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Limiting Amplitude principle in the scattering by wedges

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SUMMARY

We consider a nonstationary scattering of plane waves by a wedge. We prove that the Sommerfeld-type integral, constructed in (*Math. Meth. Appl. Sci.* 2005; **28**:147–183; *Proc. Int. Seminar* '*Day on Diffraction-2003*', University of St. Petersburg, 2003; 151–162), is a classical smooth solution from a functional space, and prove the Limiting Amplitude principle. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: scattering; wedge; Limiting Amplitude principle

1. INTRODUCTION

This paper is a continuation of our paper [1], but the exposition is independent. In Reference [1], we have proved the Sommerfeld–Malyuzhinets-type integral representation and uniqueness of the solution of a nonstationary scattering problem by a wedge using the method of the complex characteristics [2].

Here we give the next steps in our program of mathematical foundation of scattering by wedges [1]. Namely, we prove the existence of the solution and the Limiting

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Amplitude principle. We show that the Sommerfeld–Malyuzhinets-type integral, obtained in References [1,3], belongs to an appropriate functional space, and the Limiting Amplitude principle holds.

For the corresponding stationary diffraction problem, the integral representation of the solution appeared first in the Sommerfeld paper [4], for the diffraction by a half-plane, and in Malyuzhinets paper [5] for a general angle (see also References [5–8] and a survey [9]). However, its relations to the corresponding nonstationary problem has been never done rigorously.

The explicit formulas for the scattering of the incident wave of the Heaviside type were obtained in the papers [10,11]. This case is the particular case of the problem considered in this paper when the frequency is zero, and the profile function is the Heaviside one. The method [10,11] is based on the special 'polar coordinates' in xyt-space. The problem also was considered in Reference [12], where the explicit formulas were obtained using the method of the ramified solutions of the wave equations. The uniqueness and the relation with the corresponding stationary diffraction problem have not been considered.

For the first time, the Limiting Amplitude principle has been considered in Reference [13] for a concrete solution, without a detailed analysis of its smoothness and uniqueness. We give a detailed analysis of the existence and uniqueness of a solution from an appropriate functional class. The solution is smooth if the profile function is smooth. Furthermore, we prove the Limiting Amplitude principle for the solution which is defined uniquely by its features, and not by an explicit formula.

The paper concerns two-dimensional scattering of plane waves by a wedge

$$W := \{y=(y_1,y_2) : y_1=\rho\cos\theta,\ \ y_2=\rho\sin\theta,\ \ \rho\geqslant 0,\ \ 0\leqslant\theta\leqslant\phi\}$$

of an opening $\phi\in(0,\pi)$. We consider an incident plane wave $u_{\rm in}(y,t)$ of the form

$$u_{\rm in}(y,t)=\mathrm{e}^{\mathrm{i}(k_0\cdot y-\omega_0 t)}f(t-n_0\cdot y)\quad \text{for } t\in\mathbb{R} \text{ and } \ y\in Q:=\mathbb{R}^2\backslash W \tag{1}$$

Here the frequency $\omega_0>0$ and the wave vector $k_0\in\mathbb{R}^2$, $\omega_0=|k_0|$ and $n_0=k_0/\omega_0$, $a\cdot b$ stands for the scalar product in $\mathbb{R}^2$. The profile $f\in C^\infty(\mathbb{R})$, and for some $\tau_0>0$

$$f(s)=\begin{cases}0, & s\leqslant 0\\ 1, & s\geqslant\tau_0\end{cases} \tag{2}$$

Denote

$$n_0=(\cos\alpha,\ \sin\alpha) \tag{3}$$

(see Figure 1).

We consider the case

$$\max(0,\phi-\pi/2)<\alpha<\min(\pi/2,\phi) \tag{4}$$

for example (see Figure 1). Physically, in this case the front of the incident wave $u_{\rm in}$ is identically zero on the wedge W at the moment $t=0$ and is reflected by both sides of the wedge for $t>0$. Other cases can be considered similarly.

Remark 1.1
By symmetry, we can assume $0<\alpha<\phi/2$, hence $0<\alpha<\pi/2$.

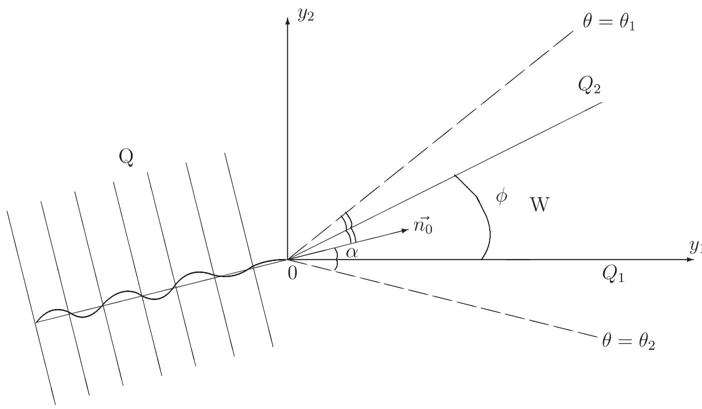

Figure 1. Scattering on a wedge.

We consider the following wave problem in Q with the Dirichlet boundary conditions:

$$\left\{\begin{array}{ll} \Box u(y,t)=0, & y\in Q \\ u(y,t)=0, & y\in \partial Q \end{array}\right| \quad t\in\mathbb{R} \tag{5}$$

where $\Box = \partial_t^2 - \Delta$. We will state the result also for the case of the Neumann boundary conditions. We include the incident wave $u_{\rm in}$ in the statement of the problem through the initial condition

$$u(y,t)=u_{\rm in}(y,t), \quad y\in Q, \quad t<0 \tag{6}$$

It is possible since $u_{\rm in}(y,t)$ is a solution to problem (5) for $t<0$: the boundary conditions in (5) hold for $t<0$ since $u_{\rm in}$ is then identically zero in a neighbourhood of ∂Q. Equivalently, $u(y,t)$ is the solution of the Cauchy problem for system (5) with the initial conditions

$$\left\{\begin{array}{l} u(0,y)=u_{\rm in}(0,y) \\ \dot u(0,y)=\dot u_{\rm in}(0,y) \end{array}\right| \quad y\in Q \tag{7}$$

Let us denote the scattered and diffracted waves by

$$u_{\rm s}(y,t):=u(y,t)-u_{\rm in}(y,t), \quad u_{\rm d}(y,t):=u_{\rm s}(y,t)-u_{\rm r}(y,t) \tag{8}$$

The reflected wave $u_{\rm r}(y,t)$ is given by

$$u_{\rm r}(y,t):=\begin{cases} u_{{\rm r},1}(\rho,\theta,t), & \phi\leqslant\theta\leqslant\theta_1 \\ 0, & \theta_1<\theta<\theta_2 \\ u_{{\rm r},2}(\rho,\theta,t), & \theta_2\leqslant\theta\leqslant 2\pi \end{cases} \tag{9}$$

where $y=\rho e^{i\theta}$,

$$\theta_1:=2\phi-\alpha, \quad \theta_2=2\pi-\alpha \tag{10}$$

(see Figure 1), and

$$u_{\mathrm{r},1}(\rho,\theta,t) = -\,\mathrm{e}^{\mathrm{i}(k_1\cdot y-\omega_0 t)} f(t - n_1\cdot y), \quad u_{\mathrm{r},2}(\rho,\theta,t) = -\,\mathrm{e}^{\mathrm{i}(k_2\cdot y-\omega_0 t)} f(t - n_2\cdot y)$$
$$k_1 = \omega_0 n_1, \quad n_1 = (\cos\theta_1,\ \sin\theta_1), \quad k_2 = \omega_0 n_2, \quad n_2 = (\cos\theta_2,\ \sin\theta_2) \tag{11}$$

(cf. with n_0 in (3)).

In the present paper, we prove the existence of the nonstationary problem (5), (6) in the functional space $\mathscr{E}_{\epsilon,N}$. We will prove that this solution (total field u) is represented as the sum of the incident wave u_{in}, reflected wave u_{r} and the diffracted wave u_{d}. We find the Sommerfeld–Malyuzhinets-type representation for the diffracted wave u_{d}. This representation allows us to prove that, the total field u belongs to an appropriate functional class. We use the representation for the proof the Limiting Amplitude principle. The comments on previous works in the directions can be found in Reference [1].

The plan of our paper is the following. In Section 2, we give the main definitions and formulate the main result. In Section 3, we reduce the problem to the stationary one. In Sections 4 and 5, we describe some important properties of the Sommerfeld–Malyuzhinets-type integrals. In Section 6, we solve the stationary diffraction problem with parameter. In Sections 7–9, we prove the estimates for densities of the stationary diffracted wave. In Section 10, we solve the nonstationary scattering problem.

In Section 11, we give a Sommerfeld–Malyuzhinets-type explicit expression for the diffracted wave. In Sections 12–14, we prove that the obtained solution belongs to the functional space $\mathscr{E}_{\epsilon,N}$ and we find the exact values of ϵ and N. In Section 15, we prove the Limiting Amplitude principle.

In the appendix, we collect well known estimates of the Cauchy-type oscillatory integrals.

2. NOTATIONS AND MAIN RESULT

2.1. Notations

1. Let us denote $\mathbb{C}^+ := \{\omega\in\mathbb{C} : \operatorname{Im}\omega > 0\}$. Let us consider a function $u(y,t)\in C(\overline{\mathbb{R}^+}\times\overline{Q})$ s.t. $|u(y,t)|\leqslant C(1+|t|)^N$ for some C,N and $u(y,t)=0,\ t<T(y)$. (Note that $T(y)=n_0\cdot y$ by (1), (2) for $u=u_{\mathrm{in}}$.) Define the Fourier–Laplace transform

$$\hat{u}(y,\omega) := F_{t\to\omega}[u](\omega) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} u(x,t)\,\mathrm{d}t, \quad \operatorname{Im}\omega > 0$$

Obviously, $\hat{u}(y,\omega)$ is an analytic function in $\omega\in\mathbb{C}^+$ for each $y\in\overline{Q}$.

2. We denote

$$\hat{g}(\omega) := \hat{f}(\omega-\omega_0) = \int_0^{\infty} \mathrm{e}^{\mathrm{i}(\omega-\omega_0)s} f(s)\,\mathrm{d}s \tag{12}$$

Since $f(s)$ satisfies (2), we have $f'(s)\in C_0^{\infty}(\mathbb{R}), \operatorname{supp} f'\subset[0,\tau_0]$ and

$$\hat{g}(\omega) = \frac{g_1(\omega)}{\omega-\omega_0}, \quad \omega\in\mathbb{C}^+ \tag{13}$$

where

$$\hat{g}_1(\omega) = \mathrm{i}\hat{h}(\omega - \omega_0), \quad \omega \in \mathbb{C} \tag{14}$$

with $h(s) := f'(s)$.

Definition 2.1
For an open set $V \subset \mathbb{C}^n$ we denote by $H(V)$ the set of analytic functions in V.

By the Paley–Wiener theorem

$$\hat{g}_1 \in H(\mathbb{C}) \tag{15}$$

Moreover, $\hat{g} \in S(\mathbb{R})$, and for any $k \in \mathbb{N}$, $N > 0$

$$\hat{g}_1^{(k)}(\omega)| \leqslant C_{k,N}(1 + |\omega|)^{-N}, \quad \omega \in \overline{\mathbb{C}^+} \tag{16}$$

Denote by $g_1(t)$ the inverse Fourier–Laplace transform of the function $\hat{g}_1(\omega)$. Then (14) implies that

$$g_1(t) = \mathrm{i}e^{-\mathrm{i}\omega_0 t} f'(t), \quad t \in \mathbb{R} \tag{17}$$

and (2) implies that

$$\operatorname{supp} g_1 \subset [0, \tau_0] \tag{18}$$

3. Denote by $\mathscr{C}$ the Sommerfeld-type contour

$$\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$$

where $\mathscr{C}_1 = \{\beta_1 - \mathrm{i}\pi/2 : \beta_1 \geqslant 1\} \cup \{1 + \mathrm{i}\beta_2 : -5/2\pi \leqslant \beta_2 \leqslant -\pi/2\} \cup \{\beta_1 - 5/2\pi\mathrm{i} : \beta_1 \geqslant 1\}$ and $\mathscr{C}_2$ is the symmetric to $\mathscr{C}_1$ with respect to $-3\pi/2$ (see Figure 2). We choose the orientation of the contours $\mathscr{C}_{1,2}$ counterclock wise.

Let us denote (see Reference [1])

$$\check{\Gamma}_1^- = \{w \in \mathbb{C} : \operatorname{Im}(-\mathrm{i}\omega \sinh w) = 0, \ 0 \in \check{\Gamma}_1^-\}$$

It is easy to check that for $\omega_2 > 0$

$$\check{\Gamma}_1^- = \left\{ w = (w_1 + \mathrm{i}w_2) : w_{1,2} \in \mathbb{R}, \ w_2 = \arctan\left(\frac{\omega_1}{\omega_2} \tanh w_1\right) \right\}$$

with the gauge $\arctan 0 = 0$. Also, we define the contours

$$\gamma(v) \equiv \check{\Gamma}_1^- + \mathrm{i}v$$

for $v \in \mathbb{R}$

4. We denote by $\dot{\overline{Q}} = \overline{Q} \backslash 0$, $\{y\} := |y|/(1 + |y|)$, $y \in \mathbb{R}^2$ or $y \in \mathbb{R}$. Let us fix some $\epsilon \geqslant 0$ and $N \geqslant 0$.

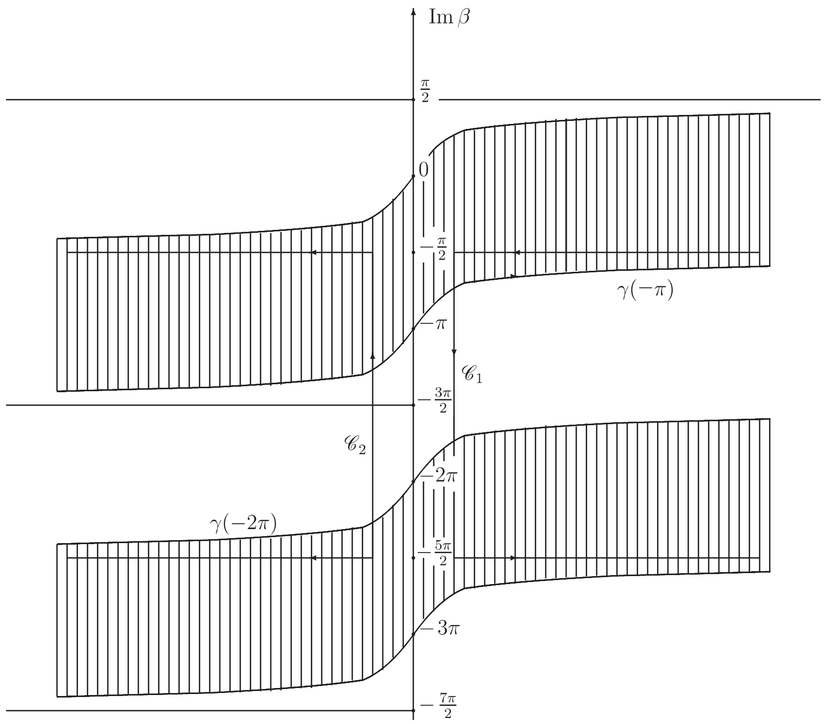

Figure 2. Sommerfeld–Malyuzhinets-type contour.

Definition 2.2

(i) E_ϵ is the space of functions $u(y)\in C(\overline{Q})\cap C^1(\dot{\overline{Q}})$ with finite norm

$$|u|_\epsilon = \sup_{y\in Q}(|u(y)| + \{y\}^\epsilon|\nabla u(y)|) < \infty$$

(ii) $\mathscr{E}_{\epsilon,N}$ is the space of functions $u(y,t)\in C(\overline{Q\times\mathbb{R}^+})$, with $\nabla u(y,t)\in C(\dot{\overline{Q}}\times\overline{\mathbb{R}^+})$ with finite norm

$$\|u\|_{\epsilon,N} := \sup_{t\geqslant 0}\sup_{y\in Q}(|u(y,t)| + (1+t)^{-N}\{y\}^\epsilon|\nabla_y u(y,t)|) < \infty \tag{19}$$

Note that for $\epsilon<1$ the functions $u\in E_\epsilon$ have the finite local energy

$$\int_{x\in Q:|x|<R}(|\nabla u(y)|^2 + |u(y)|^2)\,\mathrm{d}y < \infty, \quad R>0$$

Remark 2.3
Obviously, if $u(y,t)\in\mathscr{E}_{\epsilon,N}$, then $\hat{u}(y,\omega)\in E_\epsilon$ for $\omega\in\mathbb{C}^+$.

