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A Method of Complex Characteristics for Elliptic Problems in Angles and its Applications

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This paper is dedicated to M. I. Vishik on the occasion of his 80th birthday

ABSTRACT. We develop a method of complex characteristics for boundary value problems for second order elliptic operators with constant coefficients in angles. The method was proposed in 1973 for strongly elliptic second order operators and solutions from the Sobolev space H^s with $s > 3/2$. In the present paper we extend the method to arbitrary elliptic second order irreducible operators and to all solutions from the space of tempered distributions. Main ingredient of the extension is a development of the theory of "smooth" pseudodifferential operators of Vishik and Eskin. The method uses complex Fourier transform, Paley-Wiener theory and Malyshev's automorphic function method on a Riemann surface of complex characteristics of elliptic operators. Various applications of the method are described.

1. Introduction. Exact solutions of boundary value problems in angles

Exact solutions of boundary value problems (b.v.p.) in angles for partial differential equations with constant coefficients are of great importance while dealing with various problems of mathematical physics. For example, such problems appear in the study of diffraction by wedges, guided water waves on a sloping beach, elasticity problems in wedge-shaped domains, etc. In **[17, 18]** a new method of complex characteristics was proposed for explicit solution of general b.v.p. of the following type

$$Au(x) = \sum_{|\alpha|\le 2} a_\alpha \partial_x^\alpha u(x) = 0,\ x \in Q, \tag{1.1}$$

$$B_l u(x) = \sum_{|\alpha|\le m_l} b_{l\alpha} \partial_x^\alpha u(x) = f_l(x),\ x \in \Gamma_l,\ l = 1, 2. \tag{1.2}$$

Here $Q \subset \mathbb{R}^2$ is an open angle of magnitude $\Phi \in (0, \pi)$ with the sides Γ_l, $l = 1, 2$, and $f_l(x)$ are tempered distributions on Γ_l. All the coefficients $a_\alpha, b_{l\alpha}$, are complex,

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$m_l \geq 0$, and the operator A in **[17, 18]** is *strongly elliptic*, i.e.

$$|\tilde{A}(z)| \geq C(1+|z|)^2, \ z \in \mathbb{C}^2, \tag{1.3}$$

where the symbol $\tilde{A}(z) = \sum_{|\alpha|\leq 2} a_\alpha(-iz)^\alpha$, $z \in \mathbb{R}^2$. Boundary conditions satisfy the *strong* Shapiro-Lopatinskii condition

$$|\tilde{B}_l(z)| \geq C(1+|z|)^{m_l}, \ z \in \mathcal{C}_l, \quad l = 1, 2, \tag{1.4}$$

where $\tilde{B}_l(z) = \sum_{|\alpha|\leq m_l} b_{l\alpha}(-iz)^\alpha$, $z \in \mathbb{C}^2$ and $\mathcal{C}_l$ is the contour $\{z \in \mathbb{C}^2 : \tilde{A}(z) = 0, \quad \Im z_l \leq 0\}$.

The method of **[17, 18]** uses the complex Fourier transformation in two variables and the Paley-Wiener theory **[19]**, the theorem on division **[18]** of Bogoliubov-Vladimirov type **[3]**, and Malyshev's method of automorphic functions on the Riemann surface V of the complex characteristics of the operator A:

Definition 1.1. $V = V(A)$ is the set $\{z \in \mathbb{C}^2 : \ \tilde{A}(z) = 0\}$.

Note that in (1.4) the contour $\mathcal{C}_l$ lies on the Riemann surface V.

The general strategy of the method is the following:

I. We derive an algebraic relation on the Riemann surface V between the Fourier transforms of the four Cauchy data of the solution $u(x)$: two at the side Γ_1 and two at the side Γ_2.

II. We couple this relation to the boundary conditions (1.2) which allows us to reduce the relation to one algebraic equation on V with two unknown Cauchy data.

III. We eliminate one unknown function using Malyshev's method of automorphic functions **[30]** and get a functional equation with a shift.

IV. We reduce the functional equation to a Riemann-Hilbert problem which is solved explicitly. Thus, we reconstruct all the Cauchy data, hence the solution.

The complex Fourier transformation in both variables is suggested by the Wiener-Hopf method. However, our method *does not* use a factorization and so it is *not* an extension of the Wiener-Hopf technique (which does not provide the solution to (1.1), (1.2)).

The idea to consider the whole Riemann surface comes from the standard approach based on the analysis of the "stable" and "unstable" complex roots of the symbol $\tilde{A}(z)$ with $z_1 \in \mathbb{R}$ and $\Im z_2 < 0$, $\Im z_2 > 0$, respectively, and also with $z_2 \in \mathbb{R}$ and $\Im z_1 < 0$, $\Im z_1 > 0$.

The algebraic relation between the Cauchy data on V is a natural generalization of the well-known compatibility conditions on real characteristics for hyperbolic PDE.

In **[34, 35]** Merzon extended the method of complex characteristics **[17, 18]** from strongly elliptic operators A to arbitrary second order elliptic operators with an irreducible symbol.

Definition 1.2. A second order operator $A = \sum_{|\alpha|\leq 2} a_\alpha \partial_x^\alpha$ is irreducible if its symbol $\tilde{A}(z)$ is an irreducible polynomial of two variables over $\mathbb{C}$.

After a suitable (complex) affine transformation $(z_1, z_2) = C(z_1, z_2) + B$ such a symbol becomes $z_1^2 + z_2^2 + a$ with $a \neq 0$. Therefore, the (completed) Riemann surface V is isomorphic to $\mathbb{C} \setminus 0$ which is important for the method **[17, 18, 35]**.

Examples 1.3. i) Strongly elliptic operators are irreducible.
ii) The Helmholtz operator $A = \Delta + \omega^2$ is irreducible for any complex $\omega \neq 0$.

In [**34**] and [**35**] Merzon obtained a suitable generalization of the division theorem of [**18**] and the solution to the stationary diffraction problem for the Helmholtz operator

$$A = \Delta + \omega^2 \tag{1.5}$$

with $\omega \in \mathbb{R}$, which is not strongly elliptic. The method of [**17, 18, 34, 35**] is applicable to the Helmholtz operator (1.5) with complex ω which is necessary for diffraction problems.

Finally, in [**20**] the method of complex characteristics was extended to arbitrary angles $\Phi \in (\pi, 2\pi)$. In this case the straightforward application of the Paley-Wiener theory is impossible, and the extension uses an additional duality arguments for the analytic continuation on the Riemann surface V.

In the present paper we extend the method of complex characteristics to all solutions from the space of tempered distributions and arbitrary elliptic irreducible second order operators A. For this purpose we develop a suitable variant of the famous "smoothness property" (or "transmission property") of elliptic pseudodifferential operators with rational symbols discovered initially by Vishik and Eskin [**53**] (see also [**8**]). Note that in [**17, 18**] the solutions from the Sobolev class $H^s(Q)$ were considered with $s > 1/2$, and in [**17, 18, 34, 35**] arbitrary solutions from the space of tempered distributions were considered for the Helmholtz operator (1.5).

Our paper is organized as follows. In Section 2 we expose the extension of the method of complex characteristics to all solutions from the space of tempered distributions and arbitrary elliptic irreducible second order operators A. We describe concisely all steps: the complex Fourier transform, functional equation on the Riemann surface, Malyshev's automorphic function method and the reduction to the Riemann-Hilbert problem. We use the results of [**17, 18**] whenever it is possible. We describe the method and do not formulate its final result as a general theorem: our aim is to derive the equations and notations for concrete applications in the next sections. In Sections 3 and 4 we explain the applications of the method to the solution of the Ursell problem and to the solution of the Neumann problem in angles in Sobolev classes. These results were obtained originally in [**22**] and [**38**]. For the first time we give the derivation of basic equations of [**22, 38**] by the method of complex characteristics.

Let us describe briefly some applications of the method of complex characteristics.

I. In [**17, 18**] the method was used for the analysis of the Fredholm property of boundary value problems on manifolds with edges. The method provides a criterion for the problem to be of Fredholm type.

II. In [**35**] the method was used for the proof of the limiting absorption principle for general b.v.p. for the Helmholtz operator in angles.

III. In [**36, 22**] the authors solved the problem of completeness of Ursell's trapped modes, i.e., it was proved that there are no edge waves apart from those found by Ursell (see below).

IV. In [**38**] Zhevandrov and Merzon analyzed the following natural question: what happens to the Ursell modes if one assumes that the fluid is situated under another liquid layer whose density is much smaller than that of the lower layer, e.g., under a layer of oil or air. In that paper they were confronted with the following mathematical problem: does the solution of the Neumann problem for the Helmholtz equation in an angle with b.c. from $H^{-1/2}(\Gamma_1 \cup \Gamma_2)$, which is given by the Sommerfeld integral, belong to $H^1(Q)$? The answer in the framework of these integrals apparently turns out to present serious difficulties due to slow convergence of the Sommerfeld integrals, and the solutions obtained by Roseau show that this is not always the case. On the other hand, the solution obtained by means of the method of complex characteristics has the form of a two-dimensional Fourier transform. It is tractable by means of the standard integral operator technique and can be shown to coincide with the Sommerfeld integral with a specific kernel [**21, 22, 57**].

V. Recently the method has been applied to the proof of the limiting amplitude principle in Sommerfeld diffraction by wedges [**37**] (see below).

In closing this very brief overview, we would like to emphasize that exact solutions of b.v.p. in angles are also extremely important for b.v.p. in angular domains with curved faces, because the latter, by means of a perturbation technique, can be reduced to recurrent systems of nonhomogeneous problems of type (1.1), (1.2). For example, acoustic and electromagnetic diffraction problems for wedges with curved faces were treated in [**7, 2**]; high-frequency asymptotics of edge waves on a beach of nonconstant slope (whose existence was proved in [**4**]) were established in [**39**]. Note that in those papers the method was not used directly. However, its ideology proved out to be of great use when investigating the analytical properties of solutions to the corresponding functional equations.

Let us comment on previous general approaches to exact solutions of boundary value problems in angles.

In 1895 Macdonald [**28**] gave an integral representation of the Green function for the Dirichlet and Neumann b.v.p. for the Poisson equation in angles of magnitude $\Phi \in (0, 2\pi]$ [**28**].

In 1896 Sommerfeld [**49**] obtained the two Green functions for the Helmholtz equation in angles of magnitude $\Phi \in (0, 2\pi]$ with rational π/Φ. He used integral representations of Macdonald's type and invented a method of multivalued solutions, which is a generalization of the method of images by means of Riemann surfaces.

Later Macdonald, Carlslaw, Bromwich, Herglotz and others have extended Sommerfeld's results to arbitrary $\Phi \in (0, 2\pi]$ by some generalizations of Sommerfeld's approach (see the survey [**41**]). Their methods extend the known method of images from integer π/Φ to arbitrary Φ by means of an integral representation of the Hankel functions [**41**]. For the method of images, a fundamental solution to the Helmholtz equation is selected satisfying Sommerfeld's radiation condition at infinity, so the corresponding solution of the b.v.p. satisfies the radiation condition. However, the limiting amplitude principle is not proved for the constructed solution. In other words, the relation of the stationary diffraction theory to the nonstationary one is not established for wedges up to now.

In 1958 Malyuzhinets extended Sommerfeld's results to diffraction by wedges with impedance b.c. [**29**]: he constructed solutions to the Helmholtz equation in an angle with the Robin b.c. Malyuzhinets' approach assumes the Sommerfeld type representations for solutions. Hence this method gives some particular solutions and does not give all solutions from the appropriate functional class.

In 1969 Shilov [**47**] applied the complex Fourier transformation in one variable to the mixed problem in angles with the Cauchy boundary condition on one side. For the first time, Shilov had established the algebraic relations on some lines on the Riemann surface of complex characteristics. However, he did not solve this equation. A similar analysis has been done by Sobolev [**48**].

In 1970 Malyshev invented a new powerful method of automorphic functions for solving boundary value problems for difference equations on a lattice in a quadrant [**30**]. This method uses complex Fourier transformation in two variables on the lattice and Galois theory in the ring of analytic functions on a Riemann surface. It reduces the boundary value problem to a functional equation with a shift.

Similar equations with a shift were derived previously in some diffraction problems by Malyuzhinets in [**29**]. These equations have been solved in some particular cases (for rational angles etc.) in [**1, 51**].

In 1971 Maz'ya and Plamenevskii [**31**] found an explicit solution to a model problem for the Laplace equation with oblique derivative b.c. in wedges of arbitrary magnitude $\Phi \in (0, 2\pi)$. Their method uses the real Fourier transformation and reduces the problem to a Riemann-Hilbert problem with discontinuous b.c., which is solved explicitly. In [**32**] these authors extended the method of [**31**] to general b.v.p. (1.1), (1.2) with real coefficients $a_\alpha, b_{l\alpha}$ and the angle of magnitude $\Phi \in (0, 2\pi)$.

REMARK 1.4. The method of [**31, 32**] uses essentially the fact that the operator A is strongly elliptic and all coefficients $a_\alpha, b_{l\alpha}$ are real. Therefore, that method cannot be applied to diffraction problems for the Helmholtz operator $A = \Delta + \omega^2$ with $\Re\omega \neq 0$.

In 1992 Eskin [**12**] solved general boundary value problems for the wave equation in an angle. His approach contains the following steps: i) derivation of an algebraic equation with two unknown functions on the Riemann surface, ii) elimination of one unknown function by means of the conformal automorphisms of the Riemann surface which gives an equation with a shift for one function, iii) reduction to a Riemann-Hilbert problem, etc.

In 1998 Meister, Penzel, Speck and Teixeira [**33**] obtained some delicate results for boundary value problems for the Helmholtz equation in a quadrant using the Wiener-Hopf factorization method.

We mention also different methods of qualitative analysis of problems in angles developed by Kondratiev [**23**], Kozlov, Maz'ya and Rossmann [**25**], Nazarov and Plamenevskij [**40**]. Kondratiev's method uses the classical Mellin transform. For the operators A, B_1, B_2 with homogeneous symbols this transform reduces the two-dimensional problem in an angle to a one-dimensional problem with a parameter which can be solved explicitly. Note a very efficient application of the method [**23**] to perfectly conducting waveguides [**5, 6**]. Costabel and Dauge [**9**] consider elliptic systems in the sense of Agmon-Douglis-Nirenberg in plane domains with corners; they decompose the solution into the regular and singular parts which depend smoothly on parameters. Kozlov [**24**] analyzes singularities of the solution to the Dirichlet problem in neighborhoods of corner points in weighted Sobolev and Hölder spaces.

In closing, let us comment on different previous approaches to the Ursell and diffraction problems. Apparently, the first trapped mode in an angle had been found by Stokes [**50**] in 1846. This solution describes the so-called "edge wave" (guided water wave) on a beach of constant slope. In this case the domain is an angle of magnitude $\Phi \in (0, \pi/2)$ and the edge wave is an eigenfunction of a b.v.p. for the Helmholtz equation with mixed boundary conditions (b.c.): the Neumann b.c. on one side of the angle and a Robin b.c. on the other side; the latter contains the spectral parameter.

In 1952 Ursell [**52**] found a family of trapped modes on a sloping beach with arbitrary angle magnitude $\Phi \in (0, \pi/2)$; this family includes the Stokes mode. In 1952 Peters [**43**] obtained solutions describing waves on a sloping beach which correspond to the continuous spectrum of the problem. He looked for the solution in a prescribed form of a line integral and reduced the problem to a linear algebraic equation.

In 1958 Roseau [**46**] constructed a new family of solutions to Ursell's problem which are singular at the vertex of the angle. The method of Roseau is very similar to Peters' method. Whitham [**54**] showed that it is possible to obtain the Ursell modes using the same ideas that were used by Peters and Roseau. In 1989 Evans [**13**] developed Whitham's method and extended Ursell's result to Robin b.c. on both sides of the angle. Packham [**42**] obtained similar results using Sommerfeld-type representation which was already used by Williams in 1959 [**55**] in the context of electromagnetic waves. Williams' method is very close to the one used previously by Malyuzhinets. The strongest known results on the completeness of Ursell's trapped modes were established by Lehman and Lewy [**27**].

Note that any methods based on integral representations cannot imply a complete solution to the problem as they do not describe all solutions *a priori*. Note that the paths of integration in all these integral representations lie on the Riemann surface of complex characteristics. A generalization of Sommerfeld's representation was established for all solutions in [**21**].

The complete solution to the Ursell problem has been established in [**36**] and [**22**] by the method of complex characteristics. The method gives *all* solutions in the class of tempered distributions and does not assume in advance any integral representation of the solution. *A priori*, there are infinitely many solutions with arbitrary order of singularity and the method allows one to select the solution needed by its functional properties without any assumptions about its explicit form.

