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Global Attractor for a Nonlinear Oscillator Coupled to the Klein–Gordon Field

Alexander Komech & Andrew Komech

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Abstract

The long-time asymptotics is analyzed for all finite energy solutions to a model $\mathbf{U}(1)$-invariant nonlinear Klein–Gordon equation in one dimension, with the nonlinearity concentrated at a single point: *each finite energy solution* converges as $t \to \pm\infty$ to the set of all "nonlinear eigenfunctions" of the form $\psi(x)e^{-i\omega t}$. The *global attraction* is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersive radiation.

We justify this mechanism by the following novel strategy based on *inflation of spectrum by the nonlinearity*. We show that any *omega-limit trajectory* has the time spectrum in the spectral gap $[-m, m]$ and satisfies the original equation. This equation implies the key *spectral inclusion* for spectrum of the nonlinear term. Then the application of the Titchmarsh convolution theorem reduces the spectrum of each omega-limit trajectory to a single harmonic $\omega \in [-m, m]$.

The research is inspired by Bohr's postulate on quantum transitions and Schrödinger's identification of the quantum stationary states to the nonlinear eigenfunctions of the coupled $\mathbf{U}(1)$-invariant Maxwell–Schrödinger and Maxwell–Dirac equations.

1. Introduction

The long-time asymptotics for nonlinear wave equations have been the subject of intensive research, starting with the pioneering papers by Segal [Seg63a, Seg63b], Strauss [Str68], and Morawetz and Strauss [MS72], where the nonlinear scattering and the local attraction to zero were considered. The asymptotic stability of solitary waves has been studied since the 1990s by Soffer and Weinstein [SW90, SW92], Buslaev and Perelman [BP93, BP95], and then by others. The existing results suggest that the set of orbitally stable solitary waves typically forms a local attractor, that is, attracts finite energy solutions that were initially close to it.

In this paper, we consider the global attractor for all finite energy solutions. For the first time, we prove that in a particular $\mathbf{U}(1)$-invariant dispersive Hamiltonian system the global attractor is finite-dimensional and is formed by solitary waves. The investigation is inspired by Bohr's quantum transitions ("quantum jumps"). Namely, according to BOHR's postulates [Boh13], an unperturbed electron lives forever in a *quantum stationary state* $|E\rangle$ that has a definite value E of the energy. Under an external perturbation, the electron can jump from one state to another:

$$|E_-\rangle \longmapsto |E_+\rangle. \tag{1.1}$$

The postulate suggests the dynamical interpretation of the transitions as long-time attraction

$$\Psi(t) \longrightarrow |E_\pm\rangle, \qquad t \to \pm\infty \tag{1.2}$$

for any trajectory $\Psi(t)$ of the corresponding dynamical system, where the limiting states $|E_\pm\rangle$ generally depend on the trajectory. Then the quantum stationary states should be viewed as the points of the *global attractor* $\mathscr{S}$ which is the set of all limiting states (see Figure 1). Following de Broglie's ideas, Schrödinger identified the stationary states $|E\rangle$ as the solutions of the wave equation that have the form

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \qquad \omega = E/\hbar, \tag{1.3}$$

where $\hbar$ is Planck's constant. Then the attraction (1.2) takes the form of the long-time asymptotics

$$\psi(x,t) \sim \psi_\pm(x,t) = \phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{1.4}$$

that hold for each finite energy solution. However, because of the superposition principle, the asymptotics of type (1.4) are generally impossible for the linear autonomous Schrödinger equation of type

$$(i\partial_t - V(x))\psi(x,t) = (-i\nabla - \boldsymbol{A}(x))^2\psi(x,t), \tag{1.5}$$

where $V(x)$ and $\boldsymbol{A}(x)$ are scalar and vector potentials of a static external Maxwell field. An adequate description of this process requires us to consider the Schrödinger (or Dirac) equation coupled to the Maxwell system which governs the time evolution of the Maxwell 4-potential $A(x,t) = (V(x,t), \boldsymbol{A}(x,t))$. This coupling is inevitable indeed, because, again by Bohr's postulates, the transitions (1.1) are followed by electromagnetic radiation responsible for the atomic spectra. The coupled Maxwell–Schrödinger system was initially introduced in [Sch26]. It is a $\mathbf{U}(1)$-invariant nonlinear Hamiltonian system. Its global well-posedness was considered in [GNS95]. We might expect the following generalization of asymptotics (1.4) for solutions to the coupled Maxwell–Schrödinger (or Maxwell–Dirac) equations:

$$(\psi(x,t), A(x,t)) \sim \left(\phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, A_{\omega_\pm}(x)\right), \qquad t \to \pm\infty. \tag{1.6}$$

The asymptotics of this form are not available yet in the context of coupled systems. Let us mention that the existence of the solitary waves for the coupled Maxwell–Dirac equations was established in [EGS96].

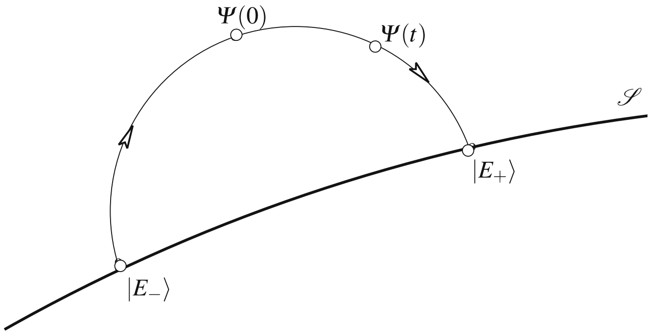

Fig. 1. Attraction of any trajectory $\Psi(t)$ to the set of solitary waves as $t \to \pm\infty$

The asymptotics (1.6) would mean that the set of all solitary waves

$$\{(\phi_\omega(x), A_\omega(x)) : \omega \in \mathbb{C}\}$$

forms a global attractor for the coupled system. Similar convergence to a global attractor is well known for dissipative systems, like Navier–Stokes equations (see [BV92, Hen81, Tem97]). In this context, the global attractor is formed by the *static stationary states*, and the corresponding asymptotics (1.4) only hold for $t \to +\infty$ (and with $\omega_+ = 0$).

Our main impetus for writing this paper was the natural question of whether dispersive Hamiltonian systems could, in the same spirit, possess finite dimensional global attractors, and whether such attractors are formed by the solitary waves. We prove such a global attraction for a model nonlinear Klein–Gordon equation

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}. \tag{1.7}$$

Here $m > 0$, $\psi(x,t)$ is a continuous complex-valued wave function, and F is a nonlinearity. The dots stand for the derivatives in t, and the primes for the derivatives in x. All derivatives and the equation are understood in the sense of distributions. Equation (1.7) describes the linear Klein–Gordon equation coupled to the nonlinear oscillator. We assume that equation (1.7) is $\mathbf{U}(1)$-invariant; that is,

$$F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \qquad \psi \in \mathbb{C}, \quad \theta \in \mathbb{R}.$$

Note that the group $\mathbf{U}(1)$ is also the (global) gauge group of the coupled Maxwell–Schrödinger and Maxwell–Dirac equations, with the representation given by

$$(\psi(x), A(x)) \mapsto (e^{i\theta}\psi(x), A(x)).$$

This gauge symmetry leads to the charge conservation and to the existence of the solitary wave solutions of the form (1.6) (see [EGS96]). We clarify the special role of the "nonlinear eigenfunctions", or solitary waves, of equation (1.7) which are finite energy solutions of type (1.3):

$$\psi_\omega(x,t) = \phi_\omega(x)e^{-i\omega t}, \quad \omega \in \mathbb{C}. \tag{1.8}$$

We prove that indeed they form the global attractor for all finite energy solutions to (1.7).

Equation (1.7) has the following key features of the coupled Maxwell–Schrödinger and Maxwell–Dirac equations: (i) the linear part of this equation has a dispersive character; (ii) it is a nonlinear Hamiltonian system; (iii) it is $\mathbf{U}(1)$-invariant. We suggest that just these features are responsible for the global attraction, such as (1.4), (1.6), to "quantum stationary states".

Let us introduce the set of all solitary waves.

Definition 1.1. Let $\mathscr{S}$ be the set of all functions $\phi_\omega(x) \in H^1(\mathbb{R})$ with $\omega \in \mathbb{C}$, so that $\phi_\omega(x)e^{-i\omega t}$ is a solution to (1.7).

Here $H^1(\mathbb{R})$ denotes the Sobolev space. Generically, the quotient $\mathscr{S}/\mathbf{U}(1)$ is isomorphic to a finite union of one-dimensional intervals. We will give an explicit construction of the set of all solitary waves for equation (1.7); see Proposition 2.1 and its proof in Appendix A. Let us mention that there are numerous results on the existence of solitary wave solutions of the form $\phi(x)e^{-i\omega t}$ to nonlinear Hamiltonian systems with $\mathbf{U}(1)$ symmetry [Str77, BL83a, BL83b, BL84, CV86, ES95]. Typically, such solutions exist for ω from an interval or a collection of intervals of the real line.

Our main result is the following long-time asymptotics: in the case when the nonlinearity F is polynomial of order strictly greater than 1, we prove the attraction of any finite energy solution to the set $\mathscr{S}$ of all solitary waves:

$$\psi(\cdot, t) \longrightarrow \mathscr{S}, \qquad t \to \pm\infty, \tag{1.9}$$

where the convergence holds in local energy seminorms. In the linear case, when $F(\psi) = a\psi$ with $a \in \mathbb{R}$, there is generally no attraction to $\mathscr{S}$; instead, we show that the global attractor is the linear span of all solitary waves, $\langle\mathscr{S}\rangle$. See Theorem 2.3.

Remark 1.1. Although we proved the attraction (1.9) to $\mathscr{S}$, we have not proved the attraction to a particular solitary wave, falling short of proving (1.4). Hypothetically, a solution can be drifting along $\mathscr{S}$, keeping asymptotically close to it, but never approaching a particular solitary wave.

Remark 1.2. The requirement that the nonlinearity F is polynomial allows us to apply the Titchmarsh convolution theorem that is vital to the proof. We do not know whether this requirement could be dropped.

Let us mention related earlier results:

(i) The asymptotics of type (1.4) were discovered first with $\psi_\pm = 0$ in the scattering theory [Str68, MS72, Str78, GS79, Kla82, GV85, Hör91]. In this case, the attractor $\mathscr{S}$ consists of the zero solution only, and the asymptotics mean well-known *local energy decay*.

(ii) The *global attraction* of type (1.4) with $\psi_\pm \neq 0$ and $\omega_\pm = 0$ was established in [Kom91, Kom95, KV96, KSK97, Kom99, KS00] for a number of nonlinear wave problems. There the attractor $\mathscr{S}$ is the set of all *static* stationary states. Let us mention that this set could be infinite and contain continuous components.

(iii) First results on the asymptotics of type (1.4), with $\omega_\pm \neq 0$, were obtained for nonlinear $\mathbf{U}(1)$-invariant Schrödinger equations in the context of asymptotic stability. This establishes asymptotics of type (1.4) but only for solutions close to the solitary waves, proving the existence of a *local attractor*. This was first done in [SW90, BP93, SW92, BP95], and then developed in [PW97, SW99, Cuc01a, Cuc01b, BS03, Cuc03] and other papers.

The *global attraction* (1.9) to the solitary waves with $\omega \neq 0$ was announced for the first time in [Kom03] for equation (1.7). In the present paper we give the detailed proofs, and also add the well-posedness result which is not trivial since the Dirac delta-function $\delta(x)$ does not belong to $L^2(\mathbb{R})$.

Let us mention that the attraction (1.4) for equation (1.7) with $m = 0$ was proved in [Kom91, Kom95]; in that case $\omega_\pm = 0$. Our proofs for $m > 0$ are quite different from [Kom91, Kom95], and are based on a nonlinear spectral analysis of *omega-limit trajectories* for $t \to +\infty$ (and similarly for $t \to -\infty$). First, we prove that their time spectrum is contained in a finite interval $[-m, m]$, since the spectral density is absolutely continuous for $|\omega| > m$ and the corresponding component of the solution disperses completely. Second, the nonlinear equation (1.7) implies the crucial spectral inclusion: the nonlinearity does not inflate the spectrum of any omega-limit trajectory. Finally, the Titchmarsh convolution theorem allows us to reduce the spectrum of the omega-limit trajectory to a single harmonic $\omega_+ \in [-m, m]$. This implies the attraction (1.9).

Remark 1.3. The global attraction (1.4), (1.6) for $\mathbf{U}(1)$-invariant equations suggests the corresponding extension to general $\mathbf{G}$-invariant equations ($\mathbf{G}$ being the Lie group):

$$\psi(x,t) \sim \psi_\pm(x,t) = e^{\mathbf{\Omega}_\pm t}\phi_\pm(x), \qquad t \to \pm\infty, \tag{1.10}$$

where $\mathbf{\Omega}_\pm$ belong to the corresponding Lie algebra and $e^{\mathbf{\Omega}_\pm t}$ are corresponding one-parameter subgroups. Respectively, the global attractor would consist of the solitary waves (1.10). In particular, for the unitary group $\mathbf{G} = \mathbf{SU}(3)$, the asymptotics (1.10) relates the “quantum stationary states” to the structure of the corresponding Lie algebra $\mathbf{su}(3)$. On a seemingly related note, let us mention that according to GELL-MANN–NE’EMAN theory [GMN64] there is a correspondence between the Lie algebras and the classification of the elementary particles which are the “quantum stationary states”. The correspondence has been confirmed experimentally by the discovery of the omega-minus Hyperon.

The plan of the paper is as follows. In Sect. 2 we state the main assumptions and results. Section 3 describes the exclusion of dispersive components from the solution. In Sect. 5 we state the spectral properties of all omega-limit trajectories and apply the Titchmarsh convolution theorem. For completeness, we also give the exhaustive treatment of the linear case, when $F(\psi) = a\psi$ with $a \in \mathbb{R}$; see Sect. 6. In Appendix A, we collect the properties of the solitary waves. In Appendix B, we describe properties of quasimeasures and corresponding multiplicators. The global well-posedness of equation (1.7) in $H^1(\mathbb{R})$ is proved in Appendix C.

2. Main results

Model

We consider the Cauchy problem for the Klein–Gordon equation with the nonlinearity concentrated at a point:

$$\begin{cases} \ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ \psi|_{t=0} = \psi_0(x), \quad \dot{\psi}|_{t=0} = \pi_0(x). \end{cases} \tag{2.1}$$

If we identify a complex number $\psi = u + iv \in \mathbb{C}$ with the two-dimensional vector $(u, v) \in \mathbb{R}^2$, then, physically, equation (2.1) describes small crosswise oscillations of the infinite string in three-dimensional space (x, u, v) stretched along the x-axis. The string is subject to the action of an "elastic force" $-m^2\psi(x,t)$ and coupled to a nonlinear oscillator of force $F(\psi)$ attached at the point $x = 0$.