5. We introduce the Sommerfeld–Malyuzhinets integral kernel:

$$H(\beta) = H(\beta,\alpha,\Phi) := \coth(q(\beta+\pi\mathrm{i}/2-\mathrm{i}\alpha)) - \coth(q(\beta-3\pi\mathrm{i}/2+\mathrm{i}\alpha)), \quad \beta\in\mathbb{C} \tag{20}$$

where

$$q := \frac{\pi}{2\Phi}, \quad \Phi := 2\pi - \phi \tag{21}$$

For $\omega \in \overline{\mathbb{C}^+}$ let us denote by $\mathscr{S}(\rho, \theta, \omega)$ the Sommerfeld–Malyuzhinets-type integral

$$\mathscr{S}(\rho, \theta, \omega) := \frac{\mathrm{i}}{4\Phi} \int_{\mathscr{C}} \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta)\, \mathrm{d}\beta, \quad \rho \geqslant 0, \ \ \phi \leqslant \theta \leqslant 2\pi \tag{22}$$

which absolutely converges (see Section 4).

2.2. *The main result*

We have proved in Reference [1] the uniqueness of solution $u(y,t) \in \mathscr{E}_{\epsilon,N}$ of problem (5), (6), with $0 < \epsilon < 1$ and $N > 0$. Namely we proved that the solution (if exists) is given by

$$u(y,t) = F^{-1}_{\omega \to t}[\hat{u}(\rho, \theta, \omega)], \quad \rho > 0, \ \ \phi < \theta < 2\pi \tag{23}$$

in the polar coordinates $y = \rho \mathrm{e}^{\mathrm{i}\theta}$, where

$$\hat{u}(\rho, \theta, \omega) = \hat{g}(\omega) \mathscr{S}(\rho, \theta, \omega), \quad \omega \in \mathbb{C}^+ \tag{24}$$

Now let us define the *limiting amplitude* $A(\rho, \theta)$ by

$$A(\rho, \theta) := \frac{\mathrm{i}}{4\Phi} \int_{\mathscr{C}} \mathrm{e}^{-\omega_0 \rho \sinh \beta} H(\beta + \mathrm{i}\theta)\, \mathrm{d}\beta \tag{25}$$

The main result of this paper is the following.

Theorem 2.4

(i) Let the incident wave profile $f(s)$ be a smooth function (see (2)). Then the function $u(y,t)$, defined by (23), belongs to the space $C^\infty(\dot{\overline{Q}} \times \mathbb{R}) \cap \mathscr{E}_{\epsilon,N}$ with $\epsilon = N = 1 - (\pi/\Phi)$, and is a solution to the scattering problem (5), (6).

(ii) The Limiting Amplitude principle holds: for any $\rho_0 > 0$

$$u(\rho, \theta, t) - \mathrm{e}^{-\mathrm{i}\omega_0 t} A(\rho, \theta) \to 0, \quad t \to \infty \tag{26}$$

uniformly for $\rho \in [0, \rho_0]$ and $\theta \in [\phi, 2\pi]$.

Remark 2.5

Similar results hold for the problem of type (5) with the Neumann boundary value conditions. Proofs for this alternative problem can be done by the same methods. In this case the density $H(\beta)$ in expression (22) is replaced by

$$H(\beta) = H_n(\beta, \alpha, \Phi) := \coth(q(\beta + \pi\mathrm{i}/2 - \mathrm{i}\alpha)) + \coth(q(\beta - 3\pi\mathrm{i}/2 + \mathrm{i}\alpha)), \quad \beta \in \mathbb{C}$$

In this case, integral (22) absolutely converges for $\phi < \theta < 2\pi$.

3. FOURIER–LAPLACE TRANSFORM

Here we recall some notations from Reference [1]. Let us consider a solution $u(y,t)$ to problem (5), (6). We apply the Fourier–Laplace transform to Equation (5) and get the Helmholtz stationary equation with a parameter. First, we reduce the problem to the zero initial conditions to get the homogeneous Helmholtz equation. Namely, define the *scattered wave* by

$$u_{\mathrm{s}}(y,t) := u(y,t) - u_{\mathrm{in}}(y,t), \quad y \in Q, \quad t \in \mathbb{R} \tag{27}$$

where $u(y,t)$ is a solution to problem (5), (6). Then (6) implies that

$$u_{\mathrm{s}}(y,t) \equiv 0, \quad y \in Q, \quad t \leqslant 0 \tag{28}$$

Furthermore, $u_{\mathrm{s}}(y,t)$ is a solution to the problem

$$\left\{ \begin{array}{l} \Box u_{\mathrm{s}} = 0, \quad y \in Q \\ u_{\mathrm{s}}|_{\partial Q} = -(u_{\mathrm{in}})|_{\partial Q} \end{array} \right| \quad t \in \mathbb{R} \tag{29}$$

Remark 3.1
Let us note that $u_{\mathrm{in}}(y,t) \in \mathscr{E}_{0,0}$. Therefore, the condition $u(y,t) \in \mathscr{E}_{\epsilon,N}$ is equivalent to $u_{\mathrm{s}}(y,t) \in \mathscr{E}_{\epsilon,N}$.

Hence, we get obviously from (28)

Lemma 3.2
The Fourier–Laplace transform $\hat{u}_{\mathrm{s}}(y,\omega,)$ is an analytic function in $\omega \in \mathbb{C}^+$ with values in E_ϵ.

In particular, $\hat{u}_{\mathrm{s}}(y,\omega)$ is a continuous function of $(y,\omega) \in \overline{Q} \times \mathbb{C}^+$. Let us apply the Fourier–Laplace transform in time to problem (29). First, we write the Fourier–Laplace transform of the function u_{in} given by (1):

$$\hat{u}_{\mathrm{in}}(y,\omega) = \hat{g}(\omega)\mathrm{e}^{\mathrm{i}\omega\rho\cos(\theta-\alpha)} \tag{30}$$

Let us split $\partial Q = Q_1 \cup Q_2$ where $Q_1 := \{y = (y_1, y_2) \in \partial Q : y_2 = 0\}$ and $Q_2 := \{y = (y_1, y_2) \in \partial Q : y_1 = \rho\cos\phi, y_2 = \rho\sin\phi\}$, $\rho \geqslant 0$. Note that Q_1 is given by $\theta = 0$ and $y_1 = \rho$. Similarly, Q_2 is given by $\theta = \phi = 2\pi - \Phi$ and $y_2 = -\rho\sin\Phi$. Hence, the Dirichlet data of $\hat{u}_{\mathrm{in}}(\cdot,\omega)$ are equal to

$$\hat{u}_{\mathrm{in}}(y,\omega) = \hat{g}(\omega)\mathrm{e}^{\mathrm{i}\omega y_1\cos\alpha}, \quad y \in Q_1, \quad \hat{u}_{\mathrm{in}}(y,\omega) = \hat{g}(\omega)\mathrm{e}^{\mathrm{i}\omega y_2\cos(\alpha+\Phi)/\sin\Phi}, \quad y \in Q_2$$

where $\hat{g}(\omega)$ is defined by (12). Therefore the scattering problem (29) is reduced to the following stationary problem.

Lemma 3.3
Let $u_s(y,t) \in \mathscr{E}_{\epsilon,N}$ be a solution to problem (29) with $\epsilon, N > 0$. Then

(i) The function $\hat{u}_s(y,\omega)$ is a solution to the boundary value problem with a parameter $\omega \in \mathbb{C}^+$,

$$\begin{cases} (-\Delta - \omega^2)\hat{u}_s(y,\omega) = 0, & y \in Q \\ \hat{u}_s(y,\omega) = -\hat{g}(\omega)e^{i\omega y_1 \cos\alpha}, & y \in Q_1 \\ \hat{u}_s(y,\omega) = -\hat{g}(\omega)e^{-i\omega y_2[\cos(\alpha+\Phi)/\sin\Phi]}, & y \in Q_2 \end{cases} \tag{31}$$

(ii) The function $\hat{u}_s(\cdot,\omega) \in E_\epsilon$ for $\omega \in \mathbb{C}^+$.

So, the nonstationary problem (29) is reduced to the stationary one, (31), with parameter $\omega \in \mathbb{C}^+$.

4. ON SOMMERFELD–MALYUZHINETS INTEGRALS

Let us examine the convergence of integral (22) and its derivatives in ω and ρ, θ. First, let us prove the exponential decay of the function H. The poles of the function H from (20) are given by

$$\beta_k' = -i\pi/2 + i\alpha - 2i\Phi k, \quad \beta_k'' = 3/2\pi i - i\alpha - 2i\Phi k, \quad k \in \mathbb{Z} \tag{32}$$

For $\delta > 0$ denote $\mathbb{C}_\delta := \{\beta \in \mathbb{C} : |\beta - \beta_k| \geqslant \delta, \ \forall k \in \mathbb{Z}\}$.

Lemma 4.1
For any $\delta > 0$ the estimate holds

$$|H(\beta,\alpha,\Phi)| \leqslant C_\delta e^{-(\pi/\Phi)|\mathrm{Re}\,\beta|}, \quad \beta \in \mathbb{C}_\delta \tag{33}$$

Proof
This follows from the representation of (29) in the form

$$H(\beta,\alpha,\Phi) = \frac{\sinh[iq(2\pi-\alpha)]}{\sinh[q(\beta + i\pi/2 - i\alpha)]\sinh[q(\beta - 3/2\pi i + i\alpha)]}$$

and (21). □

Further, let us consider $\omega := \omega_1 + i\omega_2$ with $\omega_1 \in \mathbb{R}$ and $\omega_2 \geqslant 0$ and $\beta = \beta_1 + i\beta_2 \in \mathbb{C}$ with $\beta_{1,2} \in \mathbb{R}$. Let us note that

$$|e^{-\omega\rho\sinh\beta}| = e^{-\omega_1\rho\sinh\beta_1\cos\beta_2 + \omega_2\rho\cosh\beta_1\sin\beta_2}$$

Hence, estimate (33) implies for every $\delta > 0$,

$$|e^{-\omega\rho\sinh\beta}H(\beta + i\theta,\alpha,\Phi)| \leqslant C_\delta e^{-\omega_1\rho\sinh\beta_1\cos\beta_2 + \omega_2\rho\cosh\beta_1\sin\beta_2 - (\pi/\Phi)|\beta_1|}, \quad \beta \in \mathbb{C}_\delta \tag{34}$$

Denote $\Sigma := \mathbb{R}^+ \times \mathbb{R} \times \mathbb{C}^+$.

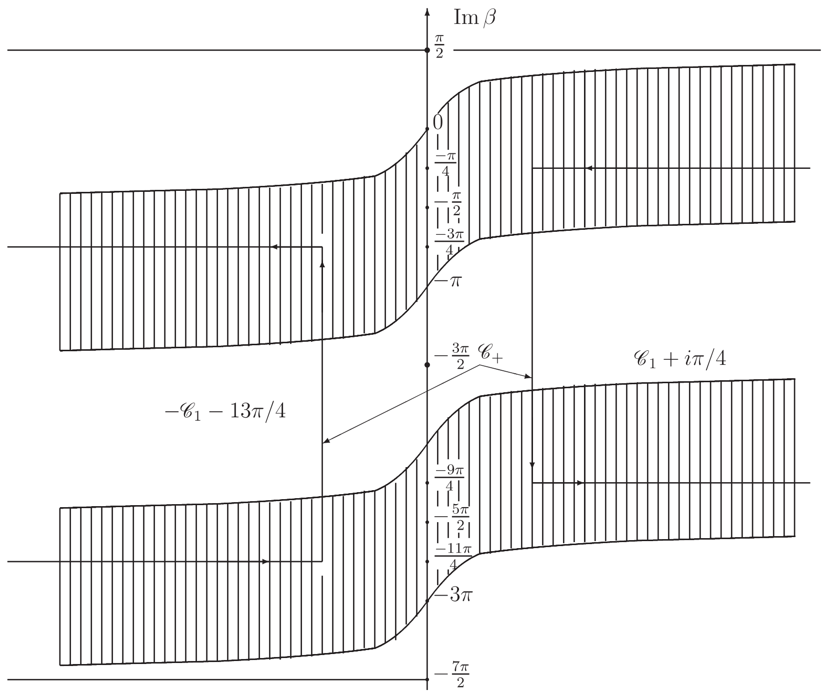

Figure 3. Contour $\mathscr{C}_+$.

Lemma 4.2

(i) Integral (22) converges absolutely and uniformly for $(\rho,\theta,\omega)\in\overline{\Sigma}$, and can be rewritten as follows:

$$\mathscr{S}(\rho,\theta,\omega)=\frac{\mathrm{i}}{4\Phi}\begin{cases}\displaystyle\int_{\mathscr{C}_+} \mathrm{e}^{-\omega\rho\sinh\beta}H(\beta+\mathrm{i}\theta,\alpha,\Phi)\,\mathrm{d}\beta, & \mathrm{Re}\,\omega\geqslant 0\\ \displaystyle\int_{\mathscr{C}_-} \mathrm{e}^{-\omega\rho\sinh\beta}H(\beta+\mathrm{i}\theta,\alpha,\Phi)\,\mathrm{d}\beta, & \mathrm{Re}\,\omega\leqslant 0\end{cases} \tag{35}$$

where $\mathscr{C}_+ := (\mathscr{C}_1+\mathrm{i}\pi/4)\cup(-\mathscr{C}_1-13\pi/4)$ and $\mathscr{C}_- := (\mathscr{C}_1-\mathrm{i}\pi/4)\cup(-\mathscr{C}_1-11\pi/4)$ (see Figure 3).

(ii) The function $\mathscr{S}(\rho,\theta,\omega)$ is continuous in $\overline{\Sigma}$.

(iii) The function $\mathscr{S}(\rho,\theta,\omega)$ is analytic in $\omega\in\mathbb{C}^+$ and smooth in $(\rho,\theta)\in\dot{\overline{Q}}$.

(iv) The function $\mathscr{S}(\rho,\theta,\omega)\in C^\infty(\dot{\overline{Q}}\times(\mathbb{R}\backslash\{0\}))$.

Proof

(i) For large $\beta=\beta_1+\mathrm{i}\beta_2\in\mathscr{C}$ we have either $\beta=\beta_1-\pi\mathrm{i}/2$, or $\beta=\beta_1-5\pi\mathrm{i}/2$ with $\beta_1\in\mathbb{R}$. Then $\beta_2=-\pi/2$ or $\beta_2=-5\pi/2$, so $\cos\beta_2=0$ and $\sin\beta_2=-1$. Therefore, estimate (34) implies that

$$|\mathrm{e}^{-\omega\rho\sinh\beta}H(\beta+\mathrm{i}\theta,\alpha,\Phi)|\leqslant C\mathrm{e}^{-\omega_2\rho\cosh\beta_1-(\pi/\Phi)|\beta_1|},\quad \beta\in\mathscr{C},\quad \beta_1\geqslant 1 \tag{36}$$

Hence, for $\omega_2,\ \rho\geqslant 0$ integral (22) converges absolutely and uniformly.

Let us prove (35), for example, for $\omega_1 = \mathrm{Re}\,\omega \geqslant 0$. The regions of decay of the exponent (34) are shown by dash in Figures 2 and 3. Hence, by the Cauchy Residue theorem we can deform the contour $\mathscr{C}$ to $\mathscr{C}_+$ in integral (22). Then we obtain

$$\mathscr{S}(\rho,\theta,\omega) = \frac{\mathrm{i}}{4\Phi}\int_{\mathscr{C}_+} \mathrm{e}^{-\omega\rho\sinh\beta} H(\beta+\mathrm{i}\theta,\alpha,\Phi)\,\mathrm{d}\beta, \quad \rho \geqslant 0, \quad \theta \in [\phi, 2\pi] \tag{37}$$

(ii) The continuity of the function $\mathscr{S}$ follows by the Lebesgue Dominated Convergence theorem from estimate (36) and the continuity of the integrand in (22).

(iii) For ω_2, $\rho > 0$ all the derivatives of integral (22) converge absolutely and uniformly by (36).

(iv) For example, let us consider $\omega_1 = \mathrm{Re}\,\omega > 0$. Then $\mathscr{S}(\rho,\theta,\omega)$ admits representation (37). Let $\beta \in \mathscr{C}_1 + \mathrm{i}\pi/4$. For large $\beta = \beta_1 + \mathrm{i}\beta_2$ we have either $\beta = \beta_1 - \pi\mathrm{i}/4$, or $\beta = \beta_1 - 9\pi\mathrm{i}/4$ with $\beta_1 \in \mathbb{R}$. Then $\beta_2 = -\pi/4$ or $\beta_2 = -9\pi/4$, so $\cos\beta_2 = 1/\sqrt{2}$ and $\sin\beta_2 = -1/\sqrt{2}$. Therefore, estimate (34) implies that

$$|\mathrm{e}^{-\omega\rho\sinh\beta} H(\beta+\mathrm{i}\theta,\alpha,\Phi)| \leqslant C\mathrm{e}^{-\omega_1\rho\cosh\beta_1/\sqrt{2}-(\pi/\Phi)|\beta_1|}, \quad \beta\in\mathscr{C}_1, \quad \beta_1 > 0$$

since $\omega_2 \geqslant 0$, $\rho > 0$. Similarly, the same estimate holds for $\beta \in -\mathscr{C}_1 - 13\pi/4$. Hence, the formal derivatives of integral (37) with respect to ω, ρ, θ converge absolutely and uniformly. □

5. STATIONARY PROBLEM

Let us consider the function $\hat{u}(\rho,\theta,\omega)$ defined by (24).