Sommerfeld's famous solution to the diffraction problem in wedges historically initiated an intensive development of the analysis of problems in angles. The limiting amplitudes for Dirichlet or Neumann b.c. were given in [**49**] in 1896. However, the limiting amplitude principle has not been proved for about 100 years. In the paper [**45**] the limiting amplitude principle has been considered for a particular explicit solution to the nonstationary diffraction problem. However, the existence and uniqueness of the solution from a functional class has not been discussed.

In [**37**] the limiting amplitude principle for a wedge is stated for a sufficiently general class of incident plane waves of sinusoidal type with Dirichlet or Neumann b.c. For the first time the solution is considered in an exact functional class, and the existence and uniqueness of the Cauchy problem is proved. The proofs use the method of complex characteristics [**17**]–[**20**]. This method allows one to analyze the limiting amplitudes of solutions for diffraction by wedges with arbitrary linear b.c. (1.2) with constant coefficients.

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2. Method of complex characteristics

In this section we describe briefly the method of complex characteristics. The method reduces general b.v.p. in a plane angle for second order elliptic equation to some equations on the boundary and solves them explicitly. The main steps of the method are the following:

I. Extension to the plane
II. Existence of boundary values
III. Reformulation via the Cauchy data
IV. Boundary conditions via the Cauchy data
V. Boundary densities via the Cauchy data
VI. Relation between the Cauchy data on complex characteristics
VII. Undetermined algebraic equation on the Riemann surface
VIII. Malyshev's automorphic functions method
IX. Reduction to the Riemann-Hilbert problem

We consider a b.v.p. of type (1.1), (1.2) for an elliptic operator A, i.e., its principal part A_2 satisfies

$$\tilde{A}_2(z) := \sum_{|\alpha|=2} a_\alpha (iz)^\alpha \neq 0, \;\; z \in \mathbb{R}^2 \setminus 0, \tag{2.1}$$

for the case $\Phi < \pi$ for simplicity. Then by a (real) linear change of variables, we can transform the angle Q to the first quadrant $K = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$.

I. Extension to the plane. First, we have to justify the statement of b.v.p. (1.1), (1.2) for the solution of "arbitrary singularity". More precisely, we consider equation (1.1) in the sense of distributions. Namely, our goal is to consider all solutions of the Schwartz class $S'(K)$: $u(x) \in S'(K)$ means that $u(x)$ is the restriction of a distribution $U(x) \in S'(\mathbb{R}^2)$ to the open region K. It is easy to prove that we can assume $U(x) \in S'(\overline{K})$, i.e. $U(x) \in S'(\mathbb{R}^2)$ and $\operatorname{supp} U(x) \subset \overline{K}$. Then equation (1.1) implies

$$AU(x) = \gamma(x), \;\; x \in \mathbb{R}^2, \tag{2.2}$$

where the "boundary density" $\gamma \in S'(\partial K)$, i.e. $\gamma \in S'(\mathbb{R}^2)$ and $\operatorname{supp}\gamma \subset \partial K$. Conversely, equation (2.2) with $\gamma \in S'(\partial K)$ implies (1.1) for $u := U|_K$. Therefore, we have

PROPOSITION 2.1. *Equation (1.1) for $u \in S'(K)$ is equivalent to (2.2) with $U \in S'(\overline{K})$ and the boundary density $\gamma \in S'(\partial K)$: the equivalence is given by $u = U|_K$.*

II. Existence of boundary values. Next we have to justify the statement of boundary conditions (1.2). We understand boundary conditions (1.2) in the following sense. Let $u(x)$ be a solution of the homogeneous equation (1.1). Then, by ellipticity, $u \in C^\infty(K)$ and its traces on the rays $x_i = \varepsilon$, $\varepsilon > 0$, parallel to the sides of the angle are well-defined. It is easy to see that these traces belong to $S'(\mathbb{R}^+)$ (with the natural identification of $\mathbb{R}^+$ and the rays mentioned above). The same is valid for the normal derivatives of u on these rays, and for all derivatives of u on these rays as well. Let us prove the existence of the limits in the sense of distributions as $\varepsilon \to 0+$.

Recall that $S'(\mathbb{R}^+)$ is the space of restrictions of distributions from $S'(\mathbb{R})$ to $\mathbb{R}^+$. More precisely, $S'(\mathbb{R}^+)$ is the factorspace $S'(\mathbb{R})/S'(\overline{\mathbb{R}^-})$ where $S'(\overline{\mathbb{R}^\pm}) = \{f \in$

$S'(\mathbb{R})$: supp $f \subset \overline{\mathbb{R}^{\pm}}\}$. Note that the solution $u(x)$ is smooth in K due to the ellipticity of the operator A.

DEFINITION 2.2. For any $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 = 0, 1, 2, \ldots$ set

$$\left\{ \begin{array}{l} \partial_x^\alpha u(x_1, 0+) = \lim\limits_{\varepsilon \to 0+} \partial_x^\alpha u(x_1, \varepsilon),\ x_1 > 0, \\ \partial_x^\alpha u(0+, x_2) = \lim\limits_{\varepsilon \to 0+} \partial_x^\alpha u(\varepsilon, x_2),\ x_2 > 0, \end{array} \right. \tag{2.3}$$

if the convergence holds in $S'(\mathbb{R}^+)$.

The limits exist in the case of elliptic operators.

PROPOSITION 2.3. *Let A be an elliptic second order operator (2.1). Then for any solution $u(x) \in S'(K)$ to equation (1.1), the limits (2.3) exist for each $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 = 0, 1, 2, \ldots$.*

PROOF. We develop a variant of the theory of the "smoothness property" [**53**] (or "transmission" property, see [**8**]). First let us consider the simplest case when the operator A is strongly elliptic (see (1.3)). Then there exists a unique fundamental solution $E(x) \in S'(\mathbb{R}^2)$ of A, which is smooth for $x \neq 0$ and decays exponentially as $|x| \to \infty$. Therefore (2.2) implies the representation of $U(x)$ as the potential of the boundary density γ

$$U(x) = E * \gamma(x) := \langle E(x-y), \gamma(y) \rangle,\ x \in \mathbb{R}^2. \tag{2.4}$$

Then the existence of the limits (2.3) would follow by a modification of the methods of [**8, 53**]. Below we give a suitable modification for the general case of an elliptic operator A.

For the general case, we modify slightly the arguments connected with the convolution (2.4). Since the boundary density $\gamma(x) \in S'(\partial K)$, it admits a representation

$$\gamma(x) = \sum_{0 \le k \le s} g_k(x_1) \delta^{(k)}(x_2) + \sum_{0 \le k \le s} h_k(x_2) \delta^{(k)}(x_1),\ x \in \mathbb{R}^2, \tag{2.5}$$

where $s < \infty$ corresponds to the singularity of the solution $u(x)$ (or $U(x)$), and $g_k, h_k \in S'(\overline{\mathbb{R}^+})$. Let us define the Fourier transform as $F_{x \to z}[\phi](z) := \tilde{\phi}(z) = \int e^{izx} \phi(x) dx$ for test functions $\phi \in C_0^\infty(\mathbb{R}^2)$, and as the extension by continuity for tempered distributions. In the Fourier representation, (2.2) and (2.5) become

$$\tilde{\gamma}(z) = \tilde{A}(z) \tilde{U}(z) = \sum_{0 \le k \le s} \tilde{g}_k(z_1)(-iz_2)^k + \sum_{0 \le k \le s} \tilde{h}_k(z_2)(-iz_1)^k,\ z \in \mathbb{R}^2. \tag{2.6}$$

As the operator A is elliptic, there exist an $R > 0$ such that

$$\tilde{A}(z) \neq 0,\ |z| \ge R. \tag{2.7}$$

Choose a cutoff function $\psi(x) \in C^\infty(\mathbb{R}^2)$ such that $\psi(z) = 0$, $|z| \le R$ and $\psi(z) = 1$, $|z| \ge 2R$. Then (2.6) implies

$$\begin{aligned} \tilde{U}(z) &= \psi(z) \left(\sum_{0 \le k \le s} \tilde{g}_k(z_1)(-iz_2)^k + \sum_{0 \le k \le s} \tilde{h}_k(z_2)(-iz_1)^k \right) / \tilde{A}(z) \\ &\quad + (1 - \psi(z)) \tilde{U}(z), \\ &=: \tilde{U}_1(z) + \tilde{U}_2(z),\ z \in \mathbb{R}^2. \end{aligned} \tag{2.8}$$

Obviously, $U_2(x) := F^{-1}_{z\to x}\Big[(1-\psi(z))\tilde U(z)\Big]$ is a smooth function in $\mathbb{R}^2$ with power bounds for each derivative. Hence, for this function all limits (2.3) exist in $S'(\overline{\mathbb{R}^+})$. It remains to analyze the function

(2.9)

$$U_1(x) := F^{-1}_{z\to x}\left[\psi(z)\left(\sum_{0\le k\le s}\tilde g_k(z_1)(-iz_2)^k + \sum_{0\le k\le s}\tilde h_k(z_2)(-iz_1)^k\right)/\tilde A(z)\right],$$

where $x\in\mathbb{R}^2$. Let us denote $E_\psi(x) := F^{-1}_{z\to x}\Big[\psi(z)/\tilde A(z)\Big]$. Obviously, $\psi(z)/\tilde A(z)$ is the symbol of a *classical* pseudodifferential operator (PDO) [**16**]. For any multi-index α and any $N>0$, we have by the standard technique of PDO:

$$\left\{\begin{array}{l} E_\psi(x)\in C^\infty(\mathbb{R}^2\setminus\{0\}),\\ |\partial_x^\alpha E_\psi(x)|\le C_\alpha |x|^{-1-|\alpha|},\ \ |x|<1,\\ |\partial_x^\alpha E_\psi(x)|\le C_{\alpha,N}(1+|x|)^{-N},\ \ |x|>1. \end{array}\right. \tag{2.10}$$

Therefore, (2.9) can be rewritten as the convolution (cf. (2.4))

$$\begin{aligned} U_1(x) &= \langle \sum_{0\le k\le s} g_k(y_1)\delta^{(k)}(y_2), E_\psi(x-y)\rangle \\ &\quad +\langle \sum_{0\le k\le s} h_k(y_2)\delta^{(k)}(y_1), E_\psi(x-y)\rangle \\ &=: U_{11}(x)+U_{12}(x),\ \ x\in\mathbb{R}^2. \end{aligned} \tag{2.11}$$

Let us prove the existence of limits (2.3) on the side $\Gamma_1=\{(0,x_1)\in\mathbb{R}^2 : x_1>0\}$. For the function $U_{12}(x)$ all the limits exist as its any derivative admits a power bound x_1^{-p} in the halfplane $x_1>0$ by (2.10). Hence, we get from (2.8) and (2.11) that

$$U(x)=U_{11}(x)+R(x),\ \ x\in\mathbb{R}^2, \tag{2.12}$$

where $R(x) := U_{12}(x)+U_2(x)$ is smooth in the halfplane $x_1>0$, and its any derivative admits a power bound x_1^{-p} in the halfplane $x_1>0$. Hence, the function $R(x)$ admits an extension to a smooth function of $x_2\in\mathbb{R}$ with values in $S'(\overline{\mathbb{R}^+})$, so for $R(x)$ all the limits (2.3) exist in $S'(\mathbb{R}^+)$.

For the analysis of U_{11}, we return to the Fourier transform since the convolution with E_ψ is a *classical* pseudodifferential operator,

$$\tilde U_{11}(z)=\psi(z)\left(\sum_{0\le k\le s}\tilde g_k(z_1)(-iz_2)^k/\tilde A(z)\right),\ \ z\in\mathbb{R}^2. \tag{2.13}$$

For any $k,N=0,1,2,\ldots$, we have the expansions (cf. [**8, 53**])

$$\begin{aligned} \psi(z)(-iz_2)^k/\tilde A(z) &= \sum_{j=0}^{k-2}P_{kj}(z_1)(-iz_2)^{k-2-j} \\ &\quad +\sum_{j=1}^{N-1}Q_{kj}(z_1)(z_2+i)^{-j}+R_{kN}(z_1,z_2), \end{aligned} \tag{2.14}$$

where $z \in \mathbb{R}^2$. Here P_{kj}, Q_{kj} are polynomials of degrees $\le j$, $\le k-2+j$, respectively, and the remainder R_{kN} admits the bound

$$|\partial_{z_1}^m R_{kN}(z_1, z_2)| \le C_{m,N}(1+|z_1|)^{2N-2-m}(1+|z_2|)^{-N} \tag{2.15}$$

for any $m = 0, 1, 2, \ldots$. Therefore, for each fixed $z_2 \in \mathbb{R}$, $R_{kN}(z_1, z_2)$ is a continuous multiplicator in $S'(\mathbb{R})$ with the following finite seminorms: for all $M > 0$ and some $p_M := -2N + 2 + m$,

$$n_M(z_2) := \sup_{|\alpha| \le M} \sup_{z_1 \in \mathbb{R}} |\partial_{z_1}^\alpha R_{kN}(z_1, z_2)|(1+|z_1|)^{-p_M} < \infty.$$

Moreover, estimates (2.15) imply that the seminorms are summable in $z_2 \in \mathbb{R}$ if $N \ge 2$. Therefore, performing the partial inverse Fourier transform $F^{-1}_{z_2 \to x_2}$ in (2.14), we get for $N \ge 2$

$$\begin{aligned} F^{-1}_{z_2 \to x_2}\left[\psi(z)(-iz_2)^k/\tilde{A}(z)\right] &= \sum_{j=0}^{k-2} P_{kj}(z_1)\delta^{(k-2-j)}(x_2) \\ &+ \sum_{j=1}^{N-1} Q_{kj}(z_1) C_j x_2^{j-1}\theta(x_2)e^{-x_2} \\ &+ R^*_{kN}(z_1, x_2), \ (z_1, x_2) \in \mathbb{R}^2. \end{aligned} \tag{2.16}$$

Here the function $R^*_{kN}(z_1, x_2)$ is a continuous function with power bounds for each derivative in z_1 as $N \ge 2$. Therefore, for each fixed x_2, $R^*_{kN}(z_1, x_2)$ is a continuous multiplicator in $S'(\mathbb{R})$ which depends continuously on $x_2 \in \mathbb{R}$. Hence, applying $F^{-1}_{z \to x}$ to (2.13), we get

$$\begin{aligned} U_{11}(x) &= \sum_{0 \le k \le s} \Big[\sum_{j=0}^{k-2} \delta^{(k-2-j)}(x_2) F^{-1}_{z_1 \to x_1} P_{kj}(z_1)\tilde{g}_k(z_1) \\ &+ \sum_{j=1}^{N-1} x_2^{j-1}\theta(x_2)e^{-x_2} F^{-1}_{z_1 \to x_1} Q^*_{kj}(z_1)\tilde{g}_k(z_1) \\ &+ F^{-1}_{z_1 \to x_1} R^*_{kN}(z_1, x_2)\tilde{g}_k(z_1)\Big], \ x \in \mathbb{R}^2. \end{aligned} \tag{2.17}$$

Therefore, the function $U_{11}(x_1, x_2)$ is a continuous function of $x_2 \in \mathbb{R}$ with values in $S'(\mathbb{R})$. □

III. Reformulation via the Cauchy data. Our general strategy is the following:

A. We plan to derive the solution $U(x)$ from equation (2.2) by the Fourier transform. Hence, we have to express the boundary density $\gamma(x)$ via the boundary data $f_1(x_1)$, $f_2(x_2)$ of problem (1.1), (1.2).

B. To do so, we would like to express γ and the boundary conditions (1.2) via the Cauchy data of the solution $u(x)$,

$$\left\{ \begin{array}{l} u_{10}(x_1) = u(x_1, 0+), \ u_{11}(x_1) = \dfrac{\partial}{\partial x_2} u(x_1, 0+), \ x_1 > 0, \\ u_{20}(x_2) = u(0+, x_2), \ u_{21}(x_2) = \dfrac{\partial}{\partial x_1} u(0+, x_2), \ x_2 > 0, \end{array} \right. \tag{2.18}$$

which exist by Proposition 2.3.

C. The final step is the derivation of the Cauchy data in terms of the functions f_1 and f_2. Then the boundary density $\gamma(x)$ is also known, and the solution $U(x)$ can be found from equation (2.2) by the Fourier transform.

IV. Boundary conditions via the Cauchy data. In the simplest cases boundary conditions just specify some of the Cauchy data. For instance, for the Dirichlet problem we have $u_{l0}(x_l) = f_l$, $l = 1, 2$.

