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To the memory of Yulian Radvogin

On Attraction to Solitons in Relativistic Nonlinear Wave Equations.

A.I.Komech [1,2]
Wolfgang Pauli Institute
c/o Institute of Mathematics, Vienna University
Vienna A-1090, Austria
e-mail: komech@mat.univie.ac.at

N.J. Mauser
Wolfgang Pauli Institute
c/o Institute of Mathematics, Vienna University
Vienna A-1090, Austria
e-mail: mauser@courant.nyu.edu

A.P. Vinnichenko [2]
A.M.Liapunov Institute
of Applied and Computer Mathematics
Moscow State University
Moscow, Russia
e-mail: arkadi_vinn@mtu-net.ru

Abstract We describe our numerical experiments on soliton-type asymptotics of solutions to relativistic nonlinear wave equations.

We propose a unifying conjecture on minimal global attractors and soliton-type asymptotics of the solutions to nonlinear G-invariant wave and Klein-Gordon Eqns, with a general Lie group G, in an infinite space. For the case of relativistic equations, invariant with respect to the Lorentz group, the conjecture reads as follows: *every finite energy solution decays to the sum of a finite combination of solitons and a dispersive wave.*

The asymptotics are inspired by the Bohr Quantum Transitions and de Broglie's Wave-Particle Duality. We discuss the physical motivations and suggestions, list known results corresponding to four simplest Lie groups, and describe our numerical observations for the Lorentz-invariant equations. Mathematical proof of the observed asymptotics is still open problem.

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0 Introduction: Quantum Mechanics and Attractors

Quantum Mechanics inspires the investigation of the attractors of the nonlinear wave equations. We keep in mind the following well known quantum phenomena:

I. Transitions between Quantum Stationary States or "quantum jumps" predicted by N.Bohr in 1913:

$$|E_-\rangle \mapsto |E_+\rangle \tag{0.1}$$

where $|E_\pm\rangle$ stands for Quantum Stationary State with the energy $E_\pm$.
II. Wave-particle duality predicted by L. de Broglie in 1922: diffraction of electrons discovered experimentally by C.Davisson and L.Germer in 1927, etc. by C.Davisson and L.Germer in 1927, etc.
III. Gell-Mann – Ne'eman classification of the elementary particles, [16].

On the other hand, since 1925, basic quantum phenomena are described by partial differential equations, like the Schrödinger, Klein-Gordon, Dirac, Yang-Mills Eqns, etc, for a wave function $\psi(x,t)$, [52]. In particular,

- Schrödinger has identified Quantum Stationary States with the wave functions $\psi(x)e^{i\omega t}$.
- Elementary Particles seem to correspond to the "solitary waves" $\psi(x-vt)e^{i\Phi(x,t)}$.

The identifications suggest the following mathematical conjectures:

I. The transitions (0.1) can be treated mathematically as the long-time asymptotics

$$\psi(x,t) \sim \psi_\pm(x)e^{i\omega_\pm t}, \qquad t \to \pm\infty, \tag{0.2}$$

where the limit wave functions $\psi_\pm(x)e^{i\omega_\pm t}$ correspond to the stationary states $|E_\pm\rangle$.
II. The wave-particle duality can be treated mathematically as the soliton-type asymptotics

$$\psi(x,t) \sim \sum_{k=1}^{N_\pm} \psi_\pm^k(x - v_\pm^k t)e^{i\Phi_\pm^k(x,t)}, \quad t \to \pm\infty. \tag{0.3}$$

The asymptotics (0.2) would mean that the set of all Quantum Stationary States is the *point attractor* of the dynamical equations. The attraction might clarify Schrödinger's identification of the Quantum Stationary States with eigenfunctions. The asymptotics (0.3) claim an inherent mechanism of the "reduction of wave packets" in the Davisson-Germer experiment and clarify the description of the electron beam by plane waves. This description plays the key role in quantum mechanical scattering problems.

In particular, it is instructive to explain the distinguished role of the exponential function $e^{i\omega t}$ and traveling waves appearing in (0.2) and (0.3). We suggest that the role is provided by the symmetry of corresponding dynamical equations: the (global) gauge-invariance w.r.t. the group $U(1)$ for the asymptotics (0.2) and translation-invariance for (0.3). The suggestion is inspired by the fact that the function $e^{i\omega t}$ is a one-parametric subgroup of the corresponding symmetry group $U(1)$, and the traveling waves also correspond to one-parametric subgroup of translations.
III. A natural extension of the asymptotics (0.2) to the equations with a higher symmetry group G would look

$$\psi(\cdot,t) \sim e^{i\Omega_\pm t}\psi_\pm(\cdot), \qquad t \to \pm\infty, \tag{0.4}$$

where $i\Omega_\pm$ is (a representation of) an element of the Lie algebra $\mathbf{G}$ of the group G. For example, $\Omega_\pm$ is an Hermitian $N \times N$ matrix if $G = U(N)$. The matrix $\Omega_\pm$ and the function $\psi_\pm$ give a solution to the corresponding *eigenmatrix stationary problem* (cf. (1.11) below). This correspondence is confirmed by the Gell-Mann – Ne'eman parallelism between the classification of the elementary particles and Lie algebras, [16], since the elementary particles appear to be the quantum stationary states.

Our numerical analysis suggests that the asymptotics (0.3) hold in *local energy seminorms* round the solitons. To state the asymptotics in the *global energy norm*, we have to modify (0.3) adding a *dispersive wave*:

$$\psi(x,t) \sim \sum_{k=1}^{N_\pm} \psi_\pm^k(x - v_\pm^k t) e^{i\Phi_\pm^k(x,t)} + W(t)\psi_\pm, \qquad t \to \pm\infty. \tag{0.5}$$

Here $W(t)$ is the dynamical group of the corresponding linear *free equation*, and $\psi_\pm$ are the *asymptotic scattering states*.

Remark 0.1 *The term $W(t)\psi_\pm$ represents the dispersive wave which brings the energy to infinity. This radiation plays the role of a dissipation in the Hamilton equations. Yulian Radvogin for the first time have analyzed numerically the details of the dispersive wave. The analysis have influenced strongly on the investigations of the soliton asymptotics (0.5). Some part of the Radvogin's results is published in [4].*

We have written three numerical codes in BorlandPascal and Delphi for the numerical simulation of the solutions to the 1D nonlinear relativistic wave and Klein-Gordon equations with polynomial nonlinear interaction. We have performed plenty of numerical experiments with different polynomial nonlinear interactions and initial functions. All the experiments demonstrate the universal character of the asymptotics (0.5) and provide a lot of interesting mathematical details. We discuss the details of two numerical examples represented by fig.1 and fig.2.
Example I. Fig. 1 represents the scattering of the *kink solutions*.
i) The kinks are represented by the yellow "straight-line" strips, with an oscillatory boundary. The velocities of the kinks are shown. The oscillations of the solitons correspond to the eigenfrequency $\omega_1 > 0$ from the discrete spectrum of the linearized equation.
ii) The dispersive wave is represented by the hyperbolic lines outside the kinks. It decays to a discrete set of the wave packets (see fig.1) corresponding to the frequencies $2\omega_1, 3\omega_1, ...$ generated by the nonlinear term.
iii) The frequencies $2\omega_1, 3\omega_1, ...$ are embedded in the continuous spectrum that causes the radiation of the wave packets. The frequency ω_1 does not radiate since ω_1 is outside the continuous spectrum.
iv) The radiation of the dispersive wave causes the local convergence to the kinks. Thus, the global soliton-type asymptotics are provided by the following two reasons:
I. *nonlinear multiplication of the frequencies* and **II.** *linear dispersion.*

This combined mechanism has been formalized in [36] for the proof of the convergence to the solitary manifold of all finite energy solutions of a nonlinear Klein-Gordon equation with the point nonlinear interaction. The key role in the formalization plays the Titchmarsh Convolution Theorem of classical Harmonic Analysis, [24, Thm 4.3.3].