We define $\Psi(t) = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix}$ and write the Cauchy problem (2.1) in the vector form:

$$\dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Psi(t) + \delta(x) \begin{bmatrix} 0 \\ F(\psi) \end{bmatrix}, \quad \Psi|_{t=0} = \Psi_0 \equiv \begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix}. \tag{2.2}$$

We will assume that the oscillator force F admits a real-valued potential:

$$F(\psi) = -\nabla U(\psi), \quad \psi \in \mathbb{C}, \qquad U \in C^2(\mathbb{C}), \tag{2.3}$$

where the gradient is taken with respect to $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$. Then equation (2.2) formally can be written as a Hamiltonian system,

$$\dot{\Psi}(t) = J\, D\mathscr{H}(\Psi), \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $D\mathscr{H}$ is the variational derivative of the Hamilton functional

$$\mathscr{H}(\Psi) = \frac{1}{2} \int_{\mathbb{R}} \left(|\pi|^2 + |\psi'|^2 + m^2|\psi|^2 \right) dx + U(\psi(0)), \quad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix}. \tag{2.4}$$

We assume that the potential $U(\psi)$ is $\mathbf{U}(1)$-invariant, where $\mathbf{U}(1)$ stands for the unitary group $e^{i\theta}$, $\theta \in \mathbb{R} \bmod 2\pi$: namely, we assume that there exists $u \in C^2(\mathbb{R})$ such that

$$U(\psi) = u(|\psi|^2), \qquad \psi \in \mathbb{C}. \tag{2.5}$$

Remark 2.1. In the context of the model of the infinite string in $\mathbb{R}^3$ that we described after (2.1), the potential $U(\psi)$ is rotation invariant with respect to the x-axis.

Conditions (2.3) and (2.5) imply that

$$F(\psi) = \alpha(|\psi|^2)\psi, \qquad \psi \in \mathbb{C}, \tag{2.6}$$

where $\alpha(\cdot) = 2u'(\cdot) \in C^1(\mathbb{R})$ is real-valued. Therefore,

$$F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \qquad \theta \in \mathbb{R}, \quad \psi \in \mathbb{C}. \tag{2.7}$$

Then the Nöther theorem formally implies that the functional

$$\mathscr{Q}(\Psi) = \frac{i}{2}\int_{\mathbb{R}} \left(\overline{\psi}\pi - \overline{\pi}\psi\right)\,dx, \qquad \Psi = \begin{bmatrix}\psi(x)\\ \pi(x)\end{bmatrix}, \tag{2.8}$$

is conserved for solutions $\Psi(t)$ to (2.2).

We introduce the phase space $\mathscr{E}$ of finite energy states for equation (2.2). Denote by L^2 the complex Hilbert space $L^2(\mathbb{R})$ with the norm $\|\cdot\|_{L^2}$, and denote by $\|\cdot\|_{L^2_R}$ the norm in $L^2(-R, R)$ for $R > 0$.

Definition 2.1.

(i) $\mathscr{E}$ is the Hilbert space of the states $\Psi = \begin{bmatrix}\psi(x)\\ \pi(x)\end{bmatrix}$, with the norm

$$\|\Psi\|^2_{\mathscr{E}} := \|\pi\|^2_{L^2} + \|\psi'\|^2_{L^2} + m^2\|\psi\|^2_{L^2}. \tag{2.9}$$

(ii) $\mathscr{E}_F$ is the space $\mathscr{E}$ endowed with the Fréchet topology defined by the seminorms

$$\|\Psi\|^2_{\mathscr{E},R} := \|\pi\|^2_{L^2_R} + \|\psi'\|^2_{L^2_R} + m^2\|\psi\|^2_{L^2_R}, \qquad R > 0. \tag{2.10}$$

The equation (2.2) is formally a Hamiltonian system with the phase space $\mathscr{E}$ and the Hamilton functional $\mathscr{H}$. Both $\mathscr{H}$ and $\mathscr{Q}$ are continuous functionals on $\mathscr{E}$. Let us note that $\mathscr{E} = H^1 \oplus L^2$, where H^1 denotes the Sobolev space

$$H^1 = H^1(\mathbb{R}) = \{\psi(x) \in L^2(\mathbb{R}) :\ \psi'(x) \in L^2(\mathbb{R})\}.$$

We introduced into (2.9), (2.10) the factor $m^2 > 0$; this provides the convenient relation $\mathscr{H}(\Psi) = \frac{1}{2}\|\Psi\|^2_{\mathscr{E}} + U(\psi(0))$. The space $\mathscr{E}_F$ is metrizable (but not complete).

Global well-posedness

To have *a priori* estimates available for the proof of the global well-posedness, we assume that

$$U(\psi) \geqq A - B|\psi|^2 \quad \text{for } \psi \in \mathbb{C}, \quad \text{where } A \in \mathbb{R} \text{ and } 0 \leqq B < m. \tag{2.11}$$

Theorem 2.1. *Let $F(\psi)$ satisfy conditions* (2.3) *and* (2.5):

$$F(\psi) = -\nabla U(\psi), \qquad U(\psi) = u(|\psi|^2), \qquad u(\cdot) \in C^2(\mathbb{R}).$$

Additionally, assume that (2.11) *holds. Then*:

(i) *For every $\Psi_0 \in \mathscr{E}$ the Cauchy problem* (2.2) *has a unique solution $\Psi(t) \in C(\mathbb{R}, \mathscr{E})$.*

(ii) *The map $W(t) :\ \Psi_0 \mapsto \Psi(t)$ is continuous in $\mathscr{E}$ and $\mathscr{E}_F$ for each $t \in \mathbb{R}$.*

(iii) *The energy is conserved*:

$$\mathscr{H}(\Psi(t)) = \text{const}, \quad t \in \mathbb{R}. \tag{2.12}$$

(iv) *The following a priori bound holds*:

$$\|\Psi(t)\|_{\mathscr{E}} \leqq C(\Psi_0), \qquad t \in \mathbb{R}. \tag{2.13}$$

We prove this theorem in Appendix C.

Remark 2.2. The value of the charge is also conserved: $\mathscr{Q}(\Psi(t)) = \text{const},\ t \in \mathbb{R}$.

Solitary waves and the main theorem

Definition 2.2.

(i) The solitary waves of equation (2.2) are solutions of the form

$$\Psi(t) = \Phi_\omega e^{-i\omega t}, \text{ where } \omega \in \mathbb{C}, \ \Phi_\omega = \begin{bmatrix} \phi_\omega \\ -i\omega\phi_\omega \end{bmatrix}, \ \phi_\omega \in H^1(\mathbb{R}). \tag{2.14}$$

(ii) The solitary manifold is the set $\mathbf{S} = \{\Phi_\omega : \omega \in \mathbb{C}\}$ of all amplitudes Φ_ω.

Identity (2.7) implies that the set $\mathbf{S}$ is invariant under multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$. Let us note that since $F(0) = 0$ by (2.6), for any $\omega \in \mathbb{C}$ there is a zero solitary wave with $\phi_\omega(x) \equiv 0$.

Note that, according to (2.6), $\alpha(|C|^2) := F(C)/C \in \mathbb{R}$ for any $C \in \mathbb{C}\backslash 0$. We will need to distinguish the cases when F is linear and nonlinear; for this, we introduce the following definition.

Definition 2.3. The function $F(\psi)$ is *strictly nonlinear* if the equation $\alpha(C^2) = a$ has a discrete (or empty) set of positive roots C for each particular $a \in \mathbb{R}$.

Lemma 2.1. *If $F(\psi)$ is strictly nonlinear in the sense of* Definition 2.3*, then nonzero solitary waves exist only for $\omega \in \mathbb{R}$.*

We prove this Lemma in Appendix A.

Proposition 2.1 (Existence of solitary waves)**.** *Assume that $F(\psi)$ satisfies* (2.7) *and that one of two following conditions holds*:

(i) *$F(\psi)$ is strictly nonlinear in the sense of* Definition 2.3;
(ii) *$F(\psi) = a\psi$ with $a \in \mathbb{R}$.*

Then all nonzero solitary wave solutions to (2.2) *are given by* (2.14) *with*

$$\phi_\omega(x) = Ce^{-\kappa|x|}, \tag{2.15}$$

where $\kappa > 0$, $\omega \in \mathbb{C}$, and $C \in \mathbb{C}\backslash 0$ satisfy the following relations:

$$\alpha(|C|^2) = 2\kappa, \qquad \kappa^2 = m^2 - \omega^2. \tag{2.16}$$

Additionally, if $F(\psi)$ is strictly nonlinear, then $\omega \in (-m, m)$.

We prove this Proposition in Appendix A.

Remark 2.3. Let us denote $\kappa_C = \alpha(|C|^2)/2$ and $\omega_C^\pm = \pm\sqrt{m^2 - \kappa_C^2}$ for $C \in \mathbb{C}$. Then the relation (2.16) demonstrates that the set of all solitary waves can be parametrized as follows:

(i) When $F(\psi)$ is strictly nonlinear, in the sense of Definition 2.3, the profile function $\phi_C(x) = Ce^{i\theta}e^{-\kappa_C|x|}$, with $C \geqq 0$ and $\theta \in [0, 2\pi]$, corresponds to the solitary waves with $\omega = \omega_C^\pm$ as long as $\kappa_C \in (0, m]$ (so that $\phi_C \in H^1$ and ω is real in agreement with Lemma 2.1).

(ii) When $F(\psi) = a\psi$ with $a \in \mathbb{R}$, we see from (2.16) that $\kappa_C = a/2$ is constant. If $a > 0$, the profile function $\phi_C(x) = Ce^{-a|x|/2}$, with $C \in \mathbb{C}$, corresponds to the solitary waves with $\omega = \pm\sqrt{m^2 - \frac{a^2}{4}}$. The restriction $\kappa_C \in (0, m]$ no longer applies since the value of ω may be imaginary. (This is different from the case of strictly nonlinear F, when imaginary values of ω are prohibited by Lemma 2.1.) If $a \leqq 0$, then there is only the zero solitary wave solution.

As we mentioned before, we need to assume that the nonlinearity is polynomial. This assumption is crucial in our argument: it will allow us to apply the Titchmarsh convolution theorem. Now all our assumptions on F can be summarized as follows.

Assumption 2.1.

$$F(\psi) = -\nabla U(\psi), \qquad U(\psi) = \sum_{n=0}^{N} u_n |\psi|^{2n}, \tag{2.17}$$

where $u_n \in \mathbb{R},\ u_N > 0,\ N \geqq 2.$

This assumption guarantees that the nonlinearity F satisfies (2.3) and (2.5), and also the bound (2.11) from Theorem 2.1. Moreover, Assumption 2.1 implies that F is strictly nonlinear in the sense of Definition 2.3. By Lemma 2.1, this in turn implies that all nonzero solitary waves correspond to $\omega \in \mathbb{R}$.

Our main result is the following theorem.

Theorem 2.2 (Main Theorem)**.** *Let the nonlinearity $F(\psi)$ satisfy* Assumption 2.1. *Then for any $\Psi_0 \in \mathscr{E}$ the solution $\Psi(t) \in C(\mathbb{R}, \mathscr{E})$ to the Cauchy problem* (2.2) *with $\Psi(0) = \Psi_0^-$ converges to* $\mathbf{S}$ *in the space $\mathscr{E}_F$:*

$$\Psi(t) \xrightarrow{\mathscr{E}_F} \mathbf{S}, \quad t \to \pm\infty. \tag{2.18}$$

Let us note that the convergence to the set $\mathbf{S}$ in the space $\mathscr{E}_F$ is equivalent to

$$\lim_{t\to\pm\infty} \rho(\Psi(t), \mathbf{S}) = 0, \tag{2.19}$$

where ρ is a metric in the space $\mathscr{E}_F$ and $\rho(\Psi(t), \mathbf{S}) := \inf_{\Phi\in\mathbf{S}} \rho(\Psi(t), \Phi)$.

Let us also give the corresponding result for the linear case, when $F(\psi) = a\psi$ with $a \in \mathbb{R}$. We restrict our consideration to the case when $a < 2m$. It is in this case that condition (2.11) is satisfied. We do not consider the case $a \geqq 2m$, since in this case the solutions are generally not bounded in $\mathscr{E}$ norm (see Remark 6.1), while our arguments rely significantly on the bounds (2.13). This case will be considered in more detail elsewhere.

Theorem 2.3 (Linear case)**.** *Assume that $F(\psi) = a\psi$, where $a < 2m$. Then for any $\Psi_0 \in \mathscr{E}$ the solution $\Psi(t) \in C(\mathbb{R}, \mathscr{E})$ to the Cauchy problem* (2.2) *with $\Psi(0) = \Psi_0$ converges in the space $\mathscr{E}_F$ to the linear span of* $\mathbf{S}$*, which we denote by $\langle\mathbf{S}\rangle$:*

$$\Psi(t) \xrightarrow{\mathscr{E}_F} \langle\mathbf{S}\rangle, \quad t \to \pm\infty. \tag{2.20}$$

Remark 2.4. In Sect. 6 we will show that:

(i) If $0 < a < 2m$, then $\langle \mathbf{S} \rangle \neq \mathbf{S}$. Particular solutions show that the attraction (2.18) does not hold in general (see Remark 6.3) and has to be substituted by (2.20).
(ii) If $a \leqq 0$, $\langle \mathbf{S} \rangle = \mathbf{S} = \{0\}$.

Strategy of the proof

For $m = 0$ the global attraction of type (2.18) is proved in [Kom95], where the proof was based on the direct calculation of the energy radiation for the wave equation. For the Klein–Gordon equation with $m > 0$, the dispersive relation $\omega^2 = k^2 + m^2$ results in the group velocities $v = \omega'(k) = k/\sqrt{k^2 + m^2}$, so every velocity $0 \leqq |v| < 1$ is possible. This complicates considerably the investigation of the energy propagation, so the approach in [Kom95] built on the fact that the group velocity was $|v| = 1$ no longer works. To overcome this difficulty, we introduce a new approach based on the nonlinear spectral analysis of the solution.

We prove the absolute continuity of the spectrum of the solution for $|\omega| > m$. This observation is similar to the well-known Kato theorem. The proof is not obvious and relies on the complex Fourier–Laplace transform and the Wiener–Paley arguments.

We then split the solution into two components: dispersive and bound, with the frequencies $|\omega| > m$ and $\omega \in [-m, m]$, respectively. The dispersive component is an oscillatory integral of plane waves, while the bound component is a superposition of exponentially decaying functions. The stationary phase argument leads to a local decay of the dispersive component, due to the absolute continuity of its spectrum. This reduces the long-time behavior of the solution to the behavior of the bound component.

Next, we establish the spectral representation for the bound component. For this, we need to know an optimal regularity of the corresponding spectral measure; we have found out that the spectral measure belongs to the space of *quasimeasures* which are Fourier transforms of bounded continuous functions [Gau66]. The spectral representation implies compactness in the space of quasimeasures, which in turn leads to the existence of *omega-limit trajectories* for $t \to \infty$.