For a general elliptic operator A, boundary conditions (1.2) can be expressed via the Cauchy data of the solution by the Cauchy-Kovalevskaya method in the following form:

$$\begin{cases} B_{10}u_{10}(x_1) + B_{11}u_{11}(x_1) = f_1(x_1), \; x_1 \in \mathbb{R}^+, \\ \\ B_{20}u_{20}(x_2) + B_{21}u_{21}(x_2) = f_2(x_2), \; x_2 \in \mathbb{R}^+. \end{cases} \tag{2.19}$$

Here B_{lk} are some differential operators, $B_{lk} = \sum_{n \le m_l} b_{lkn}\partial^n_{x_l}$, with the symbols $\tilde{B}_{lk}(z_l) = \sum_{n \le m_l} b_{lkn}(-iz_l)^n$, and

$$\begin{cases} \tilde{B}_{10}(z_1) + \tilde{B}_{11}(z_1)(-iz_2) = \tilde{B}_1(z) \bmod \tilde{A}(z), \\ \\ \tilde{B}_{20}(z_2) + \tilde{B}_{21}(z_2)(-iz_1) = \tilde{B}_2(z) \bmod \tilde{A}(z). \end{cases} \tag{2.20}$$

Therefore, the Shapiro-Lopatinskii condition (1.4) implies

$$B_{10}(z_1) + B_{11}(z_1)(-iz_2) \not\equiv 0, \qquad B_{20}(z_2) + B_{21}(z_2)(-iz_1) \not\equiv 0. \tag{2.21}$$

Let us fix arbitrary extensions $\mathrm{v}_{lk}(x_l) \in S'(\overline{\mathbb{R}^+})$ of the Cauchy data $u_{lk}(x_l)$ by zero for $x_l < 0$. Then (2.19) becomes

$$\begin{cases} B_{10}\mathrm{v}_{10}(x_1) + B_{11}\mathrm{v}_{11}(x_1) = f_1^0(x_1) + \sum\limits_{|\alpha| \le s_1} C_{1\alpha}\delta^{(\alpha)}(x_1), \; x_1 \in \mathbb{R}, \\ \\ B_{20}\mathrm{v}_{20}(x_2) + B_{21}\mathrm{v}_{21}(x_2) = f_2^0(x_2) + \sum\limits_{|\alpha| \le s_2} C_{1\alpha}\delta^{(\alpha)}(x_2), \; x_2 \in \mathbb{R}, \end{cases} \tag{2.22}$$

where $f_l^0(x_l) \in S'(\overline{\mathbb{R}^+})$ stand for some fixed extensions of $f_l^0(x_l)$ by zero for $x_l < 0$, and $C_{l\alpha}$ are some complex constants.

V. Boundary densities via the Cauchy data. For a general continuation $U \in S'(\mathbb{R}^2)$ of the solution u, the distribution γ cannot be expressed via the Cauchy data as the continuation U can contain "spurious" distributions with support in ∂K. We will construct a "canonical" extension $U^0(x) \in S'(\mathbb{R}^2)$ for which the corresponding density γ^0 is expressed uniquely through the Cauchy data $u_{lk}(x_l)$ modulo a finite sum $\sum_{\alpha \le s} C_\alpha \delta^{(\alpha)}(x)$. The magnitude of s depends on the order of the singularity of the solution. Then the choice of the constants C_α determines the solution uniquely. Let us emphasize however, that the constants are not arbitrary. We will discuss the choice of the constants below.

Rewrite the operator A in two different forms,

$$A = \sum_{j \le 2} A_{1j}\partial^j_{x_2} = \sum_{j \le 2} A_{2j}\partial^j_{x_1} \tag{2.23}$$

where $A_{lj} = A_{lj}(\partial_{x_l})$ are some differential operators of order $\le 2 - j$.

PROPOSITION 2.4. *Consider equation (1.1) with an elliptic operator A. Then for any solution $u(x) \in S'(K)$, there exists an extension $U^0(x) \in S'(\mathbb{R}^2)$ such that* *with some* $s > 0$ *and* $C_\alpha \in \mathbb{C}$,

$$\gamma^0(x) := AU^0(x) = \sum_{k=0,1} \sum_{1+k\le j\le 2} A_{1j} \mathrm{v}_{1k}(x_1)\delta^{(j-1-k)}(x_2) + \sum_{k=0,1} \sum_{1+k\le j\le 2} A_{2j} \mathrm{v}_{2k}(x_2)\delta^{(j-1-k)}(x_1) + \sum_{|\alpha|\le s} C_\alpha \delta^{(\alpha)}(x),\ x \in \mathbb{R}^2, \tag{2.24}$$

where v_{lj} *are the extensions of the Cauchy data fixed above.*

PROOF. First, consider a simple case when the solution $u(x)$ is a smooth function in $\overline{K}$. Then define U^0 as the continuation by zero:

$$U^0(x) := \begin{cases} u(x),\ x \in K, \\ 0,\ x \in \mathbb{R}^2 \setminus K. \end{cases} \tag{2.25}$$

Then (2.24) follows by the well-known formula for the distribution derivative of a piecewise continuous function. Hence, the density γ and the solution u are determined uniquely by the Cauchy data.

Now consider the general case when the solution $u(x) \in S'(K)$. Let us start with an arbitrary extension $U \in S'(\mathbb{R}^2)$. Then (2.2) holds with a density $\gamma \in S'(\mathbb{R}^2)$ of the form (2.5).

We can reduce the expression (2.5) for γ by means of the division by the operator A. Namely, divide the first sum in (2.6) by the symbol $\tilde{A}(z_1, z_2)$ in the module of polynomials in z_2 with coefficients in $S'(\mathbb{R})$ which depend on z_1. Similarly, we divide the second sum in (2.6) by the symbol $\tilde{A}(z_1, z_2)$ in the module of polynomials in z_1 with coefficients in $S'(\mathbb{R})$ which depend on z_2. Then the result is

$$\tilde{\gamma}(z) = \tilde{A}(z)\tilde{U}(z) = \tilde{A}(z)\tilde{S}(z) + \sum_{k=0,1} \tilde{G}_k(z_1)(-iz_2)^k + \sum_{k=0,1} \tilde{H}_k(z_2)(-iz_1)^k, \tag{2.26}$$

where $z \in \mathbb{R}^2$, $\tilde{S}(z)$ is a distribution of the form (2.5), and $G_k, H_k \in S'(\overline{\mathbb{R}^+})$. Then

$$\gamma(x) = AU(x) = AS(x) + \sum_{k=0,1} G_k(x_1)\delta^{(k)}(x_2) + \sum_{k=0,1} H_k(x_2)\delta^{(k)}(x_1),\ x \in \mathbb{R}^2, \tag{2.27}$$

where $S \in S'(\mathbb{R}^2)$, $\operatorname{supp} S \subset \partial K$. Finally, define the canonical extension by $U^0 := U - S$. Then

$$\gamma^0(x) := AU^0(x) = \sum_{k=0,1} G_k(x_1)\delta^{(k)}(x_2) + \sum_{k=0,1} H_k(x_2)\delta^{(k)}(x_1),\ x \in \mathbb{R}^2. \tag{2.28}$$

For the simplification of the exposition consider further the operator $A = \Delta + a$. Then the assertion (2.24) that we have to prove becomes

$$\gamma^0(x) := AU^0(x) = \delta(x_2)\mathrm{v}_{11}(x_1) + \delta'(x_2)\mathrm{v}_{10}(x_1) + \delta(x_1)\mathrm{v}_{21}(x_2) + \delta'(x_1)\mathrm{v}_{20}(x_2) + \sum_{|\alpha|\le s} C_\alpha \delta^{(\alpha)}(x),\ x \in \mathbb{R}^2. \tag{2.29}$$

We deduce (2.29) from (2.28) verifying that the distributions G_k and H_k are some extensions of the Cauchy data, i.e.

$$\left\{ \begin{array}{l} G_{1-k}(x_1) = \mathrm{v}_{1k}(x_1), \ x_1 > 0, \\ \\ H_{1-k}(x_2) = \mathrm{v}_{2k}(x_2), \ x_2 > 0, \end{array} \right. \qquad k = 0, 1. \tag{2.30}$$

To verify the first line, we will prove the following expansion in the halfplane $x_2 > 0$:

$$\frac{\partial^2 U^0(x)}{\partial x_2^2} = G_1(x_1)\delta'(x_2) + G_0(x_1)\delta(x_2) + r(x), \quad x_1 > 0, \tag{2.31}$$

where $r(x)$ is a regular function of $x_2 \in \mathbb{R}$ with values in $S'(\mathbb{R}^+)$.

Lemma 2.5. *Expansion (2.31) holds with the remainder $r(x)$ which is a piecewise continuous function of $x_2 \in \mathbb{R}$ with values in $S'(\mathbb{R}^+)$ in the following sense: for any test function $\phi(x) \in C_0^\infty(\mathbb{R}^2)$ with support in the halfplane $x_2 > 0$*

$$\langle r(x), \phi(x) \rangle = \int\limits_{\mathbb{R}^2} \langle r(\cdot, x_2), \phi(\cdot, x_2) \rangle dx_2, \tag{2.32}$$

where $r(\cdot, x_2)$ is a continuous function of $x_2 \in \overline{\mathbb{R}^\pm}$ with values in $S'(\overline{\mathbb{R}^+})$.

Proof. We follow the arguments (2.12)-(2.17). Namely, the Fourier transform of (2.28) gives an equation similar to (2.6). Therefore, U^0 admits the expansion $U^0 = \mathrm{v}_{11} + R^0$ of type (2.12), where

$$\mathrm{v}_{11}(z) = \psi(z) \left(\sum_{k=0,1} \tilde{G}_k(z_1)(-iz_2)^k / \tilde{A}(z) \right), \ z \in \mathbb{R}^2. \tag{2.33}$$

Any derivative of $R^0(x)$ admits a power bound x_1^{-p} in the halfplane $x_1 > 0$. Hence, the function $R^0(x)$ admits an extension to a smooth function of $x_2 \in \mathbb{R}$ with values in $S'(\overline{\mathbb{R}^+})$, so for $\partial_{x_2}^2 R^0(x)$ the expansion of type (2.31) holds.

Therefore, it suffices to prove the expansion (2.31) for v_{11}. For this purpose we apply the arguments (2.13)-(2.17) to $\partial_{x_2}^2 U_{11}(x)$. First, (2.33) gives for the Fourier transform of this function,

$$(-iz_2)^2 \tilde{\mathrm{v}}_{11}(z) = \psi(z) \left(\sum_{k=0,1} \tilde{G}_k(z_1)(-iz_2)^{k+2} / \tilde{A}(z) \right), \ z \in \mathbb{R}^2. \tag{2.34}$$

Further, for $A = \Delta + a$ and $k = 0, 1$, we have similarly to (2.14)

$$\psi(z)(-iz_2)^{k+2}/\tilde{A}(z) \sim (-iz_2)^k + \sum_{j \geq 1} Q_{kj}(z_1)(z_2 + i)^{k-2j}, \ |z_2| \to \infty, \tag{2.35}$$

where Q_{kj} are some polynomials. Then (2.31) and (2.32) follow from the expansion of type (2.17) for $\partial_{x_2}^2 \mathrm{v}_{11}(x)$. □

Finally, it is easy to check that (2.31) together with Lemma 2.5 imply the first line of (2.30), and the second line follows similarly. Now (2.30) implies that

$$G_{1-k}(x_1) = \mathrm{v}_{1k}(x_1) + \sum_{|\alpha| \leq s_1} D_{1\alpha} \delta^{(\alpha)}(x_1),$$

$$H_{1-k}(x_2) = \mathrm{v}_{2k}(x_2) + \sum_{|\alpha| \leq s_2} D_{2\alpha} \delta^{(\alpha)}(x_2).$$

Substituting these expressions in (2.28), we get (2.24). Therefore, Proposition 2.4 is proved for the operator $A = \Delta + a$. The proof for general elliptic operators is similar. $\square$

VI. Relation between the Cauchy data on complex characteristics. We planned to find the Cauchy data of the solution of the boundary value problem (1.1), (1.2) to express the boundary density $\gamma^0(x)$ in (2.24) and then to find the solution $u(x)$.

Boundary conditions in the form (2.19) give two equations for the Cauchy data. So we need at least two additional equations for the Cauchy data. It is quite surprising that the single equation (2.2) provides the missing algebraic equations for the Cauchy data.

Let us apply the complex Fourier transform to (2.24). Recall that $\operatorname{supp} U^0 \subset \overline{K}$ and $\operatorname{supp} \gamma^0 \subset \partial K$. Therefore, by the Paley-Wiener theorem [**19**, Thm. II.5.2, p.161], $\tilde{U}^0(z)$ and $\tilde{\gamma}^0(z)$ are analytic functions of two complex variables in the tube domain $\mathbb{C}K^* := \{z \in \mathbb{C}^2 : \Im z_1 > 0, \Im z_2 > 0\}$, and

$$\tilde{U}^0(z) = \langle U^0(x), e^{ixz} \rangle, \qquad \tilde{\gamma}^0(z) = \langle \gamma^0(x), e^{ixz} \rangle, \quad z \in \mathbb{C}K^*. \tag{2.36}$$

Hence, (2.2) implies

$$\tilde{A}(z)\tilde{U}^0(z) = \tilde{\gamma}^0(z), \;\; z \in \mathbb{C}K^*. \tag{2.37}$$

Finally, this equation implies the key relation (we call it the "connection equation") on the Riemann surface V of the complex characteristics of the operator A (see Definition 1.1),

$$\tilde{\gamma}^0(z) = 0, \;\; z \in V^* := V \cap \mathbb{C}K^*. \tag{2.38}$$

It relates the Cauchy data by (2.24). Together with the boundary conditions (2.19), equation (2.38) allows us to express the Cauchy data via the functions f_1, f_2.

Relation (2.38) is the necessary condition for the existence of the solution $U(x) \in S'(\overline{K})$ to equation (2.2) with a given distribution $\gamma(x)$. It is very important that equation (2.38) is indeed equivalent to equation (2.2).

PROPOSITION 2.6. *Let $A \neq 0$ be an arbitrary differential operator with constant coefficients in $\mathbb{R}^2$ with an irreducible symbol, and let a tempered distribution $\gamma^0(x) \in S'(\overline{K})$ satisfy relation (2.38). Then the equation $AU^0(x) = \gamma^0(x)$, $x \in \mathbb{R}^2$, admits a unique solution $U^0(x) \in S'(\overline{K})$.*

PROOF. Equation (2.37) determines an analytic function $\tilde{U}^0(z)$ uniquely in $\mathbb{C}K^*$ since the irreducible symbol $\tilde{A}(z)$ is not identically zero in $\mathbb{C}K^*$:

$$\tilde{U}^0(z) = \tilde{\gamma}^0(z)/\tilde{A}(z), \;\; z \in \mathbb{C}K^* \setminus V. \tag{2.39}$$

A priori, the quotient may be nonanalytic in $\mathbb{C}K^*$ if the symbol $\tilde{A}(z)$ vanishes at some points there. However, as the relation (2.38) holds, the quotient (2.39) is an analytic function in $\mathbb{C}K^*$ as the symbol $\tilde{A}(z)$ is irreducible.

Furthermore, we have to justify yet that the analytic quotient corresponds to a tempered distribution $U^0(x) \in S'(\overline{K})$. This follows by the general theorem on division from [**18, 34**]: the quotient (2.39) is the Fourier transform of the (unique) $U^0(x) \in S'(\overline{K})$ as the quotient is analytic in $\mathbb{C}K^*$. $\square$

Below, we use the identity (2.38) together with the boundary conditions (2.19) as an algebraic system, which allows us to reconstruct $\gamma^0(x)$ in terms of the functions f_1 and f_2. Then the solution $u(x)$ also is determined uniquely:

$$u(x) = F^{-1}_{z\to x}[\tilde{\gamma}^0(z)/\tilde{A}(z)], \; x \in K. \tag{2.40}$$

VII. Undetermined algebraic equation on the Riemann surface. Substituting the expression (2.24) for γ^0 into (2.38), we get an algebraic equation for the Fourier transforms of the Cauchy data

$$\begin{aligned} \tilde{\gamma}^0(z) \;\; := & \sum_{k=0,1}\sum_{1+k\le j\le 2} \tilde{A}_{1j}(z_1)\tilde{\mathrm{v}}_{1k}(z_1)(-iz_2)^{j-1-k} \\ & + \sum_{k=0,1}\sum_{1+k\le j\le 2} \tilde{A}_{2j}(z_2)\tilde{\mathrm{v}}_{2k}(z_2)(-iz_1)^{j-1-k} \\ & + \sum_{|\alpha|\le s} C_\alpha(-iz)^\alpha = 0, \quad z\in V^*. \end{aligned} \tag{2.41}$$

Similarly, boundary conditions (2.22) give two algebraic equations after the complex Fourier transformation:

$$\begin{cases} \tilde{B}_{10}(z_1)\tilde{\mathrm{v}}_{10}(z_1)+\tilde{B}_{11}(z_1)\tilde{\mathrm{v}}_{11}(z_1)=\tilde{f}_1^0(z_1)+\sum\limits_{|\alpha|\le s_1} C_{1\alpha}(-iz_1)^\alpha, \; \Im z_1>0, \\ \tilde{B}_{20}(z_2)\tilde{\mathrm{v}}_{20}(z_2)+\tilde{B}_{21}(z_2)\tilde{\mathrm{v}}_{21}(z_2)=\tilde{f}_2^0(z_2)+\sum\limits_{|\alpha|\le s_2} C_{2\alpha}(-iz_2)^\alpha, \; \Im z_2>0. \end{cases} \tag{2.42}$$

REMARK 2.7. Equations (2.41), (2.42), generally speaking, are not solvable for an arbitrary choice of the constants $C_\alpha, C_{l\alpha}$. The constants are determined from these equations simultaneously with the unknown functions $\tilde{\mathrm{v}}_{lk}$.