Lemma 5.1
For $\omega \in \mathbb{C}^+$, the function $\hat{u}(\cdot,\omega) \in C^\infty(\dot{\overline{Q}})$ is a classical solution to the stationary problem

$$\left.\begin{cases} (-\Delta - \omega^2)\hat{u}(y,\omega) = 0, & y \in \dot{\overline{Q}} \\ \hat{u}(\cdot,\omega)|_{\partial Q} = 0 \end{cases}\right| \omega \in \mathbb{C}^+ \tag{38}$$

Proof
First we note that $\hat{u}(\cdot,\omega) \in C^\infty(\dot{\overline{Q}})$ for $\omega \in \mathbb{C}^+$ by Lemma 4.2(iii) and (24).

The Helmholtz equation in (38) follows by the differentiation of the Sommerfeld integral (22) after the change of variable $\beta \to \beta' - \mathrm{i}\theta$ since $(\Delta + \omega^2)\mathrm{e}^{-\omega\rho\sinh(\beta-\mathrm{i}\theta)} = 0$.

It remains to check that $\hat{u}$ satisfies the boundary conditions in (38). Substitute $\theta = \phi$ and 2π in (22), and use obvious identities

$$H(\beta + \mathrm{i}\phi) = H(-\beta - 3\mathrm{i}\pi + \mathrm{i}\phi), \quad H(\beta + 2\pi\mathrm{i}) = H(-\beta - \mathrm{i}\phi)$$

Then the boundary conditions in (38) follow from the central symmetry of the contour $\mathscr{C}$ with respect to the point $-3\pi/2$. □

Let us *define*

$$\hat{u}_{\mathrm{s}}(\rho,\theta,\omega) = \hat{u}(\rho,\theta,\omega) - \hat{u}_{\mathrm{in}}(\rho,\theta,\omega) \tag{39}$$

similarly to (27).

Corollary 5.2
For $\omega \in \mathbb{C}^+$, the function

$$\hat{u}_{\mathrm{s}}(\cdot,\cdot,\omega) \in C^\infty(\dot{\overline{Q}})$$

and is a classical solution to the stationary problem (31).

Proof
The function $-u_{\mathrm{in}}(\rho,\theta,\omega) = -\hat{g}(\omega)\mathrm{e}^{\mathrm{i}\omega\rho\cos(\theta-\alpha)} \in C^\infty(\overline{Q})$ and is a solution of problem (31) for $\omega \in \mathbb{C}^+$. Therefore, the corollary follows from Lemma 5.1. □

6. INCIDENT, REFLECTED AND DIFFRACTED WAVE

In Sections 7–14, we prove the statement (i) of Theorem 2.4 for the function $u(y,t)$ defined by (23): the function is a solution to problem (5), (6) and belongs to $\mathscr{E}_{\epsilon,N}$ with ϵ, N defined in Theorem 2.4. We will prove that $u(\rho,\theta,t)$ is a smooth function and satisfies (5), (6) in the classical sense (see Corollary 10.3). The inclusion will be proved in Sections 11–14.

We will deduce (6) from (28) by the Paley–Wiener theorem, using the estimates of $\hat{u}_{\mathrm{s}}(\rho,\theta,\omega)$ for $\omega \in \mathbb{C}^+$. Note that $\mathscr{S}(\rho,\omega,\theta)$ and $\hat{u}(\rho,\omega,\theta) = \hat{u}_{\mathrm{s}}(\rho,\omega,\theta) + \hat{u}_{\mathrm{in}}(\rho,\omega,\theta)$ are not bounded for $\omega \in \mathbb{C}^+$ since $u_{\mathrm{in}}(x,t) \not\equiv 0$ for $t<0$. Therefore, we have to extract first the incident wave from integral (22).

The contour $\mathscr{C}$ in integral (22) crosses 'bad zones' between $\gamma(-\pi)$ and $\gamma(-2\pi)$, where $\mathrm{Re}(\omega\sinh\beta) < 0$, and the exponent $\mathrm{e}^{-\omega\rho\sinh\beta}$ is growing for $\mathrm{Im}\,\omega \to +\infty$. We will see that this growing part of the integral just corresponds to the incident wave.

To extract the incident wave, we will split the function $\mathscr{S}(\rho,\theta,\omega)$, in (24), into three summands

$$\mathscr{S} = \mathscr{S}_{\mathrm{in}} + \mathscr{S}_{\mathrm{r}} + \mathscr{S}_{\mathrm{d}} \tag{40}$$

Namely, let us define the functions

$$\left\{\begin{array}{lcl} \mathscr{S}_{\mathrm{d}}(\rho,\theta,\omega) & := & \dfrac{\mathrm{i}}{4\Phi}\displaystyle\int_{\mathscr{C}_0} \mathrm{e}^{-\omega\rho\sinh\beta} H(\beta+\mathrm{i}\theta,\alpha,\Phi)\,\mathrm{d}\beta, \quad \theta \neq \theta_{1,2} \\ \mathscr{S}_{\mathrm{in}}(\rho,\theta,\omega) & := & \mathrm{e}^{\mathrm{i}\omega\rho\cos(\theta-\alpha)} \\ \mathscr{S}_{\mathrm{r}}(\rho,\theta,\omega) & := & \begin{cases} -\mathrm{e}^{\mathrm{i}\omega\rho\cos(\theta-\theta_1)}, & \phi \leqslant \theta \leqslant \theta_1 \\ 0, & \theta_1 < \theta < \theta_2 \\ -\mathrm{e}^{\mathrm{i}\omega\rho\cos(\theta-\theta_2)}, & \theta_2 \leqslant \theta \leqslant 2\pi \end{cases} \end{array}\right| \quad \omega \in \overline{\mathbb{C}^+} \tag{41}$$

From the definition of the contour $\mathscr{C}_0$ and estimate (33) it follows that the integral in (41) converges absolutely for $\omega \in \overline{\mathbb{C}^+}$ and defines a continuous function of $\omega \in \overline{\mathbb{C}^+}$.

Let us note that

$$\hat{u}_{\mathrm{in}}(y,\omega) = \hat{g}(\omega)\mathscr{S}_{\mathrm{in}}(\rho,\theta,\omega) \tag{42}$$

by (30). Similarly, calculating the Fourier–Laplace transform of $u_{\mathrm{r}}(\rho,\theta,t)$, defined by (9), we obtain

$$\hat{u}_{\mathrm{r}}(y,\omega)=\begin{cases}\hat{u}_{\mathrm{r},1}(\rho,\theta,\omega), & \phi\leqslant\theta\leqslant\theta_1\\ 0, & \theta_1<\theta<\theta_2\\ \hat{u}_{\mathrm{r},2}(\rho,\theta,\omega), & \theta_2\leqslant\theta\leqslant 2\pi\end{cases}$$

where

$$\hat{u}_{\mathrm{r},1}(\rho,\theta,\omega)=-\mathrm{e}^{\mathrm{i}\omega\rho\cos(\theta-\theta_1)}\hat{g}(\omega),\quad \hat{u}_{\mathrm{r},2}(\rho,\theta,\omega)=-\mathrm{e}^{\mathrm{i}\omega\rho\cos(\theta-\theta_2)}\hat{g}(\omega) \tag{43}$$

Therefore,

$$\hat{u}_{\mathrm{r}}(\rho,\theta,\omega)=\hat{g}(\omega)\mathscr{S}_{\mathrm{r}}(\rho,\theta,\omega) \tag{44}$$

Remark 6.1
Let (40) hold. Then (39), (24) and (42) imply that

$$\hat{u}_{\mathrm{s}}=\hat{u}-\hat{u}_{\mathrm{in}}=\hat{g}(\mathscr{S}-\mathscr{S}_{\mathrm{in}})=\hat{g}\mathscr{S}_{\mathrm{s}} \tag{45}$$

where

$$\mathscr{S}_{\mathrm{s}}:=\mathscr{S}-\mathscr{S}_{\mathrm{in}}=\mathscr{S}_{\mathrm{r}}+\mathscr{S}_{\mathrm{d}} \tag{46}$$

Hence, (8) and (44) imply that

$$\hat{u}_{\mathrm{d}}=\hat{u}_{\mathrm{s}}-\hat{u}_{\mathrm{r}}=\hat{g}(\mathscr{S}_{\mathrm{r}}+\mathscr{S}_{\mathrm{d}}-\mathscr{S}_{\mathrm{r}})=\hat{g}\mathscr{S}_{\mathrm{d}}$$

We will call $\mathscr{S}$, $\mathscr{S}_{\mathrm{d}}$, $\mathscr{S}_{\mathrm{in}}$, $\mathscr{S}_{\mathrm{r}}$, $\mathscr{S}_{\mathrm{s}}$ as *densities* of the total, diffracted, incident, reflected and diffracted waves, respectively. The incident part $\mathscr{S}_{\mathrm{in}}(\rho,\theta,\cdot)$ is unbounded in $\mathbb{C}^+$ while $\mathscr{S}_{\mathrm{r}}(\rho,\theta,\cdot)$ and $\mathscr{S}_{\mathrm{d}}(\rho,\theta,\cdot)$ are bounded in $\mathbb{C}^+$ as we will check in Sections 8 and 9. Hence, $\hat{u}_{\mathrm{s}}(\rho,\theta,\cdot)$ is bounded in $\mathbb{C}^+$.

7. PROOF OF THE SPLITTING

To prove splitting (40), we will deform the contour of integration $\mathscr{C}$ in (22). Namely, define the contour

$$\mathscr{C}_0:=\gamma_1\cup\gamma_2,\quad \gamma_1:=\{\beta_1-\mathrm{i}\pi/2,\beta_1\in\mathbb{R}\},\quad \gamma_2:=\{\beta_1-5\mathrm{i}\pi/2,\beta_1\in\mathbb{R}\} \tag{47}$$

We direct the contour $\mathscr{C}_0$ such that the strip between γ_1 and γ_2 remains from the left (see Figure 4).

We are going to deform the contour $\mathscr{C}$ in (22) to the contour $\mathscr{C}_0$. Then the integral also changes and the difference is the sum of residues between $\mathscr{C}$ and $\mathscr{C}_0$ by the Cauchy Residue theorem.

Let us determine the poles and residues. By (32), the poles of the function $H(\beta+\mathrm{i}\theta,\alpha,\Phi)$ as the function of β are

$$\beta'_k(\theta):=-\mathrm{i}\pi/2+\mathrm{i}\alpha-2\mathrm{i}\Phi k-\mathrm{i}\theta,\quad \beta''_k(\theta)=3\mathrm{i}\pi/2-\mathrm{i}\alpha-2\mathrm{i}\Phi k-\mathrm{i}\theta,\quad k\in\mathbb{Z}$$

We have to take into account the poles between γ_1 and γ_2, i.e. $\operatorname{Im}\beta\in[-5\pi/2,-\pi/2]$.

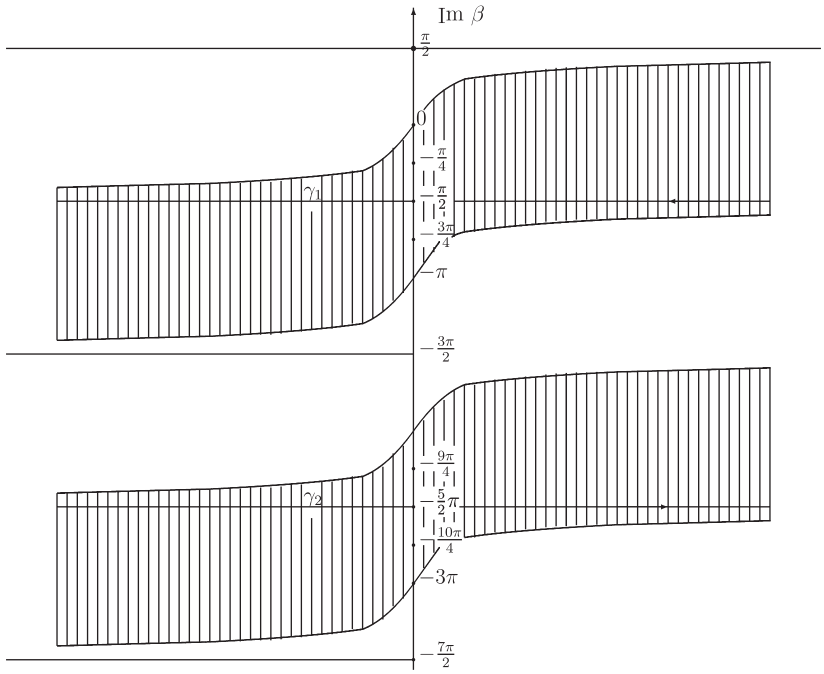

Figure 4. Contour $\mathscr{C}_0$.

The pole $\beta'_k(\theta)$ belongs to the interval $[-5i\pi/2, -i\pi/2]$ only for $k=0$ since $\theta \in [\phi, 2\pi]$. So,

$$\beta'_0(\theta) := -i\frac{\pi}{2} + i\alpha - i\theta \in [-5\pi/2 + i\alpha; -i\pi/2 + i\alpha - i\phi], \quad \theta \in [\phi, 2\pi] \tag{48}$$

By (4), we get from (48) that

$$\beta'_0(\theta) \in [-5\pi/2; -i\pi/2], \quad \theta \in [\phi, 2\pi] \tag{49}$$

As we will see further, the residue in this pole corresponds to the incident wave u_{in}.

Similarly, the pole $\beta''_k(\theta) \in [-5i\pi/2, -i\pi/2]$ only for $k=0$ and 1. For $k=0$

$$\beta''_0(\theta) = 3i\pi/2 - i\alpha - i\theta \in [-i\pi/2 - i\alpha, -i\pi/2], \quad \theta_2 \leqslant \theta \leqslant 2\pi$$

where θ_2 is defined in (10). Then Remark 1.1 implies that

$$\begin{cases} \beta''_0(\theta) \in [-5i\pi/2; -i\pi/2), & \theta \in [\theta_2; 2\pi] \\ \beta''_0(\theta_2) = -i\pi/2 \end{cases}$$

As we will see further the residue in this pole corresponds to the reflected wave from the face $\theta = 2\pi$ of the angle. For $k=1$

$$\beta''_1(\theta) = -5i\pi/2 + 2i\phi - i\alpha - i\theta \in [-5i\pi/2 + i\phi - i\alpha, -i\pi/2], \quad \phi \leqslant \theta \leqslant \theta_1$$

where θ_1 is defined in (10). Then Remark 1.1 implies that

$$\begin{cases} \beta_1''(\theta) \in (-5\mathrm{i}\pi/2; -\mathrm{i}\pi/2], & \theta \in [\phi, \theta_1) \\ \beta_1''(\theta_1) = -5\mathrm{i}\pi/2 \end{cases} \tag{50}$$

As we will see further, this pole corresponds to the wave reflected from the face $\theta = \phi$.

Definition 7.1
Critical directions correspond to the angles θ_2 and θ_1 defined by (10). Critical rays l_1, l_2 are the rays in Q, corresponding to the critical direction.

Estimate (33) implies that

$$[\mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta, \alpha, \Phi)| \leqslant C(\theta)\mathrm{e}^{-(\pi/\Phi)|\mathrm{Re}\,\beta|}, \quad \omega \in \mathbb{R}, \ \beta \in \mathscr{C}_0, \ \theta \in [\phi, 2\pi], \ \theta \neq \theta_{1,2}$$

Thus integral (41) converges for $\theta \neq \theta_1$, θ_2, and diverges for $\theta = \theta_1, \theta_2$ since the integrand has the poles on the contour of integration $\mathscr{C}_0$ then. Hence, the integral is a discontinuous function at $\theta = \theta_1$ and θ_2.

