AN EIGENVALUE PROBLEM FOR THE ASSOCIATED ASKEY–WILSON POLYNOMIALS

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Abstract. To derive an eigenvalue problem for the associated Askey–Wilson polynomials, we consider an auxiliary function in two variables which is related to the associated Askey–Wilson polynomials introduced by Ismail and Rahman. The Askey–Wilson operator, applied in each variable separately, maps this function to the ordinary Askey–Wilson polynomials with different sets of parameters. A third Askey–Wilson operator is found with the help of a computer algebra program which links the two, and an eigenvalue problem is stated.

1. Introduction

Throughout this paper, we use the standard notation for the q-shifted factorials:

$$(a;q)_n := \prod_{j=0}^{n-1}\left(1-aq^j\right), \qquad (a_1,a_2,\dots,a_r;q)_n := \prod_{k=1}^{r}(a_k;q)_n\,,$$

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n\,, \qquad (a_1,a_2,\dots,a_r;q)_\infty := \prod_{k=1}^{r}(a_k;q)_\infty\,,$$

provided $|q|<1$. The basic hypergeometric series is defined by (cf. [9])

$${}_r\varphi_s\left(\begin{matrix} a_1,a_2,\dots,a_r \\ b_1,\dots,b_s\end{matrix}\,;\,q\,,\,z\right) := \sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n}{(q,b_1,b_2,\dots,b_s;q)_n}\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} z^n.$$

If $0<|q|<1$, the series converges absolutely for all z if $r\le s$, and for $|z|<1$ if $r=s+1$.

The Askey–Wilson polynomials are the most general extension of the classical orthogonal polynomials [1], [2], [6], [11], [12], [18]. They are most conveniently given in terms of a ${}_4\varphi_3$-series,

$$\begin{aligned} p_n(x) &= p_n(x;a,b,c,d) = p_n(x;a,b,c,d|q) \\ &= a^{-n}\,(ab,ac,ad;q)_n\,{}_4\varphi_3\left(\begin{matrix} q^{-n},\ abcdq^{n-1},\ az,\ a/z \\ ab,\ ac,\ ad\end{matrix}\,;q,\,q\right), \end{aligned}$$

where $x=\left(z+z^{-1}\right)/2$, and $|z|<1$. In this normalization, the Askey–Wilson polynomials are symmetric in all four parameters due to Sears' transformation [6].

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The Askey–Wilson polynomials satisfy the 3-term recurrence relation

$$2x\ p_n(x;a,b,c,d) = A_n\, p_{n+1}(x;a,b,c,d)\ +\ B_n\, p_n(x;a,b,c,d) + \ C_n\, p_{n-1}(x;a,b,c,d), \tag{1.1}$$

where

$$A_n = \frac{a^{-1}(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{(1-abcdq^{2n-1})(1-abcdq^{2n}-q^{2n})}, \tag{1.2}$$

$$C_n = \frac{a(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})(1-q^n)}{(1-abcdq^{2n-1})(1-abcdq^{2n})}, \tag{1.3}$$

$$B_n = a + a^{-1} - A_n - C_n. \tag{1.4}$$

The weight function with respect to which the polynomials $p_n(x)$ are orthogonal was found by Askey and Wilson in [6]. The Askey–Wilson divided difference operator is defined by

$$\begin{aligned} L(x)u &:= L\,(s;a,b,c,d)\,u\,(s) \\ &= \frac{\sigma\,(-s)\,\nabla x\,(s)\,u\,(s+1) + \sigma\,(s)\,\Delta x\,(s)\,u\,(s-1) - [\sigma\,(s)\,\Delta x\,(s) + \sigma\,(-s)\,\nabla x\,(s)]\,u\,(s)}{\Delta x\,(s)\,\nabla x\,(s)\,\nabla x_1\,(s)}, \end{aligned} \tag{1.5}$$

where $\sigma\,(s) = q^{-2s}\,(q^s-a)\,(q^s-b)\,(q^s-c)\,(q^s-d)$ and, by definition,

$$x(s) = \frac{1}{2}\left(q^s + q^{-s}\right)\ , \qquad x_1(s) = x\left(s+\frac{1}{2}\right),$$

$$\Delta f(s) = f(s+1) - f(s), \qquad \nabla f(s) = f(s) - f(s-1).$$

(We follow the notation in [7] and [8].) We will make use of an analogue of the power series expansion method, where a function is expanded in terms of generalized powers. For a positive integer m, the generalized powers are defined by

$$[x(s)-x(z)]^{(m)} = \prod_{n=0}^{m-1}[x_n(s) - x_n(z-k)], \qquad x_n(z) = x\left(z+\frac{n}{2}\right) \tag{1.6}$$

(see [16, Exercises 2.9–2.11, 2.25] and [17] for more details).

