumptions (2.10). Respectively, our results can not be applied directly to  $g_\pm$ . We reduce the problem by a convolution with a smooth function  $\theta \in D \equiv C_0^\infty(\mathbf{R}^3)$ : we consider Gaussian processes  $u_\pm$  corresponding to the measures  $g_\pm$  and define the “smoothened” measures  $g_\pm^\theta$  as the distributions of the convolutions  $u_\pm * \theta$ . The measures  $g_\pm^\theta$  satisfy all our assumptions, and the convergence  $g_t^\theta \rightharpoonup g_\infty^\theta$  follows from (1.6). This implies the weak convergence of the measures  $g_t \rightharpoonup g_\infty$  since  $\theta$  is arbitrary. We show that the limit energy current for  $g_\infty$  is formally

$$\bar{j}_\infty = -\infty \cdot (0, 0, T_+ - T_-).$$

The infinity means the “ultraviolet divergence”. This relation is meaningful in the case of smoothened measures  $g_\infty^\theta$ ,

$$\bar{j}_\infty^\theta = -C_\theta \cdot (0, 0, T_+ - T_-),$$

if  $\theta(x)$  is axially symmetric with respect to  $Ox_3$ ;  $C_\theta > 0$  if  $\theta(x) \not\equiv 0$ . This corresponds to the Second Law of thermodynamics.

We prove the convergence (1.6) in three steps using the strategy of [10,18,19].

- I. The family of measures  $\mu_t$ ,  $t \geq 0$ , is weakly compact in an appropriate Fréchet space.

II. The correlation functions converge to a limit: for  $i, j = 0, 1$ ,

$$Q_t^{ij}(x, y) = \int Y^i(x)Y^j(y)\mu_t(dY) \rightarrow Q_\infty^{ij}(x, y), \quad t \rightarrow \infty. \quad (1.8)$$

III. The characteristic functionals converge to the Gaussian:

$$\hat{\mu}_t(\Psi) = \int \exp(i\langle Y, \Psi \rangle)\mu_t(dY) \rightarrow \exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\right\}, \quad t \rightarrow \infty, \quad (1.9)$$

where  $\Psi$  is an arbitrary element of the dual space, and  $\mathcal{Q}_\infty$  is the quadratic form with the integral kernel  $(Q_\infty^{ij}(x, y))_{i,j=0,1}$ .

Property I follows from the Prokhorov Compactness Theorem by using methods of [23]. First, one proves a uniform bound for the mean local energy with respect to the measure  $\mu_t$ . The conditions of the Prokhorov theorem then follow from Sobolev's Embedding Theorem. We deduce the uniform bound from the explicit expression for the correlation functions  $Q_t^{ij}(x, y)$ . The expression follows from the Kirchhoff formula for the solutions to (1.1). In particular, in the case  $u_0(x) \equiv 0$ , we have

$$u(x, t) = \frac{1}{4\pi t} \int_{S_t(x)} v_0(x') dS(x'), \quad (1.10)$$

where  $dS(x')$  is the Lebesgue measure on the sphere  $S_t(x) : |x' - x| = t$ .

Property II also follows from explicit formulas for  $Q_t^{ij}(x, y)$ . The formula (1.10) allows to express the correlation functions  $Q_t^{ij}(x, y)$  in terms of integrals over spheres of radius  $t$ . In the limit,  $t \rightarrow \infty$ , the spheres become the planes. Respectively,  $Q_\infty^{ij}(x, y)$  is expressed in terms of integrals of the Radon transform of initial correlation functions  $Q_0^{ij}(x, y)$ . We reduce the expressions to some convolutions.

*Remark 1.1.* The dynamics (1.1) is translation invariant, and its Fourier transform has a very simple form. However, the proof of (1.8) in Fourier transform is not transparent and requires additional efforts since our main assumption (1.4) is stated in the coordinate space.

*Remark 1.2.* Our proof of the convergence (1.8) in Sections 5 and 6 does not allow a simplification in the particular case of the Gibbs measures (1.7). This is related to the slow long-range decay of the correlation function  $Q_0^{00}(x, y) \sim |x - y|^{-1}$ ,  $|x - y| \rightarrow \infty$ .

We deduce Property III using the method of [10]. The method is based on a modification of the Bernstein 'room-corridor' argument, and it is suggested by the structure of the Kirchhoff formula (1.10): roughly speaking, (1.10) is

“the sum” of weakly dependent random values divided by the square root of their “number”. This observation allows us to reduce the proof of (1.9) to the Lindeberg Central Limit Theorem, similarly to [10]. We do not consider the case  $n = 2$ : it requires a different approach since the strong Huyghen’s principle breaks down.

Let us note that our mixing condition is weaker than that in [10]: this is necessary in the application to the Gibbs measures (1.7). Namely, we introduce different mixing coefficients for partial derivatives of the random solution at  $t = 0$ : we assume that the long-range decay of the mixing coefficients depends on the order of the derivatives. Respectively, our proof requires new tools (see Sections 7, 9, 10). For instance, the splitting (7.15) and the bound (9.4) play a crucial role.

All the three steps I–III of the argument rely on the mixing condition. Simple examples show that the convergence to a Gaussian measure may fail when the the mixing condition fails (see [10]).

In conclusion, we extend the convergence in (1.6) to the equations with variable coefficients, that are constant outside a finite region. The extension follows immediately from our result for constant coefficients, using method of [10]. The method is based on the scattering theory for the solutions of infinite global energy, which is constructed in [10].

The paper is organized as follows. In Section 2 we formally state our main result. We apply it to the Gibbs measure in Section 3. Sections 4–10 deal with the case of constant coefficients: the compactness (Property I) and the convergence (1.8) are proved in Sections 4–6. In Section 7 we introduce the ‘room-corridor’ method, in Section 8 we prove the convergence (1.9), and in Sections 9, 10 we check the Lindeberg condition. In Section 11 we establish the convergence (1.6) for variable coefficients. Appendix 11 concerns the Radon transform and convolution, and Appendix A concerns the Gaussian measures in the weighted Sobolev spaces.

Let us note that equation (1.1) describes a continuous  $n$ -dimensional family of harmonic oscillators. Therefore, our result is an extension of the results [2, 21] that concern the infinite one-dimensional chains of harmonic oscillators.

Our formulas for the limit correlation functions correspond to the discrete one-dimensional version [2]. For instance, the position-momentum correlations have a power long-range decay. On the other hand, in [16, 20] the limit correlation functions are constructed for the finite chains of  $N$  oscillators with the “Langevin” boundary value conditions. In the limit  $N \rightarrow \infty$  the correlation functions have an exponential long-range decay. This means that this limit leads to another stationary measure of the infinite chain, different from [2, 21].

The convergence to statistical equilibrium for the wave equation is established in [10] (see also [18, 19]) for the case of a translation-invariant initial measure  $\mu_0$ . This corresponds to our result in the particular case  $T_- = T_+$ . The similar result has been proved for the Klein–Gordon equation, [9, 14]. If

the initial measure  $\mu_0$  coincides with one of the equilibrium limit measures  $\mu_\infty$ , the corresponding random solution  $Y(t)$  is mixing in time, [6–8].

## 2. Main results

### 2.1. Notations

We assume that the initial datum  $Y_0$  belongs to the phase space  $\mathcal{H}$  defined below.

**Definition 2.1.**  $\mathcal{H} \equiv H_{loc}^1(\mathbf{R}^3) \oplus H_{loc}^0(\mathbf{R}^3)$  is the Fréchet space of pairs  $Y \equiv (u(x), v(x))$  of real functions  $u(x), v(x)$ , endowed with the local energy seminorms

$$\|Y\|_R^2 = \int_{|x| < R} (|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2) dx < \infty, \quad \forall R > 0. \quad (2.1)$$

Proposition 2.1 follows from [15, Theorems V.3.1, V.3.2] as the speed of propagation for equation (1.1) is finite.

#### Proposition 2.1.

- i) For any  $Y_0 \in \mathcal{H}$  there exists a unique solution  $Y(t) \in C(\mathbf{R}, \mathcal{H})$  to the Cauchy problem (1.2).
- ii) For any  $t \in \mathbf{R}$ , the operator  $U(t) : Y_0 \mapsto Y(t)$  is continuous in  $\mathcal{H}$ .
- iii) The energy inequalities hold for any  $R > 0$ ,

$$\|U(t)Y_0\|_R \leq C(t)\|Y_0\|_{R+|t|}, \quad t \in \mathbf{R}. \quad (2.2)$$

Let us choose a function  $\zeta(x) \in C_0^\infty(\mathbf{R}^3)$  with  $\zeta(0) \neq 0$ . Denote by  $H_{loc}^s(\mathbf{R}^3)$ ,  $s \in \mathbf{R}$ , the local Sobolev spaces, i.e. the Fréchet spaces of distributions  $u \in D'(\mathbf{R}^3)$  with the finite seminorms

$$\|u\|_{s,R} := \|\Lambda^s(\zeta(x/R)u)\|_{L^2(\mathbf{R}^3)},$$

where  $\Lambda^s v := F_{k \rightarrow x}^{-1}(\langle k \rangle^s \hat{v}(k))$ ,  $\langle k \rangle := \sqrt{|k|^2 + 1}$ , and  $\hat{v} := Fv$  is the Fourier transform of a tempered distribution  $v$ . For  $\psi \in D \equiv C_0^\infty(\mathbf{R}^3)$  define  $F\psi(k) = \int \exp(ik \cdot x)\psi(x) dx$ .

**Definition 2.2.** For  $s \in \mathbf{R}$  denote  $\mathcal{H}^s \equiv H_{loc}^{1+s}(\mathbf{R}^3) \oplus H_{loc}^s(\mathbf{R}^3)$ .

Using the standard techniques of pseudodifferential operators and Sobolev's theorem (see, e.g. [13]), it is possible to prove that  $\mathcal{H}^0 = \mathcal{H} \subset \mathcal{H}^{-\varepsilon}$  for every  $\varepsilon > 0$ , and the embedding is compact. We denote by  $\langle \cdot, \cdot \rangle$  the scalar product in real Hilbert space  $L^2(\mathbf{R}^3)$  or in  $L^2(\mathbf{R}^3) \otimes \mathbf{R}^N$  or in its various extensions.

## 2.2. Random solution. Convergence to equilibrium

Let  $(\Omega, \sigma, \mathbf{P})$  be a probability space with the expectation  $\mathbf{E}$ , let  $\mathcal{B}(\mathcal{H})$  denote the Borel  $\sigma$ -algebra in  $\mathcal{H}$ . We assume that  $Y_0 = Y_0(\omega, x)$  in (1.2) is a measurable random function with values in  $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ . In other words,  $(\omega, x) \mapsto Y_0(\omega, x)$  is a measurable map  $\Omega \times \mathbf{R}^3 \rightarrow \mathcal{H}$  with respect to the (completed)  $\sigma$ -algebra  $\Sigma \times \mathcal{B}(\mathbf{R}^3)$  and  $\mathcal{B}(\mathcal{H})$ . Then  $Y(t) = U(t)Y_0$  is also a measurable random function with values in  $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ , due to Proposition 2.1. We denote by  $\mu_0(dY_0)$  the Borel probability measure in  $\mathcal{H}$  that is the distribution of  $Y_0$ . Without loss of generality, we assume  $(\Omega, \Sigma, \mathbf{P}) = (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_0)$  and  $Y_0(\omega, x) = \omega(x)$  for  $\mu_0(d\omega) \times dx$ -almost all  $(\omega, x) \in \mathcal{H} \times \mathbf{R}^3$ .

**Definition 2.3.**  $\mu_t$  is the Borel probability measure in  $\mathcal{H}$  that is the distribution of  $Y(t)$ :

$$\mu_t(B) = \mu_0(U(-t)B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbf{R}. \quad (2.3)$$

Our main goal is to derive the convergence of the measures  $\mu_t$  as  $t \rightarrow \infty$ . We establish the weak convergence of  $\mu_t$  in the Fréchet spaces  $\mathcal{H}^{-\varepsilon}$  with any  $\varepsilon > 0$ :

$$\mu_t \xrightarrow{\mathcal{H}^{-\varepsilon}} \mu_\infty \quad \text{as } t \rightarrow \infty, \quad (2.4)$$

where  $\mu_\infty$  is the Borel probability measure in the space  $\mathcal{H}$ . By definition, this means the convergence

$$\int f(Y) \mu_t(dY) \rightarrow \int f(Y) \mu_\infty(dY) \quad \text{as } t \rightarrow \infty \quad (2.5)$$

for any bounded continuous functional  $f(Y)$  in the space  $\mathcal{H}^{-\varepsilon}$ .

**Definition 2.4.** The correlation functions of the measure  $\mu_t$  are defined by

$$Q_t^{ij}(x, y) \equiv \mathbf{E} (Y^i(x, t) Y^j(y, t)), \quad i, j = 0, 1, \quad (2.6)$$

for almost all  $x, y \in \mathbf{R}^3 \times \mathbf{R}^3$  if the expectations in the right-hand side are finite.

We set  $\mathcal{D} = D \oplus D$ , and  $\langle Y, \Psi \rangle = \langle Y^0, \Psi^0 \rangle + \langle Y^1, \Psi^1 \rangle$  for  $Y = (Y^0, Y^1) \in \mathcal{H}$ , and  $\Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$ . For a Borel probability measure  $\mu$  in the space  $\mathcal{H}$  we denote by  $\hat{\mu}$  the characteristic functional (Fourier transform)

$$\hat{\mu}(\Psi) \equiv \int \exp(i\langle Y, \Psi \rangle) \mu(dY), \quad \Psi \in \mathcal{D}.$$

A measure  $\mu$  is called Gaussian (with zero expectation) if its characteristic functional has the form

$$\hat{\mu}(\Psi) = \exp\left(-\frac{1}{2}Q(\Psi, \Psi)\right), \quad \Psi \in \mathcal{D},$$

where  $\mathcal{Q}$  is a real nonnegative quadratic form in  $\mathcal{D}$ . A measure  $\mu$  is called translation-invariant if

$$\mu(T_h B) = \mu(B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \quad h \in \mathbf{R}^3,$$

where  $T_h Y(x) = Y(x - h)$ .