Lemma 7.2

(i) Splitting (40) holds,

$$\mathscr{S}(\rho, \theta, \omega) = \mathscr{S}_{\mathrm{in}}(\rho, \theta, \omega) + \mathscr{S}_{\mathrm{r}}(\rho, \theta, \omega) + \mathscr{S}_{\mathrm{d}}(\rho, \theta, \omega) \tag{51}$$

(ii) For $(\rho, \theta) \in \overline{Q}$ with $\theta \neq \theta_{1,2}$ we have

$$\mathscr{S}_{\mathrm{d}}(\rho, \theta, \cdot) \in H(\mathbb{C}^+), \quad \mathscr{S}_{\mathrm{d}}(\rho, \theta, \omega) \in C^\infty((\dot{\overline{Q}} \backslash \{l_1 \cup l_2\}) \times (\mathbb{R} \backslash 0)) \tag{52}$$

Proof
Let us deform the contour $\mathscr{C}$ to the contour $\mathscr{C}_0$ in integral (22). Using (49), (50) and first formula from (41), we obtain by the Cauchy Residue theorem, that

$$\mathscr{S}(\rho, \theta, \omega) = \mathscr{S}_{\mathrm{d}}(\rho, \theta, \omega)$$
$$+ \begin{cases} \dfrac{\pi}{2\Phi} \mathrm{res}_{\beta = \beta_0'(\theta)} \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta, \alpha, \Phi) + \dfrac{\pi}{2\Phi} \mathrm{res}_{\beta = \beta_1''(\theta)} & \\ \quad \times \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta, \alpha, \Phi), & \phi < \theta < \theta_1 \\ \dfrac{\pi}{2\Phi} \mathrm{res}_{\beta = \beta_0'(\theta)} \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta, \alpha, \Phi), & \theta_1 < \theta < 2\pi \\ \dfrac{\pi}{2\Phi} \mathrm{res}_{\beta = \beta_0'(\theta)} \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta, \alpha, \Phi) + \dfrac{\pi}{2\Phi} \mathrm{res}_{\beta = \beta_0''(\theta)} & \\ \quad \times \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta, \alpha, \Phi), & \theta_2 < \theta < 2\pi \end{cases}$$

Calculating the residues, we obtain (51). Statement (52) follows from (51), Lemma 4.2(iii), (iv), and expressions (41) for S_{in} and S_{r}. The lemma is proved. □

Remark 7.3
The proof shows that the incident wave corresponds to the residue at $\beta = \beta_0'(\theta)$, and the reflected one to the residues at $\beta = \beta_0''(\theta)$, $\beta_1''(\theta)$.

8. DENSITY OF THE DIFFRACTED WAVE

Remind that one of our aims is to prove (28). We will deduce it by the Paley–Wiener theorem using the estimates for the density $\mathscr{S}_{\rm s}$ of the diffracted wave (46). It suffices to estimate the density of the diffracted wave $\mathscr{S}_{\rm d}$ that we prove in this section.

Let us note that the reflected density $\mathscr{S}_{\rm r}$ is obviously discontinuous at the critical directions $\theta = \theta_1$ and θ_2, while the incident density $\mathscr{S}_{\rm in}$ and the total density $\mathscr{S}$ are smooth everywhere. Therefore, the diffracted density $\mathscr{S}_{\rm d} = \mathscr{S} - \mathscr{S}_{\rm r} - \mathscr{S}_{\rm in}$ also is discontinuous at $\theta = \theta_1$ and θ_2.

Theorem 8.1
The density $\mathscr{S}_{\rm d}$ satisfies the following estimates:

$$|\mathscr{S}_{\rm d}(\rho,\theta,\omega)| \leqslant C, \quad \omega \in \overline{\mathbb{C}^+},\ \ \rho \geqslant 0,\ \ \theta \in [\phi, 2\pi],\ \ \theta \neq \theta_{1,2} \tag{53}$$

Proof
The contour $\mathscr{C}_0$ in integral (41) consists of two parts γ_1 and γ_2. Hence it suffices to prove that the function

$$M_1(\rho,\theta,\omega) := \frac{\rm i}{4\Phi}\int_{\gamma_1} {\rm e}^{-\omega\rho\sinh\beta} H(\beta + {\rm i}\theta)\,{\rm d}\beta \tag{54}$$

satisfies estimate (53), since the function

$$M_2(\rho,\theta,\omega) := \frac{\rm i}{4\Phi}\int_{\gamma_2} {\rm e}^{-\omega\rho\sinh\beta} H(\beta + {\rm i}\theta)\,{\rm d}\beta$$

can be bounded similarly. We will omit below the indices α, Φ in the expression for H.

So, we consider the function M_1. For $\beta \in \gamma_1$ we have $\beta := -{\rm i}\pi/2 + \beta_1$ where $\beta_1 \in \mathbb{R}$. Substituting this expression in integral (54), and changing β_1 by β, we obtain

$$M_1(\rho,\theta,\omega) := -\frac{\rm i}{4\Phi}\int_{\mathbb{R}} {\rm e}^{{\rm i}\omega\rho\cosh\beta} H(-{\rm i}\pi/2 + \beta + {\rm i}\theta)\,{\rm d}\beta$$

Inequality (33) implies that the integral is bounded if the function $\beta \longmapsto H(-{\rm i}\pi/2 + \beta + {\rm i}\theta)$ does not have any poles in $\beta \in \mathbb{R}$ for $\theta \in [\phi, 2\pi]$. In particular, this estimate holds for θ such that $|\theta - \theta_k| \geqslant \delta$, $k = 1, 2$. On the other hand, the function H has a pole $\beta = 0$ when $\theta = \theta_2$. Then the estimate depends on the derivative of the integrand. The differentiation gives the factor $\omega\rho$ and we do not obtain the uniform estimate of type (53). Therefore we need more delicate methods. We use the method of the steepest descent [14,15]. Namely, we represent the function M_1 in the following form:

$$M_1(\rho,\theta,\omega) := M_3(\rho,\theta,\omega) + M_4(\rho,\theta,\omega)$$

where

$$M_3(\rho,\theta,\omega) := \int_{|\beta| \leqslant 1} {\rm e}^{{\rm i}\omega\rho\cosh\beta} H(-{\rm i}\pi/2 + \beta + {\rm i}\theta)\,{\rm d}\beta$$

$$M_4(\rho,\theta,\omega) := \int_{|\beta| \geqslant 1} {\rm e}^{{\rm i}\omega\rho\cosh\beta} H(-{\rm i}\pi/2 + \beta + {\rm i}\theta)\,{\rm d}\beta$$

The function M_4 satisfies an estimate of type (53) by (33) since

$$|e^{i\omega\rho\cosh\beta}| \leqslant e^{-\omega_2\rho}, \quad \beta\in\mathbb{R}, \qquad \omega=\omega_1+i\omega_2, \quad \omega_2\geqslant 0$$

It remains to prove an estimate of type (53) for the function M_3. Representation (20) gives

$$H(-i\pi/2+\beta+i\theta)=\coth q(\beta+i\theta-i\alpha)-\coth q(-2\pi i+\beta+i\theta+i\alpha), \quad \beta\in\mathbb{R}$$

By (4) the function $\coth q(\beta+i\theta-i\alpha)$ is analytic for $\beta\in[-1,1]$. Finally, it suffices to prove estimate (53) for the function

$$M_5(\rho,\theta,\omega):=\int_{|\beta|\leqslant 1} e^{i\omega\rho\cosh\beta}\coth q(-2\pi i+\beta+i\theta+i\alpha)\,d\beta$$

The integrand has a pole in the point $\beta=0$ for $\theta=\theta_2$ (see (10)). Let us consider $\theta=\theta_2+\varepsilon$, $|\varepsilon|\leqslant\varepsilon_0$, where $\varepsilon_0>0$ is sufficiently small. Then we can rewrite the function M_5 as the function of ρ,ε,ω:

$$M_6(\rho,\varepsilon,\omega):=M_5(\rho,\theta_2+\varepsilon,\omega)=\int_{|\beta|\leqslant 1} e^{i\omega\rho\cosh\beta}\coth q(\beta+i\varepsilon)\,d\beta$$

First, we represent M_6 in the following form:

$$M_6(\rho,\varepsilon,\omega):=\int_{|\beta|\leqslant 1} e^{i\omega_1\rho\cosh\beta}(e^{-\omega_2\rho\cosh\beta}-e^{-\omega_2\rho})\coth q(\beta+i\varepsilon)\,d\beta$$
$$+e^{-\omega_2\rho}\int_{|\beta|\leqslant 1} e^{i\omega_1\rho\cosh\beta}\coth q(\beta+i\varepsilon)\,d\beta, \quad \omega_1\in\mathbb{R},\ \omega_2\geqslant 0,\ \rho\geqslant 0$$

Note that

$$|(e^{-\omega_2\rho\cosh\beta}-e^{-\omega_2\rho})\coth q(\beta+i\varepsilon)|\leqslant C(\varepsilon_0), \quad \omega_2\geqslant 0,\ \beta\in[-1,1]$$

for $|\varepsilon|<\varepsilon_0$. Therefore, it suffices to prove an estimate of type (53) for the function

$$M_7(\rho,\varepsilon,\omega):=\int_{|\beta|\leqslant 1} e^{i\omega\rho\cosh\beta}\coth q(\beta+i\varepsilon)\,d\beta, \quad \omega\in\mathbb{R},\ \rho\geqslant 0 \tag{55}$$

Since $\coth q(\beta+i\varepsilon)\sim 1/q(\beta+i\varepsilon)$ for $|\beta|\leqslant 1$, estimate (53) for M_7 follows from Proposition A.1(ii) of the appendix.

9. DERIVATIVES OF THE DENSITY OF DIFFRACTED WAVE

To prove Theorem 2.4(i) it is necessary to prove, in particular that $u(y,t)\in C^\infty(\dot{\overline{Q}}\times\mathbb{R})$. We will prove the smoothness in Section 10, using the estimates for derivatives of the density $\mathscr{S}_s$ of the scattered wave. The estimates will be proved in the present section. Note that the estimates do not follow directly from representations (23), (24) since the function $\mathscr{S}(\rho,\theta,\omega)$ is growing exponentially in $\omega\in\mathbb{C}^+$. The growth is related to the density $\mathscr{S}_{in}$ of the incident wave in (41). So, to bound the derivatives, we extract the incident wave as above. We also extract the reflected wave since the estimates for its derivatives are obvious.

The following theorem will be proved similarly to Theorem 8.1. Let us denote by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Theorem 9.1
The density of the diffracted wave $\mathscr{S}_{\mathrm{d}}$ has all derivatives and the estimates hold

$$\left| \frac{\partial^\alpha}{\partial \rho^{\alpha_1} \partial \theta^{\alpha_2}} \mathscr{S}_{\mathrm{d}}(\rho, \theta, \omega) \right| \leqslant C(\delta, \alpha)(1 + |\omega|)^{|\alpha|} \rho^{\alpha_2}, \quad \omega \in \overline{\mathbb{C}^+} \setminus \{0\}$$
$$\rho \geqslant \delta > 0, \ \ \theta \in [\phi, 2\pi], \ \ \theta \neq \theta_{1,2}, \ \ C(\delta, \alpha) > 0 \tag{56}$$

for $\alpha := (\alpha_1, \alpha_2) \in \mathbb{N}_0 \times \mathbb{N}_0, \quad |\alpha| \geqslant 0$.

Proof
I. The estimates for the case $|\alpha| = 0$ are proved in the previous section. First, we prove that derivatives (56) exist for $\alpha \neq 0$ and find an appropriate expression for them. In contrast to the case $\alpha = 0$, the integrals which express the derivatives, do not converge absolutely on the contour $\mathscr{C}_0$ for $\omega \in \mathbb{R}$. Therefore, we have to modify the proof for the case $\alpha \neq 0$.

Let us fix a function $\kappa(\beta) \in C^\infty(\mathbb{R})$ such that

$$\kappa(\beta) := \begin{cases} 0, & \beta \leqslant 1 \\ \pi/4, & \beta \geqslant 2 \end{cases}$$

Let $\mathscr{B}_\pm$ be the contours in $\mathbb{C}: \mathscr{B}_+ := \{\beta_1 \pm \mathrm{i}\kappa(\pm\beta_1) - \mathrm{i}\pi/2, \ \pm\beta_1 \geqslant 0\}$; $\mathscr{B}_- := \{\beta_1 \mp \mathrm{i}\kappa(\pm\beta_1) - \mathrm{i}\pi/2, \pm\beta_1 \geqslant 0\}$. We direct the contour $\mathscr{B}_+$ similarly to γ_1 and the contour $\mathscr{B}_\pm - 2\pi\mathrm{i}$ similarly to γ_2 (see Figure 5).

Then we obtain the modified representation for $\mathscr{S}_{\mathrm{d}}$:

$$\mathscr{S}_{\mathrm{d}}(\rho, \theta, \omega) = \frac{\mathrm{i}}{4\Phi} \int_{\mathscr{B}_\pm \cup (\mathscr{B}_\pm - 2\pi\mathrm{i})} \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta) \, \mathrm{d}\beta$$
$$\omega \in \overline{\mathbb{C}^+}, \ \ \rho \geqslant 0, \ \ \phi \leqslant \theta \leqslant 2\pi, \ \ \theta \neq \theta_{1,2} \tag{57}$$

by the definition of $\mathscr{S}_{\mathrm{d}}$ in (41), estimate (34), and the Cauchy theorem.

Here the sign '+' is taken for $\operatorname{Re}\omega \geqslant 0$, and the sign '−' is taken for $\operatorname{Re}\omega < 0$. Differentiating formally the integral with respect to ρ, we obtain the absolutely convergent integral

$$\frac{\partial^{\alpha_1}}{\partial \rho^{\alpha_1}} \mathscr{S}_{\mathrm{d}}(\rho, \theta, \omega) = \frac{\mathrm{i}}{4\Phi} (-\omega)^{\alpha_1} \int_{\mathscr{B}_\pm \cup (\mathscr{B}_\pm - 2\pi\mathrm{i})} \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta)(\sinh \beta)^{\alpha_1} \, \mathrm{d}\beta \tag{58}$$

where the contour $\mathscr{B}_+ \cup (\mathscr{B}_+ - 2\pi\mathrm{i})$ is chosen for $\operatorname{Re}\omega \geqslant 0$ and the contour $\mathscr{B}_- \cup (\mathscr{B}_- - 2\pi\mathrm{i})$ is chosen for $\operatorname{Re}\omega < 0$.

The contour of integration $\mathscr{B}_+ \cup (\mathscr{B}_+ - 2\pi\mathrm{i})$ lies in the shaded regions of the decay of the exponent if $\operatorname{Re}\omega$, $\operatorname{Im}\omega \geqslant 0$ and $\omega \neq 0$. Similarly, the contour of integration $\mathscr{B}_- \cup (\mathscr{B}_- - 2\pi\mathrm{i})$ lies in the shaded regions of the decay of the exponent if $\operatorname{Re}\omega < 0$ and $\operatorname{Im}\omega > 0$. Therefore, integral (58) converges uniformly for $\rho \geqslant \delta > 0$, $\theta \in [0, 2\pi]$, $|\theta - \theta_{1,2}| \geqslant \nu > 0$. Hence, the formal differentiation (58) is justified.

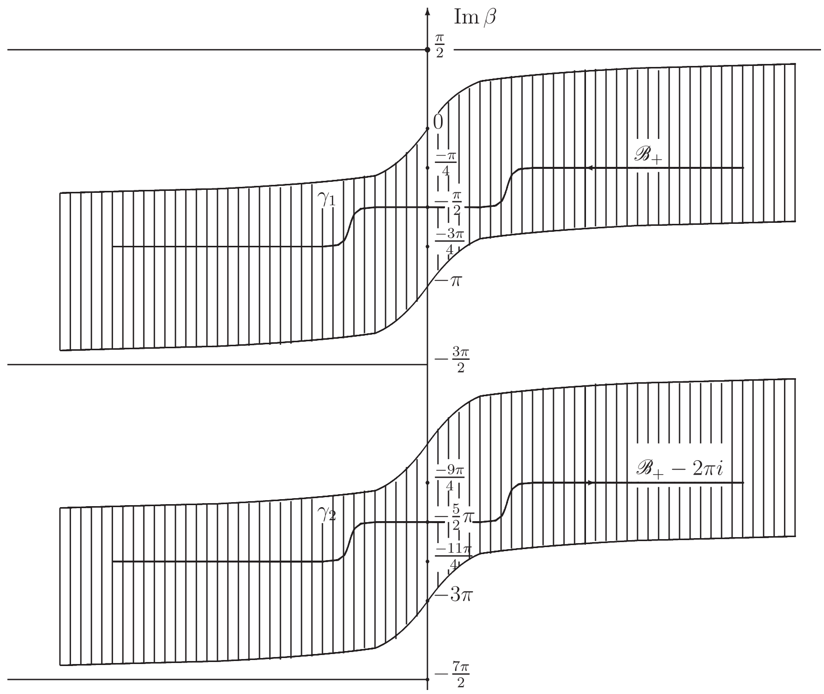

Figure 5. Contours $\mathscr{B}_+, \mathscr{B}_+ - 2\pi\mathrm{i}$. The case $\mathrm{Re}\,\omega \geqslant 0$.