Further, we prove that an omega-limit trajectory itself satisfies the nonlinear equation (1.7), and this implies the crucial spectral inclusion: the spectrum of the nonlinear term is included in the spectrum of the omega-limit trajectory. We then reduce the spectrum of this limiting trajectory to a single harmonic $\omega_+ \in [-m, m]$ using the TITCHMARSH convolution theorem [Tit26] (see also [Lev96, p.119] and [Hör90, Theorem 4.3.3]). In turn, this means that any omega-limit trajectory lies in the manifold $\mathbf{S}$ of the solitary waves, which proves that $\mathbf{S}$ is the global attractor.

Empirically, the last part of our argument is a contemplation of the radiative mechanism based on the *inflation of spectrum* by the nonlinearity: a low-frequency perturbation of the stationary state does not radiate the energy until it generates (via a nonlinearity) "a spectral line" embedded in the continuous spectrum outside $[-m, m]$. This embedded spectral line gives rise to the wave packets which bring the energy to infinity. This radiative mechanism has been originally observed in the numerical experiments with the nonlinear relativistic Ginzburg–Landau equation (see [KMV04]). The spectral inclusion for the omega-limit trajectories expresses their *nonradiative nature*: the limiting trajectory cannot radiate since the initial energy was bounded.

3. Separation of dispersive components

It suffices to prove Theorem 2.2 for $t \to +\infty$; we will only consider the solution $\psi(x,t)$ restricted to $t \geqq 0$. In this section we eliminate two dispersive components from $\psi(x,t)$.

First dispersive component

Let us split the solution $\Psi(t) = \begin{bmatrix}\psi(x,t)\\ \pi(x,t)\end{bmatrix}$ into $\Psi(t) = \Psi_1(t) + \Psi_2(t)$, where $\Psi_1(t) = \begin{bmatrix}\psi_1(x,t)\\ \pi_1(x,t)\end{bmatrix}$ and $\Psi_2(t) = \begin{bmatrix}\psi_2(x,t)\\ \pi_2(x,t)\end{bmatrix}$ are defined for $t \geqq 0$ as solutions to the following Cauchy problems:

$$\dot{\Psi}_1(t) = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Psi_1(t), \qquad \Psi_1|_{t=0} = \Psi_0, \tag{3.1}$$

(3.2)

$$\dot{\Psi}_2(t) = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Psi_2(t) + \delta(x) \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \qquad \Psi_2|_{t=0} = 0, \tag{3.3}$$

where $\Psi_0 = \begin{bmatrix}\psi_0\\ \pi_0\end{bmatrix}$ is the initial data from (2.2), and

$$f(t) := F(\psi(0,t)), \qquad t \geqq 0. \tag{3.4}$$

Note that $\psi(0,\cdot) \in C_b(\overline{\mathbb{R}^+})$ by the Sobolev embedding since $\Psi \in C_b(\overline{\mathbb{R}^+}, \mathscr{E})$ by Theorem 2.1 (iv). Hence, $f(\cdot) \in C_b(\overline{\mathbb{R}^+})$. On the other hand, since $\Psi_1(t)$ is a finite energy solution to the free Klein–Gordon equation, we also have

$$\Psi_1 \in C_b(\overline{\mathbb{R}^+}, \mathscr{E}). \tag{3.5}$$

Hence, the function $\Psi_2(t) = \Psi(t) - \Psi_1(t)$ also satisfies

$$\Psi_2 \in C_b(\overline{\mathbb{R}^+}, \mathscr{E}). \tag{3.6}$$

Lemma 3.1. *There is a local decay of* Ψ_1 *in* $\mathscr{E}_F$ *seminorms. That is,* $\forall R > 0$,

$$\|\Psi_1(t)\|_{\mathscr{E},R} \to 0, \qquad t \to \infty. \tag{3.7}$$

Proof. We have to prove that

$$\|\Psi_1(t)\|^2_{\mathscr{E},R} = \int\limits_{|x|<R} \left(|\pi_1(x,t)|^2 + |\psi_1'(x,t)|^2 + m^2|\psi_1(x,t)|^2\right) dx \tag{3.8}$$

goes to zero as t tends to infinity. Fix a cut-off function $\zeta(x) \in C_0^\infty(\mathbb{R})$ with $\zeta(x) = 1$, $|x| \leqq 1$ and $\zeta(x) = 0$, $|x| \geqq 2$. For $r > 0$, let $\zeta_r(x) = \zeta(x/r)$. Denote by $\Phi_r(t)$ and $\Theta_r(t)$ the solutions to the free Klein–Gordon equation with the initial data $\zeta_r \Psi_0$ and $(1 - \zeta_r)\Psi_0$, respectively, so that $\Psi_1(t) = \Phi_r(t) + \Theta_r(t)$. Then there exists $C_r > 0$ that depends on r so that $\|\Phi_r(t)\|^2_{\mathscr{E},R} \leqq C_r(1+t)^{-1}$ for $t > 0$, since the solution Φ_r is represented by the integral with the Green function, which is the Bessel function decaying like $(1+t)^{-1/2}$ (see e.g. [Kom94, (2.7′), Chapter I]). We then have:

$$\|\Psi_1(t)\|^2_{\mathscr{E},R} \leqq C_r(1+t)^{-1} + C\|\Theta_r(t)\|^2_{\mathscr{E},R} \tag{3.9}$$

where $r > 0$ could be arbitrary. To conclude that the left-hand side of (3.9) goes to zero, it remains to note that

$$\|\Theta_r(t)\|_{\mathscr{E},R} \leqq \|\Theta_r(t)\|_{\mathscr{E}} = \|\Theta_r(0)\|_{\mathscr{E}} \tag{3.10}$$

where the last relation is due to the energy conservation for the free Klein–Gordon equation, and that the right-hand side of (3.10) could be made arbitrarily small if $r > 0$ is taken sufficiently large. □

Complex Fourier–Laplace transform

Let us analyze the complex Fourier–Laplace transform of $\psi_2(x,t)$:

$$\tilde{\psi}_2(x,\omega) = \mathscr{F}^+_{t\to\omega}[\psi_2(x,\cdot)] := \int_0^\infty e^{i\omega t}\psi_2(x,t)\,dt, \quad \omega \in \mathbb{C}^+, \tag{3.11}$$

where $C^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Due to (3.6), $\tilde{\psi}_2(\cdot,\omega)$ is an H^1-valued analytic function of $\omega \in \mathbb{C}^+$. Equation (3.3) for ψ_2 implies that

$$-\omega^2\tilde{\psi}_2(x,\omega) = \tilde{\psi}_2''(x,\omega) - m^2\tilde{\psi}_2(x,\omega) + \delta(x)\tilde{f}(\omega), \quad \omega \in \mathbb{C}^+.$$

Hence, the solution $\psi_2(x,\omega)$ is a linear combination of the fundamental solutions which satisfy

$$G_\pm''(x,\omega) + (\omega^2 - m^2)G_\pm(x,\omega) = \delta(x), \qquad \omega \in \mathbb{C}^+.$$

These solutions are given by $G_\pm(x,\omega) = \dfrac{e^{\pm ik(\omega)|x|}}{\pm 2ik(\omega)}$, where $k(\omega)$ stands for the analytic function

$$k(\omega) = \sqrt{\omega^2 - m^2}, \qquad \operatorname{Im} k(\omega) > 0, \qquad \omega \in \mathbb{C}^+, \tag{3.12}$$

which we extend to $\omega \in \overline{\mathbb{C}^+}$ by continuity. We use the standard "limiting absorption principle" for the selection of the fundamental solution: since $\tilde{\psi}_2(\cdot,\omega) \in H^1$ for $\omega \in \mathbb{C}^+$, only G_+ is appropriate, because for $\omega \in \mathbb{C}^+$ the function $G_+(\cdot,\omega)$ is in H^1 while G_- is not. Thus,

$$\tilde{\psi}_2(x,\omega) = -\tilde{f}(\omega)G_+(x,\omega) = -\tilde{f}(\omega)\frac{e^{ik(\omega)|x|}}{2ik(\omega)}, \qquad \omega \in \mathbb{C}^+. \tag{3.13}$$

Define $\tilde{z}(\omega) := \mathscr{F}^{+}_{t\to\omega}[z(t)]$ with $z(t) := \psi_2(0,t)$. Then $\tilde{z}(\omega) = -\tilde{f}(\omega)/(2ik(\omega))$, and (3.13) becomes

$$\tilde{\psi}_2(x,\omega) = \tilde{z}(\omega)e^{ik(\omega)|x|}, \qquad \omega \in \mathbb{C}^+. \tag{3.14}$$

Let us extend $\psi_2(x,t)$ and $f(t)$ by zero for $t < 0$:

$$\psi_2(x,t) = 0 \quad \text{and} \quad f(t) = 0 \quad \text{for} \quad t < 0. \tag{3.15}$$

Then

$$\psi_2 \in C_b(\mathbb{R}, H^1) \tag{3.16}$$

by (3.6) since $\psi_2(x,0+) = 0$ by initial conditions in (3.3). The Fourier transform $\hat{\psi}_2(\cdot,\omega) := \mathscr{F}_{t\to\omega}[\psi_2(\cdot,t)]$ is a tempered H^1-valued distribution of $\omega \in \mathbb{R}$ by (3.6). The distribution $\hat{\psi}_2(\cdot,\omega)$ is the boundary value of the analytic function $\tilde{\psi}_2(\cdot,\omega)$, in the following sense:

$$\hat{\psi}_2(\cdot,\omega) = \lim_{\varepsilon\to 0+} \tilde{\psi}_2(\cdot,\omega+i\varepsilon), \qquad \omega \in \mathbb{R}, \tag{3.17}$$

where the convergence is in the space of tempered distributions $\mathscr{S}'(\mathbb{R}_\omega, H^1)$. Indeed, $\tilde{\psi}_2(\cdot,\omega+i\varepsilon) = \mathscr{F}_{t\to\omega}[\psi_2(\cdot,t)e^{-\varepsilon t}]$ and $\psi_2(\cdot,t)e^{-\varepsilon t} \underset{\varepsilon\to 0+}{\longrightarrow} \psi_2(\cdot,t)$ where the convergence holds in $\mathscr{S}'(\mathbb{R}_t, H^1)$ by (3.15). Therefore, (3.17) holds by the continuity of the Fourier transform $\mathscr{F}_{t\to\omega}$ in $\mathscr{S}'(\mathbb{R})$.

Similarly to (3.17), the distributions $\hat{z}(\omega)$ and $\hat{f}(\omega)$, $\omega \in \mathbb{R}$, are the boundary values of the analytic in $\mathbb{C}^+$ functions $\tilde{f}(\omega)$ and $\tilde{z}(\omega)$, $\omega \in \mathbb{C}^+$, respectively:

$$\hat{z}(\omega) = \lim_{\varepsilon\to 0+} \tilde{z}(\omega+i\varepsilon), \qquad \hat{f}(\omega) = \lim_{\varepsilon\to 0+} \tilde{f}(\omega+i\varepsilon), \quad \omega \in \mathbb{R}, \tag{3.18}$$

since the functions $z(t)$ and $f(t)$ are bounded for $t \geqq 0$ and zeros for $t < 0$. The convergence holds in the space of tempered distributions $\mathscr{S}'(\mathbb{R})$.

Let us justify that the representation (3.14) for $\hat{\psi}_2(x,\omega)$ is also valid when $\omega \in \mathbb{R}$ if the multiplication in (3.14) is understood in the sense of quasimeasures (see Appendix B).

Proposition 3.1. *For any fixed $x \in \mathbb{R}$, the identity*

$$\hat{\psi}_2(x,\omega) = \hat{z}(\omega)e^{ik(\omega)|x|}, \quad \omega \in \mathbb{R}, \tag{3.19}$$

holds in the sense of tempered distributions. The right-hand side is defined as the product of quasimeasure $\hat{z}(\omega)$ by the multiplicator $e^{ik(\omega)|x|}$.

Proof. The representation (3.19) for $\omega \neq \pm m$ follows from (3.14)–(3.18) since $e^{ik(\omega)|x|}$ is a smooth function of $\omega \in \overline{\mathbb{C}^+}$ outside the points $\pm m$.

So, we only need to justify (3.14) in a neighborhood of each point $\omega = \pm m$. The main problem is the low regularity of $k(\omega) = \sqrt{\omega^2 - m^2}$ at the points $\pm m$. Choose a cut-off function $\zeta(\omega) \in C_0^\infty(\mathbb{R})$ such that

$$\zeta|_{[-m-1,m+1]} \equiv 1. \tag{3.20}$$

Let us note that $\psi_2(x,\cdot) \in C_b(\mathbb{R})$ for each particular $x \in \mathbb{R}$ by (3.16) and the Sobolev embedding theorem. Therefore, for each $x \in \mathbb{R}$, $\hat{\psi}_2(x,\cdot)$ belongs to the space $\mathscr{QM}(\mathbb{R})$ of quasimeasures which are defined as functions with bounded continuous Fourier transform (see Definition B.1). In particular, $\hat{z}(\cdot)$ is a quasimeasure since it is the Fourier transform of the function $z(t) := \psi_2(0,t)$. On the other hand, the function $e^{ik(\omega)|x|}\zeta(\omega)$ is a multiplicator in $\mathscr{QM}(\mathbb{R})$ by Lemma B.3 (i) and Lemma B.2 (i) (see Appendix B). Let us now prove that

$$\hat{\psi}_2(x,\omega)\zeta(\omega) = \hat{z}(\omega)e^{ik(\omega)|x|}\zeta(\omega), \quad \omega \in \mathbb{R} \tag{3.21}$$

in the sense of quasimeasures. We define $\mu_\varepsilon(\omega) := \tilde{z}(\omega + i\varepsilon) = \mathscr{F}_{t\to\omega}[z(t)e^{-\varepsilon t}]$ for $\varepsilon > 0$. Then $\mu_\varepsilon(\omega) \in \mathscr{QM}(\mathbb{R})$, and $\mu_\varepsilon(\omega) \overset{\mathscr{QM}}{\longrightarrow} \mu(\omega) := \hat{z}(\omega)$ as $\varepsilon \to 0+$ since $z(t)e^{-\varepsilon t} \overset{C_{b,F}}{\longrightarrow} z(t)$ by (3.15) (see Definition B.2).

Let us denote $M_{x,\varepsilon}(\omega) = e^{ik(\omega+i\varepsilon)|x|}\zeta(\omega)$ for $\omega \in \mathbb{R}$ and $\varepsilon \geqq 0$. By Lemmas B.2 (ii) and B.3 (ii), $M_{x,\varepsilon}(\omega)$ are multiplicators in the space of quasimeasures. This implies that

$$\tilde{z}(\omega + i\varepsilon)e^{ik(\omega+i\varepsilon)|x|}\zeta(\omega) \overset{\mathscr{QM}}{\longrightarrow} \hat{z}(\omega)e^{ik(\omega)|x|}\zeta(\omega) \quad \text{as} \quad \varepsilon \to 0+. \tag{3.22}$$

On the other hand, the left-hand side converges to the left-hand side of (3.21) by (3.14) and (3.17). □

Absolutely continuous spectrum

We study the regularity of the spectral density $\hat{z}(\omega)$ from (3.19). Denote

$$\Omega_\delta := (-\infty, -m-\delta) \cup (m+\delta, \infty), \qquad \delta \geqq 0. \tag{3.23}$$

Note that $\overline{\Omega_0} = (-\infty, -m] \cup [m, \infty)$ coincides with the continuous spectrum of the free Klein–Gordon equation, and the function $\omega k(\omega)$ is positive for $\omega \in \Omega_0$.