### 2.3. Mixing condition

Let  $O(r)$  denote the set of all pairs of open subsets  $\mathcal{A}, \mathcal{B} \subset \mathbf{R}^3$  of distance  $\rho(\mathcal{A}, \mathcal{B}) \geq r$ , let  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  with integers  $\alpha_i \geq 0$ . Denote by  $\sigma_{i\alpha}(\mathcal{A})$  the  $\sigma$ -algebra of the subsets in  $\mathcal{H}$  generated by all linear functionals

$$Y \mapsto \langle D^\alpha Y^i, \psi \rangle = \int_{\mathbf{R}^3} D^\alpha Y^i(x) \psi(x) dx, \quad |\alpha| \leq 1 - i, \quad i = 0, 1,$$

where  $\psi \in D$  with  $\text{supp } \psi \subset \mathcal{A}$ . For  $d = 0, 1$  let  $\sigma_d$  be the  $\sigma$ -algebra generated by  $\sigma_{i\alpha}$  with  $i + |\alpha| \geq d$ , i.e.

$$\sigma_d \equiv \bigvee_{i+|\alpha| \geq d} \sigma_{i\alpha}, \quad d = 0, 1.$$

We define the Ibragimov–Linnik mixing coefficient of a probability measure  $\mu_0$  on  $\mathcal{H}$  (cf. [11, Definition 17.2.2]) for  $d_1, d_2 = 0, 1$  as follows:

$$\phi_{d_1, d_2}(r) \equiv \sup_{(\mathcal{A}, \mathcal{B}) \in O(r)} \sup_{\substack{A \in \sigma_{d_1}(\mathcal{A}), B \in \sigma_{d_2}(\mathcal{B}) \\ \mu_0(B) > 0}} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$

**Definition 2.5.** The measure  $\mu_0$  satisfies the strong uniform Ibragimov–Linnik mixing condition if for any  $d_1, d_2 = 0, 1$

$$\phi_{d_1, d_2}(r) \rightarrow 0, \quad r \rightarrow \infty. \quad (2.7)$$

Below we specify the rate of the decay.

### 2.4. Main theorem

Let  $\nu_d \in C[0, \infty)$  denote some continuous nonnegative nonincreasing functions in  $[0, \infty)$  ( $d = 0, 1, 2$ ) with the finite integrals,

$$\int_0^\infty (1+r)^{d-1} \nu_d(r) dr < \infty. \quad (2.8)$$

We also denote  $\nu(r) = \nu_2(r)$ . We assume that the measure  $\mu_0$  satisfies the following conditions S0–S3:



S0.  $\mu_0$  has the zero expectation value,

$$\mathbb{E} Y_0(x) = 0, \quad x \in \mathbf{R}^3. \quad (2.9)$$

S1. The correlation functions of  $\mu_0$  have the form (1.4).

S2. The following derivatives are continuous and the bounds hold,

$$|D_{x,y}^{\alpha,\beta} Q_0^{ij}(x,y)| \leq \begin{cases} C\nu_d(|x-y|) & \text{if } d = 0 \text{ or } 1, \\ C\nu_2(|x-y|) & \text{if } 2 \leq d \leq 4, \end{cases} \quad (2.10)$$

$$d = i + j + |\alpha| + |\beta|.$$

S3. The measure  $\mu_0$  satisfies the *strong uniform* Ibragimov–Linnik mixing condition, and for  $d_1, d_2 = 0, 1$

$$\phi_{d_1, d_2}(r) \leq C\nu_d^2(r), \quad d = d_1 + d_2. \quad (2.11)$$

*Remark 2.1.* Condition S2 implies (1.3). Condition S3 implies estimates (2.10) with  $i + |\alpha| \leq 1, j + |\beta| \leq 1$ .

*Remark 2.2.* Conditions S2 and S3 allow various modifications. We choose the variant which allow an application to the case of the Gibbs measures (1.7) (see the next section). Our mixing condition S3 is weaker than the mixing condition [10] which corresponds to S3 with the functions  $\nu_{0,1}(r) \leq \nu_2(r)$ . On the other hand, the estimates (2.10) with  $d > 2$  are not required in [10].

Let  $\mathcal{E}(x) = -(4\pi|x|)^{-1}$  be the fundamental solution of the Laplacian, i.e.  $\Delta\mathcal{E} = \delta(x)$  for  $x \in \mathbf{R}^3$ , and  $P(x) = -iF^{-1}\text{sgn } k_3/|k|$  where  $F^{-1}$  is the inverse Fourier transform. Define, for almost all  $x, y \in \mathbf{R}^3$ , the matrix-valued function

$$Q_\infty(x, y) = (Q_\infty^{ij}(x, y))_{i,j=0,1} = (q_\infty^{ij}(x - y))_{i,j=0,1}, \quad (2.12)$$

where

$$q_\infty^{00} = \frac{1}{4} [q_+^{00} + q_-^{00} - \mathcal{E} * (q_+^{11} + q_-^{11}) + P * (q_+^{01} - q_-^{01} - q_+^{10} + q_-^{10})], \quad (2.13)$$

$$q_\infty^{10} = -q_\infty^{01} = \frac{1}{4} [q_+^{10} + q_-^{10} - q_+^{01} - q_-^{01} + P * (q_+^{11} - q_-^{11} - \Delta q_+^{00} + \Delta q_-^{00})], \quad (2.14)$$

$$q_\infty^{11} = -\Delta q_\infty^{00} = \frac{1}{4} [q_+^{11} + q_-^{11} - \Delta(q_+^{00} + q_-^{00}) + P * \Delta(q_+^{10} - q_-^{10} - q_+^{01} + q_-^{01})]. \quad (2.15)$$

The definition of the convolutions with  $P$  in formulas (2.13)–(2.15) is adjusted in Appendix 11 (formula (6.10)).

Denote by  $\mathcal{Q}_\infty(\Psi, \Psi)$  the real quadratic form in  $\mathcal{D}$  defined by

$$\mathcal{Q}_\infty(\Psi, \Psi) = \sum_{i,j=0,1} \int_{\mathbf{R}^3 \times \mathbf{R}^3} Q_\infty^{ij}(x, y) \Psi^i(x) \Psi^j(y) dx dy. \quad (2.16)$$

Our main result is the following theorem.

**Theorem 2.1.** *Let S0–S3 hold. Then*

- i) *the convergence in (2.4) holds for any  $\varepsilon > 0$ ;*
- ii) *the limiting measure  $\mu_\infty$  is a Gaussian equilibrium measure on  $\mathcal{H}$ ;*
- iii) *the limiting characteristic functional has the form*

$$\hat{\mu}_\infty(\Psi) = \exp\left(-\frac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\right), \quad \Psi \in \mathcal{D},$$

where  $\mathcal{Q}_\infty$  is the quadratic form with the integral kernel  $Q_\infty(x, y)$  defined in (2.12)–(2.15).

Theorem 2.1 can be deduced from Propositions 2.2 and 2.3 below, by using the same arguments as in [23, Theorem XII.5.2].

**Proposition 2.2.** *The family of the measures  $\{\mu_t, t \geq 0\}$  is weakly compact in  $\mathcal{H}^{-\varepsilon}$  with any  $\varepsilon > 0$ .*

**Proposition 2.3.** *For any  $\Psi \in \mathcal{D}$ ,*

$$\hat{\mu}_t(\Psi) \equiv \int \exp(i\langle Y, \Psi \rangle) \mu_t(dY) \longrightarrow \exp\left(-\frac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\right), \quad t \rightarrow \infty. \quad (2.17)$$

Proposition 2.2 is proved in Section 4 for a simple particular case, and in Section 6 for the general case. Proposition 2.3 is proved in Sections 7, 8.

## 2.5. Examples

### 2.5.1. Gaussian measures

We construct the Gaussian initial measures  $\mu_0$  satisfying S0–S3. Let us take some Gaussian measures  $\mu_\pm$  in  $\mathcal{H}$  with correlation functions  $q_\pm^{ij}(x - y)$  which are zero for  $i \neq j$ , while for  $i = 0, 1$ ,

$$\begin{aligned} q_\pm^{ii}(z) &= F^{-1} \hat{q}_\pm^{ii}(k), & (2.18) \\ (1 + |k|)^s \partial_k^\gamma \hat{q}_\pm^{ii}(k) &\in L^1(\mathbf{R}^3), \quad 0 \leq d = 2i + s \leq 4, \quad |\gamma| \leq 1 + d, \\ \hat{q}_\pm^{ii}(k) &\geq 0. \end{aligned}$$

Then  $\mu_{\pm}$  satisfy S0–S2 with the functions  $\nu_d(r) = C(1+r)^{-1-d}$  for a sufficiently large  $C > 0$ . Let us take the functions  $\zeta_{\pm} \in C^{\infty}(\mathbf{R})$  such that

$$\zeta_{\pm}(s) = \begin{cases} 1, & \text{for } \pm s > a, \\ 0, & \text{for } \pm s < -a. \end{cases}$$

Let us introduce  $(Y_-, Y_+)$  as a unit random function in the probability space  $(\mathcal{H} \times \mathcal{H}, \mu_- \times \mu_+)$ . Then  $Y_{\pm}$  are Gaussian independent vectors in  $\mathcal{H}$ . Define  $\mu_0$  as the distribution of the random function

$$Y_0(x) = \zeta_-(x_3)Y_-(x) + \zeta_+(x_3)Y_+(x). \quad (2.19)$$

Then correlation functions of  $\mu_0$  are

$$Q_0^{ij}(x, y) = q_-^{ij}(x - y)\zeta_-(x_3)\zeta_-(y_3) + q_+^{ij}(x - y)\zeta_+(x_3)\zeta_+(y_3), \quad i, j = 0, 1, \quad (2.20)$$

where  $x = (x_1, x_2, x_3)$ ,  $y = (y_1, y_2, y_3) \in \mathbf{R}^3$ ,  $q_{\pm}^{ij}$  are the correlation functions of the measures  $\mu_{\pm}$ . Then S0 and S1 hold, and S2 follows for  $\mu_0$  with the same functions  $\nu_d(r)$  as for  $\mu_{\pm}$ . Let us assume, in addition to (2.19), that

$$q_{\pm}^{ii}(x) = 0, \quad |x| \geq r_0. \quad (2.21)$$

Then the mixing condition (2.7) holds since  $\phi_{d_1, d_2}(r) = 0$ ,  $r \geq r_0$ , and S3 follows. For instance, (2.19) and (2.21) hold if  $\hat{q}_{\pm}^{ii}(k_1, k_2, k_3) = f(k_1)f(k_2)f(k_3)$  with

$$f(z) = ((1 - \cos(r_0 z / \sqrt{3})) / z^2)^N, \quad z \in \mathbf{R},$$

where  $N \geq 0$  is an integer,  $2N - s > 1$  ( $s = 4 - 2i$ ).

### 2.5.2. Non-Gaussian measures

Let us choose some odd nonconstant functions  $f^0, f^1 \in C^4(\mathbf{R})$  with bounded derivatives. Let us define  $\mu_0^*$  as the distribution of the random function

$$(f^0(Y^0(x)), f^1(Y^1(x))),$$

where  $(Y^0, Y^1)$  is a random function with the Gaussian distribution  $\mu_0$  from the previous example. Then S0, S1 and S3 hold for  $\mu_0^*$  with some appropriate functions  $\nu_d$  since corresponding mixing coefficients  $\phi_{d_1, d_2}^*(r) = 0$  for  $r \geq r_0$ . Therefore, S0 implies for the corresponding correlation functions  $Q_0^*(x, y) = 0$  for  $|x - y| \geq r_0$ , so S2 also holds. The measure  $\mu_0^*$  is not Gaussian since the functions  $f^0, f^1$  are bounded and nonconstant.

## 3. Application to Gibbs measures

We apply Theorem 2.1 to the case when  $\mu_{\pm}$  are the Gibbs measures (1.7) corresponding to different positive temperatures  $T_- \neq T_+$ .

### 3.1. Gibbs measures

We will define the Gibbs measures  $g_{\pm}$  as the Gaussian measures with the correlation functions (cf. (1.7))

$$\begin{aligned} q_{\pm}^{00}(x-y) &= -T_{\pm}\mathcal{E}(x-y), \\ q_{\pm}^{11}(x-y) &= T_{\pm}\delta(x-y), \\ q_{\pm}^{01}(x-y) &= q_{\pm}^{10}(x-y) = 0, \end{aligned} \quad (3.1)$$

where  $x, y \in \mathbf{R}^3$ . The correlation functions  $q_{\pm}^{ij}$  do not satisfy condition S2 because of singularity at  $x = y$ . The singularity means that the measures  $g_{\pm}$  are not concentrated in the space  $\mathcal{H}$ . Let us introduce appropriate functional spaces for measures  $g_{\pm}$ . First, let us define the weighted Sobolev space with any  $s, \alpha \in \mathbf{R}$ .

**Definition 3.1.**  $H_{s,\alpha}(\mathbf{R}^3)$  is the Hilbert space of the distributions  $u \in S'(\mathbf{R}^3)$  with the finite norm

$$\|u\|_{s,\alpha} \equiv \|\langle x \rangle^{\alpha} \Lambda^s u\|_{L_2(\mathbf{R}^3)} < \infty, \quad \Lambda^s u \equiv F^{-1}[\langle k \rangle^s \hat{u}(k)]. \quad (3.2)$$

Let us fix arbitrary  $s, \alpha < -3/2$ .