2. The Associated Askey–Wilson Polynomials

The associated Askey–Wilson polynomials, $p_n^\alpha(x) = p_n^\alpha(x;a,b,c,d) = p_n^\alpha(x;a,b,c,d|q)$, were introduced by Ismail and Rahman in [10]. They are solutions of the 3-term recurrence relation

$$2x\ p_n^\alpha(x;a,b,c,d) = A_{n+\alpha}\ p_{n+1}^\alpha(x;a,b,c,d)\ +\ B_{n+\alpha}\ p_n^\alpha(x;a,b,c,d) + \ C_{n+\alpha}\ p_{n-1}^\alpha(x;a,b,c,d), \tag{2.1}$$

where $0<\alpha<1$, with initial values $p_{-1}^\alpha(x) = 0$, $p_0^\alpha(x) = 1$, and $A_{n+\alpha}$, $B_{n+\alpha}$, $C_{n+\alpha}$ are given as in (1.2)–(1.4) with n replaced by $n+\alpha$. The two linearly independent solutions to (1.1) found in [10] are

$$\begin{aligned} R_{n+\alpha} = {}& \frac{(abq^{n+\alpha}, acq^{n+\alpha}, adq^{n+\alpha}, bcdq^{n+\alpha}/z;q)_\infty}{(bcq^{n+\alpha}, bdq^{n+\alpha}, cdq^{n+\alpha}, azdq^{n+\alpha};q)_\infty}\left(\frac{a}{z}\right)^{n+\alpha} \\ & \times\ {}_8W_7(bcd/qz; b/z, c/z, d/z, abcdq^{n+\alpha-1}, q^{-\alpha-n}; q, qz/a) \end{aligned} \tag{2.2}$$

and

$$S_{n+\alpha}=\frac{(abcdq^{2n+2\alpha},bzq^{n+\alpha+1},czq^{n+\alpha+1},dzq^{n+\alpha+1},bcdzq^{n+\alpha+1};q)_\infty}{(bcq^{n+\alpha},bdq^{n+\alpha},cdq^{n+\alpha},q^{n+\alpha+1},bcdzq^{2n+2\alpha+1};q)_\infty}(az)^{n+\alpha}$$
$$\times\ {}_8W_7(bcdzq^{2n+2\alpha};bcq^{n+\alpha},bdq^{n+\alpha},cdq^{n+\alpha},q^{n+\alpha+1},zq/a;q,az). \quad (2.3)$$

The weight function for the associated Askey–Wilson polynomials and an explicit polynomial representation were found by Ismail and Rahman in [10]. The latter is given by

$$\begin{aligned}p_n^\alpha(x)&=p_n^\alpha(x;a,b,c,d|q)\\&=\sum_{k=0}^{n}\frac{(q^{-n},abcdq^{2\alpha+n-1},abcdq^{2\alpha-1},ae^{i\theta},ae^{-i\theta};q)_k}{(q,abq^\alpha,acq^\alpha,adq^\alpha,abcdq^{\alpha-1};q)_k}\,q^k\\&\quad\times\ {}_{10}W_9(abcdq^{2\alpha+k-1};q^\alpha,bcq^{\alpha-1},bdq^{\alpha-1},cdq^{\alpha-1},q^{k+1},abcdq^{2\alpha+n+k-1},q^{k-n};q,a^2).\end{aligned}\quad (2.4)$$

There is another useful representation of the associated Askey–Wilson polynomials in terms of a double series due to Rahman,

$$\begin{aligned}p_n^\alpha(x)&=p_n^\alpha(x;a,b,c,d|q)\\&=\frac{(abcdq^{2\alpha-1},q^{\alpha+1};q)_n}{(q,abcdq^{\alpha-1};q)_n}q^{-\alpha n}\sum_{k=0}^{n}\frac{(q^{-n},abcdq^{2\alpha+n-1};q)_k}{(q^{\alpha+1},abq^\alpha;q)_k}\\&\quad\times\frac{(aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_k}{(acq^\alpha,acq^\alpha;q)_k}\sum_{j=0}^{k}\frac{(q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_j}{(q,abcdq^{2\alpha-2},aq^\alpha e^{i\theta},aq^\alpha e^{-i\theta};q)_j}q^j,\end{aligned}\quad (2.5)$$

where $x=\cos\theta$ (see [9, Exercises 8.26–8.27] and [14], [15]). This formula will be the starting point for our investigation.

3. An Overview of the Main Result

To construct an eigenvalue problem for the associated Askey–Wilson polynomials, let us consider an auxiliary function $u_n^\alpha(x,y)$ in two variables, which for $x=y$ coincides with the associated Askey–Wilson polynomials (up to a factor). We observe that the Askey–Wilson operator $L_0(x)$ (in one variable x) maps $u_n^\alpha(x,y)$ to the n-th degree ordinary Askey–Wilson polynomial (up to some factors). A similar result is obtained for the operator $L_1(y)$ applied to $u_n^\alpha(x,y)$ with respect to the second independent variable y. We will find an operator $L_2(x)$, which maps certain multiples of $(L_1(y)+\lambda)\,u_n^\alpha(x,y)$ to $(L_0(x)+\lambda)u_n^\alpha(x,y)$. As a result, we obtain an eigenvalue problem of the form

$$\begin{aligned}&\frac{(aq^s,aq^{-s};q)_\infty}{(aq^{\alpha+s-1},aq^{\alpha-s-1};q)_\infty}(L_2(x)+\lambda)\frac{(aq^{\alpha+s},aq^{\alpha-s};q)_\infty}{(aq^s,aq^{-s};q)_\infty}\,(L_1(y)+\mu_\alpha)\,u_n^\alpha(x,y)\\&\qquad=\frac{4q^{9/2}}{(1-q)^2\gamma}(L_0(x)+\lambda_{\alpha+n})u_n^\alpha(x,y)\end{aligned}\quad (3.1)$$

related to the associated Askey–Wilson polynomials of Ismail and Rahman (see Theorem 1 below for an exact statement). We shall use the normalization

$$p_n(x;a,b,c,d)={}_4\varphi_3\left(\begin{matrix}q^{-n},\ abcdq^{n-1},\ aq^s,\ aq^{-s}\\ ab,\ ac,\ ad\end{matrix};q,\ q\right)\quad (3.2)$$

for the ordinary Askey–Wilson polynomials throughout this paper.