Proposition 3.2. *The distribution $\hat{z}(\omega)$ is absolutely continuous for $\omega \in \Omega_0$, and $\hat{z} \in L^1(\overline{\Omega_0})$. Moreover,*

$$\int_{\Omega_0} |\hat{z}(\omega)|^2 \omega k(\omega)\, d\omega < \infty. \tag{3.24}$$

Proof. Let us first explain the main idea of the proof. By (3.19), the function $\psi_2(x,t)$ *formally* is a *linear combination* of the functions $e^{ik|x|}$ with the *amplitudes* $\hat{z}(\omega)$:

$$\psi_2(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{z}(\omega)e^{ik(\omega)|x|}e^{-i\omega t}\, d\omega, \qquad x \in \mathbb{R}. \tag{3.25}$$

For $\omega \in \Omega_0$, the functions $e^{ik(\omega)|x|}$ are of infinite L^2 norm, while $\psi_2(\cdot,t)$ is of finite L^2 norm. This is possible only if the amplitude is absolutely continuous in Ω_0: for example, if we took $\hat{z}(\omega) = \delta(\omega - \omega_0)$ with $\omega_0 \in \Omega_0$, then $\psi_2(\cdot,t)$ would be of infinite L^2 norm.

For a rigorous proof, we use the Paley–Wiener arguments. Namely, the Parseval identity and (3.6) imply that

$$\int_{\mathbb{R}} \|\tilde{\psi}_2(\cdot,\omega+i\varepsilon)\|^2_{H^1}\,d\omega = 2\pi\int_0^\infty e^{-2\varepsilon t}\|\psi_2(\cdot,t)\|^2_{H^1}\,dt \leqq \frac{C}{\varepsilon},\quad \varepsilon>0. \tag{3.26}$$

On the other hand, we can calculate the term in the left-hand side of (3.26) exactly. First, according to (3.14),

$$\tilde{\psi}_2(\cdot,\omega+i\varepsilon) = \tilde{z}(\omega+i\varepsilon)e^{ik(\omega+i\varepsilon)|x|},$$

and hence (3.26) results in

$$\varepsilon\int_{\mathbb{R}} |\tilde{z}(\omega+i\varepsilon)|^2\|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1}\,d\omega \leqq C,\qquad \varepsilon>0. \tag{3.27}$$

Here is a crucial observation about the norm of $e^{ik(\omega+i\varepsilon)|x|}$.

Lemma 3.2.

(i) *For* $\omega\in\mathbb{R}$,

$$\lim_{\varepsilon\to 0+}\varepsilon\|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1} = n(\omega) := \begin{cases}\omega k(\omega), & |\omega|>m\\ 0, & |\omega|<m\end{cases}, \tag{3.28}$$

where the norm in H^1 *is chosen to be* $\|\psi\|_{H^1} = \left(\|\psi'\|^2_{L^2}+m^2\|\psi\|^2_{L^2}\right)^{1/2}$.

(ii) *For any* $\delta>0$ *there exists* $\varepsilon_\delta>0$ *such that for* $|\omega|>m+\delta$ *and* $\varepsilon\in(0,\varepsilon_\delta)$,

$$\varepsilon\|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1} \geqq n(\omega)/2. \tag{3.29}$$

Proof. Let us compute the H^1 norm using the Fourier space representation. Setting $k_\varepsilon = k(\omega+i\varepsilon)$, so that $\operatorname{Im} k_\varepsilon>0$, we get $\mathscr{F}_{x\to k}\left[e^{ik_\varepsilon|x|}\right] = 2ik_\varepsilon/(k_\varepsilon^2-k^2)$ for $k\in\mathbb{R}$. Hence,

$$\|e^{ik_\varepsilon|x|}\|^2_{H^1} = \frac{2|k_\varepsilon|^2}{\pi}\int_{\mathbb{R}}\frac{(k^2+m^2)dk}{|k_\varepsilon^2-k^2|^2} = -4\operatorname{Im}\left[\frac{(k_\varepsilon^2+m^2)\bar{k}_\varepsilon}{k_\varepsilon^2-\bar{k}_\varepsilon{}^2}\right]. \tag{3.30}$$

The last integral is evaluated using the Cauchy theorem. Substituting the expression $k_\varepsilon^2 = (\omega+i\varepsilon)^2-m^2$, we get:

$$\|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1} = \frac{1}{\varepsilon}\operatorname{Re}\left[\frac{(\omega+i\varepsilon)^2\overline{k(\omega+i\varepsilon)}}{\omega}\right],\quad \varepsilon>0,\ \omega\in\mathbb{R},\ \omega\neq 0. \tag{3.31}$$

The relation (3.28) follows since the function $k(\omega)$ is real for $|\omega|>m$, but is purely imaginary for $|\omega|<m$.

The second statement of the lemma follows since $n(\omega)>0$ for $|\omega|>m$, and $n(\omega)\sim|\omega|^2$ for $|\omega|\to\infty$. □

Remark 3.1. Note that $n(\omega)$ in (3.28) is zero for $|\omega| < m$, since in that case the function $e^{ik(\omega)|x|}$ decays exponentially in x and the H^1 norm of $e^{ik(\omega+i\varepsilon)|x|}$ remains finite when $\varepsilon \to 0+$.

Substituting (3.29) into (3.27), we get:

$$\int_{\Omega_\delta} |\tilde{z}(\omega + i\varepsilon)|^2 \omega k(\omega)\, d\omega \leqq 2C, \qquad 0 < \varepsilon < \varepsilon_\delta, \tag{3.32}$$

with the same C as in (3.27), and the region Ω_δ defined in (3.23). We conclude that for each $\delta > 0$ the set of functions

$$g_{\delta,\varepsilon}(\omega) = \tilde{z}(\omega + i\varepsilon)|\omega k(\omega)|^{1/2}, \qquad \varepsilon \in (0, \varepsilon_\delta),$$

defined for $\omega \in \Omega_\delta$, is bounded in the Hilbert space $L^2(\Omega_\delta)$, and, by the Banach Theorem, is weakly compact. The convergence of the distributions (3.18) implies the following weak convergence in the Hilbert space $L^2(\Omega_\delta)$:

$$g_{\delta,\varepsilon} \rightharpoonup g_\delta, \qquad \varepsilon \to 0+, \tag{3.33}$$

where the limit function $g_\delta(\omega)$ coincides with the distribution $\hat{z}(\omega)|\omega k(\omega)|^{1/2}$ restricted onto Ω_δ. It remains to note that the norms of all functions g_δ, $\delta > 0$, are bounded in $L^2(\Omega_\delta)$ by (3.32), and hence (3.24) follows. Finally, $\hat{z}(\omega) \in L^1(\overline{\Omega_0})$ by (3.24) and the Cauchy–Schwarz inequality.

Let us denote

$$\hat{z}_d(\omega) := \begin{cases} \hat{z}(\omega), & \omega \in \Omega_0, \\ 0, & \omega \in \mathbb{R} \setminus \Omega_0. \end{cases} \tag{3.34}$$

Then, by Proposition 3.2, $\hat{z}_d(\omega) \in L^1(\mathbb{R})$.

Second dispersive component

Proposition 3.2 and the representation (3.25) suggest that we introduce the function

$$\psi_d(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{z}_d(\omega) e^{ik(\omega)|x|} e^{-i\omega t}\, d\omega, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{3.35}$$

with $\hat{z}_d$ defined by (3.34). The Fourier transform of $\psi_d(x,t)$ is given by the formula similar to (3.19):

$$\hat{\psi}_d(x,\omega) = \hat{z}_d(\omega) e^{ik(\omega)|x|}, \qquad x \in \mathbb{R}, \qquad \omega \in \mathbb{R}. \tag{3.36}$$

We will show that $\psi_d(x,t)$ is a dispersive component of the solution $\psi(x,t)$ in the following sense.

Proposition 3.3. *$\psi_d(\cdot,t)$ is a bounded continuous H^1-valued function:*

$$\psi_d(\cdot,t)\in C_b(\mathbb{R},H^1). \tag{3.37}$$

The local energy decay holds for $\psi_d(\cdot,t)$: for any $R>0$,

$$\left\|\begin{bmatrix}\psi_d(\cdot,t)\\ \dot\psi_d(\cdot,t)\end{bmatrix}\right\|_{\mathscr{E},R}\to 0,\qquad t\to\infty. \tag{3.38}$$

Proof. Changing the variable, we rewrite (3.35) as follows:

$$\psi_d(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat z(\omega(k))e^{ik|x|}e^{-i\omega(k)t}\,\frac{k\,dk}{\omega(k)},\qquad x\in\mathbb{R}, \tag{3.39}$$

where $\omega(k)=\sqrt{k^2+m^2}$ is the branch analytic for $\operatorname{Im}k>0$ and continuous for $\operatorname{Im}k\geqq 0$. Note that the function $\omega(k)$, $k\in\mathbb{R}\backslash 0$, is the inverse function to $k(\omega)$ defined on $\overline{\mathbb{C}^+}$ (see (3.12)) and restricted onto $\Omega_0=(-\infty,-m)\cup(m,\infty)$. Let us introduce the functions

$$\psi_\pm(x,t):=\frac{1}{2\pi}\int_{\mathbb{R}}\hat z(\omega(k))e^{\pm ikx}e^{-i\omega(k)t}\,\frac{kdk}{\omega(k)},\qquad x\in\mathbb{R},\quad t\geqq 0. \tag{3.40}$$

Both functions $\psi_\pm(x,t)$ are solutions to the free Klein–Gordon equation on the whole real line. The (free Klein–Gordon) energy of each solution is finite, since

$$\begin{aligned}\left\|\begin{bmatrix}\psi_\pm(\cdot,0)\\ \dot\psi_\pm(\cdot,0)\end{bmatrix}\right\|^2_{\mathscr{E}}&=\int_{\mathbb{R}}(\omega^2(k)+|k|^2+m^2)|\hat z(\omega(k))|^2\frac{k^2}{\omega^2(k)}\,dk\\ &=\int_{\mathbb{R}}2|\hat z(\omega(k))|^2k^2\,dk=2\int_{\Omega_0}|\hat z(\omega)|^2\omega k(\omega)\,d\omega<\infty.\end{aligned}$$

In the last inequality, we used (3.24). Hence, both ψ_- and ψ_+ are bounded continuous H^1-valued functions:

$$\psi_\pm\in C_b(\mathbb{R},H^1), \tag{3.41}$$

and for any $R>0$

$$\left\|\begin{bmatrix}\psi_\pm(\cdot,t)\\ \dot\psi_\pm(\cdot,t)\end{bmatrix}\right\|_{\mathscr{E},R}\to 0,\qquad t\to\infty \tag{3.42}$$

by the same arguments as in the proof of Lemma 3.1. The function $\psi_d(x,t)$ coincides with $\psi_+(x,t)$ for $x\geqq 0$ and with $\psi_-(x,t)$ for $x\leqq 0$:

$$\psi_d(x,t)=\psi_\pm(x,t),\qquad \pm x\geqq 0.$$

Moreover, $\psi_-(0-,t)=\psi_+(0+,t)$, so $\psi_d(x,t)$ has no jump at $x=0$ and therefore $\psi_d'(x,t)$ is square-integrable over the whole x-axis. Therefore, (3.37) follows from (3.41), and (3.38) follows from (3.42). □

4. Bound component

Spectral representation

We introduce the bound component of the solution $\psi(x,t)$ by

$$\psi_b(x,t)=\psi_2(x,t)-\psi_d(x,t)=\psi(x,t)-\psi_1(x,t)-\psi_d(x,t),\quad x\in\mathbb{R},\quad t\in\mathbb{R}. \tag{4.1}$$

Then (3.16) and Proposition 3.3 imply that

$$\psi_b\in C_b(\mathbb{R},H^1). \tag{4.2}$$

By (4.2), the function

$$z_b(t):=\psi_b(0,t)=\psi_2(0,t)-\psi_d(0,t)$$

is bounded and continuous. Therefore, its Fourier transform $\hat{z}_b\in\mathscr{S}'(\mathbb{R})$ is a quasi-measure:

$$\hat{z}_b=\hat{z}-\hat{z}_d\in\mathscr{Q}\mathscr{M}(\mathbb{R}),\qquad \operatorname{supp}\hat{z}_b\subset[-m,m], \tag{4.3}$$

where the last inclusion follows from (3.34). Now (3.19), (3.36), and (4.1) imply the multiplicative relation

$$\hat{\psi}_b(x,\omega)=\hat{z}_b(\omega)e^{ik(\omega)|x|}. \tag{4.4}$$

We denote

$$\kappa(\omega):=-ik(\omega)=\sqrt{m^2-\omega^2},\qquad \operatorname{Re}\kappa(\omega)\geqq 0\quad\text{for}\quad \operatorname{Im}\omega\geqq 0, \tag{4.5}$$

where $k(\omega)$ was introduced in (3.12). Let us note that $\operatorname{Re}\kappa(\omega)\geqq 0$ and that $\kappa(\omega)>0$ for $\omega\in(-m,m)$. We rewrite (4.4) as

$$\hat{\psi}_b(x,\omega)=\hat{z}_b(\omega)e^{-\kappa(\omega)|x|},\qquad \omega\in\mathbb{R}. \tag{4.6}$$

Therefore, (4.3) implies that $\hat{\psi}_b(x,\omega)$ for any fixed $x\in\mathbb{R}$ is a quasimeasure with the support $\operatorname{supp}\hat{\psi}_b(x,\cdot)\subset[-m,m]$, and finally,

$$\psi_b(x,t)=\frac{1}{2\pi}\langle\hat{z}_b(\omega)e^{-\kappa(\omega)|x|},e^{-i\omega t}\rangle,\qquad x\in\mathbb{R},\quad t\in\mathbb{R}. \tag{4.7}$$

Compactness

We are going to prove a compactness of the set of translations of the bound component, $\{\psi_b(x, s+t) \colon s \geqq 0\}$. We will derive the compactness from the following uniform estimates which we deduce from (4.3), (4.7) by Lemma B.2.

Proposition 4.1.