**Definition 3.2.**  $G_{s,\alpha}$  is the Hilbert space  $H_{s+1,\alpha}(\mathbf{R}^3) \oplus H_{s,\alpha}(\mathbf{R}^3)$ , with the norm

$$\|Y\|_{s,\alpha} \equiv \|u\|_{s+1,\alpha} + \|v\|_{s,\alpha} < \infty, \quad Y = (u, v).$$

Introduce the Gaussian Borel probability measures  $g_{\pm}^0(du)$ ,  $g_{\pm}^1(dv)$  in spaces  $H_{s+1,\alpha}(\mathbf{R}^3)$  and  $H_{s,\alpha}(\mathbf{R}^3)$ , respectively, with characteristic functionals

$$\begin{aligned} \hat{g}_{\pm}^0(\psi) &= \int \exp\{i\langle u, \psi \rangle\} g_{\pm}^0(du) = \exp\left\{\frac{\langle \Delta^{-1}\psi, \psi \rangle}{2\beta_{\pm}}\right\}, \\ \hat{g}_{\pm}^1(\psi) &= \int \exp\{i\langle v, \psi \rangle\} g_{\pm}^1(dv) = \exp\left\{-\frac{\langle \psi, \psi \rangle}{2\beta_{\pm}}\right\}, \end{aligned}$$

$\psi \in D$ . By the Minlos theorem, [3], the Borel probability measures  $g_{\pm}^0$ ,  $g_{\pm}^1$  exist in the spaces  $H_{s+1,\alpha}(\mathbf{R}^3)$ ,  $H_{s,\alpha}(\mathbf{R}^3)$ , respectively, because *formally* (see Appendix A)

$$\int \|u\|_{s+1,\alpha}^2 g_{\pm}^0(du) < \infty, \quad \int \|v\|_{s,\alpha}^2 g_{\pm}^1(dv) < \infty, \quad s, \alpha < -3/2. \quad (3.3)$$

Finally, we define the Gibbs measures  $g_{\pm}(dY)$  as the Borel probability measures  $g_{\pm}^0(du) \times g_{\pm}^1(dv)$  in  $G_{s,\alpha}$ . Let  $g_0(dY)$  be the Borel probability measure in  $G_{s,\alpha}$  that is constructed as in the example of the previous section with  $\mu_{\pm}(dY) = g_{\pm}(dY)$ . It satisfies S0 and S1 with  $q_{\pm}^{ij}$  from (3.1). However,  $g_0$  does

not satisfy S2. Therefore, Theorem 2.1 cannot be applied directly to  $\mu_0 = g_0$ .  $G_{s,\alpha} \subset \mathcal{H}^s$  by the standard arguments of pseudodifferential equations, [13]. The next lemma follows by Fourier transform from the finite speed of propagation for wave equation.

**Lemma 3.1.** *The operators  $U(t) : Y_0 \mapsto Y(t)$  allow a continuous extension  $\mathcal{H}^s \mapsto \mathcal{H}^s$ .*

### 3.2. Convergence to equilibrium

Let  $Y_0$  be the random function with the distribution  $g_0$ , hence  $Y_0 \in G_{s,\alpha}$  a.s. Denote by  $g_t$  the distribution of  $U(t)Y_0$ .

**Theorem 3.1.** *Let  $s < -5/2$ . Then there exists a Gaussian Borel probability measure  $g_\infty$  in  $\mathcal{H}^s$  such that*

$$g_t \xrightarrow{\mathcal{H}^s} g_\infty, \quad t \rightarrow \infty. \quad (3.4)$$

*Proof.* Let us fix an  $s < -5/2$  and introduce the random function  $Y_0^s := \Lambda^s Y_0$ ,  $Y_0^s \in G_{0,\alpha}$  a.s. Let us denote by  $g_t^s$  the distribution of  $U(t)(\Lambda^s Y_0)$ ,  $t \in \mathbf{R}$ . Then  $g_t^s = g_t \Lambda^{-s}$ , and  $g_t = g_t^s \Lambda^s$  since  $\Lambda^s(U(t)Y_0) = U(t)(\Lambda^s Y_0)$ . Let us denote by  $Q_t^s(x, y)$  the (matrix) correlation function of measure  $g_t^s$ .

The measure  $g_0^s$  obviously satisfies S0. The correlation function  $Q_0^s(x, y)$  also satisfies S1 with a suitable modification: (1.4) holds up to  $\delta(1 + |x| + |y|)^{-N}$  with any  $\delta, N > 0$  and with  $a = a(\delta)$ . This follows from the convolution representation  $Q_0^s(x, y) = Q_0(x, y) * (\Lambda_s(x)\Lambda_s(y))$  since  $\Lambda_s(x) \equiv F^{-1}\langle k \rangle^s$  is a function from  $L_{loc}^1(\mathbf{R}^3)$  with a rapid long-range decay. S2 also holds for  $g_0^s$  with the functions  $\nu_a(r) = C(1 + r)^{-1-d}$  for a sufficiently large  $C = C(s, T_\pm) > 0$ . It follows immediately for  $s < 0$  with sufficiently large  $|s|$  from the same convolution representation. For  $s < -5/2$  it follows by the pseudodifferential operators techniques.

Then the conclusions of Lemmas 4.1, 5.1 hold for the random function  $Y_0^s$  and the correlation functions  $Q_t^s(x, y)$  of the measures  $g_t^s$ . The proofs are almost unchanged. Hence, the convergence (2.4) holds for the Gaussian measures  $g_t^s$ : for any  $\varepsilon > 0$ ,

$$g_t^s \xrightarrow{\mathcal{H}^{-\varepsilon}} g_\infty^s, \quad t \rightarrow \infty, \quad (3.5)$$

where  $g_\infty^s$  is a Gaussian measure in  $\mathcal{H}$ . Therefore,

$$g_t \xrightarrow{\mathcal{H}^{s-\varepsilon}} g_\infty, \quad t \rightarrow \infty,$$

since  $g_t = g_t^s \Lambda^s$ . This implies Theorem 3.1.  $\square$

The limiting measure  $g_\infty$  is Gaussian with the correlation matrix  $Q_\infty = (Q_\infty^{ij}(x, y))_{i,j=0,1}$ , where

$$Q_\infty^{00}(x, y) \equiv q_\infty^{00}(x - y) = -\frac{1}{2}(T_+ + T_-)\mathcal{E}(x - y), \quad (3.6)$$

$$Q_\infty^{10}(x, y) = -Q_\infty^{01}(x, y) \equiv q_\infty^{10}(x - y) = \frac{1}{2}(T_+ - T_-)P(x - y), \quad (3.7)$$

$$Q_\infty^{11}(x, y) \equiv q_\infty^{11}(x - y) = \frac{1}{2}(T_+ + T_-)\delta(x - y). \quad (3.8)$$

The identities (3.6)–(3.8) follow formally from (3.1) and from (2.13)–(2.15). For the proof we apply (2.13)–(2.15) to the initial measure  $g_0^s$ .

### 3.3. Limit energy current density

Let  $u(x, t)$  be the random solution to (1.1) with the initial measure  $\mu_0$  satisfying S0–S3. The mean energy current density is  $\mathbf{E} j(x, t) = -\mathbf{E} \dot{u}(x, t) \nabla u(x, t)$ . Therefore, in the limit  $t \rightarrow \infty$ ,

$$\mathbf{E} j(x, t) \rightarrow \bar{j}_\infty = \nabla q_\infty^{10}(0).$$

Respectively, in the case of the ‘‘Gibbs’’ initial measure  $g_0$ , the expression (3.7) for the limiting correlation function implies *formally* that

$$\bar{j}_\infty = \frac{T_+ - T_-}{2} \nabla P(0),$$

where  $[\nabla P](z) = -F^{-1}[k \operatorname{sgn} k_3 / |k|](z)$ . Hence, formally we have the ‘‘ultraviolet diverging’’ limit mean energy current density,

$$\bar{j}_\infty = -\frac{T_+ - T_-}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{k \operatorname{sgn} k_3}{|k|} dk = -\infty \cdot (0, 0, T_+ - T_-).$$

On the other hand, for the convolution  $U(t)(Y_0 * \theta)$  the corresponding limiting mean energy current density is finite,

$$\bar{j}_\infty^\theta = -\frac{T_+ - T_-}{2(2\pi)^3} \int_{\mathbb{R}^3} |\hat{\theta}(k)|^2 \frac{k \operatorname{sgn} k_3}{|k|} dk = -C_\theta \cdot (0, 0, T_+ - T_-),$$

if  $\theta(x)$  is axially symmetric with respect to  $Ox_3$ ;  $C_\theta > 0$  if  $\theta(x) \not\equiv 0$ .

### 4. Compactness of the measures family

Proposition 2.2 can be deduced from the bound (4.1) below with the help of the Prokhorov theorem [23, Lemma II.3.1] as in [23, Theorem XII.5.2].

**Lemma 4.1.** *Let S0–S2 hold. Then the following bounds hold*

$$\sup_{t \geq 0} \mathbf{E} \|U(t)Y_0\|_R^2 < \infty, \quad R > 0. \quad (4.1)$$

*Proof.* Assumption S2 and Proposition 2.1 iii) imply by the Fubini theorem the existence of the correlation functions in (2.6), where  $Y^i(x, t)$  are the components of  $Y(x, t) = (Y^0(x, t), Y^1(x, t))$ . Therefore, Definition 2.1 implies

$$\begin{aligned} \mathbf{E} \|Y(\cdot, t)\|_R^2 &= \mathbf{E} \int_{|x| < R} |Y^0(x, t)|^2 dx + \mathbf{E} \int_{|x| < R} |\nabla Y^0(x, t)|^2 dx \\ &\quad + \mathbf{E} \int_{|x| < R} |Y^1(x, t)|^2 dx \\ &= \int_{|x| < R} Q_t^{00}(x, x) dx + \int_{|x| < R} \nabla_x \cdot \nabla_y Q_t^{00}(x, y)|_{y=x} dx \\ &\quad + \int_{|x| < R} Q_t^{11}(x, x) dx. \end{aligned} \quad (4.2)$$

We bound for example the integral of  $Q_t^{00}(x, x)$  in (4.2) in the particular case when  $Y_0^0 \equiv u_0(x) = 0$  almost surely. The general case will be considered in Section 6 as well as the bounds for two remaining integrals in (4.2). Let us assume for a moment that the function  $Y_0^1 \equiv v_0$  is continuous almost surely. Then the Kirchhoff formula (1.10) gives by the Fubini theorem,

$$Q_t^{00}(x, x) = \frac{1}{(4\pi t)^2} \int_{S_t(x) \times S_t(x)} Q_0^{11}(x', x'') dS(x') dS(x''). \quad (4.3)$$

Let us assume for a moment that

$$Q_0^{ij}(x', x'') = 0 \quad \text{for } |x' - x''| \geq r_0, \quad i, j = 0, 1. \quad (4.4)$$

Then (4.3) implies the uniform bound

$$Q_t^{00}(x, x) \leq \frac{C}{t^2} \int_{\substack{S_t(x) \times S_t(x) \\ |x' - x''| \leq r_0}} dS(x') dS(x'') \leq I = C_1 r_0^2, \quad t \in \mathbf{R}. \quad (4.5)$$

Hence, the bound follows,

$$\int_{|x| < R} Q_t^{00}(x, x) dx \leq C I R^3, \quad t \in \mathbf{R}. \quad (4.6)$$

Next we remove the additional assumption (4.4) by the following known lemma on spherical integral identity, [12].

**Lemma 4.2.** *Let  $h(r) \in C(0, +\infty)$ . Then for any  $r_0 \geq 0$  and  $x'' \in S_t(x)$  the identity holds,*

$$\int_{\{x' \in S_t(x): |x' - x''| \geq r_0\}} h(|x' - x''|) dS(x') = 2\pi \int_{r_0}^{2t} r h(r) dr. \quad (4.7)$$

Therefore, (4.3), S2 with  $d = 2$  and Lemma 4.2 with  $r_0 = 0$  imply (see (2.8)),

$$\begin{aligned} Q_t^{00}(x, x) &\leq \frac{C}{(4\pi t)^2} \int_{S_t(x) \times S_t(x)} \nu_2(|x' - x''|) dS(x') dS(x'') \\ &\leq C_1 \int_0^{2t} r \nu_2(r) dr \leq C_2 < \infty. \end{aligned} \quad (4.8)$$

Then (4.6) follows without the assumption (4.4). The assumption on the a.s. continuity of  $v_0(x)$  can be removed by a convolution with a function  $\theta \in D$ .  $\square$

## 5. Convergence of correlation functions

Here we prove the convergence (1.8) of the correlation functions of measure  $\mu_t$ . This implies the convergence of the characteristic functionals  $\hat{\mu}_t$  in the case of Gaussian measures  $\mu_0, \mu_{\pm}$ .

**Lemma 5.1.** *Let S0–S2 hold. The following convergence holds as  $t \rightarrow \infty$*

$$Q_t^{ij}(x, y) \rightarrow Q_{\infty}^{ij}(x, y), \quad \forall x, y \in \mathbf{R}^3, \quad \forall i, j = 0, 1. \quad (5.1)$$

*Proof.* We prove the lemma again for  $i = j = 0$  in the particular case,  $u_0 \equiv 0$  almost surely. The general case is considered in Section 6. Let us assume for a moment that the function  $v_0(z)$  is continuous almost surely. Then the Kirchhoff formula (1.10) and the Fubini theorem give

$$Q_t^{00}(x, y) = \mathbf{E} u(x, t) u(y, t) = \frac{1}{(4\pi t)^2} \int_{S_t(x)} dS(x') \int_{S_t(y)} Q_0^{11}(x', y') dS(y'). \quad (5.2)$$

This integral is the convolution of  $Q_0^{11}(x, y)$  in both variables  $x, y$  with a distribution of compact support. The convolution of distributions with compact support is commutative. Therefore, the assumption on the a.s. continuity of  $v_0(x)$  can be removed by a convolution with a function  $\theta \in D$ . Changing the



variables  $x' = x + \omega t$  in the right-hand side of (5.2), we get

$$\begin{aligned}
& \frac{1}{(4\pi t)^2} \int_{S_t(x)} dS(x') \int_{S_t(y)} Q_0^{11}(x', y') dS(y') \\
&= \frac{1}{(4\pi)^2} \int_{|\omega|=1, \omega_3 < 0} dS(\omega) \int_{S_t(y)} Q_0^{11}(x + \omega t, y') dS(y') \\
&\quad + \frac{1}{(4\pi)^2} \int_{|\omega|=1, \omega_3 > 0} dS(\omega) \int_{S_t(y)} Q_0^{11}(x + \omega t, y') dS(y') \\
&= I_-(t, x, y) + I_+(t, x, y). \tag{5.3}
\end{aligned}$$

Let us recall that  $\nu(r) \equiv \nu_2(r)$ .