Lemma 1. *Let $u_n^\alpha(x,y)$ be the function in the two variables x and y defined by*

$$u_n^\alpha(x,y) := \frac{(aq^s, aq^{-s}, aq^{\alpha+z}, aq^{\alpha-z};q)_\infty}{(aq^{\alpha+s}, aq^{\alpha-s}, aq^z, aq^{-z};q)_\infty} \times \sum_{m=0}^{n} \frac{(q^{-n}, \gamma q^{2\alpha+n-1}, aq^{\alpha+s}, aq^{\alpha-s};q)_m}{(q^{\alpha+1}, abq^\alpha, acq^\alpha, adq^\alpha;q)_m} q^m \times \sum_{k=0}^{m} \frac{(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1};q)_k}{(q, \gamma q^{2\alpha-2}, aq^{\alpha+z}, aq^{\alpha-z};q)_k} q^k, \tag{3.3}$$

with $x(s) = (q^s + q^{-s})/2$ and $y(z) = (q^z + q^{-z})/2$. Then $u_n^\alpha(x,y)$ satisfies an equation of the form

$$(L_0(x) + \lambda_{\alpha+n}) u_n^\alpha(x,y) = f_n^\alpha(x,y), \tag{3.4}$$

where $L_0(x) = L(s;a,b,c,d)$ is the Askey–Wilson divided difference operator in the variable x given by (1.5)*. Here,*

$$f_n^\alpha(x,y) = -\frac{4q^{3/2-\alpha}}{(1-q)^2} \frac{(aq^s, aq^{-s}, aq^{\alpha+z}, aq^{\alpha-z};q)_\infty}{(aq^{\alpha+s-1}, aq^{\alpha-s-1}, aq^z, aq^{-z};q)_\infty} \times (q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1};q)_1 \times p_n(x; aq^{\alpha-1}, bcdq^{\alpha-1}, q^{1+z}, q^{1-z}),$$

and

$$\lambda_{\alpha+n} = \frac{4q^{3/2}}{(1-q)^2} \left(1 - q^{-\alpha-n}\right)\left(1 - \gamma q^{\alpha+n-1}\right), \qquad \gamma = abcd.$$

Note that $f_n^\alpha(x,y)$ contains the n-th degree ordinary Askey–Wilson polynomial of the form (3.2) in the variable x. Our function $u_n^\alpha(x,y)$ is the Askey–Wilson polynomial when $\alpha = 0$ and a constant multiple of the associated Askey–Wilson polynomial if $x = y$.

Lemma 2. *The function $u_n^\alpha(x,y)$ satisfies another equation, namely*

$$(L_1(y) + \mu_\alpha) u_n^\alpha(x,y) = g_n^\alpha(x,y),$$

where $L_1(y) := L(y; q/a, q/b, q/c, q/d)$ is the Askey–Wilson divided difference operator in y. Here,

$$g_n^\alpha(x,y) = -\frac{4q^{9/2-\alpha}}{(1-q)^2\gamma} \frac{(aq^s, aq^{-s}, aq^{\alpha+z+1}, aq^{\alpha-z+1};q)_\infty}{(aq^{\alpha+s}, aq^{\alpha-s}, aq^z, aq^{-z};q)_\infty} \times (q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1};q)_1 \times p_n(x; aq^\alpha, bcdq^{\alpha-2}, q^{1+z}, q^{1-z})$$

and

$$\mu_\alpha = \frac{4q^{3/2}}{(1-q)^2} \left(1 - q^\alpha\right)\left(1 - q^{3-\alpha}/\gamma\right).$$

Note that $g_n^\alpha(x,y)$ contains another n-th degree Askey–Wilson polynomial (3.2) in the same variable x.

Lemma 3. *The difference differentiation formula*

$$(L\,(x)+\lambda)p_n(x;a,b,c,d)=\lambda p_n(x;a/q,bq,c,d) \tag{3.5}$$

holds for the Askey–Wilson polynomials given by (3.2)*. Here,* $L\,(x)=L\,(s;a,a/q,c,d)$ *is the Askey–Wilson divided difference operator* (1.5) *and*

$$\lambda=\frac{4q^{3/2}}{(1-q)^2}\,(1-ac/q)\,(1-ad/q)\,.$$

Lemmas 1–3 allow us to establish the eigenvalue problem (3.1) for the associated Askey–Wilson functions (3.3), see the next section.

4. Main Result

With the help of Lemmas 1–3, we now identify an operator $L_2(x)$ linking $(L_0(x)+\lambda_{\alpha+n})u_n^\alpha(x,y)$ and $(L_1(y)+\lambda_{-\alpha})u_n^\alpha(x,y)$ in such a way that an eigenvalue problem is formulated.