(i) *The function $\psi_b(x,t)$ is smooth for $x \neq 0$ and $t \in \mathbb{R}$, and the following representation holds for any fixed $x \neq 0$, $t \in \mathbb{R}$, and any nonnegative integers j, k:*

$$\partial_x^j \partial_t^k \psi_b(x,t) = \frac{1}{2\pi}\Big\langle \hat{z}_b(\omega)(-\kappa(\omega)\operatorname{sgn} x)^j e^{-\kappa(\omega)|x|},\ (-i\omega)^k e^{-i\omega t}\Big\rangle. \quad (4.8)$$

(ii) *For any $R > 0$, there is a constant $C_{j,k,R} > 0$ so that*

$$\sup_{0<|x|\leqq R}\ \sup_{t\in\mathbb{R}} |\partial_x^j \partial_t^k \psi_b(x,t)| \leqq C_{j,k,R}. \quad (4.9)$$

Remark 4.1. Let us note that the bounds (4.9) are independent of x and remain valid in the regions $x > 0$ and $x < 0$, although the derivatives $\partial_x^j \partial_t^k \psi_b(x,t)$ with $j \neq 0$ may have a jump at $x = 0$. (This is the case for the solitary waves in (2.15))

Proof.

(i) The representation (4.8) with $j = 0$ and any k follows directly from (4.3), (4.7). Further, consider, for example, $j = 1$ and $k = 0$:

$$\partial_x \psi_b(x,t) = \lim_{\varepsilon\to 0} \frac{1}{2\pi}\left\langle \hat{z}_b(\omega)\frac{e^{-\kappa(\omega)|x+\varepsilon|} - e^{-\kappa(\omega)|x|}}{\varepsilon}\zeta(\omega),\ e^{-i\omega t}\right\rangle \quad (4.10)$$

if the limit exists. Here $\zeta(\omega) \in C_0^\infty(\mathbb{R})$ is any cut-off function which satisfies (3.20). The relation (4.10) follows by Lemma B.2 (ii), if we verify that the following convergence holds in $L^1(\mathbb{R}_t)$:

$$\mathscr{F}^{-1}_{\omega\to t}\left[\frac{e^{-\kappa(\omega)|x+\varepsilon|} - e^{-\kappa(\omega)|x|}}{\varepsilon}\zeta(\omega)\right] \xrightarrow{L^1} \mathscr{F}^{-1}_{\omega\to t}[\partial_x e^{-\kappa(\omega)|x|}\zeta(\omega)]. \quad (4.11)$$

We rewrite the expression in the brackets in the left-hand side of (4.11) as

$$\frac{e^{-\kappa(\omega)|x+\varepsilon|} - e^{-\kappa(\omega)|x|}}{\varepsilon}\zeta(\omega) = \int_0^1 \partial_x e^{-\kappa(\omega)|x+\rho\varepsilon|}\zeta(\omega)\,d\rho. \quad (4.12)$$

Now the convergence (4.11) follows from the Puiseux expansion of type (B.7) for $\partial_x e^{-\kappa(\omega)|x+\rho\varepsilon|}\zeta(\omega)$.

(ii) For fixed nonnegative integers j and k, denote

$$N_x(\omega) = (-\kappa(\omega)\,\mathrm{sgn}\,x)^j e^{-\kappa(\omega)|x|}(-i\omega)^k \zeta(\omega).$$

Lemma B.3 (iii) implies that $N_x(\omega) = \hat{K}_x(\omega)$, with $K_x(\cdot) \in L^1(\mathbb{R})$. Then (4.8) becomes

$$\partial_x^j \partial_t^k \psi_b(x,t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} z_b(t-\tau) K_x(\tau) d\tau. \tag{4.13}$$

The bounds (4.9) follow by Lemma B.1 and (B.6) since $z_b(t) = \psi_b(0,t) \in C_b(\mathbb{R})$ by (4.2). □

Corollary 4.1. *By the Ascoli–Arzelà theorem, for any sequence $s_j \to \infty$ there exists a subsequence $s_{j'} \to \infty$ such that for any nonnegative integers j and k,*

$$\partial_x^j \partial_t^k \psi_b(x, s_{j'} + t) \to \partial_x^j \partial_t^k \beta(x,t), \qquad x \neq 0, \quad t \in \mathbb{R}, \tag{4.14}$$

for some $\beta \in C_b(\mathbb{R}, H^1)$. The convergence in (4.14) *is uniform in x and t as long as $|x| + |t| \leqq R$, for any $R > 0$.*

We call *omega-limit trajectory* any function $\beta(x,t)$ that can appear as a limit in (4.14). Previous analysis demonstrates that the long-time asymptotics of the solution $\psi(x,t)$ in $\mathscr{E}_F$ depends only on the bound component $\psi_b(x,t)$. By Corollary 4.1, to conclude the proof of Theorem 2.2, it suffices to check that every omega-limit trajectory belongs to the set of solitary waves; that is,

$$\beta(x,t) = \phi_{\omega_+}(x) e^{-i\omega_+ t}, \qquad x,\ t \in \mathbb{R}, \tag{4.15}$$

with some $\omega_+ \in [-m, m]$.

Spectral identity for omega-limit trajectories

Here we study the time spectrum of the omega-limit trajectories.

Definition 4.1. Let f be a tempered distribution. By Spec f we denote the support of its Fourier transform:

$$\mathrm{Spec}\, f := \mathrm{supp}\, \tilde{f}.$$

Proposition 4.2.

(i) *For any omega-limit trajectory $\beta(x,t)$, the following spectral representation holds*:

$$\beta(x,t) = \frac{1}{2\pi} \langle \hat{\gamma}(\omega) e^{-\kappa(\omega)|x|}, e^{-i\omega t} \rangle, \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}, \tag{4.16}$$

where $\hat{\gamma} \in \mathscr{Q}\mathscr{M}(\mathbb{R})$, and

$$\mathrm{supp}\, \hat{\gamma} \subset [-m, m]. \tag{4.17}$$

(ii) *The following bound holds*:

$$\sup_{t\in\mathbb{R}} \|\beta(\cdot,t)\|_{H^1} < \infty. \tag{4.18}$$

Note that, according to (4.16), $\hat{\gamma}(\omega)$ is the Fourier transform of the function $\gamma(t) := \beta(0,t)$, $t \in \mathbb{R}$.

Proof. The representation (4.7) implies that

$$\psi_b(x, s_j + t) = \frac{1}{2\pi}\langle \hat{z}_b(\omega) N_x(\omega) e^{-i\omega s_j}, e^{-i\omega t}\rangle, \qquad x \neq 0, \qquad t \in \mathbb{R}, \tag{4.19}$$

where N_x corresponds to $j = k = 0$. The convergence (4.14) and the bounds (4.9) with $j = k = 0$ imply, by Definition B.2, that

$$z_b(s_{j'} + t) \xrightarrow{C_{b,F}} \gamma(t), \qquad s_{j'} \to \infty, \tag{4.20}$$

where $\gamma(t)$, $t \in \mathbb{R}$, is some continuous bounded function. Hence, by Definition B.3,

$$\hat{z}_b(\omega) e^{-i\omega s_{j'}} \xrightarrow{\mathscr{Q}\mathscr{M}} \hat{\gamma}(\omega), \qquad s_{j'} \to \infty. \tag{4.21}$$

Now Lemma B.2(ii) and Lemma B.3(iii) imply that

$$\hat{z}_b(\omega) N_x(\omega) e^{-i\omega s_{k'}} \xrightarrow{\mathscr{Q}\mathscr{M}} \hat{\gamma}(\omega) N_x(\omega), \qquad s_{j'} \to \infty. \tag{4.22}$$

Hence, the representation (4.16) follows from (4.19), and (4.17) follows from (4.3). Finally, the bound (4.18) follows from (4.2) and (4.14). □

The relation (4.16) implies the basic spectral identity:

Corollary 4.2. *For any omega-limit trajectory $\beta(x,t)$,*

$$\operatorname{Spec} \beta(x,\cdot) = \operatorname{Spec} \gamma, \qquad x \in \mathbb{R}. \tag{4.23}$$

Remark 4.2. It is mainly for the proof of (4.23) that we develop the theory of the quasimeasures and multiplicators in Appendix B. This theory provides the compactness of the set of distributions $\{\hat{z}_b(\omega) e^{-i\omega s} : s \geqq 0\}$ in the space of quasimeasures (see (4.21)) and the spectral representation (4.16).

5. Nonlinear spectral analysis

Here we will derive (4.15) from the following identity:

$$\gamma(t) = C e^{-i\omega_+ t}, \qquad t \in \mathbb{R}, \tag{5.1}$$

which will be proved in three steps.

Step 1

The identity is equivalent to $\hat{\gamma}(\omega) \sim \delta(\omega - \omega_+)$, so we start with an investigation of Spec γ := supp $\hat{\gamma}$.

Lemma 5.1. *For omega-limit trajectories the following spectral inclusion holds:*

$$\text{Spec}\, F(\gamma(\cdot)) \subset \text{Spec}\, \gamma. \tag{5.2}$$

Proof. The convergence (4.14) and equation (2.1), together with Lemma 3.1 and Proposition 3.3(ii), imply that the limiting trajectory $\beta(x, t)$ is a solution to equation (2.1) (although $\psi_b(x, t)$ is not!):

$$\ddot{\beta}(x, t) = \beta''(x, t) - m^2 \beta(x, t) + \delta(x) F(\beta(0, t)), \qquad (x, t) \in \mathbb{R}^2. \tag{5.3}$$

Since $\beta(x, t)$ is a smooth function for $x \leqq 0$ and $x \geqq 0$, we get the following algebraic identity (cf. (A.5)):

$$0 = \beta'(0+, t) - \beta'(0-, t) + F(\gamma(t)), \quad t \in \mathbb{R}. \tag{5.4}$$

The identity implies the spectral inclusion

$$\text{Spec}\, F(\gamma(\cdot)) \subset \text{Spec}\, \beta'(0+, \cdot) \cup \text{Spec}\, \beta'(0-, \cdot). \tag{5.5}$$

On the other hand, Spec $\beta'(0+, \cdot) \cup$ Spec $\beta'(0-, \cdot) \subset$ Spec γ by (4.23). Therefore, (5.5) implies (5.2). □

Remark 5.1. The spectral inclusion (5.4) follows from the algebraic identity (5.4), which in turn is a consequence of the fact that $\beta(x, t)$ solves (2.1). We cannot prove (5.5) for the function $\psi_b(x, t)$ since generally it is *not* a solution to (2.1).

Step 2

Proposition 5.1. *For any omega-limit trajectory, the following identity holds*:

$$|\gamma(t)| = \text{const.} \qquad t \in \mathbb{R}. \tag{5.6}$$

Proof. We are going to show that (5.6) follows from the key spectral relations (4.17), (5.2). Our main assumption (2.17) implies that the function $F(t) := F(\gamma(t))$ admits the representation (cf. (2.6))

$$F(t) = \alpha(t)\gamma(t), \tag{5.7}$$

where, according to (2.17),

$$\alpha(t) = -\sum_{n=1}^{N} 2n u_n |\gamma(t)|^{2n-2}, \qquad N \geqq 2; \quad u_N > 0. \tag{5.8}$$

Both functions $\gamma(t)$ and $\alpha(t)$ are bounded continuous functions in $\mathbb{R}$ by Proposition 4.2(iii). Hence, $\gamma(t)$ and $\alpha(t)$ are tempered distributions. Furthermore, $\hat{\gamma}$ and $\hat{\overline{\gamma}}$ have the supports contained in $[-m, m]$ by (4.17). Hence, $\hat{\alpha}$ also has a bounded support since it is a sum of convolutions of finitely many $\hat{\gamma}$ and $\hat{\overline{\gamma}}$ by (5.8). Then the relation (5.7) translates into a convolution in the Fourier space, $\hat{F} = \hat{\alpha} * \hat{\gamma}/(2\pi)$, and the spectral inclusion (5.2) takes the following form:

$$\operatorname{supp} \hat{F} = \operatorname{supp}\ \hat{\alpha} * \hat{\gamma} \subset \operatorname{supp} \hat{\gamma}. \tag{5.9}$$

Let us denote $\mathbf{F} = \operatorname{supp} \hat{F}$, $\mathbf{A} = \operatorname{supp} \hat{\alpha}$, and $\boldsymbol{\Gamma} = \operatorname{supp} \hat{\gamma}$. Then the spectral inclusion (5.9) reads as

$$\mathbf{F} \subset \boldsymbol{\Gamma}. \tag{5.10}$$

On the other hand, it is well known that $\operatorname{supp} \hat{\alpha} * \hat{\gamma} \subset \operatorname{supp} \hat{\alpha} + \operatorname{supp} \hat{\gamma}$, or $\mathbf{F} \subset \mathbf{A} + \boldsymbol{\Gamma}$. Moreover, the Titchmarsh convolution theorem states that the last inclusion is exact for the ends of the supports:

Theorem 5.1 (The Titchmarsh convolution theorem). *Let $\hat{\alpha}$, $\hat{\gamma}$ be two distributions in $\mathbb{R}$ with compact supports $\mathbf{A}$ and $\boldsymbol{\Gamma}$ respectively, and $\mathbf{F} = \operatorname{supp} \hat{\alpha} * \hat{\gamma}$. Then*

$$\inf \mathbf{F} = \inf \mathbf{A} + \inf \boldsymbol{\Gamma}, \qquad \sup \mathbf{F} = \sup \mathbf{A} + \sup \boldsymbol{\Gamma}. \tag{5.11}$$

This theorem was proved first in [Tit26] for $\hat{\alpha}$, $\hat{\gamma} \in L^1(\mathbb{R})$ (see also [Lev96, p.119] and [Hör90, Theorem 4.3.3]). The Titchmarsh convolution theorem, together with (5.10), allows us to conclude that $\inf \mathbf{A} = \sup \mathbf{A} = 0$, and hence $\mathbf{A} \subset \{0\}$. Indeed, (5.10) and (5.11) result in

$$\inf \mathbf{F} = \inf \mathbf{A} + \inf \boldsymbol{\Gamma} \geqq \inf \boldsymbol{\Gamma}, \qquad \sup \mathbf{F} = \sup \mathbf{A} + \sup \boldsymbol{\Gamma} \leqq \sup \boldsymbol{\Gamma}, \tag{5.12}$$

so that $\inf \mathbf{A} \geqq 0 \geqq \sup \mathbf{A}$. Thus, we conclude that $\operatorname{supp} \hat{\alpha} = \mathbf{A} \subset \{0\}$, and therefore the distribution $\hat{\alpha}(\omega)$ is a finite linear combination of $\delta(\omega)$ and its derivatives. Then $\alpha(t)$ is a polynomial in t; since $\alpha(t)$ is bounded by Proposition 4.2(iii), we conclude that $\alpha(t)$ is constant. Now the relation (5.6) follows since $\alpha(t)$ is a polynomial in $|\gamma(t)|$, and its degree is strictly positive by (5.8). □

Remark 5.2. The boundedness of the spectrum of both $\gamma(t)$ and $\alpha(t)$ is critical for our argument, since otherwise the Titchmarsh convolution theorem does not apply. It is to ensure that the spectrum of $\alpha(t)$ is also bounded that we had to assume the polynomial character of the nonlinearity in Assumption 2.1.