**Definition 5.1.**  $C_\nu(\mathbf{R}^3)$  is the space of functions  $f(y) \in C(\mathbf{R}^3)$  such that  $|f(y)| \leq C\nu(|y|)$  with a constant  $C \in \mathbf{R}$ .

Let us define for  $f(y) \in C_\nu(\mathbf{R}^3)$

$$\mathcal{R}f(v) \equiv \frac{1}{(4\pi)^2} \int_{|\omega|=1, \pm\omega_3 > 0} dS(\omega) \int_{p \cdot \omega = v \cdot \omega} f(p) d^2p, \quad v \in \mathbf{R}^3. \tag{5.4}$$

Here  $d^2p$  is the Lebesgue measure on the plane  $p \cdot \omega = v \cdot \omega$ . Note that the integrals with  $\pm$  are identical and converge due to (2.8). Hence, the operator  $\mathcal{R} : C_\nu(\mathbf{R}^3) \rightarrow C_b(\mathbf{R}^3)$  is continuous with the obvious norm in  $C_\nu$ :  $\|f\|_{C_\nu} = \sup_{y \in \mathbf{R}^3} |f(y)|/\nu(|y|)$ .

The convergence (5.1) follows for  $i = j = 0$  from (5.2), (5.3) and Lemmas 5.2 and 5.3.  $\square$

**Lemma 5.2.** *Let S2 hold. Then for  $x, y \in \mathbf{R}^3$ ,*

$$I_\pm(t, x, y) \rightarrow \mathcal{R}q_\pm^{11}(x - y), \quad t \rightarrow \infty. \tag{5.5}$$

**Lemma 5.3.** *Let  $f(y) \in C_\nu(\mathbf{R}^3)$ . Then*

$$\mathcal{R}f = -\frac{1}{4}\mathcal{E} * f. \tag{5.6}$$

Lemma 5.3 is proved in Appendix A.

*Proof of Lemma 5.2.* For a moment we assume additionally (4.4). Denote by  $I_{11}$  the inner integral entering (5.3):

$$I_{11} \equiv I_{11}(x, y, \omega, t) = \int_{S_t(y)} Q_0^{11}(x + \omega t, y') dS(y'). \tag{5.7}$$

Change the variables  $y' = y + \omega t + p$  and denote  $R = |x - y|$ . Assumption (4.4) implies that  $Q_0^{11}(x + \omega t, y + \omega t + p) = 0$  for  $|p| \geq r_0 + R$ , hence (5.7) becomes

$$I_{11} = \int_{S_t(-\omega t) \cap B_0} Q_0^{11}(x + \omega t, y + \omega t + p) dS(p), \quad (5.8)$$

where  $B_0$  denotes the ball  $|p| \leq r_0 + R$ . The sphere  $S_t(-\omega t)$  contains the point 0, hence in a neighbourhood of the origin the sphere converges to its tangent plane  $\omega^\perp$  as  $t \rightarrow \infty$ .

Further, consider the case  $\omega_3 < 0$  and  $\omega_3 > 0$  separately. For  $\omega_3 < 0$  and sufficiently large  $t > t(\omega) > 0$ ,

$$x_3 + \omega_3 t < -a, \quad y_3 + \omega_3 t + p_3 < -a, \quad \text{for } |p| \leq r_0 + R.$$

Then S1 implies that

$$Q_0^{11}(x + \omega t, y + \omega t + p) = q_-^{11}(x - y - p). \quad (5.9)$$

Therefore, if  $\omega_3 < 0$ ,

$$I_{11} \longrightarrow \int_{\omega^\perp \cap B_0} q_-^{11}(x - y - p) d^2 p, \quad t \rightarrow \infty \quad (5.10)$$

that coincides with the inner integral in the right-hand side of (5.4), with  $f = q_-^{11}$  and  $v = x - y$ . Similarly for  $\omega_3 > 0$ . Lemma 5.2 is proved with the additional assumption (4.4). At last, Lemma 4.2 and S2 give the uniform smallness of integral (5.7) over  $|p| \geq r_0 + R$  with large  $r_0$ . Therefore, (5.10) holds for any  $\omega$  with  $\omega_3 \neq 0$ . Hence, (5.5) follows by the Lebesgue theorem on dominated convergence.  $\square$

## 6. Correlation functions in general case

We prove Lemmas 4.1 and 5.1 in the general case. Let us assume for a moment that  $u_0 \in C^1(\mathbf{R}^3)$  and  $v_0 \in C(\mathbf{R}^3)$  almost surely. Then we apply the general Kirchhoff formula for the solution  $u(x, t)$  to the Cauchy problem (1.1): formally,

$$u(x, t) = \frac{1}{4\pi t} \int_{S_t(x)} \left( v_0(x') + \frac{1}{t} u_0(x') + \nabla u_0(x') \cdot n_x(x') \right) dS(x'), \quad (6.1)$$

where  $n_x(x') = (x' - x)/|x' - x|$ . It implies similarly to (5.2),

$$\begin{aligned}
Q_t^{00}(x, y) &= \frac{1}{(4\pi t)^2} \int_{S_t(x)} dS(x') \int_{S_t(y)} \left( [Q_0^{11}(x', y') \right. \\
&\quad \left. + \nabla_{y'}(\nabla_{x'} Q_0^{00}(x', y') \cdot n_x(x')) \cdot n_y(y')] \right. \\
&\quad \left. + \frac{1}{t} [Q_0^{10}(x', y') + Q_0^{01}(x', y') + \nabla_{y'} Q_0^{00}(x', y') \cdot n_y(y') \right. \\
&\quad \left. + \nabla_{x'} Q_0^{00}(x', y') \cdot n_x(x') + \frac{1}{t} Q_0^{00}(x', y')] \right. \\
&\quad \left. + [\nabla_{x'} Q_0^{01}(x', y') \cdot n_x(x') + \nabla_{y'} Q_0^{10}(x', y') \cdot n_y(y')] \right) dS(y').
\end{aligned} \tag{6.2}$$

*Proof of Lemma 4.1 in the general case.* We will prove the uniform bounds for  $Q_t^{00}(x, x)$ ,  $\nabla_x \cdot \nabla_y Q_t^{00}(x, y)|_{x=y}$  and  $Q_t^{11}(x, x)$ . Then (4.2) implies (4.1).

Step 1. Relation (6.2) represents  $Q_t^{00}(x, y)$  as the sum of convolutions with  $D_{x,y}^{\alpha,\beta} Q_0^{kl}(x, y)$  in both variables  $x, y$ , with  $0 \leq d \equiv k+l+|\alpha|+|\beta| \leq 2$ . Therefore,  $\nabla_x \cdot \nabla_y Q_t^{00}(x, y)$  is the similar sum involving  $D_{x,y}^{\alpha,\beta} Q_0^{kl}(x, y)$  with  $2 \leq d \leq 4$ . The similar representation holds for  $Q_t^{11}(x, y)$ . Hence,  $\nabla_x \cdot \nabla_y Q_t^{00}(x, y)$  and  $Q_t^{11}(x, y)$  can be estimated by the method of the proof of Lemma 4.1 in Section 4. Indeed, due to S2 with  $d = 2$ , (2.8) and Lemma 4.2 we get (cf. formula (4.8)),

$$\begin{aligned}
&\nabla_x \cdot \nabla_y Q_t^{00}(x, y)|_{x=y} + Q_t^{11}(x, x) \\
&\leq \frac{C}{(4\pi t)^2} \int_{S_t(x) \times S_t(x)} \nu_2(|x' - x''|) dS(x') dS(x'') \leq C_1 < \infty.
\end{aligned}$$

Step 2.  $Q_t^{00}(x, y)$  requires the particular attention due to the presence in the integrand of the functions  $D_{x',y'}^{\alpha,\beta} Q_0^{kl}(x', y')$  that are estimated by  $\nu_d(|x' - y'|)$  with  $d = 0, 1$ . In this case due to (2.8) and Lemma 4.2 we have to analyze (6.2) more carefully. The corresponding contribution of  $D_{x',y'}^{\alpha,\beta} Q_0^{kl}(x', y')$  with  $d = k + l + |\alpha| + |\beta| = 0, 1$  is

$$I_t^{00}(x, y) = \frac{1}{(4\pi t)^2} \int_{S_t(x)} dS(x') \int_{S_t(y)} \frac{1}{t} [Q_0^{10}(x', y') + \dots + \frac{1}{t} Q_0^{00}(x', y')] dS(y').$$

**Lemma 6.1.** *The integral  $I_t^{00}(x, y)$  converges to zero as  $t \rightarrow \infty$ .*

*Proof.* The assumption S2 implies

$$|I_t^{00}(x, y)| \leq \frac{C}{(4\pi t)^2} \int_{S_t(x)} dS(x') \int_{S_t(y)} \frac{1}{t} [4\nu_1(|x' - y'|) + \frac{1}{t}\nu_0(|x' - y'|)] dS(y'). \tag{6.3}$$

Therefore, Lemma 4.2 implies

$$\begin{aligned} |I_t^{00}(x, y)| &\leq \frac{C}{(4\pi t)^2} \int_{S_t(x)} dS(x') \frac{1}{t} \left[ \int_0^{2t} \left( 4r\nu_1(r) + \frac{1}{t} r\nu_0(r) \right) dr \right] \\ &\leq C_1 \int_0^{2t} \left( \frac{r}{t} \nu_1(r) + \frac{r}{t^2} \nu_0(r) \right) dr. \end{aligned} \quad (6.4)$$

Now (2.8) implies the convergence to zero by the Lebesgue theorem.  $\square$

Lemma 4.1 is proved in the general case.  $\square$

*Proof of Lemma 5.1 in the general case.* We will consider  $i = j = 0$ . The other cases can be considered similarly.

Step 1. The integrals of  $D_{x', y'}^{\alpha, \beta} Q_0^{kl}(x', y')$  with  $d \equiv k + l + |\alpha| + |\beta| \leq 1$  entering (6.2), converge to zero by Lemma 6.1. For the integrals of  $D_{x', y'}^{\alpha, \beta} Q_0^{kl}(x', y')$  with  $2 \leq d \leq 4$ , the convergence follows by the method of proof of Lemma 5.2. Let us define the operator

$$\mathcal{P}f(v) := \frac{1}{(4\pi)^2} \int_{|\omega|=1, \omega_3 > 0} dS(\omega) \int_{v \cdot \omega = p \cdot \omega} \nabla f(p) \cdot \omega \, d^2 p, \quad v \in \mathbf{R}^3, \quad (6.5)$$

for the functions  $f \in C_\nu^1(\mathbf{R}^3) := \{f \in L_{\text{loc}}^1(\mathbf{R}^3) : |\nabla f(y)| \in C_\nu(\mathbf{R}^3)\}$  (cf. Definition 5.1). With the obvious norm in  $C_\nu^1$ :  $\|f\|_{C_\nu^1} = \sup_{y \in \mathbf{R}^3} |\nabla f(y)|/\nu(|y|)$ , the operator  $\mathcal{P} : C_\nu^1(\mathbf{R}^3) \rightarrow C_b(\mathbf{R}^3)$  is continuous. For instance, the operator  $\mathcal{P}$  can be applied to  $q_\pm^{kl}$  with  $1 \leq k + l \leq 2$  since  $q_\pm^{kl} \in C_\nu^1(\mathbf{R}^3)$  by S2. Similarly, the operator  $\mathcal{R}$  (see formula (5.4)) can be applied to  $D^\alpha q_\pm^{kl}$  with  $2 \leq k + l + |\alpha| \leq 4$  since  $D^\alpha q_\pm^{kl} \in C_\nu(\mathbf{R}^3)$ . Now, (6.2) and the method of proof of Lemma 5.2 imply the convergence (5.1) with  $i = j = 0$  to the limiting function

$$q_*^{00} = \mathcal{R}[q_+^{11} + q_-^{11} - \Delta(q_+^{00} + q_-^{00})] + \mathcal{P}[q_+^{01} - q_-^{01} - q_+^{10} + q_-^{10}]. \quad (6.6)$$

Step 2. It remains to prove that  $q_*^{00} = q_\infty^{00}$ . First, let us prove that

$$\mathcal{R}\Delta q_+^{00} = -\frac{1}{4}q_+^{00}. \quad (6.7)$$

In fact,  $\Delta(\mathcal{R}\Delta q_+^{00}) = -\Delta q_+^{00}/4$  due to (5.6), hence  $f(x) \equiv \mathcal{R}\Delta q_+^{00} - q_+^{00}$  is a smooth harmonic function in  $\mathbf{R}^3$ . On the other hand,  $\Delta q_+^{00} \in C_\nu(\mathbf{R}^3)$  by S2. Hence,  $g(x) \equiv \mathcal{R}\Delta q_+^{00} \in C_b(\mathbf{R}^3)$ , and moreover,

$$g(x) \rightarrow 0, \quad |x| \rightarrow \infty. \quad (6.8)$$

Indeed,

$$\left| \int_{p \cdot \omega = x \cdot \omega} \Delta q_+^{00}(p) d^2 p \right| \leq \int_{p \cdot \omega = x \cdot \omega} \nu(|p|) d^2 p = 2\pi \int_{x \cdot \omega}^{\infty} r \nu(r) dr \quad (6.9)$$

similar to (4.7) with  $t = \infty$ . This integral is bounded uniformly in  $|\omega| = 1$  and converges to zero if  $|x| \rightarrow \infty$  and  $x = |x|\theta$  with  $\theta \cdot \omega \neq 0$ . Therefore, (6.8) follows from (5.4) by the Lebesgue theorem. Further,  $|f(x)| \leq |g(x)| + \nu_0(|x|)$  again by S2. At last,  $\nu_0(r_n) \rightarrow 0$  for some sequence  $r_n \rightarrow \infty$  due to (2.8). Finally, the maximum principle and (6.8) imply for any fixed  $x \in \mathbf{R}^3$ ,

$$|f(x)| \leq \max_{|y|=r_n} |g(y)| + \nu_0(r_n) \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore,  $f(x) \equiv 0$  and (6.7) is proved. Further, let us consider the terms with  $\mathcal{P}$  in (6.6). Obviously,  $\mathcal{P}f$  is a convolution. We prove the next lemma in Appendix 11. Let us recall that  $P(x) = -iF^{-1}[\text{sgn } k_3/|k|]$ .