Now we can calculate derivatives (56). Namely, differentiating formally (57), we get that

$$\frac{\partial^\alpha}{\partial\rho^{\alpha_1}\partial\theta^{\alpha_2}}\mathscr{S}_{\mathrm{d}}(\rho,\theta,\omega) = \left(\frac{1}{\mathrm{i}}\right)^{\alpha_2}\frac{\mathrm{i}}{4\Phi}(-\omega)^{\alpha_1}\int_{\mathscr{B}_\pm\cup(\mathscr{B}_\pm-2\pi\mathrm{i})} \mathrm{e}^{-\omega\rho\sinh\beta}\frac{\partial^{\alpha_2}}{\partial\beta^{\alpha_2}} \times[H(\beta+\mathrm{i}\theta)(\sinh\beta)^{\alpha_1}]\,\mathrm{d}\beta$$

Integrating by parts, we obtain

$$\frac{\partial^\alpha}{\partial\rho^{\alpha_1}\partial\theta^{\alpha_2}}\mathscr{S}_{\mathrm{d}}(\rho,\theta,\omega) = \frac{\mathrm{i}^{\alpha_2+1}}{4\Phi}(-\omega)^{|\alpha|}\rho^{\alpha_2}\int_{\mathscr{B}_\pm\cup(\mathscr{B}_\pm-2\pi\mathrm{i})} \mathrm{e}^{-\omega\rho\sinh\beta}H(\beta+\mathrm{i}\theta) \times(\sinh\beta)^{\alpha_1}(\cosh\beta)^{\alpha_2}\,\mathrm{d}\beta \tag{59}$$

II. Now let us prove the uniform estimate (56) using representation (59). First let us bound derivative (58) for $\mathrm{Re}\,\omega \geqslant 0$. The case $\mathrm{Re}\,\omega < 0$ is analysed similarly. We will prove that

$$\left|\int_{\mathscr{B}_+} \mathrm{e}^{-\omega\rho\sinh\beta}H(\beta+\mathrm{i}\theta)(\sinh\beta)^{\alpha_1}(\cosh\beta)^{\alpha_2}\,\mathrm{d}\beta\right| \leqslant C(\delta,\alpha)\left(\frac{1+|\omega|}{|\omega|}\right)^{|\alpha|-(\pi/\Phi)}, \quad \rho\geqslant\delta,\ \ \theta\in[\phi,2\pi],\ \ \theta\neq\theta_{1,2},\ \ C(\delta,\alpha)>0 \tag{60}$$

if $\mathrm{Re}\,\omega,\ \mathrm{Im}\,\omega \geqslant 0$ and $\omega \neq 0$.

The same integral over the contour $\mathscr{B}_+ - 2\pi\mathrm{i}$ can be estimated similarly. These estimates provide (56) by (59).

To prove (60), let us split the integral into two summands:

$$A_1(\rho,\theta,\omega) := \int_{\mathscr{B}_+\cap\{|\beta_1|\geqslant 2\}} \mathrm{e}^{-\omega\rho\sinh\beta} H(\beta+\mathrm{i}\theta)(\sinh\beta)^{\alpha_1}(\cosh\beta)^{\alpha_2}\,\mathrm{d}\beta$$

and

$$A_2(\rho,\theta,\omega) := \int_{\mathscr{B}_+\cap\{|\beta_1|\leqslant 2\}} \mathrm{e}^{-\omega\rho\sinh\beta} H(\beta+\mathrm{i}\theta)(\sinh\beta)^{\alpha_1}(\cosh\beta)^{\alpha_2}\,\mathrm{d}\beta \tag{61}$$

Let us note that the function $A_2(\rho,\theta,\omega)$ is defined for $\theta\neq\theta_{1,2}$, because the poles of $H(\beta+\mathrm{i}\theta)$ lie in imaginary axis for $\theta\in[0,2\pi]$.

Let us bound the integral

$$A_1^+(\rho,\theta,\omega) := \int_{\mathscr{B}_+\cap\{\beta_1\geqslant 2\}} \mathrm{e}^{-\omega\rho\sinh\beta} H(\beta+\mathrm{i}\theta)(\sinh\beta)^{\alpha_1}(\cosh\beta)^{\alpha_2}\,\mathrm{d}\beta$$

The corresponding integral over $\mathscr{B}_+\cap\{\beta_1\leqslant 2\}$ can be bounded similarly. The definition of the contour $\mathscr{B}_+$ implies, that

$$A_1^+(\rho,\theta,\omega) := \left(\frac{\sqrt{2}}{2}\right)^{\alpha_1+\alpha_2} \int_2^\infty \mathrm{e}^{-\omega\rho(\sqrt{2}/2)(\sinh\beta-\mathrm{i}\cosh\beta)} H(\beta-\mathrm{i}\pi/4+\mathrm{i}\theta)$$
$$\times(\sinh\beta-\mathrm{i}\cosh\beta)^{\alpha_1}(\cosh\beta-\mathrm{i}\sinh\beta)^{\alpha_2}\,\mathrm{d}\beta$$

Hence, estimate (33) implies that

$$|A_1^+(\rho,\theta,\omega)| \leqslant C\int_2^\infty \mathrm{e}^{-a_1\rho(\omega_1\sinh\beta+\omega_2\cosh\beta)}\mathrm{e}^{(|\alpha|-(\pi/\Phi))\beta}\,\mathrm{d}\beta$$
$$\leqslant C_1\int_2^\infty \mathrm{e}^{-a_2(\omega_1+\omega_2)\rho\mathrm{e}^\beta}\mathrm{e}^{(|\alpha|-(\pi/\Phi))\beta}\,\mathrm{d}\beta$$

for $\omega:=\omega_1+\mathrm{i}\omega_2$, where $a_1,\ a_2>0$.

Changing the variable $\xi:=(\omega_1+\omega_2)\rho\mathrm{e}^\beta$ in this integral, and using (21) and the conditions $\phi<\pi,\ |\alpha|\geqslant 1$, we get

$$|A_1^+(\rho,\theta,\omega)| \leqslant C_1\int_{(\omega_1+\omega_2)\rho\mathrm{e}^{-2}}^\infty \mathrm{e}^{-\alpha_2\xi}\left(\frac{\xi}{(\omega_1+\omega_2)\rho}\right)^{|\alpha|-(\pi/\Phi)}\frac{\mathrm{d}\xi}{\xi} \leqslant C((\omega_1+\omega_2)\rho)^{(\pi/\Phi)-|\alpha|}$$
$$\leqslant C(\delta)|\omega|^{(\pi/\Phi)-|\alpha|},\quad \rho\geqslant\delta>0 \tag{62}$$

It remains to bound the function A_2 from (61). The integral over $\mathscr{B}_+\cap\{1\leqslant|\beta|\leqslant 2\}$ can be estimated using (34):

$$\left|\int_{\mathscr{B}_+\cap\{1\leqslant|\beta_1|\leqslant 2\}} \mathrm{e}^{-\omega\rho\sinh\beta} H(\beta+\mathrm{i}\theta)(\sinh\beta)^{\alpha_1}(\cosh\beta)^{\alpha_2}\,\mathrm{d}\beta\right| \leqslant C,\quad \rho\geqslant 0,\ \ \theta\in[\phi,2\pi] \tag{63}$$

Therefore, it remains to bound the integral

$$A_2'(\rho,\theta,\omega) := \int_{\mathscr{B}_+ \cap \{|\beta_1| \leqslant 1\}} e^{-\omega\rho\sinh\beta} H(\beta+i\theta)(\sinh\beta)^{\alpha_1}(\cosh\beta)^{\alpha_2}\,d\beta$$

Let us check the estimate of type (63) for this integral. If $\theta \in [\phi, 2\pi]$ is not close to θ'', or $\alpha_2 > 0$, estimate (63) for A_2' is obvious, since the point $-\pi i/2$ is not a pole of the function $H(\beta+i\theta)(\cosh\beta)^{\alpha_2}$ by (33) and (4). If θ is close to θ'' and $\alpha_2 = 0$, the estimate of type (63) for this integral is reduced to the estimate of the function M_7 from (55).

Now estimate (60) follows from (62), and (63). □

Corollary 9.2

Estimates (56) imply that

$$|\nabla\mathscr{S}_d(\rho,\theta,\omega)| \leqslant C(\delta)(1+|\omega|), \quad \omega \in \overline{C^+}\backslash\{0\}, \;\; \rho \geqslant \delta > 0, \;\; \theta \in [\phi, 2\pi], \;\; \theta \neq \theta_{1,2}$$

for any $\delta > 0$.

Corollary 9.3

The density $\mathscr{S}_s(\rho,\theta,\omega)$ of the scattered wave has all the derivatives with respect to ρ, θ in the region $\dot{\overline{Q}}$, and the estimates of type (56) hold for the derivatives. In particular, the estimates hold for $\nabla\mathscr{S}_s$ in this region.

Proof

Equation (46), Lemma 4.2(iv), and the definition of $\mathscr{S}_{in}$ in (41) imply that $\mathscr{S}_s(\rho,\theta,\omega) \in C^\infty(\dot{\overline{Q}} \times (\mathbb{R}\backslash\{0\}))$. On the other side, representation (46) $\mathscr{S}_s = \mathscr{S}_r + \mathscr{S}_d$, Theorem 9.1 and the definition of $\mathscr{S}_r$ in (41) imply estimates (56) for $\theta \neq \theta_{1,2}$. Hence, the estimates hold for $\theta \in [\phi, 2\pi]$. □

10. THE SCATTERED WAVE

In this section, we start to prove Theorem 2.4(i) namely that function (23) is a smooth solution to the scattering problem (5), (6).

Note that $\hat{u}_s = \hat{u} - \hat{u}_{in}$ by (45). We will study the function

$$u_s(\rho,\theta,t) := F^{-1}_{\omega\to t}[\hat{u}_s(\rho,\theta,\omega)] \tag{64}$$

where

$$\hat{u}_s = \hat{g}(\omega)\mathscr{S}_s(\rho,\theta,\omega) \tag{65}$$

by (45), and the function $\mathscr{S}_s$ is defined by (46). We will prove that $u_s(\rho,\theta,t)$ satisfies system (29), (28).

Note that by Lemma 4.2(ii), (iii), formula (46), and the definition of $\mathscr{S}_{in}$ in (41),

$$\mathscr{S}_s \in C(\overline{Q} \times \mathbb{R}), \quad \mathscr{S}_s(\cdot,\cdot,\omega) \in C^\infty(\dot{\overline{Q}}), \quad \omega \in \mathbb{C}^+$$

First, we will study the function

$$\tilde{w}_s(\rho,\theta,\omega) := \hat{g}_1(\omega)\mathscr{S}_s(\rho,\theta,\omega), \quad \omega \in \overline{\mathbb{C}^+} \tag{66}$$

where $\hat{g}_1(\omega)$ is defined in (14). Let us define the function

$$w_s(\rho,\theta,t) := F^{-1}_{\omega\to t}[\tilde{w}_s(\rho,\theta,\omega)], \quad t\in\mathbb{R} \tag{67}$$

Lemma 10.1
For all $\rho \geqslant 0$ and $\theta \in [\phi, 2\pi]$ there exists the inverse Fourier–Laplace transform $w_s(\rho,\theta,t) = F^{-1}_{\omega\to t}[\tilde{w}_s]$ of the function $\tilde{w}_s(\rho,\theta,\omega)$, and

$$w_s \in C^\infty(\dot{\overline{Q}} \times \mathbb{R}) \tag{68}$$

$$w_s(\rho,\theta,t) \in C(\overline{Q}\times\mathbb{R}), \quad |w_s(\rho,\theta,t) \leqslant C, \quad t \geqslant 0 \tag{69}$$

$$w_s(\rho,\theta,t) = 0, \quad t<0 \tag{70}$$

Proof
Lemma 4.2(iii) shows that $\mathscr{S}(\rho,\theta,\omega)$ is analytic in $\omega\in\mathbb{C}^+$. Hence, the function $\mathscr{S}_s(\rho,\theta,\cdot) = \mathscr{S}(\rho,\theta,\cdot) - \mathscr{S}_{in}(\rho,\theta,\cdot)$ is analytic in $\omega\in\mathbb{C}^+$, since $\mathscr{S}_{in}(\rho,\theta,\omega) = e^{i\omega\rho\cos(\theta-\alpha)}$ is analytic in $\omega\in\mathbb{C}$.

Furthermore, the function $\mathscr{S}_s = \mathscr{S}_d + \mathscr{S}_r$ is bounded: namely, $\mathscr{S}_d$ is bounded by Theorem 8.1, and $\mathscr{S}_r$ is bounded by (41) since

$$\left.\begin{array}{l} |e^{i\omega\rho\cos(\theta-\theta_1)}| \leqslant C, \quad \phi \leqslant \theta \leqslant \theta_1, \\ |e^{i\omega\rho\cos(\theta-\theta_2)}| \leqslant C, \quad \theta_2 \leqslant \theta \leqslant 2\pi, \end{array}\right| \quad \omega \in \overline{\mathbb{C}^+}, \quad \rho \geqslant 0$$

The estimates hold since $-\pi/2 < \theta - \theta_l < \pi/2$, $l = 1,2$. Namely, $\phi - \theta_1 < \theta - \theta_1 < 0$ and $\phi - \theta_1 = \alpha - \phi > -\pi/2$ by (10) and (4). Similarly, (4) implies that $\alpha < \pi/2$. Therefore, $0 < \theta - \theta_2 < \alpha < \pi/2$, since $\theta_2 = 2\pi - \alpha$ by (10).

Hence,

$$|\mathscr{S}_s(\rho,\theta,\omega)| \leqslant C, \quad \omega \in \overline{\mathbb{C}^+}$$

Therefore, $\tilde{w}_s(\rho,\theta,\omega)$ also is analytic in $\omega\in\mathbb{C}^+$ by (15) and

$$|\tilde{w}_s(\rho,\theta,\omega)| \leqslant C_N(1+|\omega|)^{-N}, \quad \omega \in \overline{\mathbb{C}^+}$$

by (16). Hence, w_s satisfies (69), (70) by the Paley–Wiener theorem [16].

Moreover, Corollary 9.3, formula (66), and estimate (16) imply that

$$\left|\frac{\partial^\alpha}{\partial\rho^{\alpha_1}\partial\theta^{\alpha_2}}\tilde{\omega}_s(\rho,\theta,\omega)\right| \leqslant C_N(\delta,\alpha)\rho_2^{\alpha}(1+|\omega|)^{-N}, \quad (\rho,\theta)\in\dot{\overline{Q}}$$

$$\omega \in \overline{\mathbb{C}}^+\backslash\{0\}, \quad \rho \geqslant \delta > 0, \quad \theta\in[0,2\pi], \quad C_N(\delta,\alpha) > 0$$

Therefore, (68) holds. $\square$

In the following proposition we prove, in particular that the function $u_s(\rho,\theta,t)$, defined by (64), satisfies problem (29), (28).

Proposition 10.2

(i) The function $u_s(\rho,\theta,t)$ admits the following representation:

$$u_s(\rho,\theta,t) = -\mathrm{i}\int_0^t e^{\mathrm{i}\omega_0(r-t)}\, w_s(\rho,\theta,\tau)\,d\tau, \quad (\rho,\theta)\in\overline{Q}; \quad t\in\mathbb{R} \tag{71}$$

Furthermore,

$$u_s \in C^\infty(\dot{\overline{Q}}\times\mathbb{R}) \tag{72}$$

and for $(\rho,\theta)\in Q$,

$$u_s(\rho,\theta,t)=0, \quad t<0 \qquad |u_s(\rho,\theta,t)| \leqslant C(1+t), \quad t\geqslant 0 \tag{73}$$

(ii) u_s (ρ,θ,t) is a solution of system (29) and initial conditions (28), and

$$u_s(\rho,\theta,t)\in C(\overline{Q}\times\mathbb{R}) \tag{74}$$

Proof

(i) From (65), (66), and (13) we get

$$\hat{u}_s(\rho,\theta,\omega) = \frac{\tilde{w}_s(\rho,\theta,\omega)}{\omega-\omega_0+\mathrm{i}0}, \quad \omega\in\mathbb{C}^+$$

Hence and from (67)

$$u_s(\rho,\theta,t) = F^{-1}_{\omega\to t}\left[\frac{1}{\omega-\omega_0+\mathrm{i}0}\right] * w_s(\rho,\theta,t), \quad \omega\in\mathbb{R}, \ (\rho,\theta)\in\overline{Q} \tag{75}$$

Since

$$F^{-1}_{\omega\to t}\left[\frac{1}{\omega-\omega_0+\mathrm{i}0}\right] = -\mathrm{i}e^{-\mathrm{i}\omega_0 t}\Theta(t)$$

then (75), (70) imply that

$$u_s(\rho,\theta,t) = [-\mathrm{i}e^{-\mathrm{i}\omega_0 t}\Theta(t)] * w_s(\rho,\theta,t)$$

Hence, (71) follows.

Now, (72) follows from (68) and representation (71). At last, (73) follows from (69), (70) and (71).

(ii) System (29) holds for u_s in the classical sense since $\hat{u}_s$ satisfies (31) by Corollary 5.2, and (72) holds. Identity (28) follows from (73). Finally, (74) follows from (71) and (69). □

Corollary 10.3

The function $u(\rho,\theta,t)$ defined by (27) belongs to $C^\infty(\dot{\overline{Q}}\times\mathbb{R})$ and satisfies problem (5), (6).

Proof

The inclusion follows from (72), (27) and (1). Equations (5) and (6) for u follow from (27) and (28), (29) for u_s. □

Thus, we have proved that u is the classic solution to (7), (6).