Theorem 1. *Let* $L_2(x)=L(s;aq^\alpha,aq^{\alpha-1},q^{1+z},q^{1-z})$ *be the Askey–Wilson divided difference operator defined by* (1.5) *with*

$$\sigma(s)=q^{-2s}\,(q^s-aq^\alpha)\left(q^s-aq^{\alpha-1}\right)\left(q^s-q^{1+z}\right)\left(q^s-q^{1-z}\right)$$

and

$$\lambda=\frac{4q^{3/2}}{(1-q)^2}\left(1-aq^{\alpha-z}\right)\left(1-aq^{\alpha+z}\right).$$

Then an eigenvalue problem for the associated Askey–Wilson functions $u_n^\alpha(x,y)$ *can be stated as*

$$\frac{\gamma}{q^3}\frac{(aq^s,aq^{-s};q)_\infty}{(aq^{\alpha+s-1},aq^{\alpha-s-1};q)_\infty}\,(L_2(x)+\lambda)\,\frac{(aq^{\alpha+s},aq^{\alpha-s};q)_\infty}{(aq^s,aq^{-s};q)_\infty}\,(L_1(y)+\mu_\alpha)\,u_n^\alpha(x,y)$$
$$=\frac{4q^{3/2}}{(1-q)^2}\,(L_0(x)+\lambda_{\alpha+n})\,u_n^\alpha(x,y), \tag{4.1}$$

where L_0*,* L_1*,* $\lambda_{\alpha+n}$*,* μ_α *and* $u_n^\alpha(x,y)$ *are defined as in Lemmas* 1–3.

Computational details are left to the reader. The explicit form of the difference operator in two variables on the left-hand side of the last equation has also been calculated, but it is too long to be displayed here.

5. Proofs

5.1. **Proof of Lemma 1.** Let λ_ν be an arbitrary number. We are looking for solutions of a generalization of the equation (3.4), namely,

$$(L_0(x)+\lambda_\nu)u_n^\alpha(x,y)=f_n^\alpha(x,y),$$

in terms of generalized powers (see (1.6) for the definition)

$$u_n^\alpha(x,y)=\sum_{m=0}^{n}c_m v_m[x(s)-x(\xi)]^{(\alpha+m)},$$

where

$$v_m = v_m(y) = \frac{(aq^{\alpha+z}, aq^{\alpha-z}; q)_\infty}{(aq^z, aq^{-z}; q)_\infty} \sum_{k=0}^{n} \frac{(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}; q)_k}{(q, \gamma q^{2\alpha-2}, aq^{\alpha+z}, aq^{\alpha-z}; q)_k} q^k,$$

and $\gamma = abcd$. (This is an analogue of the power series expansion; see [7], [16, Exercises 2.9–2.11], and [17] for properties of the generalized powers.)

Apply the Askey–Wilson operator to $u_n^\alpha(x,y)$ to obtain

$$(L_0(x) + \lambda_\nu) u_n^\alpha(x,y) = \lambda_\nu \sum_{m=0}^{n} c_m v_m [x(s) - x(\xi)]^{(\alpha+m)} + \sum_{m=0}^{n} c_m v_m \; L_0(x) [x(s) - x(\xi)]^{(\alpha+m)},$$

since v_m is independent of x. By [7], we have

$$\begin{aligned} L_0(x)[x(s) - x(\xi)]^{(\alpha+m)} &= \gamma(\alpha+m)\gamma(\alpha+m-1)\sigma(\xi-\alpha-m+1)[x(s) - x(\xi-1)]^{(\alpha+m-2)} \\ &\quad + \gamma(\alpha+m)\tau_{\alpha+m-1}(\xi-\alpha-m+1)[x(s) - x(\xi-1)]^{(\alpha+m-1)} \\ &\quad - \lambda_{\alpha+m}[x(s) - x(\xi)]^{(\alpha+m)}. \end{aligned}$$

We use the same notations as in [7], [16, Exercise 2.25], or [17]. Choose $a_0 := \xi - \alpha - m + 1$ to be a root of the equation $\sigma(a_0) = 0$. Then $\xi = a_0 + \alpha + m - 1$, and one obtains

$$\begin{aligned} (L_0(x) + \lambda_\nu) u_n^\alpha(x,y) &= \sum_{m=0}^{n} c_m v_m \gamma(\alpha+m) \tau_{\alpha+m-1}(a_0) [x(s) - x(a_0+\alpha+m-2)]^{(\alpha+m-1)} \\ &\quad + \sum_{m=0}^{n} c_m v_m (\lambda_\nu - \lambda_{\alpha+m}) [x(s) - x(a_0+\alpha+m-1)]^{(\alpha+m)} \\ &= c_0 v_0 \gamma(\alpha) \tau_{\alpha-1}(a_0) [x(s) - x(a_0+\alpha-2)]^{(\alpha-1)} \\ &\quad + \sum_{m=1}^{n} c_m v_m \gamma(\alpha+m) \tau_{\alpha+m-1}(a_0) [x(s) - x(a_0+\alpha+m-2)]^{(\alpha+m-1)} \\ &\quad + \sum_{m=0}^{n} c_m v_m (\lambda_\nu - \lambda_{\alpha+m}) [x(s) - x(a_0+\alpha+m-1)]^{(\alpha+m)}. \end{aligned}$$