Example II. Fig. 2 represents the *adiabatic effective dynamics* of the solitons in a slowly varying potential. Namely, for the zero potential the solitons move asymptotically along the straight lines, as in fig.1. The solution in fig.2 is close to a soliton which oscillates around the local minimum of the potential.

In Section 1 we introduce the notations and state a general conjecture, and in Section 2 we list known results. We set our discussion in a more general context to illustrate the role of translation invariance for the soliton asymptotics.

In Sections 3 resp. 4 we describe our numerical observations of the soliton-type asymptotics for the Lorentz-invariant 1D nonlinear wave and Klein-Gordon equations. We relate the details of the observations to the mathematical properties of the nonlinear equations. In Section 5 we describe our numerical observations of the adiabatic effective dynamics of the solitons in slowly varying potentials.

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1 Nonlinear Wave Equations

1.1 Minimal Global Point Attractors

We discuss *minimal global point attractors* for Hamilton nonlinear wave equations in the entire space $\mathbb{R}^d$, $d \geq 1$. For example, consider nonlinear Klein-Gordon equations of type

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m_0^2\psi(x,t) + f(x,\psi(x,t)), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where $m_0 \geq 0$, $f(x,0) \equiv 0$, $\nabla_\psi f(x,0) \equiv 0$ and $\psi \in \mathbb{R}^N$, $N \geq 1$. The case $\psi \in \mathbb{R}^n$ corresponds to $N = 2n$. We assume that

$$f(x,\psi) = -\nabla_\psi U(x,\psi), \qquad \psi \in \mathbb{R}^N. \tag{1.2}$$

Then the equation is formally a Hamilton system with the Hamilton functional

$$\mathcal{H}(\psi,\pi) = \int \Big(\frac{|\pi(x)|^2}{2} + \frac{|\nabla\psi(x)|^2}{2} + m_0^2\frac{|\psi(x)|^2}{2} + U(x,\psi(x))\Big)dx. \tag{1.3}$$

We can write the equation (1.4) as the dynamical system

$$\dot{Y}(t) = \dot{Y}(t) = \mathbf{F}(Y(t)), \quad t \in \mathbb{R}, \tag{1.4}$$

where $Y(t) = Y(x,t) := (\psi(x,t), \dot{\psi}(x,t))$. We will introduce a metric space $\mathcal{E}_F$, which is the space of finite energy states of the equation, and construct the corresponding dynamical group $U(t) : Y(0) \mapsto Y(t)$.

Definition 1.1 *([1, 22]) A subset $\mathcal{A} \subset \mathcal{E}_F$ is a* **minimal global point attractor** *of the group $U(t)$ if*
i) For any $Y \in \mathcal{E}_F$ the convergence holds

$$U(t)Y \xrightarrow{\mathcal{E}_F} \mathcal{A}, \quad t \to \pm\infty. \tag{1.5}$$

ii) The subset is invariant, i.e. $U(t)\mathcal{A} = \mathcal{A}$, $t \in \mathbb{R}$.
iii) $\mathcal{A}$ is minimal set with the properties i) and ii).

By definition, (1.5) means that

$$\rho(U(t)Y,\mathcal{A}) := \inf_{X\in\mathcal{A}} \rho(U(t)Y,X) \to 0, \quad t \to \pm\infty, \tag{1.6}$$

where ρ stands for the metric in $\mathcal{E}_F$.

1.2 G-Invariant Equations

Let G be a Lie group and

$$g \mapsto g_* \tag{1.7}$$

is a representation of G by the (linear or nonlinear) automorphisms of the phase space $\mathcal{E}_F$.

Definition 1.2 *(cf. [20]) i) The equation (1.4) (and (1.1)) is G-invariant with respect to the representation (1.7) if for any solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$, the trajectory $g_* Y(t)$ is also a solution. ii) Solitary Wave Solution of the equation (1.4) is any solution of the form (cf. (0.4))*

$$Y(t) = e^{\mathbf{g}_* t} Y_{\mathbf{g}}, \tag{1.8}$$

where $Y_{\mathbf{g}} \in \mathcal{E}_F$ and $\mathbf{g}_*$ is the representation of an element $\mathbf{g}$ of the corresponding Lie algebra $\mathbf{G}$.

Remark 1.3 i) For *linear* representations (1.7), the equation (1.4) is G-invariant if *formally*

$$\mathbf{F}(g_* Y) = g_* \mathbf{F}(Y), \qquad g \in G. \tag{1.9}$$

ii) The representation $\mathbf{g} \mapsto \mathbf{g}_*$ of the Lie algebra corresponds to the representation (1.7) of the Lie group G. Then $e^{\mathbf{g}_* t}$ represents the one-parametric subgroup $e^{\mathbf{g}t}$ of G. Hence, the solitary wave solution (1.8) can be written as

$$Y(t) = (e^{\mathbf{g}t})_* Y_{\mathbf{g}}, \tag{1.10}$$

iii) The amplitude $Y_{\mathbf{g}}$ of the solitary wave (1.8) satisfies formally the *stationary equation*

$$\mathbf{g}_* Y_{\mathbf{g}} = \mathbf{F}(Y_{\mathbf{g}}). \tag{1.11}$$

1.3 Examples of Symmetry Groups and Solitary Waves

A Every equation is invariant w.r.t. trivial symmetry group $G = \{e\}$ with the identity representation $e_* \psi := \psi$. Then the Lie algebra $\mathbf{G} = \{0\}$ and the solitary waves are the *static stationary solutions* $Y(x,t) \equiv S(x)$. The stationary equation (1.11) becomes

$$0 = \mathbf{F}(S). \tag{1.12}$$

B The translation symmetry group $G = T := \mathbb{R}^d$ with the representation $a_* \psi(x) := \psi(x - a)$, $a \in \mathbb{R}^d$. It corresponds to translation-invariant equations (1.1) with $f(x, \psi) \equiv f(\psi)$. Then the Lie algebra $\mathbf{G} = \mathbb{R}^d$ and the solitary waves are *solitons (or traveling wave) solutions* $Y(x,t) \equiv Y_v(x - vt)$. The stationary equation (1.11) becomes

$$-v \cdot \nabla Y_v = \mathbf{F}(Y_v). \tag{1.13}$$

C The rotation symmetry group $G = U(1) := \{z \in \mathbb{C} : |z| = 1\}$ with the representation $z_* \psi(x) := z\psi(x)$. It corresponds to 'phase-invariant' equations (1.1) with $f(x, \psi) \equiv a(x, |\psi|)\psi$. Then the Lie algebra $\mathbf{G} = \mathbb{R}$ and the solitary waves have the form $Y(x,t) \equiv e^{i\omega t} Y_\omega(x)$ like the Schrödinger Quantum Stationary States. The stationary equation (1.11) becomes the *nonlinear eigenvalue problem*

$$i\omega Y_\omega = \mathbf{F}(Y_\omega). \tag{1.14}$$

D The product symmetry group $G = T \times U(1)$ with the product representation $(a, z)_* \psi(x) := z\psi(x - a)$. It corresponds to equations (1.1) with $f(x, \psi) \equiv a(|\psi|)\psi$. Then the Lie algebra $\mathbf{G} = \mathbb{R}^d \times \mathbb{R}$ and the solitary waves have the form $Y(x,t) \equiv e^{i\omega t} Y_{v,\omega}(x - vt)$. The stationary equation (1.11) becomes

$$(-v \cdot \nabla + i\omega) Y_{v,\omega} = \mathbf{F}(Y_{v,\omega}). \tag{1.15}$$

Remark 1.4 *The existence of the solitary waves (1.15) is proved in [3] for a wide class of the functions $f(x, \psi) \equiv a(|\psi|)\psi$.*

1.4 On the Structure of Minimal Global Point Attractor

We will discuss the following general conjecture **G** concerning the structure of attractors of a G-invariant equation with a fixed symmetry Lie group G:

$\mathcal{G}$ *For a generic G-invariant equation, the minimal global point attractor is the set*

$$\mathcal{A} = \{Y_{\mathbf{g}} \in \mathcal{E}_F : \mathbf{g} \in \mathbf{G} \ \text{ and } \ e^{\mathbf{g}_* t} Y \ \text{ is the Solitary Wave Solution } \}. \quad (1.16)$$

Here the expression *for generic G-invariant equation* means *for almost all G-invariant equations (1.4)*, i.e. *for almost all dynamical systems (1.4) (or the functions $f(x, \psi)$) satisfying the identity (1.9).*

This general conjecture has been justified for a list of model equations with the Lie groups and their representations from previous section. Then the general conjecture $\mathcal{G}$ reads as follows

A For the trivial symmetry group $\{e\}$: the minimal global point attractor *generically* is the set of all static stationary solutions, i.e.

$$\mathcal{A} = \{Y(\cdot) \in \mathcal{E}_F : Y(x) \ \text{ is a static solution}\}. \quad (1.17)$$

B For the translation symmetry group $T = \mathbb{R}^d$: the minimalglobal point attractor *generically* is the set of all soliton solutions, i.e.

$$\mathcal{A} = \{Y_v(\cdot) \in \mathcal{E}_F : v \in \mathbb{R}^d \ \text{ and } \ Y_v(x - vt) \ \text{ is a soliton solution}\}. \quad (1.18)$$

C For the rotation symmetry group $U(1)$: the minimal global point attractor *generically* is the set

$$\mathcal{A} = \{Y_\omega(\cdot) \in \mathcal{E}_F : \omega \in \mathbb{R} \ \text{ and } \ e^{i\omega t} Y_\omega(x) \ \text{ is a solution}\}. \quad (1.19)$$

D For the product symmetry group $\mathbb{R}^d \times U(1)$: the minimal global point attractor *generically* is the set

$$\mathcal{A} = \{Y_{v,\omega}(\cdot) \in \mathcal{E}_F : v \in \mathbb{R}^d, \omega \in \mathbb{R} \ \text{ and } \ e^{i\omega t} Y_{v,\omega}(x - vt) \ \text{ is a solution}\}. \quad (1.20)$$

Remark 1.5 It is instructive to stress that the word *generically* means *for generic equations with the corresponding fixed symmetry group.* For example: the trivial group $\{e\}$ is a subgroup of $U(1)$, hence each $U(1)$-invariant equation is also $\{e\}$-invariant. Therefore, the set of all $U(1)$-invariant equations is a subset of all ($\{e\}$-invariant) equations. This would contradict the different forms of the attractors (1.19) and (1.17) if one omits the word *generic.* However, the $U(1)$-invariant equations constitute an *exceptional class* among all ($\{e\}$-invariant) equations. Therefore, one could expect much more sophisticate long-time behavior of the solutions to $U(1)$-invariant equations, hence different form of the attractor.

The case **C** corresponds to the coupled Maxwell-Dirac and Maxwell-Schrödinger equations, [12, 21], with the (global) gauge group $U(1)$. The form of the attractor (1.19) would clarify the Schrödinger identification of the "eigenfunctions" $e^{i\omega t}\psi(x)$ with Quantum Stationary States.

1.5 Asymptotics in Local and Global Norms

We suggest that the convergence to an attractor, (1.5), holds in *local energy seminorms* that defines the corresponding metric in (1.6). The attraction in a *global energy norm* generally is impossible because of energy conservation.

We also suggest long-time "scattering asymptotics" of type (0.5) in the *global energy norm.* For example, for the translation-invariant equations (1.1) (the case **B**), the suggested asymptotics read formally,

$$Y(x,t) \approx \sum_{k=1}^{N_\pm} Y_\pm^k (x - v_\pm^k t) + W(t)\Psi_\pm, \qquad t \to \pm\infty. \quad (1.21)$$

Here $W(t)$ is the dynamical group of the *free* Klein-Gordon equation (1.1) with $f(x,\psi) \equiv 0$, and $\Psi_\pm$ are the *asymptotic scattering states*. Corresponding extension to the case **D** reads

$$Y(x,t) \approx \sum_{k=1}^{N_\pm} e^{i\omega_\pm^k t} Y_\pm^k(x - v_\pm^k t) + W(t)\Psi_\pm, \quad t \to \pm\infty. \tag{1.22}$$

2 Known Results

We will refer the attraction to the set (1.17) as to "attraction of type **A**", etc.

2.1 Attraction to Static Stationary States

The attraction to the static stationary states has been established initially in the theory of attractors of dissipative systems: Navier-Stokes, diffusion-reaction, damped wave equation, etc. The attraction holds then in the global energy norm, however only for $t \to +\infty$ (see [1, 22, 23, 58] and others).

For the Hamilton equations, the first results on the attraction of type **A** and the scattering asymptotics (1.21) with $Y_\pm^k = 0$ have been obtained in linear scattering theory (see [18, 44, 46, 51, 59] and others). The results were extended to nonlinear scattering theory (see [8, 17, 19, 25, 29, 47, 50, 51, 53, 56, 57] and others). All the results concern the case of the attractor which consists of one point which is zero solution, i.e. $\mathcal{A} = \{0\}$. The attraction to the zero solution in the local energy seminorms, is equivalent to the **local energy decay**, and (1.21) reduces to the dispersive wave: $Y(x,t) \approx W(t)\Psi_\pm$.

The attraction of type **A** to a nontrivial attractor $\mathcal{A} \neq \{0\}$ has been established i) in [30]-[32] for the 1D equations (1.1) with $f(x,\psi) = 0$, $|x| > a$, and $m = 0$ (see the survey [33]), and ii) in [41] for the nonlinear system of 3D wave equation coupled to a classical particle. The system is an analog of the coupled Maxwell-Lorentz equations of Classical Electrodynamics with the Abraham model of the *extended electron*. The corresponding point attractors can contain an arbitrary finite or infinite number of isolated points, as well as continuous finite-dimensional components. The results have been extended to the Maxwell-Lorentz equations, [40].

In [42] the Liapunov-type criterion is established for the asymptotic stability of stationary states of general nonlinear Klein-Gordon equations.

2.2 Attraction to Solitary Waves

The attraction of type **B** and soliton-type asymptotics of type (1.21) have been discovered initially for the *integrable equations*: KdV, sine-Gordon, cubic Schrödinger, etc (see [48] for the survey of the results). The results have been extended in [28, 38, 39] to the (nonintegrable) 3D translation-invariant nonlinear system studied in [41]. The generalization of the results to the Maxwell-Lorentz resp. Klein-Gordon equations is done in [26] resp. [27]. In [2] the asymptotics are extended to the relativistic-invariant nonlinear 1D equations (1.1) with $f(x,\psi) = \sum_k F_k \delta(\psi - \psi_k)$ and $m = 0$.

In this paper we will describe, for the first time, our numerical experiments for 1D relativistic-invariant equations (1.1). The numerous experiments suggest that the asymptotics (1.21) and (1.22) hold for "any" equation with a positive Hamiltonian. However, the proof is still an open problem. Numerical experiments were made by F.Collino, T.Fouquet, L.Rhaouti and O.Vacus (Project ONDES, INRIA), by Yu.Radvogin (M.Keldysh Institute of Applied Mathematics, RAS) and A.Vinnichenko.

The first results on the attraction of type **C** have been established in [54, 55] for $U(1)$-invariant 3D nonlinear Schrödinger equation (see also [49]).

In [5, 6, 7] the attraction of type **D** and the asymptotics (1.22) with $N_\pm = 1$ are established for translation-invariant $U(1)$-invariant 1D nonlinear Schrödinger equations. The results [6] are extended in [10] to all dimensions $n \geq 3$.

All the results [5, 6, 7, 10, 49, 54, 55] concern initial states which are *sufficiently close to the attractor.* The attraction **C** for *all finite energy states* is established for the first time in [36] for the nonlinear $U(1)$-invariant 1D equations (1.1) with $f(x,\psi) = \delta(x)F(\psi)$ and $m > 0$.