**Lemma 6.2.** *For  $f \in D$  we have*

$$\mathcal{P}f = \frac{1}{4}P * f. \quad (6.10)$$

Let us assume for a moment that all the correlation functions  $q_{\pm}^{kl}(\cdot)$  are smooth and have a rapid decay. Then (6.6) coincides with (2.13) by (6.7) and Lemmas 5.3, 6.2. In the general case we consider the formula (6.10) as the definition of the convolutions with  $P$ , entering (2.13)–(2.15). Lemma 5.1 is proved in the general case.  $\square$

## 7. Bernstein's argument for the wave equation

In this and the subsequent section we develop a version of the Bernstein 'room-corridor' method. We use the standard integral representation for the solutions, divide the domain of integration into 'rooms' and 'corridors' and evaluate their contribution. As the result,  $\langle U(t)Y_0, \Psi \rangle$  is represented as the sum of weakly dependent random variables. We evaluate the variances of these random variables that will be important in next section.

For the wave equation the similar method has been used in [10, Section 6] for an odd  $n \geq 3$ . Our mixing condition S3 is different from [10] (see Remarks 2.1 and 2.2). Respectively, the method of [10] requires a suitable modification.

Denote by  $\mathcal{E}_t(x) \equiv \mathcal{E}(x, t) = \delta(|x|^2 - t^2)/2\pi$  the fundamental solution to the wave equation. The support of  $\mathcal{E}_t$  is the sphere  $S_t = \{x \in \mathbf{R}^3 : |x| = t\}$ . Therefore, the dynamical group  $U(t)$  of the problem (1.2) is the convolution operator

$$U(t)Y_0 = \mathcal{G}_t * Y_0, \quad t > 0, \quad (7.1)$$

where

$$\mathcal{G}_t = \begin{pmatrix} \dot{\mathcal{E}}_t & \mathcal{E}_t \\ \Delta \mathcal{E}_t & \dot{\mathcal{E}}_t \end{pmatrix}. \quad (7.2)$$

Next we introduce a ‘room-corridor’ partition of the space  $\mathbf{R}^3$ . Given  $t > 0$ , choose  $d \equiv d_t \geq 1$  and  $\rho \equiv \rho_t > 0$  and an integer  $N \equiv N_t > 0$ . Asymptotic relations between  $t$ ,  $d_t$  and  $\rho_t$  are specified below. Define

$$a_1 = -t, \quad b_1 = a_1 + d; \quad a_2 = b_1 + \rho, \quad b_2 = a_2 + d; \quad \dots, \quad b_N \equiv a_N + d = t. \quad (7.3)$$

We divide the sphere  $S_t$  by the planes orthogonal to the axis  $Ox_3$  into the slabs which we call the ‘rooms’  $R_k^t$  ( $k = 1, \dots, N$ ), separated by the ‘corridors’  $C_k^t$  ( $k = 1, \dots, N - 1$ ),

$$R_k^t = \{x \in S_t : x_3 \in [a_k, b_k]\}, \quad C_k^t = \{x \in S_t : x_3 \in [b_k, a_{k+1}]\}. \quad (7.4)$$

Here  $x = (x_1, x_2, x_3)$ ,  $d$  is the width of a room, and  $\rho$  of a corridor. Then

$$S_t = \left( \cup R_k^t \right) \cup \left( \cup C_k^t \right). \quad (7.5)$$

For any region  $\Sigma \subset S_t$  we define the distribution  $\mathcal{E}_{t,\Sigma}$  with the support in  $\Sigma$

$$\langle \mathcal{E}_{t,\Sigma}, \theta \rangle := \frac{1}{4\pi t} \int_{\Sigma} \theta(z) dS(z), \quad \theta \in D.$$

Note that for any  $t > 0$

$$\dot{\mathcal{E}}_t(x) = -\frac{t}{\pi} \delta'(|z|^2 - t^2) = \frac{1}{t} \mathcal{E}_t(x) - \nabla \left( \frac{x}{t} \mathcal{E}_t(x) \right).$$

For any region  $\Sigma \subset S_t$  we define the distribution  $\dot{\mathcal{E}}_{t,\Sigma}$ ,

$$\dot{\mathcal{E}}_{t,\Sigma}(x) := \frac{1}{t} \mathcal{E}_{t,\Sigma}(x) - \nabla \left( \frac{x}{t} \mathcal{E}_{t,\Sigma}(x) \right). \quad (7.6)$$

Then for  $\Sigma = S_t$  we have  $\dot{\mathcal{E}}_{t,\Sigma} = \dot{\mathcal{E}}_t$ . Let us denote

$$\mathcal{G}_{t,\Sigma} := \begin{pmatrix} \dot{\mathcal{E}}_{t,\Sigma} & \mathcal{E}_{t,\Sigma} \\ \Delta \mathcal{E}_{t,\Sigma} & \dot{\mathcal{E}}_{t,\Sigma} \end{pmatrix}. \quad (7.7)$$

We define the random variable

$$I_t(\Sigma) = \langle \mathcal{G}_{t,\Sigma} * Y_0, \Psi \rangle, \quad (7.8)$$

where  $\Psi \in \mathcal{D}$  is a fixed function from (2.17). For instance, define

$$r_t^k = I_t(R_t^k), \quad c_t^k = I_t(C_t^k). \quad (7.9)$$

(7.5) implies that

$$\langle U(t)Y_0, \Psi \rangle = \langle \mathcal{G}_t * Y_0, \Psi \rangle = \sum_{k=1}^{N_t} r_t^k + \sum_{k=1}^{N_t-1} c_t^k. \quad (7.10)$$

**Lemma 7.1.** *Let S0, S3 hold. The following bounds hold for  $t > 1$  and for arbitrary  $k$ :*

$$\mathbb{E} |r_t^k|^2 \leq C(\Psi) d_t/t, \quad (7.11)$$

$$\mathbb{E} |c_t^k|^2 \leq C(\Psi) \rho_t/t. \quad (7.12)$$

*Proof.* We prove the following estimate: for any region  $\Sigma \subset S_t$ ,

$$\mathbb{E} |I_t(\Sigma)|^2 \leq C(\Psi) |\Sigma|/t^2. \quad (7.13)$$

Then (7.11) and (7.12) would follow from this estimate with  $\Sigma = R_t^k$  and  $\Sigma = R_t^k$ , respectively, as  $|R_t^k| = 2\pi t d_t$  and  $|C_t^k| = 2\pi t \rho_t$ .

Now we prove (7.13). From (7.7) and (7.8) it follows that for  $\Psi = (\Psi^0, \Psi^1)$ ,  $\Psi \in \mathcal{D}$ ,

$$I(\Sigma) = \langle \dot{\mathcal{E}}_{t,\Sigma} * u_0, \Psi^0 \rangle + \langle \mathcal{E}_{t,\Sigma} * v_0, \Psi^0 \rangle - \langle \mathcal{E}_{t,\Sigma} * \nabla u_0, \nabla \Psi^1 \rangle + \langle \dot{\mathcal{E}}_{t,\Sigma} * v_0, \Psi^1 \rangle. \quad (7.14)$$

Substituting (7.6) in the first and the last terms in the right-hand side of (7.14), we get

$$\begin{aligned} I(\Sigma) &= \langle \mathcal{E}_{t,\Sigma} * u_0/t, \Psi^0 \rangle - \left\langle \left( \frac{x}{t} \mathcal{E}_{t,\Sigma} \right) * \nabla u_0, \Psi^0 \right\rangle - \langle \mathcal{E}_{t,\Sigma} * \nabla u_0, \nabla \Psi^1 \rangle \\ &\quad + \langle \mathcal{E}_{t,\Sigma} * v_0, \Psi^0 \rangle + \frac{1}{t} \langle \mathcal{E}_{t,\Sigma} * v_0, \Psi^1 \rangle + \left\langle \left( \frac{x}{t} \mathcal{E}_{t,\Sigma} \right) * v_0, \nabla \Psi^1 \right\rangle. \end{aligned}$$

Hence,

$$I_t(\Sigma) = \sum_{j=1}^M I_t^j, \quad (7.15)$$

where  $I_t^j = c_j(t) \langle \bar{\mathcal{E}}_{t,\Sigma}^j * w_j, \theta_j \rangle$ . Here  $M \leq 6$ ,  $c_j(t)$  is a bounded function for  $t \geq \delta > 0$ ,  $w_j$  is one of  $t^{-1}u_0$ ,  $\nabla u_0$  or  $v_0$ ,  $\bar{\mathcal{E}}_{t,\Sigma}^j$  is one of  $\mathcal{E}_{t,\Sigma}(z)$  or  $z\mathcal{E}_{t,\Sigma}(z)/t$ ,  $\theta_j$

is one of  $D^\alpha \Psi^l$  with  $|\alpha| \leq 1$ . Therefore, for  $t > 1$

$$\begin{aligned}
\mathbf{E} |I_t(\Sigma)|^2 &\leq C \sum_{j=1}^M \mathbf{E} |I_t^j|^2 \\
&\leq C \sum_{j=1}^M \langle \mathbf{E} [(\bar{\mathcal{E}}_{t,\Sigma}^j * w_j)(x) (\bar{\mathcal{E}}_{t,\Sigma}^j * w_j)(y)], \theta_j(x) \theta_j(y) \rangle \\
&\leq C_1 \sum_{j=1}^M \frac{1}{t^2} \left| \left\langle \int_{\Sigma} \int_{\Sigma} \left( \frac{1}{t^2} Q_0^{00}(x-z-(y-p)) \right. \right. \right. \\
&\quad \left. \left. \left. + \sum_{|\alpha|=|\beta|=1} D_{z,p}^{\alpha,\beta} Q_0^{00}(x-z-(y-p)) \right. \right. \right. \\
&\quad \left. \left. \left. + Q_0^{11}(x-z-(y-p)) \right) dS(z) dS(p), \theta_j(x) \theta_j(y) \right\rangle \right|. \quad (7.16)
\end{aligned}$$

We have

$$\text{supp } \Psi \subset B_{r_0} = \{x \in \mathbf{R}^3 : |x| \leq r_0\} \quad (7.17)$$

with an  $r_0 > 0$ . Since  $x, y \in \text{supp } \theta_j \subset \text{supp } \Psi \subset B_{r_0}$ ,  $|x - z - y + p| \geq (|z - p| - 2r_0)_+$ , where  $s_+ = \max(s, 0)$ ,  $s \in \mathbf{R}$ . Since  $\nu_d(r)$  are non-increasing functions, (7.16) and S3 imply

$$\begin{aligned}
\mathbf{E} |I_t(\Sigma)|^2 &\quad (7.18) \\
&\leq C(\Psi) \frac{1}{t^2} \int_{\Sigma} dS(z) \int_{\Sigma} \left( \frac{1}{t^2} \nu_0((|z - p| - 2r_0)_+) + \nu_2((|z - p| - 2r_0)_+) \right) dS(p).
\end{aligned}$$

Then Lemma 4.2 and (2.8) imply as in Lemma 6.1,

$$\mathbf{E} |I_t(\Sigma)|^2 \leq C \frac{|\Sigma|}{t^2} \int_0^{2t} \left( r \nu_2((r - 2r_0)_+) + \frac{r}{t^2} \nu_0((r - 2r_0)_+) \right) dr \leq C_1 \frac{|\Sigma|}{t^2}.$$

□

## 8. Convergence of characteristic functionals

In this section we complete the proof of Proposition 2.3. If  $\mathcal{Q}_\infty(\Psi, \Psi) = 0$ , Proposition 2.3 is obvious, due to (5.1). Thus, we may assume that

$$\mathcal{Q}_\infty(\Psi, \Psi) \neq 0. \quad (8.1)$$

Choose  $0 < \delta < 1$ , and

$$N_t \sim (\ln(t+1))^{1/10}, \quad \rho_t \sim t^{1-\delta}, \quad t \rightarrow \infty. \quad (8.2)$$



**Lemma 8.1.** *The following limit holds true:*

$$N_t \left( \nu_0^2(\rho_t) + \left( \frac{\rho_t}{t} \right)^{1/2} \right) + N_t^2 \left( \nu_0(\rho_t) + \frac{\rho_t}{t} \right) \longrightarrow 0, \quad t \rightarrow \infty. \quad (8.3)$$

*Proof.* Since  $\nu_d(r)$  are non-increasing functions, (2.8) implies

$$\nu_0(r) \ln(r+1) = \int_0^r \frac{\nu_0(s)}{s+1} ds \leq \int_0^r \frac{\nu_0(s)}{s+1} ds \leq C < \infty.$$

Then (8.2) implies (8.3).  $\square$

By the triangle inequality,

$$\begin{aligned} |\hat{\mu}_t(\Psi) - \hat{\mu}_\infty(\Psi)| &\leq \left| \mathbf{E} \exp\{i\langle U(t)Y_0, \Psi \rangle\} - \mathbf{E} \exp\left\{i \sum_t r_t^k\right\} \right| \\ &\quad + \left| \exp\left\{-\frac{1}{2} \sum_t \mathbf{E}(r_t^k)^2\right\} - \exp\left\{-\frac{1}{2} Q_\infty(\Psi, \Psi)\right\} \right| \\ &\quad + \left| \mathbf{E} \exp\left\{i \sum_t r_t^k\right\} - \exp\left\{-\frac{1}{2} \sum_t \mathbf{E}(r_t^k)^2\right\} \right| \\ &\equiv I_1 + I_2 + I_3, \end{aligned} \quad (8.4)$$

where the sum  $\sum_t$  stands for  $\sum_{k=1}^{N_t}$ . We are going to show that all the summands  $I_1, I_2, I_3$  tend to zero as  $t \rightarrow \infty$ .