Step 3

Now the same Titchmarsh arguments imply that $\boldsymbol{\Gamma} := \operatorname{Spec} \gamma$ is a point $\omega_+ \in [-m, m]$. Indeed, (5.6) means that $\gamma(t)\overline{\gamma}(t) \equiv C$, and hence in the Fourier transform $\hat{\gamma} * \hat{\overline{\gamma}} = 2\pi C \delta(\omega)$. Therefore, if γ is not identically zero, the Titchmarsh theorem implies that

$$0 = \sup \boldsymbol{\Gamma} + \sup(-\boldsymbol{\Gamma}) = \sup \boldsymbol{\Gamma} - \inf \boldsymbol{\Gamma}.$$

Hence $\inf \boldsymbol{\Gamma} = \sup \boldsymbol{\Gamma}$ and therefore $\boldsymbol{\Gamma} = \omega_+ \in [-m, m]$, so that $\hat{\gamma}(\omega)$ is a finite linear combination of $\delta(\omega - \omega_+)$ and its derivatives. As the matter of fact, the derivatives could not be present because of the boundedness of $\gamma(t) = \beta(0, t)$ that follows from Proposition 4.2(iii). Thus, $\hat{\gamma} \sim \delta(\omega - \omega_+)$, which implies (5.1).

Conclusion of the proof of Theorem 2.2. The representation (4.16) implies that $\beta(x,t) = Ce^{-\kappa_+|x|}e^{-i\omega_+ t}$ since $\hat{\gamma} \sim \delta(\omega - \omega_+)$. Therefore, the equation (5.3) and the bound (4.18) imply that $\beta(x,t)$ is a solitary wave. This completes the proof of Theorem 2.2. □

6. Linear case

Let us now give a complete treatment of the linear case and prove Theorem 2.3. We assume that $F(\psi)$ that enters (2.1) is given by $F(\psi) = a\psi$, where $a \in \mathbb{R}$, and $a < 2m$. Thus, the potential is given by $U(\psi) = -a|\psi|^2/2$, and we consider the equation

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + a\delta(x)\psi(0,t), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{6.1}$$

All conclusions of Theorem 2.1 on global well-posedness hold for equation (6.1) with $a < 2m$ since in this case the condition (2.11) is satisfied. Let us note that if $a \geqq 2m$, then the conclusions (i), (ii), and (iii) of Theorem 2.1 are still valid (their proofs in Appendix C apply for bounded times, and then the conclusions follow for all times by the linearity of the equation). On the other hand, the *a priori* bound (2.13) is generally violated when $a/2 \geqq m$ (see below).

Remark 6.1. Let us summarize the properties of the solitary waves for the linear case that follow from Proposition 2.1. Note that, according to (2.16), $\kappa = a/2$.

(i) For $a \leqq 0$ there are no nonzero solitary waves since we need $\kappa > 0$ for (2.15) to be from H^1.
(ii) When $a > 0$, $a \neq 2m$, all solitary waves are given by $\phi_\omega(x) = Ce^{-a|x|/2}$, where $C \in \mathbb{C}$ and $\omega = \pm\omega_a$, where $\omega_a := \sqrt{m^2 - a^2/4}$. Note that if $a > 2m$, then the values $\pm\omega_a$ are purely imaginary and the corresponding solitary waves are exponentially growing.
(iii) If $a = 2m$, then $\omega_0 = 0$ and there is a nonzero static solitary wave solution $\phi_0(x) = e^{-m|x|}$. Besides, there is secular (linearly growing) solution $te^{-m|x|}$.

Remark 6.2. When $a > 2m$, the values of ω are purely imaginary, and the $\mathscr{E}$ norm of solitary waves that correspond to $\pm\mathrm{Im}\,\omega > 0$ grows exponentially for $t \to \pm\infty$. When $a = 2m$, we have $\omega = 0$; the $\mathscr{E}$ norm of the secular solution grows linearly in time. In both cases ($a \geqq 2m$), the *a priori* bound (2.13) fails. This illustrates that condition (2.11) is sharp, since this condition fails for the potential $U(\psi) = -\dfrac{a}{2}|\psi|^2$ with $a \geqq 2m$.

Proof of Theorem 2.3. Let us prove the global attraction to the set $\langle \mathbf{S} \rangle$. We proceed as in the proof of Theorem 2.2 until we get to equation (5.3). Since now $F(\psi) = a\psi$, (5.3) takes the following form:

$$\ddot{\beta}(x,t) = \beta''(x,t) - m^2\beta(x,t) + a\delta(x)\beta(0,t), \qquad (x,t) \in \mathbb{R}^2. \tag{6.2}$$

Now we cannot use the Titchmarsh arguments, and we have to solve the equation directly to prove that

$$\begin{bmatrix} \beta(\cdot,t) \\ \dot\beta(\cdot,t) \end{bmatrix} \in \langle \mathbf{S} \rangle \qquad \text{for} \quad t \in \mathbb{R}. \tag{6.3}$$

In the Fourier transform $\hat\beta(x,\omega) = \mathscr{F}_{t\to\omega}[\beta(x,t)]$ the equation (6.2) becomes

$$-\omega^2 \hat\beta(x,\omega) = \hat\beta''(x,\omega) - m^2\hat\beta(x,\omega) + a\delta(x)\hat\beta(0,\omega), \qquad (x,\omega) \in \mathbb{R}^2. \tag{6.4}$$

On the other hand, the representation (4.16) implies that

$$\hat\beta(x,\omega) = \hat\gamma(\omega)e^{-\kappa(\omega)|x|}. \tag{6.5}$$

Substituting the above into (6.4), we obtain

$$2\kappa(\omega)\hat\gamma(\omega)\delta(x) = a\delta(x)\hat\gamma(\omega). \tag{6.6}$$

Therefore, on the support of the distribution $\hat\gamma(\omega)$, the identity holds

$$2\kappa(\omega) = a, \tag{6.7}$$

and hence $\operatorname{supp}\hat\gamma \subset \mathbf{\Gamma}_a := \{\omega \in [-m,m] : 2\kappa(\omega) = a\}$ by (4.17). Now let us consider two cases.

(i) In the case $0 < a < 2m$, according to Remark 6.1(ii), the set of finite energy solitary waves is given by

$$\mathbf{S} = \left\{ C_1 \begin{bmatrix} e^{-a|x|/2} \\ i\omega_a e^{-a|x|/2} \end{bmatrix} + C_2 \begin{bmatrix} e^{-a|x|/2} \\ -i\omega_a e^{-a|x|/2} \end{bmatrix} : \ C_1,\ C_2 \in \mathbb{C} \right\}. \tag{6.8}$$

On the other hand, the set $\mathbf{\Gamma}_a$ contains exactly two points $\pm\omega_a$ since $0 < a < 2m$. Hence, $\hat\gamma$ is a linear combination of $\delta(\omega \pm \omega_a)$ and their derivatives. The derivatives are forbidden since $\gamma(t)$ is bounded, so finally

$$\beta(x,t) = \left(C_1 e^{i\omega_a t} + C_2 e^{-i\omega_a t} \right) e^{-a|x|/2}. \tag{6.9}$$

Now (6.3) follows from (6.9).

(ii) In the case $a \leqq 0$, the set of finite energy solitary waves consists of the zero solution only by Remark 6.1(i). For $a < 0$, the set $\mathbf{\Gamma}_a$ is empty, and hence $\beta(x,t) = 0$ and (6.3) follows. When $a = 0$, we have $\omega_a = m$ and $\mathbf{\Gamma}_a = \{-m\} \cup \{m\}$. Any omega-limit point β is given by (6.9) with $a = 0$. Since $\beta(\cdot,t) \in H^1$, we conclude that $C_1 = C_2 = 0$ in (6.9), so that $\beta(x,t) = 0$ and the inclusion (6.3) follows. □

This finishes the proof of Theorem 2.3.

Remark 6.3. For $0 < a < 2m$, a particular exact solution to (6.1), for example (6.9), with $C_1 \neq 0$ and $C_2 \neq 0$, shows that in general there could be no attraction to $\mathbf{S}$.

Appendix A. Solitary waves

Here we prove Lemma 2.1 and Proposition 2.1.

Proof of Lemma 2.1. Substituting $\phi_\omega(x)e^{-i\omega t}$ into (2.1), we get the equation

$$-\omega^2\phi_\omega(x)e^{-i\omega t} = \phi''_\omega(x)e^{-i\omega t} - m^2\phi_\omega(x)e^{-i\omega t} + \delta(x)F(e^{-i\omega t}\phi_\omega(0)), \tag{A.1}$$

where $(x,t)\in\mathbb{R}\times\mathbb{R}$. We can assume that $\phi_\omega(0)\neq 0$. Indeed, if $\phi_\omega(0)=0$, then (A.1) turns into a homogeneous second-order linear differential equation, which together with the inclusion $\phi_\omega\in H^1(\mathbb{R})$ results in $\phi_\omega(x)\equiv 0$. Equation (A.1) leads to the identity $e^{-i\omega t}\Delta = F(e^{-i\omega t}\phi_\omega(0))$, with $\Delta = \phi'_\omega(0-) - \phi'_\omega(0+)$. This results in

$$\frac{\Delta}{\phi_\omega(0)} = \frac{F(e^{-i\omega t}\phi_\omega(0))}{e^{-i\omega t}\phi_\omega(0)} = \frac{F(e^{t\,\mathrm{Im}\,\omega}\phi_\omega(0))}{e^{t\,\mathrm{Im}\,\omega}\phi_\omega(0)}, \qquad t\in\mathbb{R}. \tag{A.2}$$

We used (2.7) in the last equality. The condition that $F(\psi)$ is strictly nonlinear (in the sense of Definition 2.3) implies that (A.2) only holds at discrete values of $t\,\mathrm{Im}\,\omega$; thus, $\mathrm{Im}\,\omega = 0$, finishing the proof. □

Proof of Proposition 2.1. The relation (A.1) turns into the following eigenvalue problem:

$$-\omega^2\phi_\omega(x) = \phi''_\omega(x) - m^2\phi_\omega(x) + \delta(x)F(\phi_\omega(x)), \quad x\in\mathbb{R}. \tag{A.3}$$

The phase factor $e^{-i\omega t}$ can be canceled out because either $F(\psi)=a\psi$ or, when F is strictly nonlinear, we can use (2.7) (since in this case $\omega\in\mathbb{R}$ by Lemma 2.1). Equation (A.3) implies that away from the origin we have

$$\phi''_\omega(x) = (m^2-\omega^2)\phi_\omega(x), \qquad x\neq 0,$$

and hence $\phi_\omega(x) = C_\pm e^{-\kappa_\pm|x|}$ for $\pm x > 0$, where $\kappa_\pm$ satisfy $\kappa_\pm^2 = m^2-\omega^2$. Since $\phi_\omega(x)\in H^1$, it is imperative that $\kappa_\pm > 0$; we conclude that $|\omega| < m$ and that $\kappa_\pm = \sqrt{m^2-\omega^2} > 0$. Moreover, since the function $\phi_\omega(x)$ is continuous, $C_- = C_+ = C \neq 0$ (since we are looking for nonzero solitary waves). We see that

$$\phi_\omega(x) = Ce^{-\kappa|x|}, \qquad C\neq 0, \qquad \kappa\equiv\sqrt{m^2-\omega^2} > 0. \tag{A.4}$$

Equation (A.3) implies the following gluing condition at $x=0$:

$$0 = \phi'_\omega(0+) - \phi'_\omega(0-) + F(\phi_\omega(0)). \tag{A.5}$$

This condition and (A.4) lead to the equation $2\kappa C = F(C)$ which is equivalent to (2.16) for $C\neq 0$. □

Appendix B. Quasimeasures and multiplicators

Quasimeasures

Let us denote by $\check{g}$ the inverse Fourier transform of a tempered distribution g:

$$\check{g}(t) = \mathscr{F}^{-1}_{\omega\to t}[g(\omega)].$$

Definition B.1. A tempered distribution $\mu(\omega)$ is a *quasimeasure* if $\check{\mu} \in C_b(\mathbb{R})$.

For example, any function from $L^1(\mathbb{R})$ is a quasimeasure, and so is any finite Borel measure on $\mathbb{R}$.

Lemma B.1. *Let $\mu(\omega)$ be a quasimeasure and $\varphi(\omega)$ be a test function from the Schwartz space $\mathscr{S}(\mathbb{R})$. Then*

$$|\langle \mu(\omega), \varphi(\omega)\rangle| \leqq C\|\check{\varphi}(t)\|_{L^1(\mathbb{R})}. \tag{B.1}$$

The lemma is a trivial consequence of the Parseval identity:

$$|\langle \mu(\omega), \varphi(\omega)\rangle| = 2\pi|\langle \check{\mu}(t), \check{\varphi}(t)\rangle| \leqq 2\pi\,\|\check{\mu}(t)\|_{L^\infty(\mathbb{R})}\|\check{\varphi}(t)\|_{L^1(\mathbb{R})}. \tag{B.2}$$

Definition B.2. $C_{b,F}(\mathbb{R})$ is the vector space of bounded functions $f(t) \in C_b(\mathbb{R})$ endowed with the following convergence: $f_\varepsilon(t) \overset{C_{b,F}}{\longrightarrow} f(t)$, $\varepsilon \to 0+$ if and only if

(i) $\forall T > 0, \quad \|f_\varepsilon(t) - f(t)\|_{C[-T,T]} \to 0, \varepsilon \to 0+$;
(ii) $\sup\limits_{\varepsilon\in(0,1]} \|f_\varepsilon(t)\|_{C_b(\mathbb{R})} < \infty$.

This type of convergence coincides with the convergence stated in the Ascoli–Arzelà theorem. Next we introduce the dual class of the "Ascoli–Arzelà quasi measures".

Definition B.3. $\mathscr{QM}(\mathbb{R})$ is the linear space of all quasimeasures $\mu(\omega)$ endowed with the following convergence:

$$\mu_\varepsilon(\omega) \underset{\varepsilon\to 0+}{\overset{\mathscr{QM}}{\longrightarrow}} \mu(\omega) \quad \text{if and only if} \quad \check{\mu}_\varepsilon(t) \underset{\varepsilon\to 0+}{\overset{C_{b,F}}{\longrightarrow}} \check{\mu}(t).$$

Multiplicators

Now let us give a simple characterization of multiplicators in $\mathscr{QM}(\mathbb{R})$. Let us consider a continuous function $M(\omega) \in C(\mathbb{R})$. We also denote by M the corresponding operator of multiplication:

$$M: \ \mu(\omega) \mapsto M(\omega)\mu(\omega), \qquad \mu(\omega) \in C_0^\infty(\mathbb{R}).$$

Lemma B.2.