2.3 Adiabatic Effective Dynamics

In [38, 43] an *adiabatic effective dynamics* is established, for the solitons of 3D wave equation or Maxwell field coupled to a classical particle, in a *slowly varying external potential.* The effective dynamics explains the increment of the mass of the particle caused by its interaction with the field. The effective dynamics is extended in [14, 15] to the solitons of a nonlinear Schrödinger and Hartree equations. An extension to relativistic-invariant equations is still an open problem. On the other hand, the existence of the solitons and the Einstein mass-energy identity are proved, respectively, in [3] and [11], for general relativistic-invariant nonlinear Klein-Gordon equations (1.1). Note that the existence of the solitons is also proved in [12] for relativistic-invariant nonlinear Maxwell-Dirac equations.

2.4 Open Problems

The proving of the attraction of type **A**, **B**, **C**, **D** and the asymptotics (1.21), (1.22) for the nonlinear Klein-Gordon equation (1.1) with $n > 1$ or with $n = 1$ and $f(x,\psi) \equiv f(\psi)$ are still open problems.
• The attraction of type **C**, and the asymptotics (1.22) are not proved yet for the coupled nonlinear Maxwell-Dirac and Maxwell-Schrödinger Equations [12, 21].

3 Relativistic Ginzburg-Landau Equation

We consider real solutions to 1D relativistic nonlinear wave equation of type

$$\ddot{\psi}(x,t) = \psi''(x,t) + f(\psi(x,t)), \quad x \in \mathbb{R}, \tag{3.1}$$

with the nonlinear term $f \in C^1(\mathbb{R})$. The equation is translation invariant and Lorentz invariant. Formally it is a Hamilton system with the Hamilton functional

$$\mathcal{H}(\psi,\pi) = \int [\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + U(\psi(x))]dx, \tag{3.2}$$

where the potential $U(\psi) = -\int_0^\psi f(\varphi)d\varphi + \text{const}$. We assume $U(\psi)$ be "two-well" potential similar to the Ginzburg-Landau one $U(\psi) \sim (1-\psi^2)^2$. We assume, more generally, that

$$U \in C^2(\mathbb{R}), \qquad U(a_-) = U(a_+) = 0, \qquad \text{and } \; U(\psi) > 0 \;\text{ for }\; \psi \neq a_\pm, \tag{3.3}$$

with some $a_- < a_+$. Then $f(a_\pm) = 0$ and the constant functions $s_\pm(x) := a_\pm$ are stationary finite energy solutions to (3.1).

3.1 Kink Solutions

We assume the points $a_\pm$ be nondegenerate local minima of the potential $U(\psi)$,

$$m_\pm^2 = U''(a_\pm) > 0. \tag{3.4}$$

Then there exists a "kink", i.e. a nonconstant finite energy stationary solution $s(x)$ to (3.1),

$$0 = s''(x) + f(s(x)), \quad x \in \mathbb{R}; \quad s(x) \not\equiv \text{const}; \quad H(s, 0) < \infty. \tag{3.5}$$

The moving kinks, with velocities $v \in \mathbb{R}$, exist for $|v| < 1$ and are obtained by the Lorentz transformation

$$(x, t) \mapsto \gamma_v(x - vt, t - vx), \tag{3.6}$$

where $\gamma_v = 1/\sqrt{1 - v^2}$ corresponds to the Lorentz contraction. Namely, for any shift $q \in \mathbb{R}$ the traveling kink

$$\psi(x, t) = s(\gamma_v(x - vt - q)) \tag{3.7}$$

also is a finite energy solution to (3.1).

We suppose that for the kink solutions, the asymptotics (1.21) has to be modified as follows:

$$Y(x, t) \approx \sum_{k=1}^{N_\pm} \zeta\Big(\frac{x - v_\pm^k t}{l(t)}\Big) Y_{v_\pm^k}(x - v_\pm^k t) + \eta_\pm(x, t) W(t) \Psi_\pm, \qquad t \to \pm\infty, \tag{3.8}$$

where $l(t) := \log(|t| + 2)$, the function $\zeta \in C_0^\infty(\mathbb{R})$, $\zeta(x) = 1$ for $|x| \le 1$, and

$$\sum_{k=1}^{N_\pm} \zeta\Big(\frac{x - v_\pm^k t}{l(t)}\Big) + \eta_\pm(x, t) \equiv 1.$$

The asymptotics (3.8) mean that

i) The kinks contribute to the union of intervals

$$I_\pm(t) := \cup_{k=1}^{N_\pm} \Big[v_\pm^k t - l(t), v_\pm^k t + l(t) \Big],$$

of total lenghth $\sim \log t$.

ii) Outside the intervals, the solution is close to the dispersive wave $W(t)\Psi_\pm$ as $t \to \pm\infty$.

Note that the energy of the dispersive wave in the set $I_\pm(t)$ decays to zero. This follows by the stationary phase method (see (3.25) below).

3.2 Existence of Dynamics

Let us rewrite (3.1) as first order system

$$\dot\psi(x, t) = \pi(x, t), \quad \dot\pi(x, t) = \psi''(x, t) + f(\psi(x, t)). \tag{3.9}$$

We consider the Cauchy problem for the system (3.9) with the initial conditions

$$\psi|_{t=0} = \psi_0(x), \quad \dot\psi|_{t=0} = \pi_0(x), \quad x \in \mathbb{R}. \tag{3.10}$$

Let us define the phase space $\mathcal{E}$ of finite energy states for the wave equation (3.1). For any $p \in [1, \infty]$ let us denote by L^p the space $L^p(\mathbb{R})$ endowed with the norm $\| \cdot \|_p$. For any $R > 0$ denote by $\| \cdot \|_{p,R}$ the norm in the space $L^p(-R, R)$.

Definition 3.1 *i) E is the Hilbert space of $(\psi, \pi) \in L^2 \oplus L^2$ with finite 'energy norm'*

$$\|(\psi, \pi)\|_E = \|\psi'\|_2 + \|\psi\|_\infty + \|\pi\|_2. \tag{3.11}$$

ii) E_F is the space E endowed with the (Fréchet) topology defined by the 'local energy seminorms'

$$\|(\psi, \pi)\|_{E,R} = \|\psi'\|_{2,R} + \|\psi\|_{\infty,R} + \|\pi\|_{2,R}, \qquad R > 0. \tag{3.12}$$

iii) The phase space $\mathcal{E}$ is the set of $(\psi, \pi) \in E$ with the finite energy $\mathcal{H}(\psi, \pi) < \infty$, endowed with the topology of E. The space $\mathcal{E}_F$ is the set $\mathcal{E}$ endowed with the topology of E_F.

Remark 3.2 *The space $\mathcal{E}_F$ is metrizable. For example, the convergence in $\mathcal{E}_F$ is equivalent to the convergence w.r.t. the metric*

$$\rho(Y_1, Y_2) = \sum_{R=1}^{\infty} 2^{-R} \frac{\|Y_1 - Y_2\|_{\mathcal{E},R}}{1 + \|Y_1 - Y_2\|_{\mathcal{E},R}}.$$

It is easy to check that $(s_\pm(x), 0), (s(\pm x), 0) \in \mathcal{E}$. Let $S_v = (\psi_v, \pi_v)$ denote the initial state of the soliton (3.7) with $|v| < 1$ and $q = 0$,

$$S_v(x) = (s(\gamma_v x), -\gamma_v v s(\gamma_v x)). \tag{3.13}$$

The conditions (3.3), (3.4) imply that $S_v(x - q) \in \mathcal{E}$ for any $|v| < 1$ and $q \in \mathbb{R}$. The following lemma is obvious.

Lemma 3.3 *Let the conditions (3.3), (3.4) hold. Then Hamilton functional $\mathcal{H}$ is continuous on the phase space $\mathcal{E}$.*

The existence and uniqueness of the solutions to the Cauchy problem (3.9), (3.10) is well known and can be proved by the methods [45, 50, 57].