Step (i). Equation (7.10) implies

$$I_1 = \left| \mathbf{E} \exp\left\{i \sum_t r_t^k\right\} \left( \exp\left\{i \sum_t c_t^k\right\} - 1 \right) \right| \leq \sum_t \mathbf{E} |c_t^k| \leq \sum_t (\mathbf{E} |c_t^k|^2)^{1/2}. \quad (8.5)$$

From (8.5), (7.12) and (8.2) we obtain that

$$I_1 \leq C N_t (\rho_t/t)^{1/2} \rightarrow 0, \quad t \rightarrow \infty. \quad (8.6)$$

Step (ii). By the triangle inequality,

$$\begin{aligned} I_2 &\leq \frac{1}{2} \left| \sum_t \mathbf{E}(r_t^k)^2 - Q_\infty(\Psi, \Psi) \right| \leq \frac{1}{2} |Q_t(\Psi, \Psi) - Q_\infty(\Psi, \Psi)| \\ &\quad + \frac{1}{2} \left| \mathbf{E} \left( \sum_t r_t^k \right)^2 - \sum_t \mathbf{E}(r_t^k)^2 \right| + \frac{1}{2} \left| \mathbf{E} \left( \sum_t r_t^k \right)^2 - Q_t(\Psi, \Psi) \right| \\ &\equiv I_{21} + I_{22} + I_{23}, \end{aligned} \quad (8.7)$$

where  $Q_t$  is the quadratic form with the integral kernel  $(Q_t^{ij}(x, y))$ . Equation (5.1) implies  $I_{21} \rightarrow 0, t \rightarrow \infty$ . As to  $I_{22}$ , we first obtain that

$$I_{22} \equiv \frac{1}{2} \left| \mathbf{E} \left( \sum_t r_t^k \right)^2 - \sum_t \mathbf{E}(r_t^k)^2 \right| \leq \sum_{k < l} |\mathbf{E} r_t^k r_t^l|. \quad (8.8)$$

The next lemma is a corollary of ([11, Lemma 17.2.3]).

**Lemma 8.2.** *Let  $\xi$  be a random value measurable with respect to the  $\sigma$ -algebra  $\sigma_{d_1}(\mathcal{A})$ ,  $\eta$  be a random value measurable with respect to the  $\sigma$ -algebra  $\sigma_{d_2}(\mathcal{B})$ , and  $\text{dist}(\mathcal{A}, \mathcal{B}) \geq h > 0$ .*

i) *Let  $(\mathbb{E}|\xi|^2)^{1/2} \leq a$ ,  $(\mathbb{E}|\eta|^2)^{1/2} \leq b$ . Then*

$$|\mathbb{E}\xi\eta - \mathbb{E}\xi\mathbb{E}\eta| \leq C ab \phi_{d_1, d_2}^{1/2}(h).$$

ii) *Let  $|\xi| \leq a$ ,  $|\eta| \leq b$  almost surely. Then*

$$|\mathbb{E}\xi\eta - \mathbb{E}\xi\mathbb{E}\eta| \leq C ab \phi_{d_1, d_2}(h).$$

We apply Lemma 8.2 to deduce that  $I_{22} \rightarrow 0$  as  $t \rightarrow \infty$ . Note that  $r_t^k = \langle \mathcal{G}_{t, R_t^k} * Y_0, \Psi \rangle$  is measurable with respect to the  $\sigma$ -algebra  $\sigma_{d_k}(\mathcal{A}^k)$ , where

$$\mathcal{A}^k = \{x - y : y \in R_t^k, x \in \text{supp } \Psi \subset B_{r_0}\}.$$

The distance between the different rooms  $R_t^k$  is greater or equal to  $\rho_t$  according to (7.3) and (7.4). Then  $\rho(\mathcal{A}^k, \mathcal{A}^l) \geq \rho(R_t^k, R_t^l) - 2r_0 \geq \rho_t - 2r_0$ . Hence (8.8) and S0, S3 imply, together with Lemma 8.2 i),

$$I_{22} \leq CN_t^2 \nu_0((\rho_t - 2r_0)_+) \rightarrow 0, \quad t \rightarrow \infty, \quad (8.9)$$

because of (7.11) and Lemma 8.1. Finally, it remains to check that  $I_{23} \rightarrow 0$ ,  $t \rightarrow \infty$ . By the Cauchy–Schwarz inequality,

$$\begin{aligned} I_{23} &\leq \left| \mathbb{E} \left( \sum_t r_t^k \right)^2 - \mathbb{E} \left( \sum_t r_t^k + \sum_t c_t^k \right)^2 \right| \\ &\leq N_t \sum_t \mathbb{E} |c_t^k|^2 + 2 \left( \mathbb{E} \left( \sum_t r_t^k \right)^2 \right)^{1/2} \left( N_t \sum_t \mathbb{E} |c_t^k|^2 \right)^{1/2}. \end{aligned} \quad (8.10)$$

(7.11), (8.8) and (8.9) imply  $\mathbb{E}(\sum_t r_t^k)^2 \leq C_1 + C_2 N_t^2 \nu_0((\rho_t - 2r_0)_+) \leq C_3 < \infty$ . Then (7.12), (8.10) and Lemma 8.1 imply

$$I_{23} \leq C_1 N_t^2 \rho_t / t + C_2 N_t (\rho_t / t)^{1/2} \rightarrow 0, \quad t \rightarrow \infty. \quad (8.11)$$

So,  $I_{21}$ ,  $I_{22}$ ,  $I_{23}$  tend to zero, as  $t \rightarrow \infty$ . Then (8.7) implies

$$I_2 \leq \frac{1}{2} \left| \sum_t \mathbb{E}(r_t^k)^2 - \mathcal{Q}_\infty(\Psi, \Psi) \right| \rightarrow 0, \quad t \rightarrow \infty. \quad (8.12)$$

Step (iii). It remains to verify

$$I_3 \equiv \left| \mathbb{E} \exp \left\{ i \sum_t r_t^k \right\} - \exp \left\{ -\frac{1}{2} \sum_t \mathbb{E}(r_t^k)^2 \right\} \right| \rightarrow 0, \quad t \rightarrow \infty. \quad (8.13)$$

Using Lemma 8.2, ii) we obtain

$$\begin{aligned}
& \left| \mathbf{E} \exp \left\{ i \sum_t r_t^k \right\} - \prod_{k=1}^{N_t} \mathbf{E} \exp \{ i r_t^k \} \right| \\
& \leq \left| \mathbf{E} \exp \{ i r_t^1 \} \exp \left\{ i \sum_{k=2}^{N_t} r_t^k \right\} - \mathbf{E} \exp \{ i r_t^1 \} \mathbf{E} \exp \left\{ i \sum_{k=2}^{N_t} r_t^k \right\} \right| \\
& \quad + \left| \mathbf{E} \exp \{ i r_t^1 \} \mathbf{E} \exp \left\{ i \sum_{k=2}^{N_t} r_t^k \right\} - \prod_{k=1}^{N_t} \mathbf{E} \exp \{ i r_t^k \} \right| \\
& \leq \nu_0^2 ((\rho_t - 2r_0)_+) + \left| \mathbf{E} \exp \left\{ i \sum_{k=2}^{N_t} r_t^k \right\} - \prod_{k=2}^{N_t} \mathbf{E} \exp \{ i r_t^k \} \right|.
\end{aligned}$$

We then apply Lemma 8.2, ii) recursively and get, according to Lemma 8.1,

$$\left| \mathbf{E} \exp \left\{ i \sum_t r_t^k \right\} - \prod_{k=1}^{N_t} \mathbf{E} \exp \{ i r_t^k \} \right| \leq N_t \nu_0^2 ((\rho_t - 2r_0)_+) \rightarrow 0, \quad t \rightarrow \infty. \quad (8.14)$$

It remains to verify the convergence

$$\left| \prod_{k=1}^{N_t} \mathbf{E} \exp \{ i r_t^k \} - \exp \left\{ -\frac{1}{2} \sum_t \mathbf{E} (r_t^k)^2 \right\} \right| \rightarrow 0, \quad t \rightarrow \infty. \quad (8.15)$$

According to the standard statement of the Central Limit Theorem (see, e.g., [17, Theorem 4.7]) it suffices to verify the Lindeberg condition: for any  $\varepsilon > 0$

$$\frac{1}{\sigma_t} \sum_t \mathbf{E}_{\varepsilon \sqrt{\sigma_t}} |r_t^k|^2 \rightarrow 0, \quad t \rightarrow \infty. \quad (8.16)$$

Here  $\sigma_t \equiv \sum_t \mathbf{E} (r_t^k)^2$ , and  $\mathbf{E}_\delta f \equiv \mathbf{E} (X_\delta f)$ , where  $X_\delta$  is the indicator of the event  $|f| > \delta^2$ . Note that (8.12) and (8.1) imply

$$\sigma_t \rightarrow \mathcal{Q}_\infty(\Psi, \Psi) \neq 0, \quad t \rightarrow \infty.$$

Hence it remains to verify that for any  $\varepsilon > 0$

$$\sum_t \mathbf{E}_\varepsilon |r_t^k|^2 \rightarrow 0, \quad t \rightarrow \infty. \quad (8.17)$$

We check (8.17) in Sections 9, 10. Finally, (8.4) and (8.6), (8.12)–(8.15) imply Proposition 2.3.

### 9. The Lindeberg condition

The proof of (8.17) can be reduced to the case when for some  $b \geq 0$  we have almost surely that

$$|Y_0(x)| \leq b, \quad x \in \mathbf{R}^3. \quad (9.1)$$

The general case can be covered by the standard cutoff argument in the following way. We decompose  $Y_0$  in two summands: the first one, satisfying the estimate (9.1), and the remainder. For large  $b$ , the dispersion of the remainder is small due to S2, S3 and Lemma 8.2, i), then the dispersion (7.11) of the corresponding variables  $r_t^k$  is small uniformly in  $t$ . The last fact follows from the proof of (7.11).

Further, we estimate

$$\sum_t \mathbf{E}_\varepsilon |r_t^k|^2 = \sum_t |R_t^k| \frac{1}{|R_t^k|} \mathbf{E}_\varepsilon |r_t^k|^2 \leq 4\pi t^2 \max_{k=1, \dots, N_t} \frac{1}{|R_t^k|} \mathbf{E}_\varepsilon |r_t^k|^2.$$

Therefore, it remains to prove

$$\max_{k=1, \dots, N_t} \frac{1}{|R_t^k|} \mathbf{E}_\varepsilon |r_t^k|^2 = o(t^{-2}), \quad t \rightarrow \infty. \quad (9.2)$$

The Chebyshev inequality implies

$$\mathbf{E}_\varepsilon |r_t^k|^2 \leq \frac{1}{\varepsilon^2} \mathbf{E} |r_t^k|^4. \quad (9.3)$$

Using (7.15), we get

$$\mathbf{E} |r_t^k|^4 = \mathbf{E} |I_t^1 + \dots + I_t^M|^4 \leq C(M) \mathbf{E} (|I_t^1|^4 + \dots + |I_t^M|^4). \quad (9.4)$$

Therefore, (9.2) follows from the estimate

$$\max_k \frac{1}{|R_t^k|} \mathbf{E} |\langle \bar{\mathcal{E}}_{t, R_t^k} * w_k, \theta_k \rangle|^4 = o(t^{-2}), \quad t \rightarrow \infty. \quad (9.5)$$

We prove the following proposition in the next section.

**Proposition 9.1.** *Assume that (9.1) holds, and  $w = t^{-1}u_0, \nabla u_0$  or  $v_0$ . Then for any  $\Sigma \subset S_t$  the folling bound holds*

$$\mathbf{E} |\langle \bar{\mathcal{E}}_{t, \Sigma} * w, \theta \rangle|^4 \leq C(\theta) \left(\frac{b}{t}\right)^4 |\Sigma|^2. \quad (9.6)$$

Here  $\bar{\mathcal{E}}_{t, \Sigma}^j$  is one of  $\mathcal{E}_{t, \Sigma}(z)$  or  $\frac{z}{t} \mathcal{E}_{t, \Sigma}(z)$ ;  $\theta_j$  is one of  $D^\alpha \Psi^l$  with  $|\alpha| \leq 1$ .

This proposition implies (9.5):

$$\frac{1}{|R_t^k|} \mathbf{E} |\langle \bar{\mathcal{E}}_{t, R_t^k} * w_k, \theta_k \rangle|^4 \leq \frac{1}{|R_t^k|} C(\Psi) \left(\frac{b}{t}\right)^4 |R_t^k|^2 \leq C(b, \Psi) \frac{|R_t^k|}{t^4} = o(t^{-2}),$$

since  $|R_t^k| \leq 4\pi t^2/N_t$ , where  $N_t \rightarrow \infty$ . (8.17) is proved.

## 10. The fourth order moment functions

*Proof of Proposition 9.1.* We deduce Proposition 9.1 from the bounds for the fourth order moment functions.