Let us summarize the equivalence between the algebraic system (2.41), (2.42) and the original boundary value problem (1.1), (1.2).

THEOREM 2.8. *Let the second order operator A be elliptic and irreducible, the Shapiro-Lopatinskii condition (1.4) hold, and $f_l \in S'(\mathbb{R}^+)$. Then*

i) For any solution $u \in S'(\mathbb{R}^2)$ to (1.1), (1.2), arbitrary extensions of the Cauchy data give a solution $\tilde{\mathrm{v}}_{lk}(z_l) \in F[S'(\overline{\mathbb{R}^+})]$, $l=1,2$, $k=0,1$, to system (2.41), (2.42) with some constants C_α, $C_{1\alpha}$, $C_{2\alpha}$.

ii) Conversely, for arbitrary constants C_α, $C_{1\alpha}$, $C_{2\alpha}$, any solution $\tilde{\mathrm{v}}_{lk}(z_l) \in F[S'(\overline{\mathbb{R}^+})]$, $l=1,2$, $k=0,1$, to system (2.41), (2.42), if it exists, gives a solution $u \in S'(\mathbb{R}^2)$ to (1.1), (1.2) by formula (2.40).

Below we give a complete solution to system (2.41), (2.42). In this section, we reduce the system to one functional equation with a shift on the Riemann surface V with two unknown functions.

First, we rewrite (2.42) as equations on V. Namely, introduce $V_l := \{z \in V : \Im z_l > 0\}$, $l=1,2$, and define the functions

$$\begin{cases} v_{lk}(z) := \tilde{\mathrm{v}}_{lk}(z_l), \quad k=0,1, \\ g_l(z) := \tilde{f}_l^0(z_l) + \sum\limits_{|\alpha|\le s_l} C_{l\alpha}(-iz_l)^\alpha, \end{cases} \tag{2.43}$$

where $z \in V_l$, $l = 1, 2$. Similarly, we define

$$a_{1k}(z) := \sum_{1+k \le j \le 2} \tilde{A}_{1j}(z_1)(-iz_2)^{j-1-k},$$

$$a_{2k}(z) := \sum_{1+k \le j \le 2} \tilde{A}_{2j}(z_2)(-iz_1)^{j-1-k},$$

$$b_{lk}(z) := \tilde{B}_{lk}(z_l), \quad l = 1, 2,$$

where $z \in V$, $k = 0, 1$. Finally, define $C(z) := -\sum_{|\alpha| \le s} C_\alpha(-iz)^\alpha$, $z \in V$. Now system (2.41), (2.42) becomes

$$\begin{cases} \sum_{k=0,1} a_{1k}(z)v_{1k}(z) + \sum_{k=0,1} a_{2k}(z)v_{2k}(z) = C(z), \ z \in V^*, \\ b_{10}(z)v_{10}(z) + b_{11}(z)v_{11}(z) = g_1(z), \ z \in V_1, \\ b_{20}(z)v_{20}(z) + b_{21}(z)v_{21}(z) = g_2(z), \ z \in V_2. \end{cases} \tag{2.44}$$

This is an algebraic system for four unknown functions v_{lk} which are analytic in V_l:

$$v_{lk} \in \mathcal{H}(\mathbb{C}^+), \quad l = 1, 2, \quad k = 0, 1, \tag{2.45}$$

where $\mathbb{C}^+$ stands for the complex upper plane $\Im z_l > 0$, and $\mathcal{H}(\mathbb{C}^+)$ for the space of analytic functions on $\mathbb{C}^+$.

We can eliminate two of the functions and get one algebraic equation with two unknown functions. Namely, the Shapiro-Lopatinskii condition implies (2.21) which provides that for each $l = 1, 2$ at least one of the polynomials B_{lk}, $k = 0, 1$, is not identically zero. Assume for definiteness that $B_{l0}(z_l) \not\equiv 0$, $l = 1, 2$ (example: for the Dirichlet boundary conditions, $B_{l0}(z_l) \equiv 1$ and $B_{l1}(z_l) \equiv 0$, $l = 1, 2$). Then two last equations in (2.44) allow us to express the Dirichlet data $v_{l0}(z)$ via the Neuman data $v_{l1}(z)$ for $z \in V_l$, hence for all $z \in V^*$ since $V^* = V_1 \cap V_2$. Then, substituting the expressions for $v_{l0}(z)$ into the first equation, we get the algebraic equation

$$S_1(z)v_{11}(z) + S_2(z)v_{21}(z) = F(z), \ z \in V^*, \tag{2.46}$$

with known polynomial coefficients $S_l(z)$ and the function $F(z)$ which contains known (fixed) extensions of the boundary data f_l^0 and the unknown coefficients C_α with $|\alpha| \le s$, and $C_{l\alpha}$ with $|\alpha| \le s_l$, $l = 1, 2$. The following important lemma is proved in [**18**]:

LEMMA 2.9. *Let the operator A be strongly elliptic (see (1.3)), and the Shapiro-Lopatinskii condition (1.4) hold. Then the polynomials $S_l(z)$ are not identically zero on V,*

$$S_l(z) \not\equiv 0, \ z \in V. \tag{2.47}$$

REMARK 2.10. The dimension of the space of solutions to problem (1.1), (1.2) depends on the order of singularity of the solutions considered and in general it tends to infinity when the order of singularity tends to infinity (i.e. as $s, s_1, s_2 \to \infty$).

EXAMPLE 2.11. Consider the Dirichlet problem in the angle K for the operator $A = \Delta + a$. Then we have $\mathrm{v}_{l0}(x_l) = f_l^0(x_l) + \sum_{|\alpha| \le s_l} C_{l\alpha}\delta^{(\alpha)}(x)$, and (2.29) implies an equation of type (2.46),

$$v_{11}(z) + v_{21}(z) = F(z), \ z \in V^*. \tag{2.48}$$

The algebraic equation (2.46) contains two unknown functions $v_{l1}(z)$ on the Riemann surface V^*. Hence, this is an underdetermined algebraic problem. However, each function $v_{l1}(z)$ "depends only on z_l" by definition (2.43). We will show that this property can be adjusted in a suitable way, which makes problem (2.46) well-posed.

REMARK 2.12. For problem (1.1), (1.2) in the angle Q of magnitude $\Phi > \pi$, the Paley-Wiener theorem is absent as the corresponding tube domain $\mathbb{C}Q^* = \emptyset$ in this case. Respectively, an analytic continuation of identity (2.37) is impossible, and relation (2.38) does not make sense. Nevertheless, the corresponding generalization of relations (2.41) and (2.46) was found in this case in [**20**]: the same relation holds for the analytic continuations of the functions v_{lk} along the Riemann surface V. After this observation all the remaining steps of the method are exactly the same.

VIII. Malyshev's automorphic function method. The central point of the method is the application to (2.46) of Malyshev's automorphic function method [**30**]. The main idea of the method is to express the structure of the functions $v_{l1}(z)$, $l = 1, 2$, as an invariance with respect to the transpositions h_l of the points $(z_1, z_2) \in V$ with identical coordinates z_l. Let us analyze the properties of the coordinate projections on V.

DEFINITION 2.13. $p_l : V \to \mathbb{C}$ is the map $p_l : (z_1, z_2) \mapsto z_l$, $l = 1, 2$.

Rewrite the equation of the Riemann surface V with notations (2.23) as

$$V = \{z \in \mathbb{C}^2 : \sum_{j \le 2} \tilde{A}_{1j}(z_1)(-iz_2)^j = 0\} = \{z \in \mathbb{C}^2 : \sum_{j \le 2} \tilde{A}_{2j}(z_2)(-iz_1)^j = 0\}. \tag{2.49}$$

These equations, which determine the surface V, imply that each projection p_l is a two-folded holomorphic map (the *covering*) $p_l : V \to \mathbb{C}$, and p_l^{-1} has two branching points since the symbol $\tilde{A}(z)$ is irreducible.

Hence, the equation $z_l = p_l z$ with a fixed z_l admits two roots, $z', z'' \in V$, and the roots are distinct away from the branching points.

DEFINITION 2.14. $h_l : V \to V$ is the map $h_l : z' \mapsto z''$ and $h_l : z'' \mapsto z'$, $l = 1, 2$.

The automorphism h_l is the generator of the *monodromy group* of the covering p_l. Equations (2.23) imply, by the Vieta theorem,

$$h_1 z = (z_1, -i\tilde{A}_{11}(z_1)/\tilde{A}_{12} - z_2), \qquad h_2 z = (-i\tilde{A}_{21}(z_2)\tilde{A}_{22} - z_1, z_2), \quad z \in V. \tag{2.50}$$

Introduce the exponential notation

$$v^h(z) = v(hz), \quad z \in V,$$

for a function v on V and a map $h : V \to V$. Now we can formulate the desired property of the functions $v_{lk}(z)$ as the invariance with respect to h_l:

$$v_{lk}^{h_l}(z) = v_{lk}(z), \quad z \in V_l, \qquad k = 0, 1, \quad l = 1, 2. \tag{2.51}$$

This is a system of four algebraic equations. It means that $v_{lk}(z)$ are automorphic functions with respect to the monodromy group of the covering p_l. The combined system, (2.23) with (2.51), is equivalent to system (2.41) with (2.42).

Now we return to equation (2.46) and complete it with two invariance equations for $v_{l1}(z)$:

$$v_{11}^{h_1}(z) = v_{11}(z), \quad z \in V_1, \qquad v_{21}^{h_2}(z) = v_{21}(z), \quad z \in V_2. \tag{2.52}$$

The main strategy of Malyshev's method is to apply the automorphisms h_l to (2.46) and use (2.52). However, to do this we need to know more about the domain of action of the automorphisms h_l. For this purpose, we introduce appropriate local coordinates on the Riemann surface V. More precisely, we will use the global coordinates (the *uniformization*) on the universal covering surface $\hat{V}$ of V. The general case of irreducible operators is analyzed in [**17**]; the surface V is isomorphic to $\mathbb{C} \setminus 0$, hence the universal covering $\hat{V}$ is isomorphic to $\mathbb{C}$.

Here, for being definite, we consider the operator

$$A = \{(\partial_{x_1}^2 - 2\cos\alpha \partial_{x_1}\partial_{x_x} + \partial_{x_2}^2) - m^2 \sin^2\alpha\}/\sin^2\alpha. \tag{2.53}$$

It arises from the operator of the type $\Delta - m^2$ after the linear change of coordinates transforming the plane angle Q of magnitude $\alpha \in (0, \pi)$ to the first quadrant K. Let us calculate the universal covering $\mathbb{C} \to V$ as the function $z(w)$ of $w \in \mathbb{C}$, and express the monodromy automorphisms h_l in the coordinate w.

For the operator (2.53), the symbol is $\tilde{A}(z) = (-z_1^2 + 2\cos\alpha z_1 z_2 - z_2^2 - m^2 \sin^2\alpha)/\sin^2\alpha$. Hence the equation of the Riemann surface V can be written as

$$(z_1 \sin\alpha)^2 + (z_2 - z_1\cos\alpha)^2 = -(m\sin\alpha)^2. \tag{2.54}$$

Now the uniformization is obvious:

$$z_1 \sin\alpha = im \sin\alpha \sin\phi, \quad z_2 - z_1 \cos\alpha = im\sin\alpha\cos\phi, \qquad w \in \mathbb{C}. \tag{2.55}$$

Therefore, $z_1 = im\sin\phi$ and $z_2 = z_1\cos\alpha + im\sin\alpha\cos\phi = im\sin\phi\cos\alpha + im\sin\alpha\cos\phi = im\sin(\phi+\alpha)$. We prefer the coordinate $w = -i\phi$, hence finally, we will use the uniformization

$$z_1 = z_1(w) := m\sinh w, \quad z_2 = z_2(w) := m\sinh(w + i\alpha), \qquad \phi \in \mathbb{C}. \tag{2.56}$$

We identify the regions V_1^+ and V_2^+ with the regions $\hat{V}_1^+$ and $\hat{V}_2^+$ on $\hat{V}$, where

$$\hat{V}_1^+ = \{w \in \mathbf{C} : 0 < \Im w < \pi\}, \qquad \hat{V}_2^+ = \{w \in \mathbf{C} : -\alpha < \Im w < \pi - \alpha\}. \tag{2.57}$$

Then $V^* = V_1^+ \cap V_2^+$ will be identified with the covering region (see fig. 1)

$$\hat{V}^* = \hat{V}_1^+ \cap \hat{V}_2^+ = \{w : 0 < \Im w < \pi - \alpha\}. \tag{2.58}$$

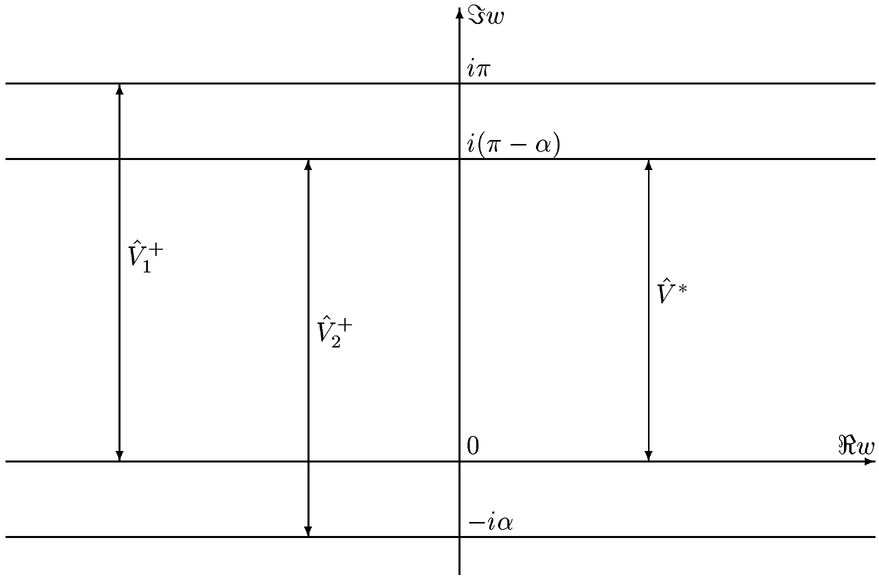

Figure 1.

Furthermore, (2.50) becomes

$$h_1(z_1, z_2) = (z_1, 2z_1 \cos\alpha - z_2),\ h_2(z_1, z_2) = (2z_2 \cos\alpha - z_1, z_2). \tag{2.59}$$

Note that the lifting $\hat{h}_l : \hat{V} \to \hat{V}$ of the generator h_l to $\hat{V}$ defined by $h_l(z(w)) = z(\hat{h}_l(w))$ is nonunique. We choose the branch which maps $\hat{V}_l^+$ (see (2.57)) onto itself for $l = 1, 2$. Then by (2.59), (2.56) $\hat{h}_l$ act as follows:

$$\hat{h}_1(w) = -w + i\pi,\ \hat{h}_2(w) = -w + i\pi - 2i\alpha. \tag{2.60}$$

The automorphism $\hat{h}_1$ is the symmetry of $\hat{V}$ with respect to the point $i\pi/2$, and $\hat{h}_2$ is the symmetry with respect to the point $i\pi/2 - i\alpha$. We note that similar automorphisms appeared already in [**43, 27, 42**] but were obtained from completely different heuristic considerations connected with an appropriate choice of the contour of integration in the integral representation of solutions to the Helmholtz equation.