Proposition 3.4 *Let the conditions (3.3), (3.4) hold. Then*
i) for every initial datum $(\psi_0(x), \pi_0(x)) \in \mathcal{E}$ there exists the unique solution $(\psi(x,t), \dot\psi(x,t)) \in C(\mathbb{R}, \mathcal{E})$ to the problem (3.9), (3.10).
ii) The energy is conserved,

$$\mathcal{H}(\psi(\cdot,t), \pi(\cdot,t)) = \text{const}, \quad t \in \mathbb{R}. \tag{3.14}$$

iii) The trajectory is bounded,

$$\sup_{t \in \mathbb{R}} \|(\psi(\cdot,t), \pi(\cdot,t)\|_E < \infty. \tag{3.15}$$

3.3 Numerical Observations

Let us describe the results of our numerical experiments and give an identification of the details in terms of equation (3.1).

We have observed the asymptotics of type (1.21) for finite energy solutions of the equations (3.1) with the polynomial potential of the Ginzburg-Landau type

$$U(\psi) = \frac{(|\psi|^2 - 1)^2}{4}. \tag{3.16}$$

Then (3.1) reads

$$\ddot\psi(x,t) = \psi''(x,t) - |\psi(x,t)|^2 \psi(x,t) + \psi(x,t), \tag{3.17}$$

For the potential (3.16) the conditions (3.3), (3.4) hold with $a_\pm = \pm 1$, and the (standing) kink solutions are

$$s(x) = \pm \tanh \tilde x, \quad \tilde x := x/\sqrt{2}, \tag{3.18}$$

up to translation. We have chosen different "smooth" initial functions ψ_0, π_0 with the following properties:

$$|\psi_0(x)|, |\pi_0(x)| \sim 1, \qquad \text{supp } \psi_0', \text{supp } \pi_0 \subset [-20, 20], \qquad |\psi(x)| \equiv 1 \text{ for } |x| \ge 20. \tag{3.19}$$

We use the numerical second order scheme with $\Delta t \sim \Delta x \sim 0.01, 0.001$. In all cases (more than 100 initial functions), we have observed the asymptotics of type (1.21) for $t \ge 100$, with the number of the solitons $N_+ = 0, 1, ..., 5$.

Example 3.5 Figure 1 represents a solution of the equation (3.1) with the potential (3.16).
• **Space and Time:** The space variable x (horizontal axis) and the time axis t (vertical axis). The space is two times contracted at time $t = 20$ and $t = 60$.
• **Colors:** The distribution of the colors corresponds to the range of the solution as follows,

ψ	$(-\infty, -1.01)$	$[-1.01, -0.99)$	$[-0.99, -0.8)$	$[-0.8, 0.8]$	$(0.8, 0.99]$	$(0.99, 1.01]$	$(1.01, \infty)$
Color	*White*	*Blue*	*Grey*	*Yellow*	*Grey*	*Red*	*White*

• **Transient Phase:** for $t \in [0, 20]$ we observe a chaotic behavior *without radiation*.
• **Asymptotic Phase:** for $t > 20$ we observe the asymptotics of type (1.21) with $N_+ = 3$.
• **Kinks** The three *Yellow color oscillating strips* represent the trajectories of the kinks. The Yellow strip around a trajectory correspond to the 'support' of a kink (with the values $\psi \in [-0.8, 0.8]$).
• **Dispersive Wave** The *hyperbolic and rectilinear* Blue-White and Red-White trajectories outside the Yellow strips represent the dispersive wave (the values $\psi \approx \pm 1$) The rectilinear trajectories mean the decay of the dispersive wave to the *wave packets*, propagating uniformly, with distinct group velocities.

3.4 Kink Oscillations and Linearized Equation

• **Lorentz-Einstein dilation** The boundaries of the yellow strips oscillate with different periods:
1) For the left kink, with the velocity $v_l \approx -0.24$, the period $T_l \approx 5.3$: about 45 of the periods between t=60 and t=300.
2) For the central kink, with the velocity $v_c \approx 0.02$, the period $T_c \approx 5.1$: about 47 of the periods between t=60 and t=300.
3) For the right kink, with the velocity $v_r \approx 0.88$, the period $T_r \approx 8.8$: about 15 of the periods between t=60 and t=190.
The ratio $T_c/T_l \approx 5.1/5.3 = 0.96$ corresponds to the Lorentz-Einstein dilation $\sqrt{1 - v_l^2}/\sqrt{1 - v_c^2} \approx \sqrt{1 - v_l} \approx 0.97$. The error $0.96 - 0.97$ is about 1%.
The ratio $T_c/T_r \approx 5.1/8.8 = 0.58$ approximately corresponds to the Lorentz-Einstein dilation $\sqrt{1 - v_r^2}/\sqrt{1 - v_c^2} \approx \sqrt{1 - v_r^2} \approx 0.48$. The error $0.58 - 0.48$ is about 17% and probably is due to the nonlinear interaction.

It is supposed that the oscillations are provided by the eigenvalue $\omega_1 = \sqrt{3/2} \approx 1.224$ of the linearized equation, and

$$T_c \approx \frac{2\pi}{\omega_1 \sqrt{1 - v_c^2}} = 5.13.... \tag{3.20}$$

The error $5.13 - 5.1$ is about 0.6%.
• **Spectrum of Linearized Equation** Let us derive the equation for small perturbation of the kink with the zero velocity. Namely, substitute $\psi(x, t) = s(x) + \varphi(x, t)$ into (3.17). Neglecting the terms $\mathcal{O}(|\varphi|^2)$, we formally obtain the linearized equation

$$\begin{aligned} \ddot{\varphi}(x, t) = -H\varphi(x, t) : &= \varphi''(x, t) - 3s^2(x)\varphi(x, t) + \varphi(x, t) \\ &= \varphi''(x, t) - 2\varphi(x, t) - V(x)\varphi(x, t), \end{aligned} \tag{3.21}$$

where the potential $V(x) := 3s^2(x) - 3 \le 0$ and the 'Schrödinger operator' $H := -\dfrac{d^2}{dx^2} + 2 + V(x)$.

The continuous spectrum of H is evidently the interval $[2, \infty)$. The discrete spectrum contains at least two points: $\lambda = 0$ and $\lambda = 3/2$. Namely,