The next theorem will complete the proof of Theorem 2.4 (i). Sections 11–14 concern the proof of the following theorem.

Theorem 10.4
The function $u_{\rm s}(\rho,\theta,t)$ belongs to the space $\mathscr{E}_{1-(\pi/\Phi),1-(\pi/\Phi)}$.

Now Theorem 2.4(i) follows since Remark 3.1 implies that $u\in\mathscr{E}_{1-(\pi/\Phi),1-(\pi/\Phi)}$.

Proof
Definition 2.2, (74) and (72) imply that it suffices to prove only estimates (19) for $u_{\rm s}$. Note, that it suffices to prove the estimate for $u_{\rm d}=u_{\rm s}-u_{\rm r}$ outside the critical directions $\theta=\theta_1,\theta_2$ since the estimates for $u_{\rm r}$ are trivial. Sections 11–14 concern the proof of estimates (19) for the diffracted wave $u_{\rm d}(y,t)$. □

11. SOMMERFELD–MALYUZHINETS REPRESENTATION FOR THE DIFFRACTED WAVE

In this section, we construct a convenient representation of the diffracted wave

$$u_{\rm d}:=F^{-1}_{\omega\to t}[\hat u_{\rm d}]$$

where

$$\hat u_{\rm d}:=\hat g\mathscr{S}_{\rm d} \tag{76}$$

The representation plays a crucial role in the proof of estimates (19) for $u_{\rm s}(\rho,\theta,t)$.

Lemma 11.1
The scattered wave $u_{\rm s}$ admits the following representation:

$$u_{\rm s}(\rho,\theta,t)=\begin{cases} u_{\rm d}(\rho,\theta,t)+u_{\rm r,1}(\rho,\theta,t), & \phi\leqslant\theta\leqslant\theta_1\\ u_{\rm d}(\rho,\theta,t), & \theta_1\leqslant\theta\leqslant\theta_2\\ u_{\rm d}(\rho,\theta,t)+u_{\rm r,2}(\rho,\theta,t), & \theta_2\leqslant\theta\leqslant 2\pi\end{cases} \tag{77}$$

Proof
First, (65), the second identity of (46), (76), the definition of $\mathscr{S}_{\rm r}$ in (41) imply that

$$\hat u_{\rm s}(\rho,\theta,\omega)=\begin{cases} u_{\rm d}(\rho,\theta,\omega)-\hat g(\omega){\rm e}^{{\rm i}\omega\rho\cos(\theta-\theta_1)}, & \phi\leqslant\theta\leqslant\theta_1\\ u_{\rm d}(\rho,\theta,\omega), & \theta_1\leqslant\theta\leqslant\theta_1\\ u_{\rm d}(\rho,\theta,\omega)-\hat g(\omega){\rm e}^{{\rm i}\omega\rho\cos(\theta-\theta_2)}, & \theta_2\leqslant\theta\leqslant 2\pi\end{cases}$$

Now representation (77) follows from (43). □

Theorem 11.2
The diffracted wave $u_{\rm d}$ admits the following integral representation:

$$u_{\rm d}(\rho,\theta,t)={\rm i}\frac{{\rm e}^{-{\rm i}\omega_0 t}}{4\Phi}\int_{\mathscr{C}_0}{\rm e}^{-\omega_0\rho\sinh\beta}H(\beta+{\rm i}\theta)f(t-{\rm i}\rho\sinh\beta)\,{\rm d}\beta,\quad \theta\neq\theta_{1,2} \tag{78}$$

where H is defined by (20), the contour $\mathscr{C}_0$ is defined by (47) and f is defined by (2).

Remark 11.3
Since for $\beta \in \mathscr{C}_0$ we have $f(t - \mathrm{i}\rho \sinh \beta) = 0$ for $\rho \geqslant t$, then

$$u_{\mathrm{d}}(\rho, \theta, \cdot) = 0, \quad \rho \geqslant t \tag{79}$$

Note, that this corresponds to the fact that the wave scattered by the vertex of the angle attains the point (ρ, θ) for the time ρ, if the velocity of the propagation of the signal is equal to 1.

Proof of Theorem 11.2
From (76), (13) we have

$$\hat{u}_{\mathrm{d}}(\rho, \theta, \omega) = \frac{1}{\omega - \omega_0} w_{\mathrm{d}}(\rho, \theta, \omega), \quad (\rho, \theta) \in \overline{Q}, \ \omega \in \overline{\mathbb{C}}^+ \setminus \{\omega_0\} \tag{80}$$

where

$$w_{\mathrm{d}}(\rho, \theta, \omega) := \hat{g}_1(\omega) \mathscr{S}_{\mathrm{d}}(\rho, \theta, \omega), \quad \omega \in \overline{\mathbb{C}}^+ \tag{81}$$

From (15), (16), Lemma 7.2, Theorem 8.1 it follows that w_{d} is analytic in $\mathbb{C}^+$, infinitely differentiable in $\mathbb{R} \setminus \{0\}$ and admits the following estimate:

$$|w_{\mathrm{d}}(\rho, \theta, \omega)| \leqslant C_N (1 + |\omega|)^{-N}, \quad (\rho, \theta) \in \overline{Q}, \ N \in \mathbb{N}$$

Hence, by the Paley–Wiener theorem we obtain that

$$\operatorname{supp} \mathrm{w}_{\mathrm{d}}(\rho, \theta, \cdot) \subset [0, \infty)$$

where

$$\mathrm{w}_{\mathrm{d}}(\rho, \theta, t) := F^{-1}_{\omega \to t}[w_{\mathrm{d}}(\rho, \theta, \omega)] \tag{82}$$

From (80) it follows that

$$u_{\mathrm{d}}(\rho, \theta, t) = -\mathrm{i}[\mathrm{e}^{-\mathrm{i}\omega_0 t} \Theta(t)] * \mathrm{w}_{\mathrm{d}}(\rho, \theta, t)] \tag{83}$$

Let us calculate $\mathrm{w}_{\mathrm{d}}(\rho, \theta, t)$ for $t \geqslant 0$, and $\theta \neq \theta_1, \theta_2$.

From (82), (81), the definition of $\mathscr{S}_{\mathrm{d}}$ in (41), estimates (33), (16) and the Fubini theorem we have for $t \geqslant 0$, $\theta \neq \theta_1, \theta_2$

$$\begin{aligned} \mathrm{w}_{\mathrm{d}}(\rho, \theta, t) &= \frac{\mathrm{i}}{8\pi\Phi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} \left[\hat{g}_1(\omega) \int_{\mathscr{C}_0} \mathrm{e}^{-\omega\rho \sinh \beta} H(\beta + \mathrm{i}\theta) \, \mathrm{d}\beta \right] \mathrm{d}\omega \\ &= \frac{\mathrm{i}}{8\pi\Phi} \int_{\mathscr{C}_0} H(\beta + \mathrm{i}\theta) \left[\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega(t - \mathrm{i}\rho \sinh \beta)} \hat{g}_1(\omega) \, \mathrm{d}\omega \right] \mathrm{d}\beta \\ &= \frac{\mathrm{i}}{4\Phi} \int_{\mathscr{C}_0} H(\beta + \mathrm{i}\theta) g_1(t - \mathrm{i}\rho \sinh \beta) \, \mathrm{d}\beta \end{aligned} \tag{84}$$

where $g_1(t) := F^{-1}_{\omega \to t}[\hat{g}_1(\omega)]$. Note that $t - \mathrm{i}\rho \sinh \beta \in \mathbb{R}$ for $\beta \in \mathscr{C}_0$ by definition (47) of the contour $\mathscr{C}_0$. Furthermore,

$$\operatorname{supp} \mathrm{w}_{\mathrm{d}}(\rho, \theta, \cdot) \subset [\rho, +\infty)$$

since $g_1(t - \mathrm{i}\rho\sinh\beta) = 0$ for $t \leqslant \rho$ by (18). Hence, (83) implies that

$$u_{\mathrm{d}}(\rho,\theta,t) = \begin{cases} -\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega_0 t}\displaystyle\int_\rho^t \mathrm{e}^{\mathrm{i}\omega_0\tau}\mathrm{w}_{\mathrm{d}}(\rho,\theta,\tau)\,\mathrm{d}\tau, & \rho \leqslant t \\ 0, & \rho > t \end{cases} \Bigg| (\rho,\theta) \in \overline{Q} \tag{85}$$

Substituting (84) in (85), using estimates (33), (16) and the Fubini theorem we obtain for $\theta \neq \theta_k$, $k = 1, 2$

$$u_{\mathrm{d}}(\rho,\theta,t) = \frac{\mathrm{e}^{-\mathrm{i}\omega_0 t}}{4\Phi}\int_{\mathscr{C}_0} H(\beta + \mathrm{i}\theta)\left[\int_\rho^t \mathrm{e}^{\mathrm{i}\omega_0 s} g_1(s - \mathrm{i}\rho\sinh\beta)\,\mathrm{d}s\right]\mathrm{d}\beta, \quad \rho \leqslant t \tag{86}$$

Remark 11.4
The diffracted wave $u_{\mathrm{d}}(\rho,\theta,t)$ vanishes for $\rho > t$. This corresponds to the Huygens principle for the scattering by the wedge.

Substituting (17) in (86), we obtain for $\theta \neq \theta_k$, $k = 1, 2$,

$$u_{\mathrm{d}}(\rho,\theta,t) = \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega_0 t}}{4\Phi}\int_{\mathscr{C}_0} \mathrm{e}^{-\omega_0\rho\sinh\beta} H(\beta + \mathrm{i}\theta)\left[\int_\rho^t f'(s - \mathrm{i}\rho\sinh\beta)\,\mathrm{d}s\right]\mathrm{d}\beta, \quad \rho \leqslant t$$

Hence, by (2)

$$u_{\mathrm{d}}(\rho,\theta,t) = \mathrm{i}\frac{\mathrm{e}^{-\mathrm{i}\omega_0 t}}{4\Phi}\int_{\mathscr{C}_0} \mathrm{e}^{-\omega_0\rho\sinh\beta} H(\beta + \mathrm{i}\theta) f(t - \mathrm{i}\rho\sinh\beta)\,\mathrm{d}\beta, \quad \rho \leqslant t \tag{87}$$

Therefore, representation (78) follows from (87), (85) and Remark 11.3. Theorem is proved. $\square$

Let us denote

$$Z(\beta,\theta) := -H(-\mathrm{i}\pi/2 + \beta + \mathrm{i}\theta) + H(-5\mathrm{i}\pi/2 + \beta + \mathrm{i}\theta) \tag{88}$$

and

$$h(\beta,\rho,t) := f(t - \rho\cosh\beta)\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}, \quad \beta \in \mathbb{R} \tag{89}$$

Corollary 11.5
The function $u_{\mathrm{d}}(\rho,\theta,t)$ admits the following representation for $\theta \neq \theta_1, \theta_2$:

$$u_{\mathrm{d}}(\rho,\theta,t) = \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega_0 t}}{4\Phi}\int_{\mathbb{R}} Z(\beta,\theta) h(\beta,\rho,t)\,\mathrm{d}\beta \tag{90}$$

Proof
Note, that $\sinh\beta = -\mathrm{i}\cosh(\mathrm{Re}\,\beta)$ for $\beta \in \mathscr{C}_0$. Making the change of variable $\mathrm{Re}\,\beta \to \beta$, we obtain from (87), representation (90) for $\theta \neq \theta_k$, $k = 1, 2$. The corollary is proved. $\square$

12. ESTIMATES FOR DIFFRACTED WAVE

In this section, we start to prove estimates (19) for $u_{\mathrm{d}}(\rho, \theta, t)$. In Sections 12 and 13, we prove that u_{d} is bounded. First, we prove that the function u_{d} is bounded beyond the critical directions. Let us choose $\delta > 0$ sufficiently small.

Lemma 12.1
The function $u_{\mathrm{d}}(\rho, \theta, t)$ satisfies the following estimate:

$$|u_{\mathrm{d}}(\rho, \theta, t)| \leqslant C_\delta, \quad t \geqslant 0 \tag{91}$$

for $|\theta_k - \theta| \geqslant \delta > 0, \quad k = 1, 2$.

Proof
Estimate (33) and (88) imply that the function $Z(\beta, \theta)$ satisfies the estimate of type (33) for $\beta \in \mathbb{R}$ and $|\theta - \theta_k| \geqslant \delta$, $k = 1, 2$. Hence, estimate (91) follows from definition (2) of the profile function f, (89) and (90). □

Next, let us prove (91) for θ close to θ_1 or θ_2.

Theorem 12.2
For $k = 1, 2$ the function $u_{\mathrm{d}}(\rho, \theta, t)$ satisfies estimate (91) for $|\theta - \theta_k| < \delta$ with some $\delta > 0$.

Proof
Let us consider the case when

$$|\theta - \theta_2| < \delta \tag{92}$$

for some $\delta > 0$. The case $|\theta - \theta_1| < \delta$ is analysed similarly. The second term on the right-hand side of (88) has not a pole for $\beta \in \mathbb{R}$ and θ satisfying (92). Hence, (33) implies that

$$|H(-5\mathrm{i}\pi/2 + \beta + \mathrm{i}\theta)| \leqslant C(\delta) \mathrm{e}^{-(\pi/\Phi)|\mathrm{Re}\,\beta|}, \quad \beta \in \mathbb{R} \tag{93}$$

Representation (90) and (88) imply that

$$u_{\mathrm{d}} = \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega_0 t}}{4\Phi}(-v_{\mathrm{d},1} + v_{\mathrm{d},2})$$

where

$$v_{\mathrm{d},1}(\rho, \theta, t) = \int_{\mathbb{R}} H(-\mathrm{i}\pi/2 + \beta + \mathrm{i}\theta) h(\beta, \rho, t)\, \mathrm{d}\beta \tag{94}$$

$$v_{\mathrm{d},2}(\rho, \theta, t) = \int_{\mathbb{R}} H(-5\mathrm{i}\pi/2 + \beta + \mathrm{i}\theta) h(\beta, \rho, t)\, \mathrm{d}\beta \tag{95}$$

It suffices to prove estimate (91) for the function $v_{\mathrm{d},1}$, since the estimate for $v_{\mathrm{d},2}$ follows from (93) and (89).

Let us split integral (94) into two integrals: one for $|\beta| \geqslant 1$ and second for $|\beta| \leqslant 1$. Estimate (91) for the function

$$v''_{\mathrm{d},1}(\rho, \theta, t) := \int_{|\beta| \geqslant 1} H(-\mathrm{i}\pi/2 + \beta + \mathrm{i}\theta) h(\beta, \rho, t)\, \mathrm{d}\beta \tag{96}$$

follows from the estimate of the function $h(\beta,\rho,\theta)$ by (33) since $|H(-\mathrm{i}\pi/2+\beta+\mathrm{i}\theta| \leqslant C_1 \mathrm{e}^{-(\pi/\Phi)\mathrm{Re}\,\beta}$ for $|\beta|\geqslant 1$. It remains to prove (91) for the function

$$v'_{\mathrm{d},1}(\rho,\theta,t) := \int_{|\beta|\leqslant 1} H(-\mathrm{i}\pi/2+\beta+\mathrm{i}\theta) h(\beta,\rho,t)\,\mathrm{d}\beta \tag{97}$$

By definition (20) we have

$$v'_{\mathrm{d},1}(\rho,\theta,t) = -v_m(\rho,\theta,t)+v_b(\rho,\theta,t)$$

where

$$\begin{aligned} v_b(\rho,\theta,t) &= \int_{|\beta|\leqslant 1} \coth q(\beta+\mathrm{i}\theta-\mathrm{i}\alpha) h(\beta,\rho,t)\,\mathrm{d}\beta \\ v_m(\rho,\theta,t) &= \int_{|\beta|\leqslant 1} \coth[q(-2\pi\mathrm{i}+\beta+\mathrm{i}\theta+\mathrm{i}\alpha)] h(\beta,\rho,t)\,\mathrm{d}\beta \end{aligned} \tag{98}$$

Condition (92) implies that

$$\theta = \theta_2 + \varepsilon$$

where $|\varepsilon| < \delta$. Substituting in representations (98) we obtain

$$\begin{aligned} \tilde{v}_b(\rho,\varepsilon,t) &:= v_b(\rho,\theta_2+\varepsilon,t) = \int_{|\beta|\leqslant 1} \coth[q(\beta+\mathrm{i}\varepsilon+2\pi\mathrm{i}-2\mathrm{i}\alpha)] h(\beta,\rho,t)\,\mathrm{d}\beta \\ \tilde{v}_m(\rho,\varepsilon,t) &:= v_m(\rho,\theta_2+\varepsilon,t) = \int_{|\beta|\leqslant 1} \coth[q(\beta+\mathrm{i}\varepsilon)] h(\beta,\rho,t)\,\mathrm{d}\beta \end{aligned} \tag{99}$$

Let us note that the function $\tilde{v}_b(\rho,\varepsilon,t)$ is regular in the point $\varepsilon=0$ since $\coth[q(\beta+2\pi\mathrm{i}-2\mathrm{i}\alpha+\mathrm{i}\varepsilon)]$ does not have a pole for $|\beta|\leqslant 1$, if $|\varepsilon|\leqslant\delta=1/2$. It follows since $\alpha<\pi/2$ by (4). Therefore estimate (91) for the function $\tilde{v}_b$ follows since function $h(\beta,\rho,t)$ is bounded.