Letting $m = k+1$, we get

$$\begin{aligned} (L_0(x) + \lambda_\nu) u_n^\alpha(x,y) &= c_0 v_0 \gamma(\alpha) \tau_{\alpha-1}(a_0) [x(s) - x(a_0+\alpha-2)]^{(\alpha-1)} \\ &\quad + \sum_{k=0}^{n-1} c_{k+1} v_{k+1} \gamma(\alpha+k+1) \tau_{\alpha+k}(a_0) [x(s) - x(a_0+\alpha+k-1)]^{(\alpha+k)} \\ &\quad + \sum_{k=0}^{n} c_k v_k (\lambda_\nu - \lambda_{\alpha+k}) [x(s) - x(a_0+\alpha+k-1)]^{(\alpha+k)}. \end{aligned} \tag{5.1}$$

Note that for

$$v_k = \sum_{l=0}^{k} e_l, \qquad e_l := \frac{(aq^{\alpha+z}, aq^{\alpha-z}; q)_\infty}{(aq^z, aq^{-z}; q)_\infty} \frac{(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1}; q)_l}{(q, \gamma q^{2\alpha-2}, aq^{\alpha+z}, aq^{\alpha-z}; q)_l} q^l$$

one has

$$v_{k+1} = v_k + e_{k+1} \qquad \text{and} \qquad v_0 = e_0.$$

After choosing $\lambda_\nu = \lambda_{\alpha+n}$, equation (5.1) becomes

$$(L_0(x)+\lambda_{\alpha+n})u_n^\alpha(x,y) = \sum_{k=-1}^{n-1} c_{k+1}e_{k+1}\gamma(\alpha+k+1)\tau_{\alpha+k}(a_0)[x(s)-x(a_0+\alpha+k-1)]^{(\alpha+k)}$$
$$+\sum_{k=0}^{n-1} c_{k+1}v_k\gamma(\alpha+k+1)\tau_{\alpha+k}(a_0)[x(s)-x(a_0+\alpha+k-1)]^{(\alpha+k)}$$
$$+\sum_{k=0}^{n-1} c_k v_k(\lambda_{\alpha+n}-\lambda_{\alpha+k})[x(s)-x(a_0+\alpha+k-1)]^{(\alpha+k)}.$$

The latter two sums vanish if

$$c_{k+1}\gamma(\alpha+k+1)\tau_{\alpha+k}(a_0) = c_k(\lambda_{\alpha+n}-\lambda_{\alpha+k}).$$

Therefore,

$$(L_0(x)+\lambda_{\alpha+n})u_n^\alpha(x,y) = \sum_{k=-1}^{n-1} c_{k+1}e_{k+1}\gamma(\alpha+k+1)\tau_{\alpha+k}(a_0)[x(s)-x(a_0+\alpha+k-1)]^{(\alpha+k)}$$
$$= \sum_{m=0}^{n} c_m e_m\gamma(\alpha+m)\tau_{\alpha+m-1}(a_0)[x(s)-x(a_0+\alpha+m-2)]^{(\alpha+m-1)} =: f_n^\alpha(x,y).$$

Finally, we show that the function $f_n^\alpha(x,y)$ is, up to a factor, the n-th ordinary Askey–Wilson polynomial. The generalized powers have the property (see [16])

$$[x(s)-x(z)]^{(n+1)} = [x(s)-x(z)][x(s)-x(z-1)]^{(n)},$$

which leads to

$$f_n^\alpha(x,y) = \sum_{m=0}^{n} c_m e_m\gamma(\alpha+m)\tau_{\alpha+m-1}(a_0)\frac{[x(s)-x(a_0+\alpha+m-1)]^{(\alpha+m)}}{[x(s)-x(a_0+\alpha+m-1)]}.$$

Moreover,

$$c_m[x(s)-x(a_0+\alpha+m-1)]^{(\alpha+m)}$$
$$= c_0\frac{(q^{-n},\gamma q^{2\alpha+n-1};q)_m}{(q^{\alpha+1},abq^\alpha,acq^\alpha,adq^\alpha;q)_m}q^m\ [x(s)-x(a_0+\alpha+m-1)]^{(\alpha+m)}$$
$$= c_0\,\varphi_m(x)\,[x(s)-x(a_0+\alpha-1)]^{(\alpha)},$$

where, by definition,

$$\varphi_m(x) := \frac{(aq^s,aq^{-s};q)_\infty}{(aq^{\alpha+s},aq^{\alpha-s};q)_\infty}\frac{(q^{-n},\gamma q^{2\alpha+n-1},aq^{\alpha+s},aq^{\alpha-s};q)_m}{(q^{\alpha+1},abq^\alpha,acq^\alpha,adq^\alpha;q)_m}q^m.$$

Therefore,

$$f_n^\alpha(x,y) = \frac{(aq^s,aq^{-s};q)_\infty}{(aq^{\alpha+s},aq^{\alpha-s};q)_\infty}\frac{(q^{-n},\gamma q^{2\alpha+n-1},aq^{\alpha+s},aq^{\alpha-s};q)_m}{(q^{\alpha+1},abq^\alpha,acq^\alpha,adq^\alpha;q)_m}q^m$$
$$\times\frac{(aq^{\alpha+z},aq^{\alpha-z};q)_\infty}{(aq^z,aq^{-z};q)_\infty}\frac{(q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_m}{(q,\gamma q^{2\alpha-2},aq^{\alpha+z},aq^{\alpha-z};q)_m}q^m$$

$$\times \frac{\gamma(\alpha+m)\tau_{\alpha+m-1}(a_0)}{[x(s)-x(a_0+\alpha+m-1)]}. \tag{5.2}$$