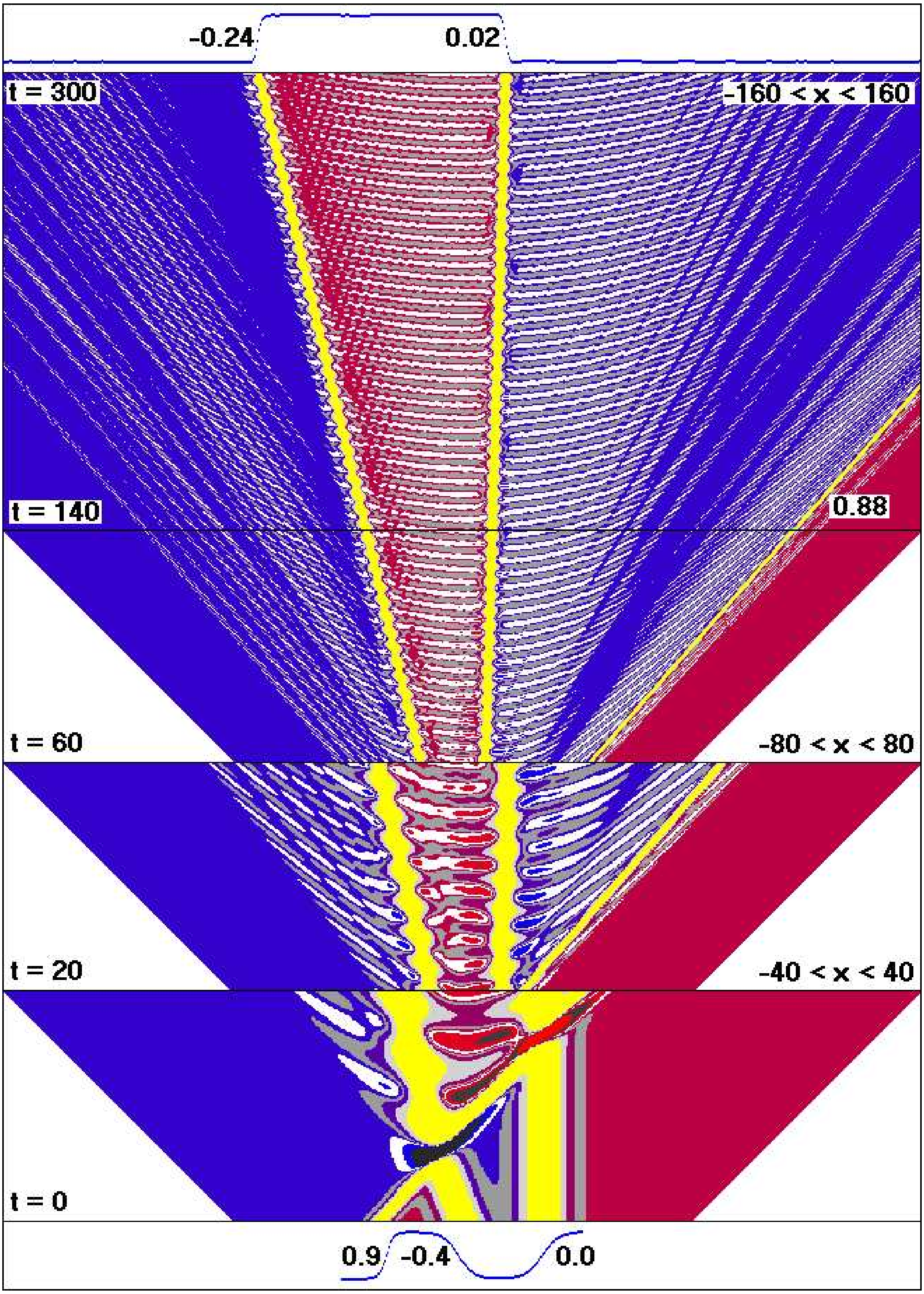

Figure 1: Formation of kinks

I. The zero point corresponds to the eigenfunction $\psi_0(x) = s'(x)$ which is the groundstate since it is positive. Namely, differentiating the stationary equation (3.5), we obtain $0 = Hs'(x)$.

II. The point $\lambda = 3/2$ is the next eigenvalue since it corresponds to the eigenfunction $\psi_1(x) = \dfrac{\sinh(\tilde{x})}{\cosh^2(\tilde{x})}$ where $\tilde{x} := x/\sqrt{2}$ (it is easy to check by direct calculation), and the eigenfunction has one zero point.

The point $\lambda = 3/2$ provides the oscillatory solutions $\mathrm{Re}\,\psi_1(x)e^{\pm i\omega_1 t}$ where $\omega_1 := \sqrt{3/2}$. We suppose that the oscillatory solutions are responsible for the oscillations of the kinks. For example, for the central kink, the deviation of the periods of the oscillations from $2\pi/(\omega_1\sqrt{1-v_c^2})$ is about 0.6%.

3.5 Dispersive Wave

• **Hyperbolic Structure** The dispersive wave is detected by the hyperbolic level lines (see fig. 1) typical for the solutions of the Klein-Gordon equation. Namely, outside the yellow strips surrounding the kinks, the observed solution is close to the stationary solutions $s_\pm(x) \equiv \pm 1$: the difference is ~ 0.01. Hence, the solution approximately satisfies the linearized equations, corresponding to $s_\pm(x)$, which is the Klein-Gordon equation (3.21) with the potential $V(x) := 3s_\pm^2(x) - 3 \equiv 0$:

$$\ddot{\varphi}(x,t) = \varphi''(x,t) - 2\varphi(x,t). \tag{3.22}$$

Each finite energy solution is a convolution with the fundamental solution which is the Bessel function of the Lorentz interval $t^2 - x^2$, [35]. Hence, the level lines of any finite energy solution are asymptotically described by the equation $t^2 - x^2 = \text{const}$.

Remark 3.6 *The identification of the dispersive wave with the Bessel functions has been checked numerically with a high precision by Yulian Radvogin. Some of the numerical results are published in [4].*

• **Dispersion Relation** The dispersive wave, outside the kinks, decays like $t^{-1/2}$. This follows by method of stationary phase [13]. Therefore, the density of energy decays like t^{-1}. Hence, the total energy between the moving kinks is about constant since the distance is of order t. We have checked numerically that the energy of the dispersive wave between the kinks does not decay to zero and converges to a nonzero limit.

This corresponds to the dispersion relation $\omega(k) = \pm\sqrt{k^2+m^2}$ of the Klein-Gordon equation. Namely, all group velocities $v \in (-1,1)$ are allowed since $v = \nabla\omega(k)$, [37]. For example, the energy of the dispersive wave between the left and central kinks, is transported by the harmonics with the wave numbers satisfying the inequalities

$$-0.24 \approx v_l < \nabla\omega(k) < v_c \approx 0.22. \tag{3.23}$$

For the linear Klein-Gordon equation (3.22), the energy in the sector (3.23) converges to a limit which generally is not zero. Namely, the total energy for the linear equation reads (cf. (3.2))

$$\frac{1}{2}\int(|\dot{\phi}(x,t)|^2 + |\phi'(x,t)|^2 + 2|\phi(x,t)|^2)dx = \frac{1}{2}\int(|\dot{\hat{\phi}}(k,t)|^2 + |k\hat{\phi}(k,t)|^2 + 2|\hat{\phi}(k,t)|^2)dk \tag{3.24}$$

by the Parseval identity, where $\hat{\phi}(k,t)$ stands for the Fourier transform

$$\hat{\phi}(k,t) := \frac{1}{\sqrt{2\pi}}\int e^{ikx}\phi(x)dx.$$

The energy in any region $v_l t + o(t) < x < v_c t + o(t)$ converges to the corresponding limit:

$$\lim_{t\to\pm\infty} \frac{1}{2} \int\limits_{v_l t+o(t)<x<v_c t+o(t)} (|\dot{\phi}(x,t)|^2 + |\phi'(x,t)|^2 + 2|\phi(x,t)|^2)dx$$

$$= \frac{1}{2} \int\limits_{v_l<\nabla\omega(k)<v_c} (|\dot{\hat{\phi}}(k,0)|^2 + |k\hat{\phi}(k,0)|^2 + 2|\hat{\phi}(k,0)|^2)dk \tag{3.25}$$

if the initial functions are sufficiently smooth. This can be proved by the methods [37, Section "Schrödinger Equation and Geometric Optics"].

• **Discrete Spectrum of Group Velocities** The dispersive wave decays to the discrete wave packets (see fig. 1). This demonstrates the discreteness of the corresponding group velocities. We suppose that the discreteness is due to the polynomial character of the nonlinear term. Namely, let us consider the solution around a moving kink. After a Lorentz transformation, we can assume that the velocity of the kink is zero. Then the oscillations of the kink (probably) correspond to the solutions $\mathrm{Re}\,\psi_1(x)e^{\pm i\omega_1 t}$ of the linearized equation. The polynomial nonlinear term produces all the frequencies $n\omega_1$ with $n = \pm2, \pm3, ...$ Therefore, it is natural to think that the observed discrete spectrum of the group velocities is described by $v_n = \pm\nabla\omega(k_n)$ where the wave numbers k_n satisfy the dispersion relation $n\omega_1 = \pm\sqrt{k_n^2 + m^2}$. However, a satisfactory numerical identification of the wave packets is not done yet.

3.6 Linear and Nonlinear Radiative Mechanism

Our experiments suggest that the attraction to the kinks is due to the radiation induced by the oscillations. This mechanism can be explained by the equation (3.17).