In contrast, the function $\tilde{v}_m(\rho,\varepsilon,t)$ is singular at the point $\varepsilon=0$. Nevertheless we will show that estimate (91) holds for this function too.

13. SINGULAR PART OF THE DIFFRACTED WAVE

First we bound function (99) in the simple case when the profile function $f(s)\equiv\Theta(s)$. Then (89) implies that

$$\tilde{v}_m(\rho,\varepsilon,t) = \int_{|\beta|\leqslant\overline{\beta}_0} \coth[q(\beta+\mathrm{i}\varepsilon)] \mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}\,\mathrm{d}\beta$$

where

$$\overline{\beta}_0 = \min\{1,\beta_0\} \tag{100}$$

and $\beta_0=\beta_0(\rho,t)\geqslant 0$ is defined by

$$t-\rho\cosh\beta_0 = 0 \tag{101}$$

Lemma 13.1
The function $\tilde{v}_m$ satisfies the following estimate:

$$|\tilde{v}_m(\rho,\varepsilon,t)| \leqslant C, \quad t \geqslant \rho \geqslant 0, \ \varepsilon \neq 0$$

Proof
By (100) it suffices to note that for any $a \in [0,1]$ the integral

$$u_1(\rho,\varepsilon,a) := \int_{|\beta| \leqslant a} \frac{e^{i\omega_0 \rho \cosh \beta}}{(\beta + i\varepsilon)} \, d\beta$$

is uniformly bounded with respect to its arguments (see Proposition A.1 in the appendix). The lemma is proved. □

Now we bound the function (99) for a general profile function f.

Theorem 13.2
The function $\tilde{v}_m(\rho,\varepsilon,t)$ satisfies the following estimate:

$$|\tilde{v}_m(\rho,\varepsilon,t)| \leqslant C, \quad \rho \geqslant 0, \ \varepsilon \neq 0 \tag{102}$$

Proof
It suffices to bound the function

$$r(\rho,\varepsilon,t) = \int_{|\beta| \leqslant 1} \frac{h(\beta,\rho,t)}{\beta + i\varepsilon} \, d\beta \tag{103}$$

with h defined in (89). We give the proof in Lemmas 13.3–13.5. By (89), it suffices to check (102) in the following three regions:

$$\mathscr{R}_1 := \{(\rho,t) : 0 \leqslant \rho \leqslant 5\tau_0, \ \rho \leqslant t\}$$
$$\mathscr{R}_2 := \{(\rho,t) : \rho \geqslant 5\tau_0, \ t - 2\tau_0 \leqslant \rho \leqslant t\}$$
$$\mathscr{R}_3 := \{(\rho,t) : 5\tau_0 \leqslant \rho \leqslant t - 2\tau_0\}$$

where τ_0 is the same as in (2).

First, we prove the uniform estimate of this function in $\mathscr{R}_1$.

Lemma 13.3
The function $r(\cdot,\varepsilon,\cdot)$ is bounded in $\mathscr{R}_1$ uniformly in $\varepsilon \neq 0$:

$$|r(\rho,\varepsilon,t)| \leqslant C, \quad (\rho,t) \in \mathscr{R}_1, \ \varepsilon \neq 0$$

Proof
Represent r in the form:

$$r(\rho,\varepsilon,t) = \int_{|\beta| \leqslant 1} \frac{h(\beta,\rho,t) - h(0,t,\rho)}{\beta + i\varepsilon} \, d\beta + h(0,t,\rho) \int_{|\beta| \leqslant 1} \frac{1}{\beta + i\varepsilon} \, d\beta$$

The second summand is bounded, because $h(0,t,\rho)$ is bounded by (89) and the integral

$$\left|\int_{|\beta|\leqslant 1} \frac{1}{\beta+\mathrm{i}\varepsilon}\,\mathrm{d}\beta\right| = |\arg(\beta+\mathrm{i}\varepsilon)|_{-1}^{1}|\leqslant\pi,\quad \varepsilon\neq 0$$

Let us consider the first summand. We have

$$h(\beta,\rho,t)-h(0,t,\rho)=h'_{\beta}(\tilde{\beta},t,\rho)\beta$$

where $|\tilde{\beta}|\leqslant 1$. Calculating the derivative of h, we get that there exists $C>0$:

$$|h(\beta,\rho,t)-h(0,t,\rho)|\leqslant C\rho,\quad 0\leqslant\rho\leqslant t,\ \ |\beta|\leqslant 1 \tag{104}$$

It implies the estimate

$$\left|\int_{|\beta|\leqslant 1} \frac{h(\beta,\rho,t)-h(0,t,\rho)}{\beta+\mathrm{i}\varepsilon}\,\mathrm{d}\beta\right| \leqslant C\rho\int_{|\beta|\leqslant 1}\frac{|\beta|}{|\beta+\mathrm{i}\varepsilon|}\,\mathrm{d}\beta\leqslant C_1$$

since $\rho\leqslant 5\tau_0$. □

Now we prove the estimate $r(\rho,\varepsilon,t)$ in $\mathscr{R}_2$ near the characteristic of the wave equation.

Lemma 13.4
The function $r(\cdot,\varepsilon,\cdot)$ is bounded in $\mathscr{R}_2$ uniformly in $\varepsilon\neq 0$:

$$|r(\rho,\varepsilon,t)|\leqslant C,\quad (\rho,t)\in\mathscr{R}_2,\ \ \varepsilon\neq 0 \tag{105}$$

Proof
We represent the function r as

$$r(\rho,\varepsilon,t):=r_1(\rho,\theta,t)+r_2(\rho,\varepsilon,t)$$

where

$$r_1(\rho,\varepsilon,t):=\int_{|\beta|\leqslant 1}\frac{\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}}{\beta+\mathrm{i}\varepsilon}\bar{f}(\beta,\rho,t)\,\mathrm{d}\beta \tag{106}$$

with

$$\bar{f}(\beta,\rho,t):=f(t-\rho\cosh\beta)-f(t-\rho)$$

and

$$r_2(\rho,\theta,t):=f(t-\rho)\int_{|\beta|\leqslant 1}\frac{\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}}{\beta+\mathrm{i}\varepsilon}\,\mathrm{d}\beta$$

Estimate (105) for the function r_2 follows from Proposition A.1 of the appendix, since the function $f(t-\rho)$ is bounded. It remains to prove estimate (105) for the function r_1.

By (2) the function $f(t-\rho\cosh\beta)$ is equal to 0 for $\cosh\beta\geqslant t/\rho$. Since $t/\rho\leqslant 1+2\tau_0/\rho$ for $(\rho,t)\in\mathscr{R}_2$, we have $f(t-\rho\cosh\beta)=0$ if $\cosh\beta\geqslant 1+2\tau_0/\rho$. Therefore we can change the interval of integration in (106) by $[-\beta_2, \beta_2]$, where $\beta_2 > 0$, and $\cosh\beta_2 = 1 + (2\tau_0/\rho)$, since $\beta_2 < 1$ for $\rho \geqslant 5\tau_0$. Thus, the function r_1 is represented in the form

$$r_1(\rho,\varepsilon,t) := \int_{|\beta|\leqslant\beta_2} \frac{e^{i\omega_0\rho\cosh\beta}}{\beta+i\varepsilon}\bar{f}(\beta,t,\rho)\,d\beta \tag{107}$$

Now we bound the integrand in (107). Using the Lagrange theorem we write

$$\left|\frac{e^{i\omega_0\rho\cosh\beta}}{\beta+i\varepsilon}\bar{f}(\rho,\beta,t)\right| \leqslant C\rho\frac{|\beta|}{|(\beta+i\varepsilon)|}|\sinh\tilde{\beta}| \tag{108}$$

where $|\tilde{\beta}|\leqslant\beta_2$. Note that $1+(2\tau_0/\rho) = \cosh\beta_2 \sim 1+(\beta_2^2/2)$, hence $\beta_2 \sim 1/\sqrt{\rho}$, since $\rho\geqslant 5\tau_0 > 0$. Therefore,

$$\sinh\tilde{\beta} \leqslant \sinh\beta_2 \sim \beta_2 \sim \frac{1}{\sqrt{\rho}}$$

Now, (108) implies that

$$\left|\frac{e^{i\omega_0\rho\cosh\beta}}{\beta+i\varepsilon}\bar{f}(\rho,\beta,t)\right| \leqslant C_1\sqrt{\rho}, \quad \rho\geqslant 5\tau_0$$

Therefore, (107) implies that

$$|r_1(\rho,\varepsilon,t)| \leqslant C\sqrt{\rho}\beta_2 \leqslant C_1 \quad \rho\geqslant 5\tau_0$$ □

It remains to prove estimate (105) in the region $\mathscr{R}_3$ beyond the characteristics. By (2) and (101) we have that

$$f(t-\rho\cosh\beta) = \begin{cases} 0, & \cosh\beta \geqslant \cosh\beta_0 \\ 1, & \cosh\beta \leqslant \dfrac{t}{\rho} - \dfrac{\tau_0}{\rho} \end{cases} \tag{109}$$

Equation

$$\cosh\beta_3 = \frac{t}{\rho} - \frac{\tau_0}{\rho} \tag{110}$$

admits a solution $\beta_3 \geqslant 0$, since $t/\rho - \tau_0/\rho \geqslant 1$ in $\mathscr{R}_3$. Then (109) implies that

$$f(t-\rho\cosh\beta) = 1 \quad \text{for } |\beta|\leqslant\beta_3 \tag{111}$$

Let us denote

$$f_1(t-\rho\cosh\beta) = \begin{cases} 0, & \beta \geqslant \overline{\beta}_0 \\ f(t-\rho\cosh\beta), & \beta\leqslant\overline{\beta}_0 \end{cases}$$

where $\overline{\beta}_0$ is defined by (100). Then we can rewrite (103) in the form:

$$r(\rho,\varepsilon,t) = \int_{|\beta|\leqslant\overline{\beta}_0} f_1(\beta,t,\rho)\frac{e^{i\omega_0\rho\cosh\beta}}{\beta+i\varepsilon}\,d\beta, \quad \rho\leqslant t \tag{112}$$

Lemma 13.5
The function $r(\cdot,\varepsilon,\cdot)$ is bounded in the region $\mathscr{R}_3$ uniformly in $\varepsilon\neq 0$:

$$|r(\rho,\varepsilon,t)|\leqslant C, \quad 5\tau_0\leqslant\rho\leqslant t-2\tau_0, \quad \varepsilon\neq 0 \tag{113}$$

Proof
Let us denote

$$\overline{\beta}_3:=\min\{1,\beta_3\}, \quad \rho\leqslant t-\tau_0 \tag{114}$$

Then for $\rho\leqslant t-2\tau_0$ we can rewrite (112) in the form:

$$r(\rho,\varepsilon,t)=\int_{|\beta|\leqslant\overline{\beta}_3}\frac{\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}}{\beta+\mathrm{i}\varepsilon}\,\mathrm{d}\beta+\int_{\overline{\beta}_3\leqslant|\beta|\leqslant\overline{\beta}_0} f_1(\beta,t,\rho)\frac{\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}}{\beta+\mathrm{i}\varepsilon}\,\mathrm{d}\beta$$

The first integral is bounded by Proposition A.1 of the appendix and (114):

$$\left|\int_{|\beta|\leqslant\overline{\beta}_3}\frac{\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}}{\beta+\mathrm{i}\varepsilon}\,\mathrm{d}\beta\right|\leqslant C, \quad \rho\geqslant 0, \quad \varepsilon\neq 0$$

For the second integral we prove that

$$|A(\rho,\varepsilon,t)|:=\left|\int_{\overline{\beta}_3\leqslant|\beta|\leqslant\overline{\beta}_0} f_1(\beta,t,\rho)\frac{\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}}{\beta+\mathrm{i}\varepsilon}\,\mathrm{d}\beta\right|\leqslant C, \quad 5\tau_0\leqslant\rho\leqslant t-2\tau_0 \tag{115}$$

Let us consider two cases.

I. Let $\overline{\beta}_3\geqslant\frac{1}{2}$. Then estimate (115) holds, since $|\beta+\mathrm{i}\varepsilon|\geqslant 1/2$, for $|\beta|\geqslant\overline{\beta}_3$ and integral (115) is bounded by a constant in view of (109).
II. Let $\overline{\beta}_3(\rho,t)\leqslant\frac{1}{2}$. Then $\beta_3\leqslant\frac{1}{2}$ by (114). Hence $\cosh\beta_3\leqslant\cosh\frac{1}{2}$ and $t/\rho\leqslant\tau_0/\rho+\cosh\frac{1}{2}$ by (110). On the other hand, $\rho\geqslant 5\tau_0$. Hence $\tau_0/\rho\leqslant\frac{1}{5}\leqslant\cosh 1-\cosh\frac{1}{2}$ and $\tau_0\rho+\cosh\frac{1}{2}\leqslant\cosh 1$. Therefore $t/\rho\leqslant\cosh 1$ and $\beta_0\leqslant 1$ by (101). Now, let us check that

$$\frac{\beta_0}{\beta_3}\leqslant 2 \tag{116}$$

Namely, (116) is equivalent to $\cosh\beta_0\leqslant 2\cosh\beta_3^2-1$. Definitions (101), (110) of β_0, β_3 imply that the last inequality is equivalent to $\rho(t+\rho)\leqslant 2(t-\tau_0)^2$, which holds by hypothesis (113). Therefore, (116) is proved. Now (115), follows since (116) implies that

$$\begin{aligned}|A(\rho,\varepsilon,t)| &\leqslant \int_{\beta_3\leqslant|\beta|\leqslant\beta_0}\left|\frac{\mathrm{d}\beta}{\beta+\mathrm{i}\varepsilon}\right|\leqslant\int_{\beta_3\leqslant|\beta|\leqslant\beta_0}\left|\frac{\mathrm{d}\beta}{\beta_3}\right| \\ &\leqslant\frac{2}{\beta_3}|\beta_0-\beta_3|\leqslant 2\left(1+\left|\frac{\beta_0}{\beta_3}\right|\right)\leqslant 6\end{aligned} \tag{117}$$

Here we used that $\beta_3=\overline{\beta}_3$, since $\overline{\beta}_3\leqslant\frac{1}{2}$ and $\overline{\beta}_0=\beta_0$ since $\beta_0<1$. Theorem 13.2 is proved. □

Thus, we have proved estimate (19) for u_{d}. In the following section we prove the estimate for ∇u_{d}.

14. DERIVATIVES OF THE DIFFRACTED WAVE

In this section, we finish the proof of Theorem 10.4. Namely, we prove estimates (19) for ∇u_{d}.

14.1. Beyond the critical directions

In this subsection, we prove that the derivatives of diffracted wave (90) satisfy estimate (19) beyond the critical directions. In the next subsection, we prove that the derivatives of diffracted wave (90) satisfy estimate (19) near the critical directions.

Proposition 14.1
Let θ satisfies the condition of Lemma 12.1 for some $\delta > 0$. Then the function $u_{\mathrm{d}}(\rho, \theta, t)$ satisfies the following estimate:

$$|\nabla u_{\mathrm{d}}(\rho, \theta, t)| \leqslant C_\delta (1 + t^\varepsilon)(1 + \rho^{-\varepsilon}), \quad 0 < \rho < t \leqslant 0 \tag{118}$$

where

$$\varepsilon = 1 - \frac{\pi}{\Phi}$$

Proof
(i) First, we check (118) for the radial derivative $(\partial/\partial\rho) u_{\mathrm{d}}(\rho, \theta, t)$. Then we prove this estimate for the angular derivative. To prove this estimate for the radial derivative we consider the radial derivative of the function $v_{\mathrm{d},1}$ from (94) only, since the estimate for the second summand from (95) is proved similarly. Differentiating this function with respect to ρ and using (89), we get

$$\begin{aligned}\frac{\partial}{\partial\rho} v_{\mathrm{d},1}(\rho, \theta, t) = \Bigg\{ \int_{\mathbb{R}} & \mathrm{e}^{\mathrm{i}\omega_0 \rho \cosh\beta} H(-\mathrm{i}\pi/2 + \beta + \mathrm{i}\theta) \cosh\beta [\mathrm{i}\omega_0 f(t - \rho\cosh\beta) \\ & - f'(t - \rho\cosh\beta)]\, \mathrm{d}\beta \Bigg\}\end{aligned} \tag{119}$$

The integrand in (119) vanished outside the interval $[-\beta_0, \beta_0]$ according to (2) and definition (101). Therefore, (33) implies the following estimate for (119):

$$\left| \frac{\partial}{\partial\rho} v_{\mathrm{d},1}(\rho, \theta, t) \right| \leqslant C_\delta \int_0^{\beta_0} \mathrm{e}^{\beta(1-(\pi/\Phi))}\, \mathrm{d}\beta \tag{120}$$

Definition (101) gives that

$$\beta_0 = \ln\left(\frac{t + \sqrt{t^2 - \rho^2}}{\rho} \right)$$

Hence, (120) implies the following estimate:

$$\left| \frac{\partial}{\partial\rho} v_{\mathrm{d},1}(\rho, \theta, t) \right| \leqslant C \left[\left(\frac{t + \sqrt{t^2 - \rho^2}}{\rho} \right)^{\varepsilon} - 1 \right], \quad \rho \leqslant t \tag{121}$$

The estimate $(t+\sqrt{t^2-\rho^2})\leqslant 2(1+t)$ implies the estimate $(t+\sqrt{t^2-\rho^2})^{\varepsilon}\leqslant C(1+t^{\varepsilon})$. Hence and from (121) we obtain

$$\left|\frac{\partial}{\partial\rho}v_{\mathrm{d},1}(\rho,\theta,t)\right|\leqslant C[(1+t^{\varepsilon})(1+\rho^{-\varepsilon})+(1+\rho^{-\varepsilon})],\quad \rho\leqslant t \tag{122}$$

Now, estimate (118) follows for $C_{\delta}:=2C$. The statement (i) is proved.