Recall that $a = q^{a_0}$ and

$$\gamma(\alpha+m) = q^{-\frac{\alpha+m-1}{2}}\,\frac{1-q^{\alpha+m}}{1-q},$$

$$x(s)-x(a_0+\alpha+m-1) = -\frac{1}{2a}q^{-\alpha-m+1}(1-aq^{\alpha-s+m-1})(1-aq^{\alpha+s+m-1}),$$

$$\tau_{\alpha+m-1}(a_0) = \frac{2}{a(1-q)}q^{-2(\alpha+m-1)+\frac{\alpha+m}{2}}(1-abq^{\alpha+m-1})(1-acq^{\alpha+m-1})(1-adq^{\alpha+m-1}),$$

which allows us to simplify the last term of (5.2) to

$$q^m\frac{\gamma(\alpha+m)\tau_{\alpha+m-1}(a_0)}{[x(s)-x(a_0+\alpha+m-1)]} = -4q^{\frac{3}{2}-\alpha}\frac{1-q^{\alpha+m}}{1-q}\,\frac{(1-abq^{\alpha+m-1})(1-acq^{\alpha+m-1})(1-adq^{\alpha+m-1})}{(1-aq^{\alpha-s+m-1})(1-aq^{\alpha+s+m-1})}.$$

Thus $f_n^\alpha(x,y)$ becomes

$$\begin{aligned} f_n^\alpha(x,y) &= \frac{-4q^{\frac{3}{2}-\alpha}}{(1-q)^2}\,\frac{(aq^s,aq^{-s},aq^{\alpha+z},aq^{\alpha-z};q)_\infty}{(aq^{\alpha+s-1},aq^{\alpha-s-1},aq^z,aq^{-z};q)_\infty}\ (q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_1 \\ &\quad\times\sum_{m=0}^{n}\frac{(q^{-n},\gamma q^{2\alpha+n-1},aq^{\alpha+s-1},aq^{\alpha-s-1};q)_m}{(q,\gamma q^{2\alpha-2},aq^{\alpha+z},aq^{\alpha-z};q)_m}q^m \\ &= \frac{-4q^{\frac{3}{2}-\alpha}}{(1-q)^2}\,\frac{(aq^s,aq^{-s},aq^{\alpha+z},aq^{\alpha-z};q)_\infty}{(aq^{\alpha+s-1},aq^{\alpha-s-1},aq^z,aq^{-z};q)_\infty}\ (q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_1 \\ &\quad\times p_n(x;aq^{\alpha-1},bcdq^{\alpha-1},q^{1+z},q^{1-z}), \end{aligned} \tag{5.3}$$

which completes the proof of the lemma.

5.2. **Proof of Lemma 2.** Consider the equation

$$(L_1(y)+\lambda_\nu)u_n^\alpha(x,y) = g_n^\alpha(x,y),$$

and rewrite $u_n^\alpha(x,y)$ in the form

$$u_n^\alpha(x,y) = \sum_{m=0}^{n} c_m^\alpha(aq^{\alpha+s},aq^{\alpha-s};q)_m\frac{(aq^s,aq^{-s};q)_\infty}{(aq^{\alpha+s},aq^{\alpha-s};q)_\infty}v_m^\alpha(y),$$

where

$$c_m^\alpha = \frac{(q^{-n},\gamma q^{2\alpha+n-1};q)_m}{(q^{\alpha+1},abq^\alpha,acq^\alpha,adq^\alpha;q)_m}q^m, \qquad \gamma = abcd,$$

and

$$v_m^\alpha(y) = \frac{(aq^{\alpha+z},aq^{\alpha-z};q)_\infty}{(aq^z,aq^{-z};q)_\infty}\sum_{k=0}^{m}\frac{(q^\alpha,abq^{\alpha-1},acq^{\alpha-1},adq^{\alpha-1};q)_k}{(q,\gamma q^{2\alpha-2},aq^{\alpha+z},aq^{\alpha-z};q)_k}q^k.$$

Apply the Askey–Wilson operator $L_1(y) := L\,(y;q/a,q/b,q/c,q/d)$ to $u_n^\alpha(x,y)$ to obtain

$$(L_1(y)+\lambda_\nu)u_n^\alpha(x,y) = \sum_{m=0}^{n} c_m^\alpha(aq^{\alpha+s},aq^{\alpha-s};q)_m\frac{(aq^s,aq^{-s};q)_\infty}{(aq^{\alpha+s},aq^{\alpha-s};q)_\infty}\ (L_1(y)+\lambda_\nu)\,v_m^\alpha(y).$$

Let

$$v_m^\alpha(y) := \sum_{k=0}^{m} \frac{c_k}{[x(s)-x(\xi)]^{(\alpha+k)}}$$

in analogy with [7]. Then

$$(L_1(y)+\lambda_\nu)v_m^\alpha(y) = \lambda_\nu \sum_{k=0}^{m} \frac{c_k}{[x(s)-x(\xi)]^{(\alpha+k)}} + \sum_{k=0}^{m} c_k\, L_1(y)\left(\frac{1}{[x(s)-x(\xi)]^{(\alpha+k)}}\right).$$