Namely, let us represent the equation as the linearized Klein-Gordon equation (3.21) excited by the source which includes the nonlinear term:

$$\ddot{\psi}(x,t) + H\psi(x,t) = -|\psi(x,t)|^2\psi(x,t) + 3\psi(x,t) + V(x)\psi(x,t), \tag{3.26}$$

It is well-known from the scattering theory [37] that the long-time asymptotics and the radiation depend on the time-spectrum of the source:

• For ω in the continuous spectrum of the linear Klein-Gordon equation, $\omega \in \mathbb{R} \setminus (-\sqrt{2}, \sqrt{2})$, the harmonics $e^{i\omega t}$ in the source generates the radiation to infinity that means the *long range scattering.*

• Otherwise, for $\omega \in (-\sqrt{2}, \sqrt{2})$, the harmonics generates the forced oscillation without radiation to infinity that means the *short range scattering.*

The numerical experiments [9] illustrate this theory for small perturbation of the standing kink (3.18). Namely

I. For small times, $t \in [0, 100]$, the time-spectrum of the solution $\psi(0,t)$ contains mainly two points $\pm\omega_1 \in (-\sqrt{2}, \sqrt{2})$, and the radiation is not observed.

II. For large times, $t \in [100, 1000]$, the time-spectrum of the solution $\psi(0,t)$ contains new points $\pm2\omega_1, \pm3\omega_1, ... \in \mathbb{R} \setminus (-\sqrt{2}, \sqrt{2})$, and the radiation of corresponding frequencies is observed. The frequencies are generated from ω_1 by the cubic polynomial in RHS of (3.26).

Our experiment also demonstrates the difference in the radiation of the dispersive wave for $t < 20$ and $t > 20$ (see fig. 1).

Summarizing: The nonlinear term translates the harmonics from the spectral gap $(-\sqrt{2}, \sqrt{2})$ into the continuous spectrum $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$. Then the linear Klein-Gordon dynamics disperses the energy at infinity. This *radiative mechanism* plays the role of a dissipation and is responsible for the convergence to the attractor in reversible Hamilton equations.

In [36] the convergence to the attractor is proved for $U(1)$-invariant 1D nonlinear Klein-Gordon equation with the nonlinear term $\delta(x)F(\psi)$ concentrated at one point $x = 0$. The function $F(\psi) = g(|\psi|^2)\psi$ where g is a polynomial of order ≥ 1. The proof is based on the detailed analysis of the radiative mechanism by the Titchmarsh Convolution theorem [24, Thm 4.3.3].

4 Relativistic Klein-Gordon Equations

4.1 Soliton Solutions

Further, we have observed the asymptotics of type (1.22) for complex solutions to relativistic 1D nonlinear Klein-Gordon equation

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - \psi(x,t) + f(\psi(x,t)), \qquad x \in \mathbb{R}. \tag{4.1}$$

We assume that $f(\psi) = -\nabla U(|\psi|)$ with a polynomial potential

$$U(|\psi|) = a|\psi|^{2m} - b|\psi|^{2n}, \tag{4.2}$$

where $a, b > 0$ and $m > n = 2, 3, ...$ Then

$$f(\psi) = 2am|\psi|^{2m-2}\psi - 2bn|\psi|^{2n-2}\psi. \tag{4.3}$$

The equation (4.1) is the Hamilton dynamical system with the Hamilton functional

$$\mathcal{H}(\psi, \pi) = \int [\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + \frac{|\psi(x)|^2}{2} + U(|\psi(x)|)]dx, \tag{4.4}$$

Further, Eqn (4.1) is translation invariant and $U(1)$-invariant, hence generally admits the solitary wave solutions $e^{i\omega t}\psi_{v,\omega}(x - vt)$. First consider the standing solitary waves, i.e. with $v = 0$. Substitution into (4.1) gives

$$-\omega^2\psi_{0,\omega}(x) = \psi''_{0,\omega}(x) - \psi_{0,\omega}(x) + f(\psi_{0,\omega}(x)), \quad x \in \mathbb{R}, \tag{4.5}$$

since $f(e^{i\theta}\psi) \equiv e^{i\theta}f(\psi)$ by (4.3). The equation (4.5) can be solved explicitly, and the finite energy solitary waves generally exist for a range of $\omega \in I \subset [-1, 1]$ and decay exponentially at infinity. For the concreteness we denote by $\phi_{0,\omega}(x)$ an even solution to (4.5).

For $|v| < 1$ the solitary waves are obtained by the Lorentz transformation (3.6): the moving solitary wave is equal to

$$\phi_{v,\omega}(x,t) := e^{i\omega\gamma_v(t-vx)}\phi_{0,\omega}(\gamma_v(x - vt)). \tag{4.6}$$

The *total energy* of the soliton coincides with the *kinetic energy* of classical relativistic particle:

$$\mathcal{H}(\phi_{v,\omega}(\cdot,t), \dot{\phi}_{v,\omega}(\cdot,t)) = \frac{E_0(\omega)}{\sqrt{1 - v^2}}, \tag{4.7}$$

where generically $E_0(\omega) > 0$. This is proved in [11] for general relativistic equations in all dimensions.

Remark 4.1 *The solution (4.6) can be rewritten in the form $e^{i\tilde{\omega}t}\psi(x - vt)$ with $\tilde{\omega} := \gamma_v\omega$ (cf. (1.20)).*

4.2 Numerical Observations

We have tried the following three combinations of a, b, m, n:

N	a	m	b	n
1	1	3	0.61	2
2	10	4	2.1	2
3	10	6	8.75	5

We have chosen different "smooth" initial functions ψ_0, π_0 with the support in a bounded interval $[-20, 20]$. We use the numerical second order scheme with $\Delta x, \Delta t \sim 0.01, 0.001$. In all cases we have observed the asymptotics of type (1.22) with the number of solitons $N_+ = 0, 1, 3$ for $t \geq 100$.

5 Adiabatic Effective Dynamics of Solitons

5.1 Effective Hamiltonian

We also consider the equation (4.1) with an external slowly varying potential,

$$\ddot{\psi}(x,t) = \psi''(x,t) - \psi(x,t) + f(\psi(x,t)) - V(x)\psi(x,t), \quad x \in \mathbb{R}, \tag{5.1}$$

The equation is the Hamilton dynamical system with the Hamilton functional

$$\mathcal{H}_V(\psi,\pi) = \int [\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + \frac{|\psi(x)|^2}{2} + U(|\psi(x)|) + V(x)\frac{|\psi(x)|^2}{2}]dx, \tag{5.2}$$

Note that the function (4.6) is not a solution to (5.1) if $V(x) \not\equiv 0$. However, we have observed numerically the solutions close to the solitary manifold for all times, i.e.

$$\psi(x,t) \approx \phi_{v(t),\omega(t)}(x - q(t), 0), \tag{5.3}$$

where $\phi_{v,\omega}$ stands for the function (4.6). The numerical experiments suggest an effective adiabatic dynamics for the parameters q, v, ω of the soliton if the potential $V(x)$ is slowly varying, i.e.

$$\varepsilon := \max |V'(x)| \ll 1. \tag{5.4}$$

Namely, let us choose the initial point from the solitary manifold, i.e.

$$\psi(x,0) = \phi_{v(0),\omega(0)}(x - q(0), 0), \qquad \dot{\psi}(x,0) = \dot{\phi}_{v(0),\omega(0)}(x - q(0), 0), \tag{5.5}$$

with some initial parameters $q(0), v(0), \omega(0)$. If $V(x) \equiv \text{const}$, the solution remains the soliton with the parameters $q(t) = q(0) + v(0)t, v(t) = v(0), \omega(t) = \omega(0)$. Further, let us assume (5.4) hold, i.e. $V(x) \approx \text{const}$. Then it is natural to expect an adiabatic effective dynamics of the parameters if the initial point is sufficiently close to the solitary manifold.