(i) *Let $\check{M}(t) \in L^1(\mathbb{R})$. Then the operator M extends to a linear continuous operator in the space of quasimeasures:*

$$M: \ \mathscr{QM}(\mathbb{R}) \to \mathscr{QM}(\mathbb{R}).$$

(ii) *Let* $\mu_\varepsilon(\omega) \stackrel{\mathscr{Q}\mathscr{M}}{\longrightarrow} \mu(\omega)$ *and* $\check{M}_\varepsilon(t) \stackrel{L^1}{\longrightarrow} \check{M}(t)$ *as* $\varepsilon \to 0+$. *Then*

$$M_\varepsilon(\omega)\mu_\varepsilon(\omega) \stackrel{\mathscr{Q}\mathscr{M}}{\longrightarrow} M(\omega)\mu(\omega), \qquad \varepsilon \to 0+\,. \tag{B.3}$$

Proof. First we define $M(\omega)\mu(\omega) := \mathscr{F}_{t\to\omega}[\big(\check{M} * \check{\mu}\big)(t)](\omega)$ that agrees with the case $\mu \in C_0^\infty(\mathbb{R})$. Then (i) follows from (ii) with $M_\varepsilon = M$ and $\mu_\varepsilon \in C_0^\infty(\mathbb{R})$. To prove (ii), we need to show that

$$\mathscr{F}^{-1}_{\omega\to t}[M_\varepsilon(\omega)\mu_\varepsilon(\omega)] = \big(\check{M}_\varepsilon * \check{\mu}_\varepsilon\big)(t) \stackrel{C_{b,F}}{\longrightarrow} \big(\check{M} * \check{\mu}\big)(t). \tag{B.4}$$

We have to check both conditions (i) and (ii) of Definition B.2 for the functions

$$f_\varepsilon(t) := \mathscr{F}^{-1}_{\omega\to t}[M_\varepsilon(\omega)\mu_\varepsilon(\omega)] = \big(\check{M}_\varepsilon * \check{\mu}_\varepsilon\big)(t),$$
$$f(t) := \mathscr{F}^{-1}_{\omega\to t}[M(\omega)\mu(\omega)] = \big(\check{M} * \check{\mu}\big)(t).$$

We have:

$$f_\varepsilon(t) - f(t) = \big(\check{M}_\varepsilon * \check{\mu}_\varepsilon\big)(t) - \big(\check{M} * \check{\mu}\big)(t) = \big((\check{M}_\varepsilon - \check{M}) * \check{\mu}_\varepsilon\big)(t) + \big(\check{M} * (\check{\mu}_\varepsilon - \check{\mu})\big)(t).$$

The first term in the right-hand side converges to zero uniformly in $t \in \mathbb{R}$ since $\check{M}_\varepsilon - \check{M} \to 0$ in L^1 while $\check{\mu}_\varepsilon \in C_b(\mathbb{R})$ and is bounded uniformly for $\varepsilon \in (0, 1)$. Let us analyze the second term,

$$\int_{\mathbb{R}} \check{M}(\tau)(\check{\mu}_\varepsilon(t-\tau) - \check{\mu}(t-\tau))\,d\tau. \tag{B.5}$$

Since $\check{M} \in L^1$, for any $\delta > 0$ there exists a finite $R > 0$ so that $\int_{|\tau|>R} |\check{M}(\tau)|\,d\tau \leqq \delta$. On the other hand, for any $T > 0$, the difference $\check{\mu}_\varepsilon(t-\tau)-\check{\mu}(t-\tau)$ is uniformly small for $|t| \leqq T$, $|\tau| < R$ and small ε. Therefore, the integral (B.5) converges to zero uniformly in $|t| \leqq T$ as $\varepsilon \to 0+$. Hence, the convergence (i) of Definition B.2 follows.

Finally, the uniform bound (ii) of Definition B.2 for the functions $f_\varepsilon(t)$ is obvious. The convergence (B.4) is proved. □

Bounds for multiplicators

Let us justify the properties of the multiplicators which we used in Sect. 4. Recall that we use the notation

$$M_{x,\varepsilon}(\omega) := e^{ik(\omega+i\varepsilon)|x|}\zeta(\omega), \qquad x \in \mathbb{R}, \quad \varepsilon \geqq 0,$$

where $\zeta(\omega) \in C_0^\infty(\mathbb{R})$ is a fixed cut-off function, and also the notation

$$N_x(\omega) := (ik(\omega)\,\mathrm{sgn}\,x)^j e^{ik(\omega)|x|}(-i\omega)^k\zeta(\omega), \qquad x \in \mathbb{R},$$

where j, k are fixed nonnegative integers.

Lemma B.3. *For any fixed $x \in \mathbb{R}$ we have:*

(i) $\check{M}_{x,\varepsilon}(t) \in L^1(\mathbb{R})$ *for any* $\varepsilon \geqq 0$.

(ii) $\check{M}_{x,\varepsilon}(t) \xrightarrow{L^1} \check{M}_{x,0}(t), \quad \varepsilon \to 0.$

(iii) $\check{N}_x \in L^1(\mathbb{R})$*, and for any* $R > 0$ *there exists* $C_{j,k,R} > 0$ *so that*

$$\sup_{|x| \leqq R} \|\check{N}_x\|_{L^1(\mathbb{R})} \leqq C_{j,k,R}. \tag{B.6}$$

Proof. For any fixed $x \in \mathbb{R}$, the Puiseux expansion holds:

$$e^{ik(\omega+i\varepsilon)|x|} \sim 1 + \sum_{\pm} \sum_{j=1}^{\infty} C_j^{\pm}(x)(\omega + i\varepsilon \mp m)^{j/2}, \qquad \omega + i\varepsilon \to \pm m, \qquad \varepsilon > 0. \tag{B.7}$$

Therefore, the function $\check{M}_{x,\varepsilon}(t)$ is smooth and decays at least like $|t|^{-3/2}$ when $t \to \infty$. This finishes the proof of the first statement of the lemma.

The second statement of the lemma follows from (B.7).

The last statement of the lemma follows by the same arguments from the Puiseux expansion for $\check{N}_x(\omega)$ similar to expansion (B.7) with $\varepsilon = 0$. □

Appendix C. Global well-posedness

Here we prove Theorem 2.1. We first need to adjust the nonlinearity F so that it becomes bounded, together with its derivatives. Define

$$\Lambda(\Psi_0) = \sqrt{\frac{\mathscr{H}(\Psi_0) - A}{m - B}}, \tag{C.1}$$

where $\Psi_0 \in \mathscr{E}$ is the initial data from Theorem 2.1 and A, B are constants from (2.11). Then we may pick a modified potential function $\widetilde{U} \in C^2(\mathbb{C}, \mathbb{R})$, $\widetilde{U}(\psi) = \widetilde{U}(|\psi|)$, so that

$$\widetilde{U}(\psi) = U(\psi) \qquad \text{for } |\psi| \leqq \Lambda(\Psi_0), \quad \psi \in \mathbb{C}, \tag{C.2}$$

$\widetilde{U}(\psi)$ satisfies (2.11) with the same constants A, B as $U(\psi)$ does:

$$\widetilde{U}(\psi) \geqq A - B|\psi|^2, \quad \text{for } \psi \in \mathbb{C}, \quad \text{where } A \in \mathbb{R} \text{ and } 0 \leqq B < m, \tag{C.3}$$

and so that $|\widetilde{U}(\psi)|$, $|\widetilde{U}'(\psi)|$, and $|\widetilde{U}''(\psi)|$ are bounded for $\psi \geqq 0$. We define

$$\widetilde{F}(\psi) = -\nabla \widetilde{U}(\psi), \qquad \psi \in \mathbb{C}, \tag{C.4}$$

where ∇ denotes the gradient with respect to $\operatorname{Re}\psi, \operatorname{Im}\psi$; then $\widetilde{F}(e^{is}\psi) = e^{is}\widetilde{F}(\psi)$ for any $\psi \in \mathbb{C}$, $s \in \mathbb{R}$.

We consider the Cauchy problem of type (2.1) with the modified nonlinearity,

$$\begin{cases} \ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)\widetilde{F}(\psi(0,t)), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ \psi|_{t=0} = \psi_0(x), \qquad \dot{\psi}|_{t=0} = \pi_0(x), \end{cases} \tag{C.5}$$

which we rewrite in the vector form in terms of $\Psi = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix}$ similarly to (2.2):

$$\dot{\Psi} = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Psi + \delta(x) \begin{bmatrix} 0 \\ \widetilde{F}(\psi) \end{bmatrix}, \qquad \Psi|_{t=0} = \Psi_0 \equiv \begin{bmatrix} \psi_0(x) \\ \pi_0(x) \end{bmatrix}. \quad (C.6)$$

This is a Hamiltonian system, with the Hamilton functional

$$\widetilde{\mathscr{H}}(\Psi) = \int\limits_{\mathbb{R}} \left(|\pi|^2 + |\nabla\psi|^2 + m^2|\psi|^2 \right) dx + \widetilde{U}(\psi(0,t)), \quad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix} \in \mathscr{E}, \quad (C.7)$$

which is Fréchet differentiable in the space $\mathscr{E} = H^1 \times L^2$. By the Sobolev embedding theorem, $L^\infty(\mathbb{R}) \subset H^1(\mathbb{R})$, and there is the following inequality:

$$\|\psi\|_{L^\infty}^2 \leqq \frac{1}{2m}(\|\psi'\|_{L^2}^2 + m^2\|\psi\|_{L^2}^2) \leqq \frac{1}{2m}\|\Psi\|_{\mathscr{E}}^2. \quad (C.8)$$

Thus, (C.3) leads to

$$\widetilde{U}(\psi(0)) \geqq A - B\|\psi\|_{L^\infty}^2 \geqq A - \frac{B}{2m}\|\Psi\|_{\mathscr{E}}^2. \quad (C.9)$$

Taking into account (C.7), we obtain the inequality

$$\|\Psi\|_{\mathscr{E}}^2 = 2\widetilde{\mathscr{H}}(\Psi) - 2\widetilde{U}(\psi(0)) \leqq 2\widetilde{\mathscr{H}}(\Psi) - 2A + \frac{B}{m}\|\Psi\|_{\mathscr{E}}^2, \quad \Psi \in \mathscr{E}, \quad (C.10)$$

which implies

$$\|\Psi\|_{\mathscr{E}}^2 \leqq \frac{2m}{m-B}\left(\widetilde{\mathscr{H}}(\Psi) - A\right), \qquad \Psi \in \mathscr{E}. \quad (C.11)$$

Lemma C.1.

(i) *There is the identity* $\widetilde{\mathscr{H}}(\Psi_0) = \mathscr{H}(\Psi_0)$.
(ii) *If* $\Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix} \in \mathscr{E}$ *satisfies* $\widetilde{\mathscr{H}}(\Psi) \leqq \widetilde{\mathscr{H}}(\Psi_0)$, *then* $\widetilde{U}(\psi(0)) = U(\psi(0))$.

Proof.

(i) According to (C.11), the Sobolev embedding (C.8), and the choice of $\Lambda(\Psi_0)$ in (C.1),

$$\|\psi_0\|_{L^\infty}^2 \leqq \frac{1}{2m}\|\Psi_0\|_{\mathscr{E}}^2 \leqq \frac{\mathscr{H}(\Psi_0) - A}{m-B} = \Lambda(\Psi_0)^2. \quad (C.12)$$

Thus, according to the choice of $\widetilde{U}$ (equality (C.2)), $\widetilde{U}(\psi_0(0)) = U(\psi_0(0))$, proving (i).

(ii) By (C.8), (C.11), the condition $\widetilde{\mathscr{H}}(\Psi) \leqq \widetilde{\mathscr{H}}(\Psi_0)$, and part (i) of the lemma, we have:

$$\|\psi\|_{L^\infty}^2 \leqq \frac{1}{2m}\|\Psi\|_{\mathscr{E}}^2 \leqq \frac{\widetilde{\mathscr{H}}(\Psi) - A}{m - B} \leqq \frac{\widetilde{\mathscr{H}}(\Psi_0) - A}{m - B} = \frac{\mathscr{H}(\Psi_0) - A}{m - B} = \Lambda(\Psi_0)^2.$$

Hence, (ii) follows by (C.2). □

Remark C.1. We will show that if $\Psi(t)$ solves (C.6), then $\widetilde{\mathscr{H}}(\Psi(t)) = \widetilde{\mathscr{H}}(\Psi_0)$, and therefore $\widetilde{U}(\psi(0,t)) = U(\psi(0,t))$ by Lemma C.1(ii). Hence, $\widetilde{F}(\psi(0,t)) = F(\psi(0,t))$ for all $t \geqq 0$, allowing us to conclude that $\Psi(t)$ solves (2.2) as well as (C.6).

Local well-posedness

The solution to the Cauchy problem

$$\dot{\Xi} = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Xi, \qquad \Xi(x,0) = \Xi_0(x) = \begin{bmatrix} \xi_0(x) \\ \eta_0(x) \end{bmatrix} \tag{C.13}$$

is represented by

$$\Xi(x,t) = W_0(t)\Xi_0 = \int_{\mathbb{R}} \begin{bmatrix} \dot{G}(x-y,t) & G(x-y,t) \\ \ddot{G}(x-y,t) & \dot{G}(x-y,t) \end{bmatrix} \begin{bmatrix} \xi_0(y) \\ \eta_0(y) \end{bmatrix} dy, \tag{C.14}$$

where $G(x,t)$ is the forward fundamental solution to the Klein–Gordon equation, $G(x,t) = \theta(t-|x|)J_0(m\sqrt{t^2-x^2})/2$, with J_0 being the Bessel function (see e.g. [Kom94]). Then the solution to the Cauchy problem (C.6) can be represented by

$$\Psi(x,t) = W_0(t)\Psi_0 + Z[\psi(0,\cdot)](t),$$
$$\text{where} \quad Z[\psi(0,\cdot)](t) := \int_0^t W_0(t-s) \begin{bmatrix} 0 \\ \delta(\cdot)\widetilde{F}(\psi(0,s)) \end{bmatrix} ds. \tag{C.15}$$

Lemma C.2. *For any nonnegative integers j and k there is a constant $C_{j,k} > 0$ such that*

$$|\partial_x^j \partial_t^k J_0(m\sqrt{t^2-x^2})| \leqq C_{j,k}(1+t)^{j+k}, \qquad |x| < t. \tag{C.16}$$

Proof. The proof immediately follows from the observation that all the derivatives of the Bessel function $J_0(z)$ are bounded for $z \in \mathbb{R}$, and that $J_0(z)$ is an absolutely converging Taylor series in even powers of z. Hence, all derivatives of the function $J_0(\sqrt{r})$ in r are bounded for $r \geqq 0$. □

The next lemma establishes the contraction principle for the integral equation (C.15).

Lemma C.3. *There exists a constant* $C > 0$ *so that for any two functions* $\Psi_k(\cdot, t) = \begin{bmatrix} \psi_k(\cdot, t) \\ \pi_k(\cdot, t) \end{bmatrix} \in C([0, 1], \mathscr{E})$, $k = 1, 2$, *we have:*

$$\|Z[\psi_1(0, \cdot)](t) - Z[\psi_2(0, \cdot)](t)\|_{\mathscr{E}} \leqq C t^{1/2} \sup_{s\in[0,t]} \|\Psi_1(\cdot, s) - \Psi_2(\cdot, s)\|_{\mathscr{E}},$$

for $0 \leqq t \leqq 1$.