By [7], we have

$$\begin{aligned} L_1(y)\left(\frac{1}{[x(s)-x(\xi)]^{(\alpha+k)}}\right) &= \frac{\gamma(\alpha+k)\gamma(\alpha+k+1)\sigma(\xi+1)}{[x(z)-x(\xi+1)]^{(\alpha+k+2)}} \\ &\qquad - \frac{\gamma(\alpha+k)\tau_{-\alpha-k-1}(\xi+1)}{[x(z)-x(\xi)]^{(\alpha+k+1)}} - \frac{\lambda_{-\alpha-k}}{[x(z)-x(\xi)]^{(\alpha+k)}} \end{aligned}$$

(see also [16, Exercise 2.25]). Upon choosing $a_0 := \xi+1$ to be a root of the equation $\sigma(a_0)=0$, we obtain

$$\begin{aligned} (L_1(y)+\lambda_\nu)v_m^\alpha(y) &= \lambda_\nu \sum_{k=0}^{m} \frac{c_k}{[x(s)-x(a_0)]^{(\alpha+k)}} \\ &\qquad - \sum_{k=0}^{m} c_k \left(\frac{\gamma(\alpha+k)\tau_{-\alpha-k-1}(a_0)}{[x(z)-x(a_0-1)]^{(\alpha+k+1)}} + \frac{\lambda_{-\alpha-k}}{[x(z)-x(a_0-1)]^{(\alpha+k)}}\right) \\ &= \sum_{k=0}^{m} \frac{c_k\,(\lambda_\nu-\lambda_{-\alpha-k})}{[x(z)-x(a_0-1)]^{(\alpha+k)}} - \sum_{k=0}^{m} \frac{c_k\,\gamma(\alpha+k)\tau_{-\alpha-k-1}(a_0)}{[x(z)-x(a_0-1)]^{(\alpha+k+1)}} \\ &= \frac{c_0\,(\lambda_\nu-\lambda_{-\alpha})}{[x(z)-x(a_0-1)]^{(\alpha)}} + \sum_{k=1}^{m} \frac{c_k\,(\lambda_\nu-\lambda_{-\alpha-k})}{[x(z)-x(a_0-1)]^{(\alpha+k)}} \\ &\qquad - \frac{c_m\,\gamma(\alpha+m)\tau_{-\alpha-m-1}(a_0)}{[x(z)-x(a_0-1)]^{(\alpha+m+1)}} - \sum_{k=0}^{m-1} \frac{c_k\,\gamma(\alpha+k)\tau_{-\alpha-k-1}(a_0)}{[x(z)-x(a_0-1)]^{(\alpha+k+1)}}. \end{aligned}$$

Now choose $\lambda_\nu = \lambda_{-\alpha}$ and let $k = l+1$. Then we obtain

$$\begin{aligned} (L_1(y)+\lambda_\nu)v_m^\alpha(y) &= -\frac{c_m\,\gamma(\alpha+m)\tau_{-\alpha-m-1}(a_0)}{[x(z)-x(a_0-1)]^{(\alpha+m+1)}} \\ &\qquad + \sum_{l=0}^{m-1} \frac{c_{l+1}\,(\lambda_{-\alpha}-\lambda_{-\alpha-l-1})}{[x(z)-x(a_0-1)]^{(\alpha+l+1)}} - \sum_{l=0}^{m-1} \frac{c_l\,\gamma(\alpha+l)\tau_{-\alpha-l-1}(a_0)}{[x(z)-x(a_0-1)]^{(\alpha+l+1)}}. \end{aligned}$$

The latter two sums vanish if

$$c_{l+1}\,(\lambda_{-\alpha}-\lambda_{-\alpha-l-1}) = c_l\,\gamma(\alpha+l)\tau_{-\alpha-l-1}(a_0).$$

In that case, we have

$$(L_1(y)+\lambda_\nu)v_m^\alpha(y) = -\frac{c_m\,\gamma(\alpha+m)\tau_{-\alpha-m-1}(a_0)}{[x(z)-x(a_0-1)]^{(\alpha+m+1)}} = -\frac{c_{m+1}\,(\lambda_{-\alpha}-\lambda_{-\alpha-m-1})}{[x(z)-x(a_0-1)]^{(\alpha+m+1)}} =: h_m^\alpha(y).$$

Here,

$$\begin{aligned}
\frac{c_{m+1}}{[x(z)-x(a_0-1)]^{(\alpha+m+1)}} &= \frac{c_0}{[x(z)-x(a_0-1)]^{(\alpha)}}\varphi_{m+1}(z),\\
\varphi_{m+1}(z) &= \frac{(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1};q)_{m+1}}{(q,\gamma q^{2\alpha-2}, aq^{\alpha+z}, aq^{\alpha-z};q)_{m+1}}q^{m+1},\\
\frac{c_0}{[x(z)-x(a_0-1)]^{(\alpha)}} &= \frac{(aq^{\alpha+z}, aq^{\alpha-z};q)_\infty}{(aq^z, aq^{-z};q)_\infty},\\
\lambda_{-\alpha}-\lambda_{-\alpha-m-1} &= \frac{4}{(1-q)^2\gamma}q^{\frac{7}{2}-\alpha-m}(1-q^{m+1})(1-\gamma q^{2\alpha+m-2})
\end{aligned}$$

and

$$h_m^\alpha(y) = -\frac{4q^{\frac{9}{2}-\alpha}}{(1-q)^2\gamma}\frac{(aq^{\alpha+z}, aq^{\alpha-z};q)_\infty}{(aq^z, aq^{-z};q)_\infty}\frac{(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1};q)_{m+1}}{(q,\gamma q^{2\alpha-2};q)_m(aq^{\alpha+z}, aq^{\alpha-z};q)_{m+1}}.$$