Now we are ready to perform Malyshev's elimination process. First, note that the function $F(z)$ in (2.46) admits a splitting

$$F(z) = F_1(z) + F_2(z), \quad z \in V^*, \tag{2.61}$$

where F_l is holomorphic on V_l^+, $l = 1, 2$. It suffices to solve equation (2.46) with each function F_l instead of F as the uniqueness can be analyzed easily. Consider, for example,

$$S_1(z)v_{11}(z) + S_2(z)v_{21}(z) = F_1(z),\ z \in V^*. \tag{2.62}$$

This implies that $v_{21}(z)$ admits a meromorphic continuation to V_1^+ as $S_2(z)$ is a polynomial function which is not identically zero in V by (2.47). Now (2.62) holds for $z \in V_1^+$:

$$S_1(z)v_{11}(z) + S_2(z)v_{21}(z) = F_1(z),\ z \in V_1^+. \tag{2.63}$$

Therefore, applying h_1 to (2.46), we get by the first equation from (2.52),

$$S_1^{h_1}(z)v_{11}(z) + S_2^{h_1}(z)v_{21}^{h_1}(z) = F_1^{h_1}(z),\ z \in V_1^+. \tag{2.64}$$

Then, eliminating $v_{11}(z)$ from (2.63) and (2.64), we get the following equation for the *meromorphic continuation* of the function v_{12},

$$Q_1(z)v_{12}(z) - Q_2(z)v_{12}^{h_1}(z) = G_1(z),\ z \in V_1^+. \tag{2.65}$$

This equation contains only one unknown function, however, this is a so-called *one-sided* problem [**58**] which connects the values of an analytic function on one side of the boundary. Hence, equation (2.65) is not a well-posed problem (see [**58**]). To get a well-posed problem, we use the last equation from (2.52): it implies that $v_{21}^{h_1}(z) = v_{21}^{h_2h_1}(z)$, $h_1z \in V_2^+$. Hence, denoting $h = h_2h_1$, we get from (2.65)

$$Q_1(z)v_{21}(z) - Q_2(z)v_{21}^{h}(z) = G_1(z),\ z \in V_1^+ \cap h_1V_2^+. \tag{2.66}$$

Let us lift this equation to $\hat{V}$. Namely, denote by $\hat{v}_{21}(w)$ the lifting of the meromorphic function $v_{21}(z_2)$ from $V_2^+ \cup V_1^+$ to $\hat{V}_2^+ \cup \hat{V}_1^+$ and by $\hat{Q}_l$ the lifting of Q_l to $\hat{V}$ by means of formulas (2.56):

$$\hat{v}_{21}(w) = v_{21}(z_2(w)), \quad \hat{Q}_l(w) = Q_l(z_1(w), z_2(w)), \quad l = 1, 2. \tag{2.67}$$

Moreover, $\hat{Q}_i$ is a $2\pi i$-periodic function of w. Further, in the variable w the shift h becomes $\hat{h}w = \hat{h}_2\hat{h}_1w = w - 2i\alpha$ by (2.60) and $V_1^+ \cap h_1V_2^+$ corresponds to $\alpha < \Im w < \pi$ with the identifications $V_l^+ \leftrightarrow \hat{V}_l^+$. Hence, (2.66) turns into the following difference equation:

$$\hat{Q}_1(w)\hat{v}_{21}(w) - \hat{Q}_2(w)\hat{v}_{21}(w - 2i\alpha) = \hat{G}_1(w),\ 0 < \Im w < \pi. \tag{2.68}$$

More precisely, (2.66) implies this identity for $\alpha < \Im w < \pi$, which extends to $0 < \Im w < \pi$ by a suitable meromorphic continuation of $\hat{v}_{21}$ to the strip $\Pi_{-2\alpha}^{\pi} := \{w \in \mathbb{C} : -2\alpha < \Im w < \pi\}$.

REMARK 2.15. The poles of $\hat{v}_{21}(z)$ in $\Pi_{-2\alpha}^{\pi}$ and their multiplicities correspond to the zeros of $S_1(z)$ and $S_2(z)$ in V.

IX. Reduction to the Riemann-Hilbert problem. Equation (2.68) is a *Hasemann problem* for the meromorphic function $\hat{v}_{12}(w)$ (see [**58**]). We reduce it to a Riemann-Hilbert problem. Namely, (2.68) is equivalent to

$$\hat{Q}_1(w)\hat{v}_{21}(w - i0) - \hat{Q}_2(w)\hat{v}_{21}(w - 2i\alpha + i0) = \hat{G}_1(w),\ \Im w = \pi, \tag{2.69}$$

where the limit values of $\hat{v}_{21}$ are understood in the sense of distributions. There exists a holomorphic map $w \mapsto t$ of the strip Π onto $\mathbb{C} \setminus I$, where I is the segment $[-1, 1]$, such that the line $\Im w = \pi$ becomes $I + i0$. Denote $\check{v}_{21}(t) = \hat{v}_{21}(w)$ in $\mathbb{C} \setminus I$, and for $t \in I$ set $\check{Q}_1(t) = \hat{Q}_1(w)$, $\check{G}_1(t) = \hat{G}_1(w)$, where $\Im w = \pi$ and $\check{Q}_2(t) = \hat{Q}_2(w)$ where $\Im w = \pi - 2\alpha$. Then (2.69) reduces to the Riemann-Hilbert problem for the meromorphic function $\check{v}_{21}(t)$ in $\mathbb{C} \setminus I$,

$$\check{Q}_1(t)\check{v}_{21}(t + i0) + \check{Q}_2(t)\check{v}_{21}(t - i0) = \check{G}(t) \text{ for } t \in (-1, 1). \tag{2.70}$$

The solution to this Riemann-Hilbert problem can be written explicitly as the function $\check{G}(t)$ is known for $t \in (-1, 1)$ and the coefficients $\check{Q}_l$ have at most finite set of zeros on I with finite multiplicities by (2.47).

REMARKS 2.16. i) The Riemann-Hilbert problem (2.70) corresponds to the strongly elliptic operator (2.53) and depends on the geometry of the regions V_l^+ and V^* on the Riemann surface V. For a general strongly elliptic operator (1.3), the corresponding Riemann-Hilbert problem has a similar structure (2.70). Then the strong Shapiro-Lopatinskii condition (1.4) together with Remark 2.15 guarantee that problems (2.70) and (1.1), (1.2) have finite-dimensional kernels and cokernels in the appropriate functional spaces (see [**17**] for details).

ii) The Helmholtz operator (1.5) with real ω is not strongly elliptic. Respectively, the geometry of the regions V_l^+ and V^* is quite different and in the corresponding Riemann-Hilbert problem the relation of type (2.70) holds only for $t \in (0,1)$, while the solution is meromorphic outside $[-1,1]$. Therefore, this Riemann-Hilbert problem is ill-posed as well as problem (1.1), (1.2). This is a standard situation for the Helmholtz equation in unbounded domains, where some additional principle must be added to make the problem well-posed: limiting absorption principle, limiting amplitude principle, or Sommerfeld radiation conditions at infinity. Merzon [**35**] has established that the limiting absorption principle makes this problem well-posed in the case of the Dirichlet and Neumann boundary conditions in (1.2). Namely, he proved that this principle implies that relation (2.70) holds for $-1 < t < 0$ with $\check{G}_1(t) = 0$.

iii) For a strongly elliptic operator A the solution $u(x)$ is given by formula (2.40) which can be represented as the convolution of type (2.4) with a tempered fundamental solution of the operator A. The Helmholtz operator is not strongly elliptic. Hence we cannot express (2.40) as a convolution. In [**21**] Merzon found an analog of (2.4) for the solutions to problem (1.1), (1.2) for the Helmholtz equation with Dirichlet and Neumann boundary conditions. Namely, he transformed the expression (2.40) to Sommerfeld-type representation for solutions satisfying the limiting absorption principle.

3. The Ursell problem

I. Statement of the problem. In this section we describe the application of the method to the proof of completeness of trapping modes of water waves on a sloping beach of angle α. This problem consists in finding certain solutions (to be specified later) of the Laplace equation in $\Omega \times \mathbb{R}$, where Ω is the plane angle $0 < y < x \tan\alpha,\ 0 < \alpha < \pi/2$, with the corresponding boundary conditions:

$$(3.1)\qquad \left\{ \begin{array}{c} \partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi = 0 \text{ in } \Omega \times \mathbb{R}, \\ \\ \partial_t^2\Phi - g\partial_y\Phi = 0 \text{ on } \Gamma_F \times \mathbb{R}, \qquad \partial_n\Phi = 0 \text{ on } \Gamma_B \times \mathbb{R}. \end{array} \right.$$

Here $\Phi(x,y,z,t)$ denotes the velocity potential of the fluid. If $v(x,y,z,t)$ is the velocity at the point $(x,y,z) \in \Omega\times\mathbb{R}$ at the moment t, then $v(x,y,z,t) = \nabla\Phi(x,y,z,t)$, $\Gamma_F = \{(x,0) : x > 0\}$ is the free surface, $\Gamma_B = \partial\Omega \setminus \overline{\Gamma}_F$ is the ocean bottom, n is the outer normal to Γ_B, g is the acceleration of gravity. We seek the solutions of problem (3.1) of the form

$$(3.2)\qquad \Phi(x,y,z,t) = \Re\{\varphi(x,y) \exp i(\sigma t - kz)\},$$

where σ is a real number, $k > 0$ and $\varphi(x, y)$ is a real-valued function. We assume φ fits the following finite energy condition:

$$\int_\Omega (|\varphi(x,y)|^2 + |\nabla\varphi(x,y)|^2)dxdy < \infty. \tag{3.3}$$

The solution corresponds to a sinusoidal wave propagating along the z-axis without decreasing or aberration. In (3.2) σ is the frequency, k is the wave number. The wave propagates with the phase velocity σ/k. We substitute the expressions (3.2) in (3.1) and put $g = 1$. After the scaling $(x, y) \mapsto (kx, ky)$, we get the boundary value problem in the plane angle Ω for $\varphi(x, y)$:

$$\left\{\begin{array}{c} \Delta\varphi - \varphi = 0 \text{ in } \Omega, \ \varphi \not\equiv 0, \\ \\ \partial_y\varphi + \sigma^2\varphi = 0 \text{ on } \Gamma_F, \ \ \partial_n\varphi = 0 \text{ on } \Gamma_B, \end{array}\right. \tag{3.4}$$

(we have denoted σ^2/k by σ^2 again). The assumption (3.3) means that φ belongs to the Sobolev space $H^1(\Omega)$, thus the traces of φ on Γ_F and Γ_B are well-defined as square summable functions; the traces of the normal derivative $\dfrac{\partial\varphi}{\partial n}$ on Γ_F and Γ_B are well defined in the sense of distributions (see Lemma 10.1 in Appendix in [**22**] for details). The following problem will be called Ursell's problem in the sequel:
Problem U. *To find all real-valued functions $\varphi(x, y) \not\equiv 0$ in Ω, $\varphi \in H^1(\Omega)$, and all numbers $\sigma^2 \geq 0$ that satisfy (3.4).*

Ursell [**52**] has found a sequence of nontrivial solutions to problem (3.4) with $\sigma = \sigma_n$, where

$$\sigma_n^2 = \sin(2n+1)\alpha, \ \ (2n+1)\alpha < \frac{\pi}{2}, \ n = 0, 1, 2, \ldots. \tag{3.5}$$

Ursell's solutions are

$$\begin{aligned} \varphi_n(x,y) &= \exp\{-[x\cos\alpha + y\sin\alpha]\} \\ &\quad + \sum_{m=1}^{n} A_{mn}\{\exp\{-[x\cos(2m-1)\alpha - y\sin(2m-1)\alpha]\} \\ &\quad + \exp\{-[x\cos(2m+1)\alpha + y\sin(2m+1)\alpha]\}\}, \end{aligned} \tag{3.6}$$

where $A_{mn} = (-1)^m \prod\limits_{r=1}^{m} [\tan(n-r+1)\alpha/\tan(n+r)\alpha]$, $n = 0, 1, 2, \ldots$, $1 \leq m \leq n$.

It is obvious that the solutions φ_n belong to $H^s(\Omega)$ for all $s \geq 0$ under the condition $(2n+1)\alpha < \pi/2$. For all $\alpha < \frac{\pi}{2}$ there exists a solution $\exp\{-[x\cos\alpha + y\sin\alpha]\}$ corresponding to $n = 0$. This solution is called the Stokes mode [**50**].

By means of the general method of Section 2 we prove that:

1) there are no other solutions of system (3.4) with values σ from (3.5) except for those found by Ursell (3.6);

2) there are no other values σ except for those found by Ursell (3.5) for which there exist nontrivial solutions of system (3.4) belonging to the space $H^1(\Omega)$.

II. Reduction to a linear algebraic equation on a Riemann surface. We reduce problem (3.4) to a linear algebraic equation on a Riemann surface of type (2.41). But, since we seek solutions of system (3.4) belonging to the space $H^1(\Omega)$, it is more convenient to make it reducing the b.v.p. to an integral identity. This is done in [**22**]. Namely, the following lemma is proved there. For a function $\varphi \in H^1(\Omega)$ denote by φ_0 its extension by zero outside Ω.

LEMMA 3.1 (see [**22**]). *Ursell's problem (U) is equivalent to the problem of finding a function $\varphi \in H^1(\Omega)$ satisfying*

$$(3.7)\qquad (\Delta\varphi_0 - \varphi_0, \psi) = \int_{\partial\Omega} \varphi \partial_n \psi ds - \sigma^2 \int_0^\infty \varphi(x,0)\psi(x,0)dx \ \forall \psi \in C_0^\infty(\mathbb{R}^2),$$

where the boundary values of φ on $\partial\Omega$ are well defined by the Sobolev trace theorem.

As in Section 2.I, introduce the linear transformation

$$(3.8)\qquad (x_1, x_2) = \mathcal{L}(x,y) := (x - y\cot\alpha, y/\sin\alpha),$$

which reduces the angle Ω to the first quadrant K. Then (3.4) becomes the following problem for the function $U^0(x_1,x_2) := \varphi_0(\mathcal{L}^{-1}(x_1,x_2))$:

$$(3.9)\qquad HU^0(x_1,x_2) = \gamma^0(x_1,x_2),\ (x_1,x_2) \in \mathbb{R}^2.$$

Here H is the operator

$$(3.10)\qquad H = \left(\Delta - 2\cos\alpha\partial_{x_1}\partial_{x_2} - \sin^2\alpha\right)/\sin^2\alpha,$$

and γ_0 is the distribution defined by

$$\begin{aligned}(3.11)\quad & (\gamma^0(x_1,x_2), \vartheta(x)) \\ &= (\sin\alpha)^{-1}\int_0^\infty U^0(0,x_2)\{\cot\alpha\partial_{x_2}\vartheta(0,x_2) - (\sin\alpha)^{-1}\partial_{x_1}\vartheta(0,x_2)\}dx_2 \\ &\quad + (\sin\alpha)^{-1}\int_0^\infty U^0(x_1,0)\{\cot\alpha\partial_{x_1}\vartheta(x_1,0) - (\sin\alpha)^{-1}\partial_{x_2}\vartheta(x_1,0) \\ &\qquad - \sigma^2\vartheta(x_1,0)\}dx_1\end{aligned}$$

for any test function $\vartheta \in C_0^\infty(\mathbb{R}^2)$.

Note that Ursell's solutions (3.6) in the variables $(x_1,x_2) \in K$ have the form

$$\begin{aligned}(3.12)\quad u_n(x_1,x_2) &= \exp\{-[x_1\cos\alpha + x_2]\} \\ &\quad + \sum_{m=1}^{n} A_{mn}\{\exp\{-[x_1\cos(2m-1)\alpha + x_2\cos 2m\alpha]\} \\ &\qquad + \exp\{-[x_1\cos(2m+1)\alpha + x_2\cos 2m\alpha]\}\}.\end{aligned}$$

The Fourier transform reduces (3.9) to the algebraic equation which is analogous to (2.37):

$$(3.13)\qquad \tilde{H}(z)\tilde{U}^0(z) = \tilde{\gamma}^0(z),\ z \in \mathbb{C}K^*.$$

Here $\tilde{H}(z)$ is the symbol of the operator (3.10):

$$(3.14)\qquad \tilde{H}(z) = (-z_1^2 - z_2^2 + 2z_1z_2\cos\alpha - \sin^2\alpha)/\sin^2\alpha.$$

Further, (3.9) implies for the distribution $\tilde{\gamma}^0(z)$ the representation

$$(3.15)\quad \tilde{\gamma}^0(z) = (\sin\alpha)^{-2}[v_1(z_1)(iz_1\cos\alpha - iz_2 - \sigma^2\sin\alpha) + v_2(z_2)(iz_2\cos\alpha - iz_1)],$$

where the functions v_l stand for the Fourier transforms of the Dirichlet data

$$(3.16)\qquad \mathrm{v}_1(x_1) = U_0(x_1, 0+),\ \mathrm{v}_2(x_2) = U_0(0+, x_2).$$

Note that the functions v_l were denoted as v_{l0} in Section 2 (see (2.43)).