(ii) Let us prove estimate (118) for the angular derivative of the function u_{d}. As in the proof (i) it suffices to prove this estimate for $(1/\rho)v_{\mathrm{d},1}(\rho,\theta,t)$. Differentiating function (94) with respect to θ, noting, that

$$\frac{\partial}{\partial\theta}H(-\mathrm{i}\pi/2+\beta+\mathrm{i}\theta)=(-\mathrm{i})\frac{\partial}{\partial\beta}H(-\mathrm{i}\pi/2+\beta+\mathrm{i}\theta)\,\mathrm{d}\beta$$

we obtain:

$$\frac{1}{\rho}\frac{\partial}{\partial\theta}v_{\mathrm{d},1}(\rho,\theta,t)=-\frac{\mathrm{i}}{\rho}\left[\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}\frac{\partial}{\partial\beta}H(-\mathrm{i}\pi/2+\beta+\mathrm{i}\theta)f(t-\rho\cosh\beta)\,\mathrm{d}\beta\right] \tag{123}$$

Integrating by parts in (123), and using that the function $f(t-\rho\cosh\beta)\in C_0^{\infty}(\mathbb{R})$ we obtain

$$\begin{aligned}\frac{1}{\rho}\frac{\partial}{\partial\theta}v_{\mathrm{d},1}(\rho,\theta,t)=\mathrm{i}\int_{\mathbb{R}}&\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}H(-\mathrm{i}\pi/2+\beta+\mathrm{i}\theta)\sinh\beta[\mathrm{i}\omega_0 f(t-\rho\cosh\beta)\\&-f'(t-\rho\cosh\beta)]\,\mathrm{d}\beta\end{aligned} \tag{124}$$

Thus, we have obtained the expression similar to (119) and applying the same arguments as in the proof of (i) we obtain estimate (118) for the derivative $(1/\rho)(\partial/\partial\theta)v_{\mathrm{d},1}(\rho,\theta,t)$. The proposition is proved. □

14.2. Near the critical directions

In this section, we obtain estimate (118) for θ close to θ_1 and θ_2.

Proposition 14.2
Estimate (118) holds for $|\theta-\theta_k|<\delta$, $k=1,2$ for sufficiently small $\delta>0$.

Proof
First, we prove estimate (118) for the radial derivative. We suppose that θ satisfies (92). The case when θ is closed to θ_1 is considered similarly. As in Proposition 14.1, we check this estimate for $\partial v_{\mathrm{d},1}/\partial\rho$ from (119). This function is the sum of the functions $w'_{\mathrm{d},1}$ and $w''_{\mathrm{d},2}$, where

$$\begin{aligned}w'_{\mathrm{d},1}(\rho,\theta,t):=\int_{|\beta|\leqslant 1}&\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}H(-\mathrm{i}\pi/2+\beta+\mathrm{i}\theta)\cosh\beta[\mathrm{i}\omega_0 f(t-\rho\cosh\beta)\\&-f'(t-\rho\cosh\beta)]\,\mathrm{d}\beta\end{aligned} \tag{125}$$

and

$$\begin{aligned}w''_{\mathrm{d},1}(\rho,\theta,t):=\int_{|\beta|\geqslant 1}&\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}H(-\mathrm{i}\pi/2+\beta+\mathrm{i}\theta)\cosh\beta[\mathrm{i}\omega_0 f(t-\rho\cosh\beta)\\&-f'(t-\rho\cosh\beta)]\,\mathrm{d}\beta\end{aligned} \tag{126}$$

The functions $w'_{\mathrm{d},1}$ and $w''_{\mathrm{d},2}$ are similar to the functions $v'_{\mathrm{d},1}$ and $v''_{\mathrm{d},1}$ from (97) and (96). The difference is that the function h from (89) is changed by the function

$$g(\beta,\rho,t) := \mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}\cosh\beta[\mathrm{i}\omega_0 f(t-\rho\cosh\beta) - f'(t-\rho\cosh\beta)], \quad \beta\in\mathbb{R}, \ \rho\leqslant t \tag{127}$$

The function $w''_{\mathrm{d},1}$ satisfies estimate (118). In effect, it is obtained from the estimate of type (120) for the function $(\partial/\partial\rho)v_{\mathrm{d},1}(\rho,\theta,t)$ from (119), where instead of the integrating over $[0,\beta_0]$ it is necessary to integrate over $[1,\beta_0]$.

Let us prove the estimate for $w'_{\mathrm{d},1}$. Similarly to (97)–(99) we reduce this problem to the estimate of the function

$$\tilde{w}_m(\rho,\varepsilon,t) := \int_{|\beta|\leqslant 1} \coth[q(\beta+\mathrm{i}\varepsilon)]g(\beta,t,\rho)\,\mathrm{d}\beta$$

Similarly to Lemma 13.1, we reduce the problem to the function

$$b(t,\rho,\varepsilon) := \int_{|\beta|\leqslant 1} \frac{g(\beta,t,\rho)}{\beta+\mathrm{i}\varepsilon}\,\mathrm{d}\beta$$

This function is similar to the function $r(\rho,\varepsilon,t)$ from (103). The difference is that the function h in (103) is changed by the function g from (127).

Proposition 14.3

$$|b(\rho,\varepsilon,t)|\leqslant C, \quad \varepsilon\neq 0, \ 0\leqslant\rho\leqslant t$$

Proof

The proof is similar to the proof of Theorem 13.2. We analyse Lemmas 13.3–13.5. The proof of Lemma 13.3 serves for the this case since g admits the estimate of type (104):

$$|g(\beta,\rho,t) - g(0,\rho,t)|\leqslant C\rho, \quad 0\leqslant\rho\leqslant t, \ |\beta|\leqslant 1$$

The proof of Lemma 13.4 also serves since the function $\bar{g} := g(t-\rho\cosh\beta) - g(t-\rho)$ admits estimate (108). Finally, since the function g is bounded, then the proof of Lemma 13.5 for $b(\rho,\varepsilon,t)$ is reduced to the estimate of the function $A(\rho,\varepsilon,t)$ from (117). Thus, the proposition and Proposition 14.2 are proved for the radial derivative.

Now, we prove the estimate for the angular derivative $(1/\rho)(\partial/\partial\theta)v_{\mathrm{d},1}(\rho,\theta,t)$. Let us consider expression (124). Similarly to the case of the radial derivative, we represent this expression as the sum of the integral of types (125) and (126). The integral over $|\beta|\geqslant 1$ is bounded similarly to the function $w''_{\mathrm{d}}(\rho,\theta,t)$. The estimate of the same integral over $|\beta|\leqslant 1$ is trivial, since $|\sinh\beta/(\beta+\mathrm{i}\varepsilon)|\leqslant C$, $|\beta|\leqslant 1$. □

Thus, we have proved completely estimate (118). Together with the results of Sections 12 and 13, it implies estimate (19) for the function u_{s}. Theorems 10.4 and 2.4(i) are proved. □

In next section we finish the proof of Theorem 2.4.

15. LIMITING AMPLITUDE PRINCIPLE

Here we prove the Limiting Amplitude principle (26), i.e.

$$u(\rho,\theta,t) \sim A(\rho,\theta)\mathrm{e}^{-\mathrm{i}\omega_0 t}, \quad t\to\infty \tag{128}$$

Let us consider the function

$$u(y,t) = u_{\text{in}}(y,t) + u_{\text{r}}(y,t) + u_{\text{d}}(y,t) \tag{129}$$

which is a solution of nonstationary problem (5), (6) given by (23), with u_{in}, u_{r}, u_{d} given by (1), (9)–(11) and (90), respectively. Equations (1), (9)–(11) imply that for any $\rho_0 > 0$

$$u_{\text{in}} = A_{\text{in}}(\rho,\theta)\text{e}^{-\text{i}\omega_0 t}, \quad u_{\text{r}} = A_{\text{r}}(\rho,\theta)\text{e}^{-\text{i}\omega_0 t}, \quad t \geqslant \tau_0 + \rho_0, \quad \rho \leqslant \rho_0 \tag{130}$$

where

$$\begin{aligned} A_{\text{in}}(\rho,\theta) &:= \text{e}^{\text{i}\omega_0\rho\cos(\theta-\alpha)} \\ A_{\text{r}}(\rho,\theta) &= \begin{cases} \text{e}^{\text{i}\omega_0\rho\cos(\theta-\theta_1)}, & \phi \leqslant \theta \leqslant \theta_1 \\ 0, & \theta_1 < \theta < \theta_2 \\ \text{e}^{\text{i}\omega_0\rho\cos(\theta-\theta_2)}, & \theta_2 \leqslant \theta \leqslant 2\pi \end{cases} \end{aligned} \tag{131}$$

and $\theta_{1,2}$ are defined by (10). It remains to check the asymptotics of type (128) for the diffracted wave $u_{\text{d}}(\rho,\theta,t)$ given by (90), with the corresponding limiting amplitude

$$A_{\text{d}}(\rho,\theta) := \frac{\text{i}}{4\Phi}\int_{\mathbb{R}} \text{e}^{\text{i}\omega_0\rho\cosh\beta} Z(\beta,\theta)\,\text{d}\beta, \quad (\rho,\theta)\in Q \tag{132}$$

where Z is defined by (88).

Theorem 15.1
For any $\rho_0 > 0$ the following asymptotics hold:

$$u_{\text{d}}(\rho,\theta,t) - A_{\text{d}}(\rho,\theta)\text{e}^{-\text{i}\omega_0 t} \to 0, \quad t \to \infty$$

uniformly in $\rho \in [0,\rho_0]$ and $\theta \in [\phi, 2\pi]$.

Proof
Equation (90) implies that

$$A_{\text{d}}(\rho,\theta,t) := \text{e}^{\text{i}\omega_0 t} u_{\text{d}}(\rho,\theta,t) = \frac{\text{i}}{4\Phi}\int_{\mathbb{R}} Z(\beta,\theta) h(\beta,\rho,\theta)\,\text{d}\beta \tag{133}$$

It remains to prove that

$$A_{\text{d}}(\rho,\theta,t) \to A_{\text{d}}(\rho,\theta), \quad t \to \infty$$

uniformly with the respect to $\rho \leqslant \rho_0$. Formulas (132) and (133) imply that

$$A_{\text{d}}(\rho,\theta,t) - A_{\text{d}}(\rho,\theta) = \frac{\text{i}}{4\Phi}\int_{\mathbb{R}} [Z(\beta,\theta)h(\beta,\rho,t) - Z(\beta,\theta)\text{e}^{\text{i}\omega_0\rho\cosh\beta}]\,\text{d}\beta$$

Let us fix $\rho_0 > 0$ and $\varepsilon > 0$. Let us choose $\overline{\beta} \geqslant 1$ such that

$$\frac{8C_1\Phi\text{e}^{-(\pi/\Phi)|\overline{\beta}|}}{\pi} < \varepsilon$$

where C_1 is the constant from (33). Then by (88), (33) and (2)

$$\int_{|\beta|\geqslant\overline{\beta}} |Z(\beta,\theta)h(\beta,\rho,t) - Z(\beta,\theta)\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}| < 8C_1 \int_{\beta\geqslant\overline{\beta}} \mathrm{e}^{-(\pi/\Phi)\beta}$$

$$\leqslant 8C_1 \frac{\Phi \mathrm{e}^{-(\pi/\Phi)\overline{\beta}}}{\pi} < \varepsilon, \quad t\in\mathbb{R} \tag{134}$$

It remains to prove the converges to zero of the integral over $|\beta|<\overline{\beta}$. First, $\cosh\beta_3(\rho,\theta)=(t-\tau_0)/\rho\geqslant\cosh\overline{\beta}$, for $t\geqslant\tau_0+\rho\cosh\overline{\beta}$, where β_3 is defined by (110). This implies that $f(t-\rho\cos\beta)=1$ for $|\beta|\leqslant\overline{\beta}$ by (111). Hence

$$\int_{|\beta|\leqslant\overline{\beta}} |Z(\beta,\theta)h(\beta,\rho,t) - Z(\beta,\theta)\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}|\,\mathrm{d}\beta = 0 < \varepsilon, \quad t\geqslant\tau_0+\rho_0\cosh\overline{\beta}, \quad \rho\leqslant\rho_0$$

Therefore, (134) implies that

$$\int_{\mathbb{R}} |Z(\beta,\theta)h(\beta,\rho,t) - Z(\beta,\theta)\mathrm{e}^{\mathrm{i}\omega_0\rho\cosh\beta}|\,\mathrm{d}\beta < \varepsilon, \quad t\geqslant\tau_0+\rho_0\cosh\beta, \quad \rho\leqslant\rho_0$$

The theorem is proved. □

Proof of Theorem 2.4
(ii) Using the Cauchy Residue theorem, we split the limiting amplitude $A(\rho,\theta)$, given by (25), into three summands, similarly to the splitting of the function $\mathscr{S}(\rho,\theta,\omega)$ in Lemma 7.2. So, we obtain that

$$A(\rho,\theta) = A_{\mathrm{in}}(\rho,\theta) + A_{\mathrm{r}}(\rho,\theta) + A_{\mathrm{d}}(\rho,\theta)$$

where A_{in}, A_{r}, A_{d} are given by (131) and (132). Here representation (132) for A_{d} is obtained similarly to the derivation from (41) of representation (90) for $\mathscr{S}_{\mathrm{d}}$. Now, statement (ii) of Theorem 2.4 follows from (129), (130), and Theorem 15.1. □

Remark 15.2
Expression (25) coincides with the well-known Sommerfeld–Malyuzhinets formula [7,8,17]. It is known (see, for example Reference [6]) that it is the unique solution of the stationary diffraction problem

$$\begin{cases} (\Delta+\omega_0^2)A(\rho,\theta)=0, & (\rho,\theta)\in Q \\ A|_{\partial Q}=0 \end{cases}$$

satisfying the following conditions:

1. $A\in C(\overline{Q})$, $A\in C^2(Q)$.
2. The Meixner condition holds in some neighbourhood of the vertex:

$$A(\rho,\theta) = O(\rho^{\pi/2\Phi}), \quad \rho\to 0$$

3. The limiting amplitude $A_{\mathrm{d}}(\rho,\theta) := A - A_{\mathrm{in}} - A_{\mathrm{r}}$ of the diffracted wave, satisfies the Sommerfeld radiation condition [6–8,17].

APPENDIX A: OSCILLATORY CAUCHY-TYPE INTEGRALS

Let us consider the following singular Fresnel integrals:

$$\psi_1(z,\varepsilon,a) := \int_{-a}^{a} \frac{\mathrm{e}^{\mathrm{i}zt^2}}{t+\mathrm{i}\varepsilon}\,\mathrm{d}t, \quad \psi_2(z,\varepsilon,a) := \int_{-a}^{a} \frac{\mathrm{e}^{\mathrm{i}z\cosh t}}{t+\mathrm{i}\varepsilon}\,\mathrm{d}t$$

We have used the following well-known estimates.

Proposition A.1 (*Bleistein* [*18*] *and Fedoryuk* [*14*])

(i) The function $\psi_1(z,\varepsilon,a)$ is uniformly bounded:

$$|\psi_1(z,\varepsilon,a)| \leqslant C, \quad z \geqslant 0, \quad 0 \leqslant a \leqslant 1, \quad \varepsilon \neq 0$$

(ii) The function $\psi_2(z,\varepsilon,a)$ is uniformly bounded:

$$|\psi_2(z,\varepsilon,a)| \leqslant C, \quad z \geqslant 0, \quad 0 \leqslant a \leqslant 1, \quad \varepsilon \neq 0$$

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