Let us determine the corresponding effective Hamilton functional. Namely, substitute the function (5.3) to the Hamiltonian (5.2), and denote the relativistic momentum $p := v/\sqrt{1 - v^2}$. Then (4.7) implies that

$$\mathcal{H}_V(\psi(\cdot,t), \dot{\psi}(\cdot,t)) \approx \frac{E_0(\omega(t))}{\sqrt{1 - v^2(t)}} + V(q(t))I(p(t),\omega(t)), \quad I(p,\omega) := \frac{1}{2}\int \phi_{v,\omega}^2(x,0)dx \tag{5.6}$$

since the soliton $\phi_{v(t),\omega(t)}(x - q(t), 0)$ is concentrated near the point $q(t)$. Then we define the effective Hamiltonian as follows:

$$\mathcal{H}_{\text{eff}}(Q, P, \Omega) := E_0(\Omega)\sqrt{1 + P^2} + V(Q)I(P, \Omega). \tag{5.7}$$

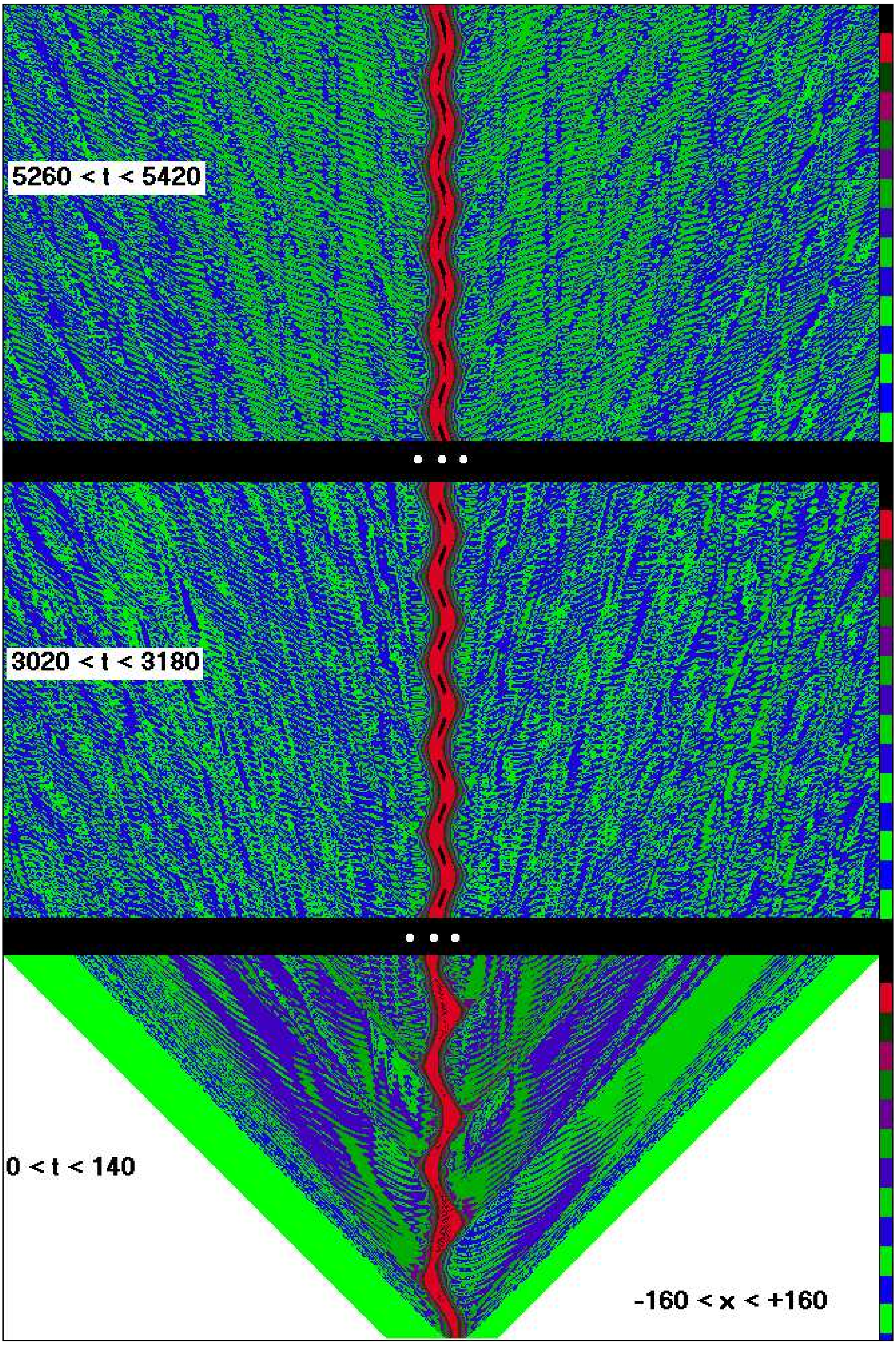

Figure 2: Adiabatic effective dynamics of a soliton

It is natural to expect that the functions $q(t), \omega(t)$ are close to the corresponding components of a trajectory of the Hamilton system

$$\left\{ \begin{array}{ll} \dot{Q} = \nabla_P \mathcal{H}_{\rm eff}(Q, P, \Omega), & \dot{P} = -\nabla_Q \mathcal{H}_{\rm eff}(Q, P, \Omega) \\ \\ \dot{\Theta} = \nabla_\Omega \mathcal{H}_{\rm eff}(Q, P, \Omega), & \dot{\Omega} = -\nabla_\Theta \mathcal{H}_{\rm eff}(Q, P, \Omega) = 0 \end{array} \right. \tag{5.8}$$

since the effective Hamiltonian does not depend on the phase variable Θ. Therefore, $\Omega = \mathrm{const}$, and $Q(t)$ is a solution of two first equations with the initial conditions $Q(0) = q(0)$, $P(0) = p(0)$ and the fixed $\Omega = \omega(0)$. Finally, we expect that $q(t)$ is close to $Q(t)$ in the adiabatic limit, i.e.

$$|q(t) - Q(t)| \le 1, \qquad\qquad |t| \le C\varepsilon^{-1}.$$

Our numerical observations confirm qualitatively the adiabatic effective dynamics. The mathematical proof is still open problem. Similar asymptotics are proved for the solitons of the nonlinear Schrödinger and Hartree Eqns, [14, 15], and for the particle coupled to scalar or Maxwell field [38, 43].

5.2 Numerical Observation

Figure 2 represents a solution of the equation (5.1) with the potential (4.2) where $a = 10$, $m = 6$ and $b = 8.75$, $n = 5$. We choose $V(x) = -0.2\cos(0.31x)$ and the following initial conditions:

$$\psi(x, 0) = \phi_{v(0),\omega(0)}(x - q(0), 0), \qquad \dot{\psi}(x, 0) = 0, \tag{5.9}$$

where $v(0) = 0$, $\omega(0) = 0.6$ and $q(0) = 5.0$. Note that the initial state does not belong to the solitary manifold (5.5) since $\omega(0) \neq 0$. The effective width (half-amplitude) of the soliton is in the interval $[4.4, 5.6]$. The width is sufficiently small w.r.t. the period $2\pi/0.31 \sim 20$ of the potential: it is confirmed by the numerical observation. Namely,

• Blue-green sites represent the amplitudes $|\psi(x, t)| < 0.01$; red sites represent the amplitudes $|\psi(x, t)| \in [0.4, 0.8]$.

• The trajectory of the soliton in the figure 2 ('red snake') is similar to the oscillation of the classical particle.

• For $0 < t < 140$ the solution is not very close to the solitary manifold and we observe an intensive radiation.

• For $3020 < t < 3180$ the solution approaches the solitary manifold, and the radiation is less intensive. The amplitude of the soliton oscillations is almost constant for a large time that corresponds to the effective Hamilton dynamics (5.8).

• On the other hand, for $5260 < t < 5420$ the amplitude of the soliton oscillations is two-times smaller. Hence, the amplitude decays on a larger time scale that contradicts the effective Hamilton dynamics (5.8). Therefore, the Hamilton dynamics may be efficient only in an adiabatic limit like $t \sim \varepsilon^{-1}$.

• The deviation from the Hamilton dynamics is caused by the radiation which plays the role of a dissipation.

• We observe the radiation with the discrete spectrum of the group velocities like fig 1. The magnitude of the solution at the soliton is of order ~ 1 while the radiation field is less 0.01, so its density of the energy is less 0.0001.

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