REMARK 3.2. Arbitrary constants C_α in (3.15) in contrast to (2.41), (2.42) do not appear due to the following reason: the integral formulation (3.7) of our problem already uses the boundary conditions. Therefore the γ^0 from (3.15) differs from (2.24) which contains all the four Cauchy data. If one applies the general method directly to problem (3.4), then the corresponding constants appear both in the right-hand side of (3.13) and in the Fourier transforms of the boundary conditions (cf. [**36**]) but afterwards these constants cancel out (since both (2.41), (2.42) contain them) and the corresponding functional equation (2.62) and our right-hand side (3.15) do not contain them.

Equation (3.13) implies the "connection equation" of type (2.38) on the Riemann surface V^*:

$$\tilde{\gamma}^0(z) = 0, \quad z \in V^*. \tag{3.17}$$

As in (2.40), the solution of problem (3.9) is expressed by

$$u(x) = F^{-1}_{z\mapsto x}\left[\tilde{\gamma}^0(z)/\tilde{H}(z)\right], \ x \in K. \tag{3.18}$$

Note that the Dirichlet data $\mathrm{v}_l(x_l) \in L^2(\overline{\mathbb{R}^+}) \equiv \{u \in L^2(\mathbb{R}), \ \operatorname{supp} u \in \overline{\mathbb{R}^+}\}$ by the Sobolev trace theorem since $\varphi \in H^1(\Omega)$. Thus, each solution to Ursell's problem **U** corresponds to a solution of the following

Problem A. To find all $\sigma^2 \in \mathbb{R}$ such that there exist two functions $\mathrm{v}_l(x_l) \in L^2(\overline{\mathbb{R}^+})$, $l = 1, 2$, at least one of them not vanishing identically, such that (3.17) holds for $\tilde{\gamma}_0(z)$ defined by (3.15), where $v_l := \hat{\mathrm{v}}_l$.

III. Difference equation for the Ursell problem. We solve problem **A** reducing the connection equation (3.17) to a difference equation of type (2.68) on the universal covering $\hat{V}$ of the Riemann surface V (see Definition 1.1). To this end we use the method of Malyshev described in Section 2, VIII.

Firstly, using (2.56), we obtain from the connection equation (3.17) and (3.15) the equation of type (2.62) in the variable w,

$$\hat{v}_2(w)(-\cosh(w + i\alpha)) + \hat{v}_1(w)(\cosh w - \sigma^2) = 0, \ w \in \hat{V}^*. \tag{3.19}$$

Apply $\hat{h}_2$ to (3.19) and eliminate $\hat{v}_2(w)$ analogously to (2.67) and (2.68). Thus we obtain the difference equation of type (2.68) with an unknown meromorphic function $\hat{v}_1(w)$ with the condition of analyticity (2.45) and the automorphy condition (2.52), (2.60) (see [**22**] for details):

$$\text{(3.20)}\begin{cases} \hat{v}_1(w)(\cosh w - \sigma^2) - \hat{v}_1(w + 2i\alpha)(\cosh(w + 2i\alpha) + \sigma^2) = 0, \ w \in \mathbb{C}, \\ \\ \hat{v}_1(-w + i\pi) = \hat{v}_1(w), \ w \in \mathbb{C}, \ \hat{v}_1 \in \mathcal{H}(\hat{V}_1^+). \end{cases}$$

Here $\mathcal{H}(O)$ is the space of all analytic functions on the open region O. Thus we have the following theorem.

THEOREM 3.3. *If σ^2, $\tilde{v}_l$, $l = 1, 2$, constitute a solution of problem* **A**, *then σ^2, $\hat{v}_1$ constitute a solution of system (3.20).*

IV. Uniqueness of Ursell's eigenvalues. We recall that we consider the solutions to the Problem A such that

$$\mathrm{v}_1 \in L^2(\overline{\mathbb{R}^+}) \tag{3.21}$$

We use the following consequence of (3.21).

LEMMA 3.4 (Paley-Wiener, [**11**]). *i) If* $\mathrm{v}(x) \in L^2(\mathbb{R})$, $\mathrm{v}(x) = 0$ *for* $x < 0$*, then for all* $\eta \geq 0$ *the following estimate is valid:*

$$\int_{\Im z = \eta} |\tilde{\mathrm{v}}(z)|^2 dz \leq C, \tag{3.22}$$

where C *does not depend on* $\eta \geq 0$.

ii) The following bound holds for any $\varepsilon > 0$*:*

$$|\tilde{\mathrm{v}}(z)| \leq C_\varepsilon (\Im z)^{-1/2}, \quad \Im z \geq \varepsilon > 0. \tag{3.23}$$

In [**22**] we construct the eigenfunctions of problem (3.20) corresponding to the eigenvalues σ_n^2 (3.5) and prove their uniqueness in the space $\hat{L}^2(\mathbb{R}^+)$ related to the space $L^2(\overline{\mathbb{R}^+})$ in the following sense.

Denote by $z_1 : \hat{V}_1^+ \mapsto \mathbb{C}^+$ the two-folded covering defined by (2.56). This map defines a bijective correspondence between the space $\mathcal{H}(\mathbb{C}^+)$ and the space of all functions $\hat{f}(w) \in \mathcal{H}(\hat{V}_1^+)$ such that $\hat{f}(h_1(w)) = \hat{f}(-w + i\pi) = \hat{f}(w)$. This correspondence is indeed bijective because $\sinh(-w + i\pi) = \sinh(w)$. Denote by $\hat{f}$ the lifting of a function $\tilde{f}(z) \in \mathcal{H}(\mathbb{C}^+)$ to $\hat{V}_1^+$ and by $\tilde{f}$ the lowering of $\hat{f} \in \mathcal{H}(V_1^+)$ to $\mathbb{C}^+$ with the help of the same map.

DEFINITION 3.5. The space $\hat{L}^2(\mathbb{R}^+)$ is defined by the following formula:

$$\hat{L}^2(\mathbb{R}^+) \equiv \{\hat{f} \in \mathcal{H}(\hat{V}_1^+) : \hat{f}(h_1(w)) = \hat{f}(w), F^{-1}_{z \mapsto x}(\tilde{f}) \in L^2(\overline{\mathbb{R}^+})\}.$$

Similarly,

$$\hat{H}^s(\mathbb{R}^+) \equiv \{\hat{f} \in \mathcal{H}(\hat{V}_1^+) : \hat{f}(h_1(w)) = \hat{f}(w), F^{-1}_{z \mapsto x}(\tilde{f}) \in L^2(\overline{\mathbb{R}^+}),$$

$$F^{-1}_{z \mapsto x}(\tilde{f})|_{\mathbb{R}^+} \in H^s(\mathbb{R}^+)\}.$$

Note that if $\hat{f} \in \hat{L}^2(\mathbb{R}^+)$ then

$$|\hat{f}(w)| \leq C_\varepsilon e^{-\frac{1}{2}|w|} \text{ for } \varepsilon < \Im w < \pi - \varepsilon. \tag{3.24}$$

Indeed, this follows from (3.23) and the equality (see (2.56))

$$\Im z_1 = \sin(\Im w) \cosh(\Re w). \tag{3.25}$$

Now we can describe the derivation of the formulas for the eigenfunctions to problem (3.20) and the proof of their uniqueness in the space introduced above.

Namely, it is possible to solve system (3.20) explicitly for Ursell's values of σ_n^2 (3.5). In this way we get the following formulas [**22**]:

$$\hat{v}_1(w) = -\frac{1}{i \sinh w - \cos \alpha} \quad \text{for} \quad \sigma_0^2 = \sin \alpha, \tag{3.26}$$

$$\hat{v}_1(w) = C \, \frac{\prod_{k=0}^{n} -i \sinh w + \cos(\alpha(2k - 2n + 1))}{\prod_{k=0}^{n+1} i \sinh w + \cos(\alpha(2k + 2n + 1))} \quad \text{for} \quad \sigma_n^2 = \sin(2n + 1)\alpha, \tag{3.27}$$

where $n = 1, 2, \ldots$. (For simplification we suppress an additional index n in the notation).

LEMMA 3.6. [**22**, Thm. 4.1] *For Ursell's values of* σ_n^2 *from (3.5) the solutions (3.26), (3.27) to the system (3.20) are unique up to a factor in the space* $\hat{L}^2(\mathbb{R}^+)$.

PROOF. We transform conformally the region $\hat{\Pi}_1 = \{w : \Re w > 0, \pi/2 - \alpha < \Im w < \pi/2 + \alpha\}$ to $\mathbb{C} \setminus [0,1]$ by the map

$$t = t(w) = \coth^2((\pi/(2\alpha))w - i\pi^2/(4\alpha)). \tag{3.28}$$

Then the first equation of (3.20) becomes the Riemann-Hilbert problem of type (2.70) with $\check{G}(t) \equiv 0$. The problem admits the solutions (3.26), (3.27). The uniqueness up to a factor follows by a detailed analysis of the asymptotics of the factorization at the end points $t = 0, 1$ (see [**22**] for details). □

Thus we have the following

COROLLARY 3.7. *Let* $\sigma_n^2 = \sin(2n+1)\alpha$ and $(2n+1)\alpha < \pi/2$. *Then (up to a constant factor) there are no nontrivial solutions to problem (3.9) in the space* $H^1(K)$ *except for the Ursell mode (3.12).*

PROOF. Let a function $U \in H^1(K)$ and its extension $U^0(x_1, x_2)$ by zero to $\mathbb{R}^2 \setminus K$ be a solution to equation (3.9) corresponding to the eigenvalue $\sigma_n^2 = \sin(2n+1)\alpha$, $n = 0, 1, \ldots$. Then the corresponding function v_1 satisfies condition (3.21). Furthermore, the corresponding $\tilde{v}_1$ is a solution to system (3.20) by Theorem 3.3. Clearly, the same statements hold for Ursell's solution (3.9). Then we apply Lemma 3.6: up to a constant factor, $\hat{v}_1(w)$ is equal to the function (3.26) if $n = 0$ or (3.27) if $n > 0$. Then the connection equation (3.19) implies that the Dirichlet datum $\mathrm{v}_2(x_2)$ of the function $U^0(x_1, x_2)$ coincides with corresponding Dirichlet datum of Ursell's mode (3.9). Finally, by (3.15), (3.18) the solution U coincides with Ursell's mode (3.12) up to a constant factor. □

V. Obtaining the Ursell modes. The Sommerfeld integral. Further we obtain from (3.26), (3.27) the expressions (3.6) for Ursell's modes. Note that Ursell [**52**] does not give a method for finding these modes. To this end we prove a rather general theorem which allows us to express the solutions of Helmholtz equations of type (3.4) by means of their Cauchy data, for example by means of the Dirichlet datum. We call this representation the Sommerfeld-type representation. We obtain the representation in the variables $(x, y) = \mathcal{L}^{-1}(x_1, x_2)$. We also use polar coordinates (ρ, θ):

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad \rho > 0, \quad 0 < \theta < \alpha. \tag{3.29}$$

To formulate this representation, let us introduce for $\varepsilon > 0$ the contours

$$\hat{\Gamma}^-_{1,\varepsilon} = \{w \in \hat{V}_1^+ \backslash \hat{V}_2^+ : \Im z_1(w) = \varepsilon\}, \quad \hat{\Gamma}^-_{2,\varepsilon} = \{w \in \hat{V}_2^+ \backslash \hat{V}_1^+ : \Im z_2(w) = \varepsilon\}, \tag{3.30}$$

where the functions $z_1(w)$ and $z_2(w)$ were defined in (2.56). We choose the directions of the contours $\hat{\Gamma}^-_{l,\varepsilon}$ in such a way that the regions $\Im z_l(w) > \varepsilon$ are to the left of the contours. For sufficiently small $\varepsilon > 0$ these contours take the following form (see fig. 2):

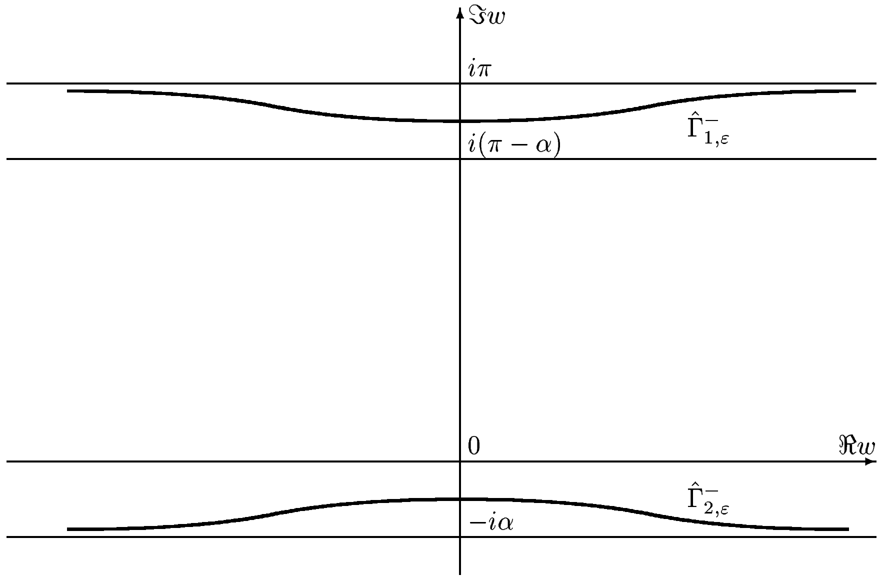

Figure 2.

THEOREM 3.8 (see [**22**], Thm. 5.1). *Let* $\varphi \in H^1(\Omega)$ *be a solution to problem (3.4), let* $u(x_1, x_2) \equiv \varphi(\mathcal{L}(x, y))$ *and let* $\hat{v}_1(w)$ *be the lifting of the Fourier transform of*

$$\mathrm{v}_1(x_1) = \begin{cases} u(x_1, 0), & x_1 > 0, \\ 0, & x_1 < 0, \end{cases}$$

to $\hat{V}_1^+$. *Then there exists an* $\varepsilon > 0$ *such that for all* $(x,\ y) \equiv (\rho\cos\theta,\ \rho\sin\theta) \in \Omega,\ \rho > 0,\ 0 < \theta < \alpha$, *the following representation is valid:*

$$\varphi(x,\ y) = \frac{1}{4\pi\sin\alpha}\int_{\hat{\Gamma}^{-}_{1,\varepsilon}\cup\hat{\Gamma}^{-}_{2,\varepsilon}} e^{-i\rho\sinh(w+i\theta)}\hat{v}_1(w)(\cosh w - \sigma^2)dw, \tag{3.31}$$

where the integral converges absolutely.

REMARK 3.9. If the function $\hat{v}_1(w)$ is periodic with period $2\pi i$, the integral (3.31) is equal to the same integral over another contour

$$\Gamma = \hat{\Gamma}^{-}_{2,\varepsilon} \cup (\hat{\Gamma}^{-}_{1,\varepsilon} - 2\pi i). \tag{3.32}$$

By means of this theorem we find the Ursell modes (3.6). We show how to find these modes in the case $n = 0$. The case $n > 1$ is analyzed analogously (see [**22**], §6). From (3.26), (3.5) it follows that

$$\hat{v}_1(w)(\cosh w - \sigma^2) = \frac{\cosh w - \sin\alpha}{i\sinh w - \cos\alpha}. \tag{3.33}$$

From (3.33), (3.31) we get the expression for $\varphi(x, y)$ in Ω:

$$\varphi(x, y) = \frac{1}{2\pi}\int_{\Gamma} e^{-i\rho\sinh(w+i\theta)}\frac{\cosh w - \sin\alpha}{i\sinh w - \cos\alpha}dw, \tag{3.34}$$

where Γ is the contour (3.32). We transform this expression to the form (3.6) (for $n = 0$) by the Cauchy residue theorem. This is possible because of the analyticity and decay of the integrand in (3.34) in the region enclosed by the contour Γ. For this purpose, note that the zeros of the integrand lying in the closure of the strip $\Pi_\alpha \equiv \{w \in \mathbf{C} : -\pi < \Im w < -\alpha\}$ are $w_0 = -i\pi/2 - i\alpha$. Calculating the integral by the Cauchy residue theorem we obtain the expression (3.6).

VI. Nonexistence of eigenvalues distinct from the Ursell ones. Finally we show how to prove, using the general method, that there are no other eigenvalues σ^2 except for the Ursell values (3.5). Here we prove only that there are no eigenvalues satisfying $\sigma^2 < \sin\alpha$. All the other cases are analyzed in [**22**], §§ 8, 9.