Define  $m_0^{(l)}(\bar{z}) := \mathbf{E} w(z_1) \cdots w(z_l)$ ,  $\bar{z} = (z_1, \dots, z_l)$ , where  $w(z_k) = v_0(z_k)$  for every  $k = 1, \dots, l$ , or  $w(z_k) = \nabla u_0(z_k)$  for every  $k = 1, \dots, l$ , or  $w(z_k) = t^{-1}u_0(z_k)$  for every  $k = 1, \dots, l$ . We have  $\text{supp } \theta \subset B_{r_0}$  for  $r_0 > 0$ . Then the left-hand side of (9.6) is estimated as follows,

$$\mathbf{E} | \langle \bar{\mathcal{E}}_{t,\Sigma} * w, \theta \rangle |^4 \leq \frac{C(\theta)}{t^4} \int_{B_{r_0}^4} \int_{\Sigma^4} |m_0^{(4)}(\bar{x} - \bar{z})| dS(\bar{z}) d\bar{x}, \quad (10.1)$$

where  $dS(\bar{z}) := dS(z_1) \dots dS(z_4)$ . Therefore, we have to prove that

$$I(\bar{x}) \equiv \int_{\Sigma^4} |m_0^{(4)}(\bar{x} - \bar{z})| dS(\bar{z}) \leq Cb^4 |\Sigma|^2, \quad \bar{x} \in B_{r_0}. \quad (10.2)$$

*Step 1.* Let us prove an estimate for the moment functions  $m_0^{(4)}(y_1, y_2, y_3, y_4)$  by the method [10, Section 6.2]. We use the mixing condition for different configurations of the points  $y_1, y_2, y_3, y_4$  in the space  $\mathbf{R}^3$ .

**Lemma 10.1.** *The following bound holds*

$$\begin{aligned} |m_0^{(4)}(y_1, y_2, y_3, y_4)| &\leq 4b^4 \left( \nu_2^2 \left( \frac{1}{3} |y_1 - y_2| \right) + \nu_0^2 \left( \frac{1}{3} |y_1 - y_2| \right) t^{-4} \right) \\ &\quad + 16b^4 \sum_{i,j=0,2} \left( \frac{1}{t^i} \nu_{2-i}^2 (|y_1 - y_3|) \cdot \frac{1}{t^j} \nu_{2-j}^2 (|y_2 - y_4|) \right. \\ &\quad \left. + \frac{1}{t^i} \nu_{2-i}^2 (|y_1 - y_4|) \cdot \frac{1}{t^j} \nu_{2-j}^2 (|y_2 - y_3|) \right). \end{aligned} \quad (10.3)$$

*Proof.* Let us divide the space  $\mathbf{R}^3$  in three regions  $I_1, I_2, I_3$  by two hyperplanes that are orthogonal to the segment  $[y_1, y_2]$  and divide it in three equal segments,  $y_1 \in I_1, y_2 \in I_3$ . At least one of the regions  $I_1, I_2, I_3$  does not contain  $y_3, y_4$ . If the points  $y_3, y_4 \notin I_1$ , then S0 and S3 imply (10.3), since

$$\begin{aligned} |m_0^{(4)}(y_1, y_2, y_3, y_4)| &= |m_0^{(4)}(y_1, y_2, y_3, y_4) - m_0^{(1)}(y_1) m_0^{(3)}(y_2, y_3, y_4)| \\ &\leq 4b^4 \left( \nu_2^2 \left( \frac{1}{3} |y_1 - y_2| \right) + \nu_0^2 \left( \frac{1}{3} |y_1 - y_2| \right) t^{-4} \right). \end{aligned}$$

The same proof is valid for the case  $y_3, y_4 \notin I_3$ . Now let us assume that  $y_3, y_4 \notin I_2$ , for instance,  $y_3 \in I_1, y_4 \in I_3$ . Then S0, S3 imply (10.3), since by Lem-

ma 8.2, ii)

$$\begin{aligned}
|m_0^{(4)}(y_1, y_2, y_3, y_4)| &\leq |m_0^{(4)}(y_1, y_2, y_3, y_4) - m_0^{(2)}(y_1, y_3)m_0^{(2)}(y_2, y_4)| \\
&\quad + |m_0^{(2)}(y_1, y_3)m_0^{(2)}(y_2, y_4)| \\
&\leq 4b^4 \left( \nu_2^2 \left( \frac{1}{3} |y_1 - y_2| \right) + \nu_0^2 \left( \frac{1}{3} |y_1 - y_2| \right) t^{-4} \right) \\
&\quad + 16b^4 \sum_{i,j=0,2} \frac{1}{t^i} \nu_{2-i}^2 (|y_1 - y_3|) \cdot \frac{1}{t^j} \nu_{2-j}^2 (|y_2 - y_4|).
\end{aligned}$$

The proof for the case  $y_3 \in I_3, y_4 \in I_1$  is the same.  $\square$

*Remark.* For a translation-invariant measure  $\mu_0$  the estimate similar to (10.3) is obtained in [1, inequality (20.42)].

Step 2. (10.3) holds with any permutations of  $y_1, y_2, y_3, y_4$  in the right-hand side. Hence

$$\begin{aligned}
|m_0^{(4)}(\bar{y})| &\leq 4b^4 \left( \nu_2^2 \left( \frac{1}{3} |y_s - y_p| \right) + \nu_0^2 \left( \frac{1}{3} |y_s - y_p| \right) t^{-4} \right) \\
&\quad + 16b^4 \sum_{i,j=0,2} \left( \frac{1}{t^i} \nu_{2-i}^2 (|y_s - y_k|) \cdot \frac{1}{t^j} \nu_{2-j}^2 (|y_p - y_l|) \right. \\
&\quad \left. + \frac{1}{t^i} \nu_{2-i}^2 (|y_s - y_l|) \cdot \frac{1}{t^j} \nu_{2-j}^2 (|y_p - y_k|) \right) \\
&\equiv M_{s,p}^1(\bar{y}) + M_{s,p}^2(\bar{y}) \tag{10.4}
\end{aligned}$$

for any permutation  $\{s, p, k, l\}$  of  $\{1, 2, 3, 4\}$ . Let us define

$$\Sigma_{s,p} := \{ \bar{z} \in \Sigma^4 \mid |z_s - z_p| = \max_{i,j} |z_i - z_j| \}.$$

Then  $(\Sigma)^4 = \cup_{(s,p)} \Sigma_{s,p}$ , where the union is taken over all the pairs  $(s, p)$  of the indexes 1, 2, 3, 4. Therefore, (10.4) implies

$$\begin{aligned}
I(\bar{x}) &\equiv \int_{\Sigma^4} |m_0^{(4)}(\bar{x} - \bar{z})| dS(\bar{z}) \\
&\leq \sum_{(s,p)} \left\{ \int_{\Sigma_{s,p}} M_{s,p}^1(\bar{x} - \bar{z}) dS(\bar{z}) + \int_{\Sigma_{s,p}} M_{s,p}^2(\bar{x} - \bar{z}) dS(\bar{z}) \right\}. \tag{10.5}
\end{aligned}$$

Here the sum is taken over all the pairs  $(s, p)$ . Every of the six terms corresponding to different pairs  $(s, p)$  in the right-hand side of (10.5) coincide. We have to estimate  $I(\bar{x})$  only for  $\bar{x} \in B_{r_0}^4$  (see (10.2)). Then  $|z_s - z_p - x_s + x_p| \geq$

$(|z_s - z_p| - 2r_0)_+$  for any  $z_s, z_p \in \mathbf{R}^3$ . Since  $\nu_d$  is a non-increasing function, (10.4), (10.5) imply

$$\begin{aligned}
I(\bar{x}) &\leq Cb^4 \int_{\Sigma_{1,2}} \left( \nu_2^2 \left( \frac{1}{3} (|z_1 - z_2| - 2r_0)_+ \right) \right. \\
&\quad \left. + \nu_0^2 \left( \frac{1}{3} (|z_1 - z_2| - 2r_0)_+ \right) t^{-4} \right) dS(\bar{z}) \\
&\quad + Cb^4 \sum_{i,j=0,2} \int_{\Sigma_{1,2}} \left( \frac{1}{t^i} \nu_{2-i}^2 ( (|z_1 - z_3| - 2r_0)_+ ) \right. \\
&\quad \left. \times \frac{1}{t^j} \nu_{2-j}^2 ( (|z_2 - z_4| - 2r_0)_+ ) \right) dS(\bar{z}) \\
&\equiv I_1 + I_2. \tag{10.6}
\end{aligned}$$

Step 3. Let us estimate  $I_1$  and  $I_2$  separately.

**Lemma 10.2.**  $I_1 \leq Cb^4 |\Sigma|^2$ .

*Proof.* The integrand in  $I_1$  does not depend on  $z_3$  and  $z_4$ . Therefore, the result of the integration in  $z_3, z_4$  is estimated by the factor  $\pi |\Sigma| |z_1 - z_2|^2$ , since  $|z_3 - z_4| \leq |z_1 - z_2|$  by the definition of  $\Sigma_{1,2}$ . Lemma 4.2 implies

$$\begin{aligned}
I_1 &\leq C_1 b^4 |\Sigma| \int_{(\Sigma)^2} \left( \nu_2^2 \left( \frac{1}{3} (|z_1 - z_2| - 2r_0)_+ \right) + t^{-4} \nu_0^2 \left( \frac{1}{3} (|z_1 - z_2| - 2r_0)_+ \right) \right) \\
&\quad \times |z_1 - z_2|^2 dS(z_1) dS(z_2) \\
&\leq C_1 b^4 |\Sigma|^2 \int_0^{2t} \left( \nu_2^2 \left( \frac{1}{3} (r - 2r_0)_+ \right) + t^{-4} \nu_0^2 \left( \frac{1}{3} (r - 2r_0)_+ \right) \right) r^3 dr. \tag{10.7}
\end{aligned}$$

Condition (2.8) implies

$$r^2 \nu_2(r) = \nu_2(r) 2 \int_0^r s ds \leq 2 \int_0^r s \nu_2(s) ds \leq C < \infty.$$

Therefore, using (2.8) again,

$$\int_0^{2t} r^3 \nu_2^2 \left( \frac{1}{3} (r - 2r_0)_+ \right) dr \leq C < \infty. \tag{10.8}$$

Finally, the integral

$$\int_0^{2t} \nu_0^2 \left( \frac{1}{3}(r - 2r_0)_+ \right) \frac{1}{t^4} r^3 dr = \int_0^{2t} \frac{\nu_0 \left( \frac{1}{3}(r - 2r_0)_+ \right)}{r} \frac{\nu_0 \left( \frac{1}{3}(r - 2r_0)_+ \right) r^4}{t^4} dr$$

is bounded: it converges to zero as  $t \rightarrow \infty$  by the Lebesgue theorem as in Lemma 6.1. Hence, Lemma 10.2 follows from (10.7) and (10.8).  $\square$

**Lemma 10.3.**  $I_2 \leq Cb^4|\Sigma|^2$ .

*Proof.* Since  $\nu_d^2(r) \leq C\nu_d(r)$  and  $\Sigma_{1,2} \subset \Sigma^4$ , we have by Lemma 4.2

$$\begin{aligned} I_2 &= Cb^4 \sum_{i,j=0,2} \int_{\Sigma_{1,2}} \frac{1}{t^i} \nu_{2-i}(|z_1 - z_3| - 2r_0)_+ \frac{1}{t^j} \nu_{2-j}(|z_2 - z_4| - 2r_0)_+ dS(\bar{z}) \\ &\leq Cb^4 \sum_{i,j=0,2} \int_{\Sigma^2} \frac{1}{t^i} \nu_{2-i}(|z_1 - z_3| - 2r_0)_+ dS(z_1) dS(z_3) \\ &\quad \times \int_{\Sigma^2} \frac{1}{t^j} \nu_{2-j}(|z_2 - z_4| - 2r_0)_+ dS(z_2) dS(z_4) \\ &\leq Cb^4 \sum_{i,j=0,2} |\Sigma| \int_0^{2t} \frac{r}{t^i} \nu_{2-i}((r - 2r_0)_+) dr \cdot |\Sigma| \int_0^{2t} \frac{r}{t^j} \nu_{2-j}((r - 2r_0)_+) dr \\ &\leq Cb^4 |\Sigma|^2. \end{aligned} \tag{10.9}$$

In the last inequality we use (2.8) and the Lebesgue theorem as in Lemma 6.1. Lemma 10.3 is proved.  $\square$

Now Lemmas 10.2, 10.3 and (10.6) imply (10.2).  $\square$

## 11. Convergence to equilibrium for variable coefficients

We extend all results of previous sections to the case of the wave equations with variable coefficients. We consider the wave equations in  $\mathbf{R}^3$  with the initial conditions

$$\begin{cases} \ddot{u}(x, t) = \sum_{j,k=1}^3 \partial_j(a_{jk}(x)\partial_k u(x, t)) - a_0(x)u(x, t), & x \in \mathbf{R}^3, t \in \mathbf{R}, \\ u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x), \end{cases} \tag{11.1}$$

where  $\partial_j \equiv \frac{\partial}{\partial x_j}$ . We assume the following properties E 1–E 3 of equation (11.1).



E1.  $a_{jk}(x) = \delta_{jk} + b_{jk}(x)$ , where  $b_{jk}(x) \in D$ ; also  $a_0(x) \in D$ .

E2.  $a_0(x) \geq 0$ , and the hyperbolicity condition holds: there exists  $\alpha > 0$  such that

$$H(x, k) \equiv \frac{1}{2} \sum_{i,j=1}^3 a_{ij}(x) k_i k_j \geq \alpha |k|^2, \quad x, k \in \mathbf{R}^3. \quad (11.2)$$

E3. Non-trapping condition holds (see [22]): for  $(x(0), k(0)) \in \mathbf{R}^3 \times \mathbf{R}^3$  with  $k(0) \neq 0$

$$|x(t)| \rightarrow \infty, \quad t \rightarrow \infty, \quad (11.3)$$

where  $(x(t), k(t))$  is a solution to the following Hamiltonian system

$$\dot{x}(t) = \nabla_k H(x(t), k(t)), \quad \dot{k}(t) = -\nabla_x H(x(t), k(t)).$$

**Example.** E1-E3 hold in the case of constant coefficients,  $a_{jk}(x) \equiv \delta_{ij}$ . For instance, E3 holds because  $\dot{k}(t) \equiv 0 \Rightarrow x(t) \equiv k(0)t + x(0)$ .