Proof. According to (C.14) and (C.15),

$$Z[\psi_1(0, \cdot)](t) - Z[\psi_2(0, \cdot)](t) = \begin{bmatrix} I(x, t) \\ \partial_t I(x, t) \end{bmatrix},$$

where

$$I(x, t) := \int_0^t G(x, t - s) \left(\widetilde{F}(\psi_1(0, s)) - \widetilde{F}(\psi_2(0, s))\right) ds.$$

First we prove the L^2 estimate for $I(x, t)$. By the Sobolev embedding theorem,

$$\begin{aligned} \|I(\cdot, t)\|_{L^2} &\leqq C \left\| \int_0^t \theta(t - s - |x|) |\widetilde{F}(\psi_1(0, s)) - \widetilde{F}(\psi_2(0, s))| \, ds \right\|_{L^2} \\ &\leqq C \sup_{z\in\mathbb{C}} |\nabla \widetilde{F}(z)| \left\| \int_0^t \theta(t - s - |x|) \, ds \right\|_{L^2} \sup_{s\in[0,t]} \|\psi_1(\cdot, s) \\ &\quad - \psi_2(\cdot, s)\|_{H^1} \\ &\leqq C' t^{3/2} \sup_{s\in[0,t]} \|\psi_1(\cdot, s) - \psi_2(\cdot, s)\|_{H^1}, \end{aligned} \tag{C.17}$$

where we took into account that $|\nabla \widetilde{F}(z)|$ is bounded due to the choice of $\widetilde{U}$.

Similarly, we derive the L^2 estimate for the derivative $\partial_x I(x, t - s)$. We first analyze

$$\partial_x G(x, t - s) = \frac{1}{2} \theta(t - s - |x|) \partial_x J_0 \left(m\sqrt{(t - s)^2 - x^2}\right) - \frac{1}{2} \operatorname{sgn} x \, \delta(t - s - |x|).$$

By Lemma C.2 for $|x| \leqq |t - s| \leqq 1$, we have $|\partial_x J_0(m\sqrt{(t - s)^2 - x^2})| \leqq C$; we conclude that $\|\partial_x I(\cdot, t)\|_{L^2}$ is bounded by

$$\begin{aligned} &\left\| \int_0^t \left[C\theta(t - s - |x|) + \frac{\delta(t - s - |x|)}{2} \right] ds \right\|_{L^2} \sup_{s\in[0,t]} \left| \widetilde{F}(\psi_1(0, s)) \right. \\ &\qquad \left. - \widetilde{F}(\psi_2(0, s)) \right| \\ &\quad \leqq C \, \|\theta(t - |x|)\|_{L^2} \sup_{s\in[0,t]} \|\psi_1(\cdot, s) - \psi_2(\cdot, s)\|_{H^1} \\ &\quad \leqq C' t^{1/2} \sup_{s\in[0,t]} \|\psi_1(\cdot, s) - \psi_2(\cdot, s)\|_{H^1}. \end{aligned} \tag{C.18}$$

The L^2 norm of $\partial_t I(x, t)$ is estimated similarly. □

For $E > 0$, let us denote $\mathscr{E}_E = \{\Psi_0 \in \mathscr{E} \colon \mathscr{H}(\Psi_0) \leqq E\}$.

Corollary C.1.

(i) *For any* $E > 0$ *there exists* $\tau = \tau(E) > 0$ *such that for any* $\Psi_0 \in \mathscr{E}_E$ *there is a unique solution* $\Psi(x,t) \in C([0,\tau], \mathscr{E})$ *to the Cauchy problem* (C.6) *with the initial condition* $\Psi(0) = \Psi_0$.

(ii) *The map* $W(t): \ \Psi_0 \mapsto \Psi(t)$, $t \in [0,\tau]$ *are continuous maps from* $\mathscr{E}_E$ *to* $\mathscr{E}$.

Smoothness of the solution

In this section, we will study the smoothness of the solution

$$\Psi(x,t) = (\psi(x,t), \pi(x,t)) \in C([0,\tau], \mathscr{E})$$

constructed in Corollary C.1(i) assuming that $\psi_0(x), \pi_0(x) \in C_0^\infty(\mathbb{R})$. According to the integral representation (C.15), $\psi(x,t)$, $t \in [0,\tau]$, can be represented as

$$\psi(x,t) = \int\limits_{\mathbb{R}} \big(\dot{G}(x-y,t)\psi_0(y) + G(x-y,t)\pi_0(y)\big)\, dy + \int\limits_0^t G(x,t-s)\widetilde{F}(\psi(0,s))\, ds. \tag{C.19}$$

First, let us prove the smoothness of the function $\psi(0,t)$.

Lemma C.4. $\psi(0,t) \in C^\infty([0,\tau])$.

Proof. The integral representation (C.19) implies that, for $t \in [0,\tau]$,

$$\psi(0,t) = \int\limits_{\mathbb{R}} \big(\dot{G}(y,t)\psi_0(y) + G(y,t)\pi_0(y)\big)\, dy + \frac{1}{2}\int\limits_0^t J_0(m(t-s))\widetilde{F}(\psi(0,s))\, ds. \tag{C.20}$$

The first integral is a smooth function. Further, from $\|\psi(\cdot,t)\|_{H^1} \leqq C < \infty$, $t \in [0,\tau]$, we conclude that $|\psi(0,t)|$ is bounded. Hence, (C.20) implies that $\psi(0,\cdot) \in C([0,\tau])$, and then by induction that $\psi(0,\cdot) \in C^\infty([0,\tau])$ since the Bessel function is smooth. □

Now, from (C.19), we conclude that $\psi(x,t)$ is smooth away from the singularities of $G(x,t)$.

Proposition C.1. *The solution* $\psi(x,t)$ *is piecewise smooth inside each of the four regions of* $[0,\tau] \times \mathbb{R}$ *cut off by the lines* $x = 0$ *and* $x = \pm t$.

Proof. The first integral in the right-hand side of (C.19) is infinitely smooth in x and t for all $x \in \mathbb{R}$, $t \geqq 0$. Now let us consider the second integral in the right-hand side of (C.19), which could be written as follows:

$$\frac{\theta(t-|x|)}{2}\int_0^{t-|x|} J_0\left(m\sqrt{(t-s)^2-x^2}\right)\widetilde{F}(\psi(0,s))\,ds. \qquad \text{(C.21)}$$

Here the function $\widetilde{F}(\psi(0,s))$ is smooth in $s \in [0,\tau]$ by Lemma C.4. By Lemma C.2, all the partial derivatives of $J_0(m\sqrt{(t-s)^2-x^2})$ in x and t are continuous and uniformly bounded for $|x| < t-s$, $t \leqq \tau$. Therefore, (C.21) is smooth, with all the derivatives uniformly bounded, in each of the regions $0 \leqq x \leqq t$, $-t \leqq x \leqq 0$. In the regions $|x| > t$, (C.21) is identically equal to zero. □

Lemma C.5. *For* $0 < t \leqq \tau$,

$$\lim_{x\to 0-}\dot{\psi}(x,t) = \lim_{x\to 0+}\dot{\psi}(x,t). \qquad \text{(C.22)}$$

Proof. We have to analyze only the contribution from the second term in the right-hand side of (C.19), that is,

$$\partial_t \int_0^t G(x,t-s)\widetilde{F}(\psi(0,s))\,ds = G(x,0+)\widetilde{F}(\psi(0,t)) + \int_0^t \dot{G}(x,t-s)\widetilde{F}(\psi(0,s))\,ds.$$

The first term in the right-hand side is equal to zero for $x \neq 0$. The second term is continuous since the Green function $G(x,t-s)$ is smooth at $x=0$ for $t-s>0$. □

Lemma C.6. *For* $0 < t \leqq \tau$,

(i) $\dot{\psi}(x,t) + \psi'(x,t)$ *is continuous across the characteristic* $x = t$.
(ii) $\dot{\psi}(x,t) - \psi'(x,t)$ *is continuous across the characteristic* $x = -t$.

Proof. The proofs for both statements of the lemma are identical; we will only prove the first statement with $x > 0$. We have to study only the contribution from the second term in the right-hand side of (C.19), i.e.

$$(\partial_t+\partial_x)\int_0^t G(x,t-s)\widetilde{F}(\psi(0,s))\,ds = \int_0^t (\partial_t+\partial_x)G(x,t-s)\widetilde{F}(\psi(0,s))\,ds. \qquad \text{(C.23)}$$

Here we took into account that, as above, $G(x,0+)\widetilde{F}(\psi(0,t)) = 0$ for $x \neq 0$. Next, the key observation is that, for $x > 0$, the derivative $\partial_t + \partial_x$ applied to $G(x,t)$, does not produce a delta-function:

$$(\partial_t+\partial_x)G(x,t) = \frac{1}{2}\left\{\theta(t-x)(\partial_t+\partial_x)J_0(m\sqrt{t^2-x^2})\right\}.$$

Hence, the integral (C.23) is continuous in x and t across the line $x = t$, $0 < t \leqq \tau$ by Lemma C.2 □

Energy conservation and global well-posedness

Lemma C.7. *For the solution to the Cauchy problem* (C.6) *with the initial data* $\Psi_0 \in \mathscr{E}$*, the energy is conserved:* $\widetilde{\mathscr{H}}(\Psi(t)) = \mathrm{const}$*,* $t \in [0, \tau]$.

Proof. We follow [Kom95]. First, we prove that the energy is conserved for the smooth initial data with compact support: $\Psi_0 = \begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix}$, with $\psi_0, \pi_0 \in C_0^\infty(\mathbb{R})$. Consider the norm (2.9),

$$\|\Psi(t)\|_{\mathscr{E}}^2 = int_{-\infty}^{\infty}[|\dot{\psi}|^2 + |\psi'|^2 + m^2|\psi|^2]\,dx, \qquad t \in [0, \tau]. \tag{C.24}$$

We split this integral into four pieces: the integration over $(-\infty, -t)$, $(-t, 0)$, $(0, t)$, and (t, ∞). By Proposition C.1, on the support of each of these integrals $\psi(x, t)$ for $t \in [0, \tau]$ is a smooth function of x and t. Then, differentiating, we may express $\partial_t \|\Psi(t)\|_{\mathscr{E}}^2$ as

$$\begin{aligned} \partial_t \|\Psi(t)\|_{\mathscr{E}}^2 = & \Big[|\dot{\psi}|^2 + |\psi'|^2 + m^2|\psi|^2\Big]_{x=-t-0}^{x=-t+0} \\ & -\Big[|\dot{\psi}|^2 + |\psi'|^2 + m^2|\psi|^2\Big]_{x=t-0}^{x=t+0} \\ & +2\int_{-\infty}^{\infty} [\dot{\psi}\ddot{\psi} + \psi'\dot{\psi}' + m^2\psi\dot{\psi}]\,dx, \quad t \in [0, \tau]. \end{aligned} \tag{C.25}$$

The terms $m^2|\psi|^2$ could be discarded due to continuity of ψ across the characteristics $x = \pm t$. Integrating by parts the terms $\psi'\dot{\psi}'$ and using the cancelations of the integrals due to equation (C.5) away from $x = 0$, we get:

$$\begin{aligned} \partial_t \|\Psi(t)\|_{\mathscr{E}}^2 = & \Big[|\dot{\psi}|^2 + |\psi'|^2 - 2\psi'\dot{\psi}\Big]_{x=-t-0}^{x=-t+0} \\ & -\Big[|\dot{\psi}|^2 + |\psi'|^2 + 2\psi'\dot{\psi}\Big]_{x=t-0}^{x=t+0} - 2\Big[\psi'\dot{\psi}\Big]_{x=0-}^{x=0+} \\ = & \Big[(\dot{\psi} - \psi')^2\Big]_{x=-t-0}^{x=-t+0} - \Big[(\dot{\psi} + \psi')^2\Big]_{x=t-0}^{x=t+0} \\ & -2\Big[\psi'\dot{\psi}\Big]_{x=0-}^{x=0+}. \end{aligned} \tag{C.26}$$

According to Lemma C.6, the first two terms in (C.26) do not give any contribution. Let us compute the contribution of the last term. According to Lemma C.5, $\dot{\psi}(0\pm, t) = \dot{\psi}(0, t)$ for $t \in [0, \tau]$, and therefore

$$\big[\psi'\dot{\psi}\big]_{x=0-}^{x=0+} = \big[\psi'(x, t)\big]_{x=0-}^{x=0+}\,\dot{\psi}(0, t) = -\widetilde{F}(\psi(0, t))\dot{\psi}(0, t) = \frac{d}{dt}\widetilde{U}(\psi(0, t)).$$

In the second equality, we computed the jump of ψ' using equation (C.5) and the piecewise smoothness of the solution. We conclude that

$$\frac{d}{dt}\left\{\frac{1}{2}\|\Psi(t)\|_{\mathscr{E}}^2 + \widetilde{U}(\psi(0, t))\right\} = 0,$$

and hence the value of the functional $\widetilde{\mathscr{H}}$ defined in (C.7) is conserved.

Since we proved the energy conservation for the initial data that constitute a dense set in $\mathscr{E}$ and since the dynamical group is continuous in $\mathscr{E}$ by Corollary C.1(ii), we conclude that the energy is conserved for arbitrary initial data from $\mathscr{E}$. $\square$

Corollary C.2.

(i) *The solution* Ψ *to the Cauchy problem* (C.6) *with the initial data* $\Psi|_{t=0} = \Psi_0 \in \mathscr{E}$ *exists globally:* $\Psi \in C_b(\mathbb{R}, \mathscr{E})$.
(ii) *The energy is conserved:* $\widetilde{\mathscr{H}}(\Psi(t)) = \widetilde{\mathscr{H}}(\Psi_0), \qquad t \geqq 0.$

Proof. Corollary C.1(i) yields a solution $\Psi \in L^\infty([0,\tau], \mathscr{E})$ with a positive $\tau = \tau(E)$. However, the value of $\mathscr{H}(\Psi(t))$ is conserved for $t \leqq \tau$ by Lemma C.7. Corollary C.1(i) allows us then to extend Ψ to the interval $[\tau, 2\tau]$, and eventually to all $t \geqq 0$. In the same way we extend the solution $\Psi(t)$ for all $t < 0$. $\square$

Conclusion of the proof of Theorem 2.1

The trajectory $\Psi = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix} \in C_b(\mathbb{R}, \mathscr{E})$ is a solution to (C.6), for which Corollary C.2(ii) together with Lemma C.1(i) imply the energy conservation (2.12). By Lemma C.1(ii), $\widetilde{U}(\psi(0,t)) = U(\psi(0,t))$, for all $t \in \mathbb{R}$. This tells us that $\psi(x,t)$ is a solution to (2.1). Finally, the *a priori* bound (2.13) follows from (C.11) and the conservation of $\mathscr{H}(\Psi(t))$. This finishes the proof of Theorem 2.1.

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Faculty of Mathematics,
Wien A-1090,
Austria.
On leave from Department of Mechanics and Mathematics,
Moscow State University, Moscow 119899, Russia.
e-mail: alexander.komech@univie.ac.at

and

Mathematics Department,
Texas A&M University,
College Station,
Texas, USA.
e-mail: comech@math.tamu.edu

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