THEOREM 3.10. *Let $0 \le \sigma^2 < \sin\alpha$. Then there are no nontrivial solutions to problem (3.20) satisfying (3.21).*

PROOF. Let $\hat{v}_1$ be a nontrivial solution to (3.20) satisfying (3.21). Denote $\hat{T}(w) \equiv \hat{v}_1(w)$. Then from (3.20) it follows that

$$(3.35)\quad \hat{T}(w)/\hat{T}(w+2i\alpha) = (\cosh(w+2i\alpha)+\sigma^2)/(\cosh w - \sigma^2) \equiv P(w),\ w \in \hat{V}_2^+.$$

The equality in (3.35) holds, in particular, for $w \in \hat{a}_+ := \{w_1 + i\pi/2 + i\alpha : w_1 > 0\}$ because $\hat{a}_+ \subset \hat{V}_2^+$ due to (2.57). Since the transform (3.28) transfers $\hat{a}_+$ onto $\check{a} := (0, 1)$, equality (3.35) is equivalent to

$$(3.36)\qquad \check{T}^+(t)/\check{T}^-(t) = \check{P}(t),\ t \in (0,\,1),\ \text{ where } \check{T}^\pm(t) = \lim_{s\to t\pm i0} \check{T}(s).$$

Here the limits exist because $\hat{T}(w)$ is analytic in $\hat{V}_1^+$ and $\hat{\Pi}_1 \subset \hat{V}_1^+$ since $\alpha < \pi/2$.

The analyticity of the function $\hat{T}(w)$ in the region $\hat{V}_1^+$ implies its analyticity in $\hat{\Pi}_1 \cup \hat{\Gamma}_1 \cup \hat{\Gamma}_2$, where $\hat{\Gamma}_1 = \{iw_2 : \pi/2 \le w_2 \le \pi/2 + \alpha\}$, $\hat{\Gamma}_2 = \{iw_2 : \pi/2 - \alpha \le w_2 \le \pi/2\}$. Taking into account the fact that the function $\hat{T}(w)$ is invariant with respect to $\hat{h}_1$, we obtain that $\check{T}(t)$ is analytic in $\overline{\mathbb{C}} \setminus [0,\,1]$.

To prove Theorem 3.10, let us consider the asymptotics of the function $\check{T}(t)$ as $t \to 0$.

First, note that $\hat{T}(i\pi/2 - i\alpha) = \hat{T}(i\pi/2 + i\alpha) = T^*$ by $\hat{T}(\hat{h}_1(w)) = \hat{T}(w)$ and (2.60). Then $T^* = 0$. Indeed, if, on the contrary, $T^* \neq 0$, then, after substituting $w = i\pi/2 - i\alpha$ in equation (3.20), we get $\hat{T}(i\pi/2 - i\alpha)(\cosh(i\pi/2 - i\alpha) - \sigma^2) = 0$. Hence $\sigma^2 = \sin\alpha$, and this contradicts the assumption of Theorem 3.10. Thus $T^* = 0$.

Recall that the function $\hat{T}(w)$ is analytic at the point $w = i\pi/2 - i\alpha$. Then we have for some integer $k \ge 1$ that $\hat{T}(w) = (w - i\pi/2 + i\alpha)^k T^1(w)$, where $T^1(w) \to \text{const} \neq 0$ as $w \to i\pi/2 - i\alpha$. Hence (3.28) implies that

$$(3.37)\qquad |\check{T}(t)| = |t|^{k/2}|\check{S}^1(t)|,\quad |\check{S}^1(t)| \to \text{const} \neq 0 \text{ as } t \to 0.$$

Consider the asymptotics of $\check{T}(t)$ as $t \to 1$. Lemma 3.4 and (3.21) imply that $\hat{T}(w)$ satisfies the estimate (3.24) in the region $\{\varepsilon < \Im w < \pi + \varepsilon\}$ and in particular in the region $\hat{\Pi}$. This estimate in the variable t (3.28) takes the form

$$(3.38)\qquad |\check{T}(t)| \le C_\alpha |t-1|^{\alpha/(2\pi)} \text{ for } |t-1| < \varepsilon,$$

because the condition $\Re w \to +\infty$ is equivalent to the condition $t \to 1$.

Now we introduce the "standard" factorization for problem (3.36).

LEMMA 3.11 (see Lemma 7.1 in [**22**]). *Let* $0 \le \sigma^2 < \sin\alpha$. *Then there exists a solution* $\check{T}_1(t)$ *to problem (3.36) such that*
i) $\check{T}_1(t)$ *is analytic and nonzero in* $\overline{\mathbb{C}} \setminus [0, 1]$,
ii) $\check{T}_1(t) \to \text{const} \neq 0$ *as* $t \to \infty$, *and*
iii)

$$|\check{T}_1(t)| \cdot |t|^{-1/2} \to C_1 \neq 0 \text{ as } t \to 0, \tag{3.39}$$

$$|\check{T}_1(t)| \cdot |t-1|^{1-\alpha/\pi} \to C_2 \neq 0 \text{ as } t \to 1. \tag{3.40}$$

Now let us finish the proof of Theorem 3.10. Compare two solutions of the factorization problem (3.36): the unknown solution $\check{T}(t)$ and the known standard solution $\check{T}_1(t)$. Namely, consider the fraction $\check{R}(t) \equiv \check{T}(t)/\check{T}_1(t)$, $t \in \overline{\mathbb{C}} \setminus [0, 1]$. From (3.36) and Lemma 3.11 it follows that the function $\check{R}(t)$ has an analytic continuation to the region $\mathbb{C} \setminus \{0, 1\}$. Lemma 3.11 ii), analyticity of $\hat{T}(w)$ at the point $w = i\pi/2$, the condition $\hat{T}(-w + i\pi) = \hat{T}(w)$ and (3.28) imply that $\check{R}(t)$ is analytic at the point $t = \infty$. From (3.37) and (3.39), we have that $t = 0$ is a removable singularity of $\check{R}(t)$. Finally, (3.38) and (3.40) imply that $\check{R}(t) \to 0$ as $t \to 1$. Thus $\check{R}(t)$ is an entire function which is equal to zero at $t = 1$ and is analytic at the point $t = \infty$. Hence $\check{R}(t) \equiv 0$ by the Liouville theorem, and thus $\check{T}(t) \equiv 0$. Theorem 3.10 is proven. □

4. The Neumann problem in an angle

In this section we describe briefly the following result. Consider the well-known Sommerfeld integral which provides a solution to the Neumann problem for the Helmholtz equation in a plane angle Ω with boundary data from the space $H^{-1/2}(\Gamma)$, where Γ is the boundary of Ω. We prove that this Sommerfeld solution belongs to the Sobolev space $H^1(\Omega)$ and depends continuously on the boundary data. For the proof we use the general method described in Section 2. Namely, we use the Fourier representation (2.40). All the details can be found in [**57**].

I. Formulation of the problem. Let Ω be a plane angle of magnitude $\alpha < \pi$ with vertex at the origin and sides $\Gamma_1 \equiv \{(x, 0) : x > 0\}$, $\Gamma_2 \equiv \{(x, x\tan\alpha) : x > 0\}$. Denote by γ_1 the trace operator of restriction of normal derivatives of functions from $D(\overline{\Omega})$ to the boundary $\Gamma \equiv \Gamma_1 \cup \Gamma_2$ of the angle. We consider the Neumann problem of the form

$$(\Delta - 1)u = 0, \quad (x, y) \in \Omega, \qquad \gamma_1 u = (g_1, g_2). \tag{4.1}$$

Problems of type (4.1) arise in various applications; see, for example, paper [**56**], where the motions of a two-layer fluid are studied.

DEFINITION 4.1. i) $H_+^{-1/2}(\mathbb{R}^+)$ is the set of functions from $H^{-1/2}(\mathbb{R})$ whose supports belong to $\overline{\mathbb{R}^+}$.
ii) $\overset{\circ}{H}{}^{1/2}(\mathbb{R}^+)$ is the subspace of $H^{1/2}(\mathbb{R}^+)$ consisting of functions $h(x)$ whose continuations by zero to $\mathbb{R}$ belong to $H^{1/2}(\mathbb{R})$ with the norm [**10, 44**]

$$\|h\|^2_{\overset{\circ}{H}{}^{1/2}(\mathbb{R}^+)} = \|h\|^2_{H^{1/2}(\mathbb{R}^+)} + \int_{\mathbb{R}^+} |h(x)|^2 |x|^{-1}\, dx,$$

iii) $H^{-1/2}(\Gamma)$ $(\Gamma = \Gamma_1 \cup \Gamma_2)$ consists of pairs $(g_1(s), g_2(t)) \in [\overset{\circ}{H}{}^{1/2}(\Gamma_1)]' \times [\overset{\circ}{H}{}^{1/2}(\Gamma_2)]'$ such that

$$g_1(s) + g_2(s) \in H_+^{-1/2}(\mathbb{R}^+). \tag{4.2}$$

We equip $H^{-1/2}(\Gamma)$ with the norm

$$\|(g_1, g_2)\|_{H^{-1/2}(\Gamma)} = \|g_1\|_{[\overset{\circ}{H}{}^{1/2}(\mathbb{R}^+)]'} + \|g_2\|_{[\overset{\circ}{H}{}^{1/2}(\mathbb{R}^+)]'} + \|g_1 + g_2\|_{H_+^{-1/2}(\mathbb{R}^+)}. \tag{4.3}$$

Here the norm in the space $[\overset{\circ}{H}{}^{1/2}(\mathbb{R}^+)]'$ is the standard norm of the dual to a Banach space, and the norm of the space $H_+^{-1/2}(\mathbb{R}^+)$ is the one induced by the norm of $H^{-1/2}(\mathbb{R})$ (see [**44**]).

It is known ([**44**], p. 189) that the operator γ_1 can be continued to a bounded operator acting from the space $E \equiv \{\varphi \in H_1(\Omega) : (\Delta - 1)\varphi \in L^2(\Omega)\}$ to $H^{-1/2}(\Gamma)$.

We assume $(g_1, g_2) \in H^{-1/2}(\Gamma)$ $(\Gamma = \Gamma_1 \cup \Gamma_2)$ and consider the problem (4.1) in the generalized sense as, for example, in [**44**]. It is not hard to reduce this problem to a problem with zero Neumann condition on one of the sides, e.g., on Γ_2. The second datum g_1 turns out to belong automatically to the space $H_+^{-1/2}(\mathbb{R}^+)$ because of the compatibility condition (3.25). We will assume in the sequel that $g_2 = 0$ and $g_1 \equiv g \in H_+^{-1/2}(\mathbb{R}^+)$.

It is known (see, for example, [**22**], [**57**]) that an explicit solution of problem (4.1) can be obtained by means of the Sommerfeld integrals. On the other hand, it is also known that there exists a unique solution from $H^1(\Omega)$, which can be proved by means of the standard variational technique. Nevertheless, it is not clear whether the latter solution is the one given by the Sommerfeld integral. In other words, does the Sommerfeld integral belong to $H^1(\Omega)$?

Seemingly, the answer must be quite simple. The following example shows that the situation is more difficult. Consider the function $\varphi \equiv e^{-c(x+y)}$ with $c = 1/\sqrt{2}$. Obviously, $\varphi \in H^1(\Omega)$ (and even $\varphi \in S(\overline{\Omega})$) and satisfies (4.1) in the angle of magnitude $\pi/4$ with $g = ce^{-cx}, x > 0$. The Sommerfeld integral for this function has the form

$$\varphi(x, y) = \frac{-1}{4\pi} \int_{C_1 \cup C_2} e^{-i\rho \sinh(w + i\vartheta)} \frac{\cosh(w + i\pi/4)}{i \sinh(w + i\pi/4) - 1} dw,$$

where ρ, ϑ are polar coordinates, $C_1 = \{w - i\pi/4, w \in \mathbb{R}\}$, $C_2 = \{w + i\pi, w \in \mathbb{R}\}$. It is easy to see that this integral converges for $0 < \vartheta < \pi/4$, but on the sides $\vartheta = 0, \pi/4$ the convergence holds only in the sense of the principal value. After differentiation, convergence is even worse. This example demonstrates the difficulties that arise when showing that the Sommerfeld integral belongs to H^1. Thus it is not surprising that only in 1998 this problem was solved in [**33**] for a right angle when the Sommerfeld integral reduces to an inverse Fourier transform. This section is a generalization of the results of [**33**] to an arbitrary angle of magnitude $< \pi$.

We bypass the difficulties connected with the direct study of the Sommerfeld integral by means of the use of the Fourier representation (2.40), which reduces to the Sommerfeld integral as in Theorem 3.8. In contrast to the Sommerfeld integral (3.31), the Fourier representation can be handled by means of the standard Fourier technique.

II. Reduction to a difference equation. In order to avoid certain technical difficulties connected with the analytic continuation of solutions of the difference equation (4.9) (see below), we will consider only the case $\pi/2 < \alpha < \pi$. After the change of the variables (3.8) $x = (x_1, x_2) = \mathcal{L}(x, y)$, which transforms the angle Ω into the first quadrant K, we obtain the following integral identity (which is simply the weak formulation of problem (4.1) and is an analog of the identity (3.7)). Denote $H = \left(\Delta - 2\cos\alpha\partial_{x_1}\partial_{x_2} - \sin^2\alpha\right)/\sin^2\alpha$.

Problem N. Find $u(x) \in H^1(K)$ and $\mathrm{v}_{1,2} \in S'(\overline{\mathbb{R}^+})$ such that

$$\int_K u(x) H\vartheta dx_1 dx_2 = -\sin\alpha\langle g(x_1), \vartheta(x_1)\rangle \tag{4.4}$$

$$+\langle \mathrm{v}_1(x_1), \cos\alpha\partial_{x_1}\vartheta_1(x_1) + \vartheta_1^1(x_1)\rangle + \langle \mathrm{v}_2(x_2), \cos\alpha\partial_{x_2}\vartheta_2(x_2) + \vartheta_2^1(x_2)\rangle$$

for all $\vartheta \in S(\overline{\Omega})$. Here $\vartheta_{1,2}, \vartheta_{1,2}^1$ are the Dirichlet and Neumann data of the function ϑ on corresponding sides of the quadrant K and $S'(\overline{\mathbb{R}^+})$ is the space of tempered distributions with supports in $\overline{\mathbb{R}^+}$.

The solution of problem **N** is connected with the solution of problem (4.1) through the change of the variables indicated above. Extending the solution $u(x)$ by zero to the function U^0 on the plane as in (2.25) and applying the complex Fourier transform (2.36) we obtain the equation in the class of functions analytic in $\mathbb{C}K^*$ (cf (2.37), (3.13)):

$$\tilde{H}(z)\tilde{U}^0(z) = \tilde{\gamma}^0(z), \quad \tilde{H}(z) = (-z_1^2 - z_2^2 + 2z_1z_2\cos\alpha - \sin^2\alpha)/\sin^2\alpha, \tag{4.5}$$

where

(4.6)

$$\tilde{\gamma}^0(z) = (\sin\alpha)^{-2}\left[(-\sin\alpha)\tilde{g}(z_1) + (iz_1\cos\alpha - iz_2)\tilde{\mathrm{v}}_1(z_1) + (iz_2\cos\alpha - iz_1)\tilde{\mathrm{v}}_2(z_2)\right].$$

Note that, as in the previous section, (4.6) does not contain arbitrary constants by the same reasons as in Remark 3.2.

Equation (4.5) implies the necessary condition (2.38) for the function $\tilde{\gamma}^0$ (cf (3.17)):

$$\tilde{\gamma}^0(z) = 0 \quad \text{for} \quad z \in V^* := \{(z_1, z_2) \in \mathbb{C}K^* : \tilde{H}(z) = 0\}. \tag{4.7}$$

If this condition holds, the solution of problem **N** has the form (2.40), (3.18):

$$u(x_1, x_2) = U^0(x_1, x_2)|_K = F^{-1}_{z\to x}\left[\tilde{\gamma}^0(z)/\tilde{H}(z)\right]\Big|_K. \tag{4.8}$$

The expression (4.6) for $\tilde{\gamma}^0(z)$ involves two unknown functions $\tilde{\mathrm{v}}_1(z_1)$ and $\tilde{\mathrm{v}}_2(z_2)$, $z_{1,2} \in \mathbb{C}^+$, which are analytic in $\mathbb{C}^+$ (see (2.45)).