We use the notation as above,  $Y(t) \equiv (u(\cdot, t), \dot{u}(\cdot, t))$ ,  $Y_0 \equiv (u_0, v_0)$ . Then (11.1) becomes

$$\dot{Y}(t) = \mathcal{F}_*(Y(t)), \quad t \in \mathbf{R}, \quad Y(0) = Y_0. \quad (11.4)$$

Proposition 2.1 holds for the solutions to the Cauchy problem (11.4) as well as for (1.2). Let  $Y_0$  in (11.4) be a measurable random function with values in  $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ , and let  $\mu_0$  be its distribution, as above. Denote by  $\mu_t$  the distribution of the solution  $Y(t)$  to the problem (11.4). Let us state the extension of main Theorem 2.1. We introduce the appropriate Hilbert spaces of initial data of the infinite energy. Let  $\delta$  be an arbitrary positive number.

**Definition 11.1.**  $\mathcal{H}_\delta$  is the Hilbert space of functions  $Y = (u, v) \in \mathcal{H}$  with the finite norm

$$\|Y\|_\delta^2 = \int e^{-2\delta|x|} (|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2) dx < \infty.$$

**Theorem 11.1.** *Let E1–E3, S0–S3 hold. Then*

- i) *the convergence (2.4) holds for any  $\varepsilon > 0$ ;*
- ii) *the limit measure  $\mu_\infty$  is a Gaussian measure on  $\mathcal{H}$ ;*

iii) *the limit characteristic functional has the form*

$$\hat{\mu}_\infty(\psi) = \exp \left\{ -\frac{1}{2} Q_\infty(W\Psi, W\Psi) \right\}, \quad \Psi \in \mathcal{D},$$

where  $W : \mathcal{D} \rightarrow \mathcal{H}'_\delta$  is a linear continuous operator for sufficiently small  $\delta > 0$ .

Theorem 11.1 follows immediately from Theorem 2.1, using the method of [10]. The method is based on the scattering theory for the solutions of infinite energy.

### Appendix A. Radon transform

*Proof of Lemma 5.3.* Since  $\int_{p \cdot \omega = z \cdot \omega} f(p) d^2 p$  is an even function with respect to  $\omega$ , it suffices to prove the next lemma.

**Lemma A.1.** *Let (2.8) hold, and  $f \in C_\nu(\mathbf{R}^3)$ . Then*

$$\frac{1}{(4\pi)^2} \int_{|\omega|=1} dS(\omega) \int_{p \cdot \omega = z \cdot \omega} f(p) d^2 p = -\frac{1}{2} \mathcal{E} * f(z), \quad \forall z \in \mathbf{R}^3. \quad (\text{A.1})$$

*Proof.* Both sides of (A.1) define the continuous operators  $C_\nu(\mathbf{R}^3) \mapsto C_b(\mathbf{R}^3)$ . Therefore, it suffices to consider  $f \in D$ . Applying the Fourier transform, we obtain with  $\rho = |k|$ ,

$$\begin{aligned} (\mathcal{E} * f)(z) &= \frac{1}{(2\pi)^3} \int \hat{\mathcal{E}}(k) \hat{f}(k) \exp\{-iz \cdot k\} d^3 k \\ &= \frac{1}{(2\pi)^3} \int_{|\omega|=1} dS(\omega) \int_0^{+\infty} \rho^2 \exp\{-i\rho z \cdot \omega\} \hat{\mathcal{E}}(\rho\omega) \hat{f}(\rho\omega) d\rho. \end{aligned} \quad (\text{A.2})$$

We substitute  $\hat{\mathcal{E}}(\rho\omega) = -\rho^{-2}$  in the right-hand side of (A.2) and get

$$(\mathcal{E} * f)(z) = -\frac{1}{(2\pi)^3} \int_{|\omega|=1} dS(\omega) \int_0^{+\infty} \exp\{-i\rho z \cdot \omega\} \hat{f}(\rho\omega) d\rho. \quad (\text{A.3})$$

Note that

$$\hat{f}(\rho\omega) = \int_{-\infty}^{+\infty} \exp\{i\rho h\} f^\#(h, \omega) dh, \quad (\text{A.4})$$

where  $f^\#(h, \omega) \equiv \int_{y \cdot \omega = h} f(y) d^2y$ . Then from (A.3), (A.4) we have

$$\begin{aligned}
(\mathcal{E} * f)(z) &= -\frac{1}{(2\pi)^3} \int_{|\omega|=1} dS(\omega) \frac{1}{2} \int_{-\infty}^{+\infty} \exp\{-iyz \cdot \omega\} dy \\
&\quad \times \int_{-\infty}^{+\infty} \exp\{i\rho h\} f^\#(h, \omega) dh \\
&= -\frac{1}{8\pi^2} \int_{|\omega|=1} dS(\omega) F_{y \rightarrow (z \cdot \omega)}^{-1} F_{h \rightarrow y} f^\#(h, \omega) \\
&= -\frac{1}{8\pi^2} \int_{|\omega|=1} f^\#(z \cdot \omega, \omega) dS(\omega).
\end{aligned}$$

Lemma A.1 is proved.  $\square$

*Proof of Lemma 6.2.* Since  $F[P](k) = -i|k|^{-1} \text{sgn } k_3$ , we have

$$\begin{aligned}
(P * f)(z) &= \frac{1}{(2\pi)^3} \int \hat{P}(k) \hat{f}(k) \exp\{-iz \cdot k\} d^{-3}k \tag{A.5} \\
&= -\frac{i}{(2\pi)^3} \int_{|\omega|=1} dS(\omega) \int_0^{+\infty} \rho \exp\{-i\rho z \cdot \omega\} \text{sgn}(\omega_3) \hat{f}(\rho\omega) d\rho \\
&= -\frac{i}{(2\pi)^3} \int_{|\omega|=1, \omega_3 > 0} dS(\omega) \int_0^{+\infty} \rho \exp\{-i\rho z \cdot \omega\} \hat{f}(\rho\omega) d\rho \\
&\quad + \frac{i}{(2\pi)^3} \int_{|\omega|=1, \omega_3 < 0} dS(\omega) \int_0^{+\infty} \rho \exp\{-i\rho z \cdot \omega\} \hat{f}(\rho\omega) d\rho.
\end{aligned}$$

In the last integral we change the variables  $\omega \rightarrow -\omega$ ,  $\rho \rightarrow -\rho$ , then apply (A.4) and get

$$\begin{aligned}
(P * f)(z) &= -\frac{i}{(2\pi)^3} \int_{|\omega|=1, \omega_3 > 0} dS(\omega) \int_{-\infty}^{+\infty} \rho \exp\{-i\rho z \cdot \omega\} \hat{f}(\rho\omega) d\rho \\
&= -\frac{i}{(2\pi)^3} \int_{|\omega|=1, \omega_3 > 0} dS(\omega) \int_{-\infty}^{+\infty} \exp\{-i\rho z \cdot \omega\} \rho d\rho
\end{aligned}$$

$$\times \int_{-\infty}^{+\infty} \exp\{i\rho h\} f^\#(h, \omega) dh. \quad (\text{A.6})$$

Note that

$$\rho \int_{-\infty}^{+\infty} e^{i\rho h} f^\#(h, \omega) dh = i \int_{-\infty}^{+\infty} e^{i\rho h} (\nabla f)^\#(h, \omega) \cdot \omega dh, \quad \rho \in \mathbf{R}. \quad (\text{A.7})$$

Indeed, applying (A.4) in the both sides of

$$F[\nabla f](\rho\omega) \cdot \omega = -i\rho F[f](\rho\omega),$$

we obtain (A.7). Finally, from (A.7) and (A.6) we get

$$\begin{aligned} (P * f)(z) &= \frac{1}{(2\pi)^3} \int_{|\omega|=1, \omega_3 > 0} dS(\omega) \int_{-\infty}^{+\infty} \exp\{-i\rho z \cdot \omega\} d\rho \\ &\quad \times \int_{-\infty}^{+\infty} \exp\{i\rho h\} (\nabla f)^\#(h, \omega) \cdot \omega dh \\ &= \frac{1}{4\pi^2} \int_{|\omega|=1, \omega_3 > 0} F_{\rho \rightarrow z \cdot \omega}^{-1} F_{h \rightarrow \rho} (\nabla f)^\#(h, \omega) \cdot \omega dS(\omega) \\ &= \frac{1}{4\pi^2} \int_{|\omega|=1, \omega_3 > 0} (\nabla f)^\#(z \cdot \omega, \omega) \cdot \omega dS(\omega) = 4\mathcal{P}f(z). \end{aligned} \quad (\text{A.8})$$

Lemma 6.2 is proved.  $\square$

## Appendix B. Gaussian measures in Sobolev's spaces

We verify (3.3). Definition (3.2) implies for  $u \in H_{s, \alpha}$ ,

$$\|u\|_{s, \alpha}^2 = \int (1 + |x|)^{2\alpha} \left( \int e^{-ix(k-\eta)} (1 + |k|)^s (1 + |\eta|)^s \hat{u}(k) \bar{\hat{u}}(\eta) dk d\eta \right) dx. \quad (\text{B.1})$$

Let  $\mu(du)$  be a Gaussian translation invariant measure in  $H_{s, \alpha}$  with a correlation function  $Q(x, y) = q(x - y)$ . Let us introduce the following correlation function

$$C(k, \eta) \equiv \int \hat{u}(k) \bar{\hat{u}}(\eta) \mu(du) \quad (\text{B.2})$$

in the sense of distributions. Since  $u(x)$  is real-valued, we get

$$C(k, \eta) = F_{x \rightarrow k} F_{y \rightarrow -\eta} Q(x, y) = C_n \delta(k - \eta) \hat{q}(k). \quad (\text{B.3})$$

Then, integrating (B.1) with respect to the measure  $\mu(du)$ , we get the formula

$$\int \|u\|_{s, \alpha}^2 \mu(du) = C_n \int (1 + |x|)^{2\alpha} dx \int (1 + |k|)^{2s} \hat{q}(k) dk. \quad (\text{B.4})$$

Substituting  $\hat{q}(k) = 1$  and  $\hat{q}(k) = |k|^{-2}$ , we get (3.3).  $\square$

## References

- [1] P. BILLINGSLEY (1968) *Convergence of Probability Measures*. Wiley, New York.
- [2] C. BOLDRIGHINI, A. PELLEGRINOTTI AND L. TRIOLO (1983) Convergence to stationary states for infinite harmonic systems. *J. Stat. Phys.* **30**, 123–155.
- [3] I.P. CORNFELD, S.V. FOMIN AND YA.G. SINAI (1981) *Ergodic Theory*. Springer, New York.
- [4] R.L. DOBRUSHIN AND YU.M. SUHOV (1978) On the problem of the mathematical foundation of the Gibbs postulate in classical statistical mechanics. In: *Mathematical Problems in Theoretical Physics*, Lect. Notes Phys. **80**, 325–340.
- [5] R.L. DOBRUSHIN AND YU.M. SUHOV (1981) Time asymptotics for some degenerate models of evolution of systems with an infinite number of particles. *J. Sov. Math.* **16**, 1277–1340.
- [6] T.V. DUDNIKOVA (1995) Ergodic phase flow of wave equation with mixing. *Moscow University Mathematics Bulletin* **50** (1), 15–19.
- [7] T.V. DUDNIKOVA AND A.I. KOMECH (1996) Ergodic properties of hyperbolic equations with mixing. *Theory Probab. Appl.* **41** (3), 436–448.
- [8] T.V. DUDNIKOVA (1997) Stabilization of space-time statistical solutions of the Klein–Gordon equation. *Russian J. Math. Phys.* **5** (2), 176–188.
- [9] T.V. DUDNIKOVA, A.I. KOMECH, E.A. KOPYLOVA AND YU.M. SUHOV On convergence to equilibrium distribution, I. The Klein–Gordon equation with mixing. To appear in *Comm. Math. Phys.*
- [10] T.V. DUDNIKOVA, A.I. KOMECH, N.E. RATANOV AND YU.M. SUHOV On convergence to equilibrium distribution, II. The wave equation in odd dimensions, with mixing. To appear in *J. Stat. Phys.*
- [11] I.A. IBRAGIMOV AND YU.V. LINNIK (1971) *Independent and Stationary Sequences of Random Variables*. Wolters–Noordhoff, Groningen.
- [12] F. JOHN (1955) *Plane Waves and Spherical Means applied to Partial Differential Equations*. Interscience Publishers, New York - London.
- [13] L. HÖRMANDER (1985) *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. Springer-Verlag.

- [14] E. A. KOPYLOVA (1986) Stabilization of statistical solutions of the Klein–Gordon equation. *Moscow University Mathematics Bulletin* **41** (2), 72–75.
- [15] V. P. MIKHAILOV (1978) *Partial Differential Equations*. Mir, Moscow.
- [16] H. NAKAZAWA (1970) On the lattice thermal conduction. *Supplement of the Progress of Theor. Phys.* **45**, 231–262.
- [17] V. V. PETROV (1995) *Limit Theorems of Probability Theory*. Clarendon Press, Oxford.
- [18] N. E. RATANOV (1984) Stabilization of statistical solutions of second order hyperbolic equations. *Russian Mathematical Surveys* **39** (1), 179–180.
- [19] N. E. RATANOV (1985) Asymptotic normality of statistical solutions of the wave equation. *Moscow University Mathematics Bulletin* **40** (4), 77–79.
- [20] Z. RIEDER, J. L. LEBOWITZ AND E. LIEB (1967) Properties of a harmonic crystal in a stationary nonequilibrium state. *J. Math. Phys.* **8** (5), 1073.
- [21] H. SPOHN AND J. LEBOWITZ (1977) Stationary non-equilibrium states of infinite harmonic systems. *Commun. Math. Phys.* **54** (2), 97–120.
- [22] B. R. VAINBERG (1989) *Asymptotic Methods in Equations of Mathematical Physics*. Gordon and Breach, New York.
- [23] M. I. VISHIK AND A. V. FURSIKOV (1988) *Mathematical Problems of Statistical Hydromechanics*. Kluwer Academic Publishers, Dordrecht.