We derive these functions by solving the connection equation (4.7). Namely, the Riemann surface $V^* := \{(z_1, z_2) \in \mathbb{C}^2 : \tilde{H}(z) = 0\}$ has the universal covering $\hat{V}$, which is isomorphic to $\mathbb{C}$ and can be uniformized by means of the parameter w via the formulas (2.56) with $m = 1$. Denote $\Pi_a^b = \{w : a < \Im w < b\}$, and let $\hat{v}_1(w)$, $\hat{v}_2(w), w \in \Pi_{-\alpha}^{\pi-\alpha}$ and $\hat{g}(w), w \in \Pi_0^\pi$, be the liftings of the functions $\tilde{\mathrm{v}}_l(z_l), l = 1, 2$, and $\tilde{g}(z_1)$ to $\hat{V}$, respectively, in the sense (2.67). The results of Section 2, VIII imply that if $u(x, y)$ is a solution to (4.1), then

i) $\hat{v}_2(w)$ is meromorphic in $\Pi_{-2\alpha}^\pi$ and the following difference equation holds (cf. (2.68)):

$$\hat{v}_2(w)\cosh(w + i\alpha) - \hat{v}_2(w - 2i\alpha)\cosh(w - i\alpha) = -2\hat{g}(w), \quad w \in \Pi_0^\pi, \tag{4.9}$$

ii) the automorphy condition (2.52) holds (see (2.60)):

$$\hat{v}_2(i\pi - 2i\alpha - w) = \hat{v}_2(w), \quad w \in \Pi^{\pi}_{-2\alpha}. \tag{4.10}$$

For us a converse statement, guaranteeing that the solution of (4.9), (4.10) gives the solution of (4.1), is important. To obtain it, we prove the following theorem. Denote

$$\hat{v}_1(w) \equiv (\hat{g}(w) + \hat{v}_2(w)\cosh(w + i\alpha))/\cosh w, \quad w \in \Pi^{\pi}_0. \tag{4.11}$$

THEOREM 4.2 (see [**57**], Thm. 6.3). *Let $\hat{v}_2(w)$ be an analytic in $\Pi^{\pi}_{-2\alpha}$ solution of problem (4.9),(4.10). Assume that $\hat{v}_l \in F\left[S'(\overline{\mathbb{R}^+})\right]$, $l = 1, 2$, and the identity (4.11) holds. Let $u(x)$ defined by (4.8) and (4.6) belong to $H^1(K)$. Then $u(x)$ solves problem* **N**.

We express the solution of problem (4.1) in terms of the function $\hat{v}_2(w)$ which solves (4.9), (4.10). Namely,

$$\hat{v}_2(w) = T(w + i\alpha - i\pi/2)/\cosh(w + i\alpha), \quad w \in \Pi^{\pi}_{-2\alpha}, \tag{4.12}$$

where

(4.13)
$$T(w) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \sin wt \frac{\tilde{g}(t)}{\sinh \alpha t}dt, \quad w \in \Pi^{\alpha}_{-\alpha}, \qquad \tilde{g}(t) = \int_{\mathcal{C}_0} \hat{g}(w + i\pi/2)e^{iwt}dw,$$

and $\mathcal{C}_\alpha \equiv \{w + i\alpha, w \in \mathbb{R}\}$. We prove that corresponding function $u(x)$ defined by (4.8), (4.6) and (4.11) belongs to $H^1(K)$. For this purpose we study carefully the integral operator (4.13). Finally, it can be shown [**22**], [**57**] that the function (4.8) reduces to the Sommerfeld integral. Thus we obtain the desired result: the Sommerfeld integral is the finite energy solution of problem (4.1).

In the next section we indicate the main steps of the proof of the facts mentioned in the last paragraph.

III. Boundedness of the integral operator. Quadratic summability of the solution $u(x_1, x_2)$ is proved in a relatively simple way by means of the use of "crude" properties of the solution $\hat{v}_2$ of the difference equation (4.9). The proof of the inclusions $\partial_{x_1,x_2}u \in L^2$ is much more involved. To prove it, we also use representation (4.8) for U^0, but after differentiation there naturally appear δ-type singularities on the boundary of K. Subtracting them, we prove that the functions $U_l(x_1, x_2) = \partial_{x_1}U^0(x_1, x_2) - \mathrm{v}_{3-l}(x_{3-l}) \times \delta(x_l), l = 1, 2$, belong to $L^2(\mathbb{R}^2)$. Passing to the Fourier transforms in the last expressions, using (4.8) and the variables $\mathbf{w} = (w_1, w_2), z_1 = \sinh w_1, z_2 = \sinh(w_2 + i\alpha)$, we obtain the following expression for, e.g., $\hat{U}_1(\mathbf{w})$: $\hat{U}_1(\mathbf{w}) = B_1(\mathbf{w}) + B_2(\mathbf{w})$, where

$$B_1(\mathbf{w}) = \frac{i(\sinh w_1 \cos\alpha - \sinh(w_2 + i\alpha))}{4\alpha\hat{H}(\mathbf{w})}\int_{\mathcal{C}_\varepsilon} R(\mathbf{w}, w')\hat{g}(w' + i\pi/2)dw', \tag{4.14}$$

$$R(\mathbf{w}, w') = \frac{\tanh w_1 \sinh 2a_1}{\cosh^2 b - \cosh^2 a_1} - \frac{\tanh(w_2 + i\alpha)\sinh 2a_2}{\cosh^2 b - \cosh^2 a_2},$$

$b = \tau w'$, $a_1 = \tau(w_1 - i\pi/2)$, $a_2 = \tau(w_2 - i\pi/2)$, $\tau = \pi/(2\alpha)$, $\hat{H}(w_1, w_2) = \tilde{H}(\sinh w_1, \sinh(w_2 + i\alpha))$ and $B_2(\mathbf{w})$ is a certain function depending on v_2 and g such that, as it is not hard to see,

$$B_2 \in L^2(\Sigma, P(\mathbf{w})) \equiv \{f(\mathbf{w}) : \int_\Sigma |f(\mathbf{w})|^2 P(\mathbf{w})|d\mathbf{w}| < \infty\},$$

$P(\mathbf{w}) = \exp(|w_1| + |w_2|)$, $\Sigma = \mathcal{C}_0 \times \mathcal{C}_\alpha$. (Note that the latter inclusion is equivalent to the inclusion $B_2 \in L^2(\mathbb{R}^2)$ in the variables x_1, x_2.) Thus we are left with the proof of the fact that $B_1 \in L^2(\Sigma, P)$ and depends continuously on the boundary datum g with respect to the norm (4.3). This, in fact, is the central technical result of the paper [**57**].

We prove that the integral operator in (4.14) is bounded as an operator from $L^2(\mathcal{C}_\varepsilon)$ to $L^2(\Sigma, P)$ for some $\varepsilon > 0$ by means of the Schur test [**15**], which, in particular, states that if $\int_{\mathbb{R}^2} |K(\mathbf{w}, w')| d\mathbf{w} \le C$ and $\int_{\mathcal{C}_\varepsilon} |K(\mathbf{w}, w')| dw' \le C$, then K is a bounded kernel in L^2. By means of this assertion and using the fact that R is close to zero on the diagonal $w_1 = w_2$ as $|\mathbf{w}| \to \infty$, we obtain the necessary estimates and this finishes the proof of our result.

References

[1] J.-M. L. Bernard, *Diffraction by a metalic wedge covered with a dielectric material*, Wave Motion **9** (1987), 543–561.

[2] J.-M. L. Bernard, *Diffraction par un dièdre à faces courbes non parfaitement conducteur*, Revue Technique Thomson-CSF **23** (1991), 321–330.

[3] N. N. Bogoliubov, V. S. Vladimirov, *Representations of n-point functions*, Proc. Steklov Inst. Math. **112** (1971), 1–18.

[4] A.-S. Bonnet-Ben Dhia, P. Joly, *Mathematical analysis of guided water waves*, SIAM. J. Appl. Math. **53** (1993), 1507–1550.

[5] A.-S. Bonnet-Ben Dhia, M. Dauge, K. Ramdani, *Spectral analysis and singularities of a non-coercive transmission problem*, C. R. Acad. Sci., Paris, Sér. I, Math. **328** (1999), 717–720.

[6] A.-S. Bonnet-Ben Dhia, K. Ramdani, *Mathematical analysis of conducting and superconducting transmission lines*, SIAM J. Appl. Math. **60** (2000), 2087–2113.

[7] V. A. Borovikov, *Diffraction by a wedge with curved faces*, Acust. Zh. **25** (1979), 825–835 (Russian).

[8] L. Boutet de Monvel, *Comportement d'un operateur pseudo-differentiel sur une variete a bord. I: La propriete de transmission, II: Pseudo-noyaux de Poisson*, J. Anal. Math. **17** (1966), 241–253, 255–304.

[9] M. Costabel and M. Dauge, *Construction of corner singularities for Agmon-Douglis-Nirenberg elliptic systems*, Comm. Partial Differential Equations **9** & **10** (1994), 1677–1726.

[10] M. Costabel and E. Stephan, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation*, Banach Center Publicat. **15** (1985), 175–251.

[11] G. I. Eskin, *Boundary Value Problems for Pseudodifferential Equations*, Amer. Math. Soc., Providence, RI, 1981.

[12] G. Eskin, *The wave equation in a wedge with general boundary conditions*, Commun. Partial Differ. Equations **17** (1992), 99–160.

[13] D. V. Evans, *Edge waves over a sloping beach*, Quart. J. Mech. Appl. Math. **42** (1989), 131–142.

[14] D. V. Evans, N. Kuznetsov, *Trapped modes*, Gravity waves in water of finite depth (Advances in Fluid Mecanics, vol. 10), ed. by J. N. Hunt, WIT Press, Southampton, 1997, pp. 127–168)

[15] P. R. Halmos and V. S. Sunder, *Bounded integral operators on L^2 spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[16] L. Hörmander, *The Analysis of Linear Partial Differential Operators III: Pseudo-differential operators*, Springer-Verlag, Berlin, 1985.

[17] A. I. Komech, *Elliptic boundary value problems on manifolds with piecewise smooth boundary*, Math. USSR Sbornik **21** (1973), 91–135.

[18] ______, *Elliptic differential equations with constant coefficients in a cone*, Moscow Univ. Math. Bull. , math.-mech. series **29** (1974), 140–145.

[19] ______, *Linear partial differential equations with constant coefficients*, Partial Differential Equations II (Encyclopaedia of Mathematical Sciences, Vol.31), Springer, Berlin, 1994, pp. 127–260.

[20] A. I. Komech, A. E. Merzon, *General boundary value problems in region with corners*, Operator Theory: Advances and Applications, vol.57, Birkhauser Verlag, Basel, 1992, pp. 171–183.

[21] ———, *Sommerfeld representation in stationary diffraction problems in angles*, Spectral and Evolutional Problems, vol. 4, Simferopol State University, Simferopol, 1995, pp.210–219.

[22] A. I. Komech, A. E. Merzon, P. N. Zhevandrov, *On completeness of Ursell's trapping modes*, Russian J. Math. Phys. **4** (1996), 55–85.

[23] V. A. Kondrat'ev, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Mosc. Math. Soc. **16** (1967), 227–313.

[24] V. Kozlov, *Singularities of solutions of the Dirichlet problem for elliptic equations in a neighborhood of corner points*, Leningrad Math. J. **4** (1990), 967–982.

[25] V. A. Kozlov, V. G. Maz'ya, J. Rossmann, *Spectral problems associated with corner singularities of solutions to elliptic equations*, Amer. Math. Soc., Providence, RI, 2001.

[26] N. G. Kuznetssov, V. G. Maz'a, B. R. Vainberg, *Lectures on linear time-harmonic water waves*, Department of Mathematics Tecnical Report Series, UNC, Charlotte, no. 28223, 2000.

[27] R. S. Lehman and H. Lewy, *Uniqueness of water waves on a sloping beach*, Comm. Pure Appl. Math. **14** (1961), 521–546.

[28] H. M. Macdonald, *The electrical distribution on a cylinder bounded by two spherical surfaces cutting at any angle*, Proc. Lond. Math. Soc. **26** (1895), 156–172.

[29] G. D. Malyuzhinets, *Excitation, reflection and emission of surface waves from a wedge with given face impedances*, *Soviet. Phys. Dokl.* **3** (1959), 752–755.

[30] V. A. Malyshev, *Random Walks, Wiener-Hopf Equations in the Quadrant of Plane, Galois Automorphisms*, Moscow University, Moscow, 1970 (Russian).

[31] V. G. Maz'ya, B. A. Plamenevskii, *Problems with oblique derivatives in regions with piecewise smooth boundaries*, Func. Anal. Appl. **5** (1971), 256–258.

[32] ———, *On boundary value problems for a second order elliptic equations in a domain with edges*, Vest. Leningr. Univ., Math. **8** (1980), 99–106.

[33] E. Meister, F. Penzel, F.-O. Speck, and F. S. Teixeira, *Some interior and exterior boundary-value problems for the Helmholtz equaton in a quadrant*, Operator Theory: Advances and Applications, vol. 102, Birkhauser Verlag, Basel, 1998, pp. 169–178.

[34] A. E. Merzon, *On the solvability of differential equations with constant coefficients in a cone*, Soviet Math. Dokl. **14** (1973), 1012–1015.

[35] ———, *General boundary value problems for the Helmholtz equations in a plane angle*, Uspekhi Math. Nauk **32** (1977), 219–220 (Russian).

[36] ———, *On Ursell's problem*, Proceedings of the Third International Conference on Mathematical and Numerical Aspects of Wave propagation, ed. by G. Cohen, INRIA, 1995, pp. 613–623.

[37] ———, *Sommerfeld solution as limiting amplitude diffraction by wedge*, Abstracts Sommerfeld '96 – Workshop, Modern mathematical methods in Diffraction theory and its Applications in Engineering, ed. E. Meister, Preprint Nr.1868 Technische Hochschule Darmstadt, Darmstadt, 1996, p. 29.

[38] A. E. Merzon, P. N. Zhevandrov, *Asymptotics of edge waves on a beach of nonconstant slope*, Uspekhi Math. Nauk **51** (1996), 220 (Russian).

[39] ———, *High-frequency asymptotics of edge waves on a beach of nonconstant slope*, SIAM J. Appl. Math, **59** (1998), 529–546.

[40] S. A. Nazarov, B. A. Plamenevskij, *Elliptic problems in domains with piecewise smooth boundaries*, De Gruyter, Berlin, 1994.

[41] F. Oberhettinger, *Diffraction of waves by a wedge*, Comm. Pure Appl. Math. **7** (1954), 551–563.

[42] B. A. Packham, *A note on generalized edge waves on a sloping beach*, Quart. J. Mech. Appl. Math. **42** (1989), 441–446.

[43] A. S. Peters, *Water waves over sloping beaches and the solution of a mixed boundary value problem for $\Delta\Phi - k^2\Phi = 0$ in a sector*, Comm. Pure Appl. Math. **5** (1952), 87–108.

[44] T. von Petersdorff, *Boundary integral equations for mixed Dirichlet, Neumann and transmission problems*, Math. Meth. Appl. Sci., **11** (1989), 185–213.

[45] G. I. Petrashen', V. G. Nikolaev, D. P. Kouzov, *On the method of series in the theory of diffraction of waves by plane corner regions*, Scientific Notes LGU, **246**, (1958), no. 5, 5–70 (Russian).

[46] M. Roseau, *Short waves parallel to the shore over a sloping beach*, Comm. Pure. Appl. Math. **11** (1958), 433–493.

[47] G. E. Shilov, *On boundary value problems in a quarter plane for partial differential equations with constant coefficients*, Sib. Math J. **11** (1961), 144–160 (Russian).

[48] S. L. Sobolev, *On mixed problem for partial differential equations with two independent variables*, Doklady Acad. Sci. USSR **122** (1958), 555–558 (Russian).

[49] A. Sommerfeld, *Mathematische theorie der diffraction*, Math. Ann. **47** (1896), 317–341.

[50] G. G. Stokes, Rep. Brit. Ass. part **1** (1846), 1 (Collected papers Vol.I, 167).

[51] A. A. Tuzhilin, *The theory of Malyuzhinets' inhomogeneous functional equations*, Differ. Equations, **9** (1973), 1580–1584.

[52] F. Ursell, *Edge waves on a sloping beach*, Proc. R. Soc. Lond. **A214** (1952), 79–97.

[53] M. I. Vishik, G. I. Eskin, *Equations in convolutions in a bounded region*, Russ. Math. Surv. **20** (1965), 85–151.

[54] G. B. Whitham, *Lectures on Wave Propagation*, Springer-Verlag, New-York, 1979.

[55] W. E. Williams, *Diffraction of an E-polarized plane wave by an imperfectly conducting wedge*, Proc. R. Soc. Lond. **A252** (1959), 376–394.

[56] P. N. Zhevandrov, A. E. Merzon, *Stability of entrained surface waves under small perturbations of the density of the upper layer*, Math. Notes, **64** (1998), 814–817.

[57] ———, *On the Neumann Problem for the Helmholtz equation in a plane angle*, Math. Meth. Appl. Sci. **23** (2000), 1401–1446.

[58] E. I. Zverovich, G. S. Litvinchuk, *Boundary-value problems with a shift for analytic functions and singular functional equations*, Russ. Math. Surv. **23** (1968), 67–124.

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