Therefore,

$$\begin{aligned}
&(L_1(y)+\lambda_\nu)u_n^\alpha(x,y)\\
&\quad= \sum_{m=0}^n c_m^\alpha (aq^{\alpha+s}, aq^{\alpha-s};q)_m \frac{(aq^s, aq^{-s};q)_\infty}{(aq^{\alpha+s}, aq^{\alpha-s};q)_\infty} L_1(y) v_m^\alpha(y)\\
&\quad= -\sum_{m=0}^n c_m^\alpha (aq^{\alpha+s}, aq^{\alpha-s};q)_m \frac{(aq^s, aq^{-s};q)_\infty}{(aq^{\alpha+s}, aq^{\alpha-s};q)_\infty}\\
&\qquad\times \frac{4q^{\frac{9}{2}-\alpha}}{(1-q)^2\gamma}\frac{(aq^{\alpha+z}, aq^{\alpha-z};q)_\infty}{(aq^z, aq^{-z};q)_\infty}\frac{(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1};q)_{m+1}}{(q,\gamma q^{2\alpha-2};q)_m(aq^{\alpha+z}, aq^{\alpha-z};q)_{m+1}}\\
&\quad= -\frac{4q^{\frac{9}{2}-\alpha}}{(1-q)^2\gamma}\frac{(aq^s, aq^{-s}, aq^{\alpha+z+1}, aq^{\alpha-z+1};q)_\infty}{(aq^{\alpha+s}, aq^{\alpha-s}, aq^z, aq^{-z};q)_\infty}\\
&\qquad\times \sum_{m=0}^n \frac{(q^{-n}, \gamma q^{2\alpha+n-1}, aq^s, aq^{-s};q)_m}{(q^{\alpha+1}, abq^\alpha, acq^\alpha, adq^\alpha;q)_m}q^m\\
&\qquad\times \frac{(1-q^\alpha)(1-abq^{\alpha-1})(1-acq^{\alpha-1})(1-adq^{\alpha-1})}{(q,\gamma q^{2\alpha-2};q)_m(aq^{\alpha+z+1}, aq^{\alpha-z+1};q)_m}\\
&\quad= -\frac{4q^{\frac{9}{2}-\alpha}}{(1-q)^2\gamma}\frac{(aq^s, aq^{-s}, aq^{\alpha+z+1}, aq^{\alpha-z+1};q)_\infty}{(aq^{\alpha+s}, aq^{\alpha-s}, aq^z, aq^{-z};q)_\infty}\\
&\qquad\times \left(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1};q\right)_1\\
&\qquad\times {}_4\varphi_3\left(\begin{matrix} q^{-n}, \gamma q^{2\alpha+n-1}, aq^{\alpha+s}, aq^{\alpha-s}\\ \gamma q^{2\alpha-2}, aq^{\alpha+z+1}, aq^{\alpha-z+1}\end{matrix};q,\ q\right)\\
&\quad= -\frac{4q^{\frac{9}{2}-\alpha}}{(1-q)^2\gamma}\frac{(aq^s, aq^{-s}, aq^{\alpha+z+1}, aq^{\alpha-z+1};q)_\infty}{(aq^{\alpha+s}, aq^{\alpha-s}, aq^z, aq^{-z};q)_\infty}\\
&\qquad\times \left(q^\alpha, abq^{\alpha-1}, acq^{\alpha-1}, adq^{\alpha-1};q\right)_1 \times\ p_n(x; aq^\alpha, bcdq^{\alpha-2}, q^{1+z}, q^{1-z}).
\end{aligned}$$

This completes the proof of the lemma.

5.3. **Proof of Lemma 3.** The structure of the Askey–Wilson operator in (1.5) and the basic hypergeometric series representation (3.2) suggest to look for a 4-term relation of the form

$$K_1\,{}_4\varphi_3\left(\begin{matrix}A,B,C,D\\F,G,H\end{matrix};q,\,q\right)+K_2\,{}_4\varphi_3\left(\begin{matrix}A,B,Cq,D/q\\F,G,H\end{matrix};q,\,q\right)\\+K_3\,{}_4\varphi_3\left(\begin{matrix}A,B,C/q,Dq\\F,G,H\end{matrix};q,\,q\right)+K_4\,{}_4\varphi_3\left(\begin{matrix}A,B,C/q,D/q\\F,G/q,H/q\end{matrix};q,\,q\right)=0, \quad (5.4)$$

for some undetermined coefficients K_1, K_2, K_3 and K_4 (up to a common factor). Doing a term-wise comparison, we may hope to find K_1, K_2, K_3, K_4 which satisfy

$$\begin{aligned}K_1(1-C)(1-D)(1-Cq^{k-1})(1-Dq^{k-1})(1-G/q)(1-H/q)&\\+K_2(1-Cq^k)(1-Cq^{k-1})(1-D/q)(1-D)(1-G/q)(1-H/q)&\\+K_3(1-Dq^k)(1-Dq^{k-1})(1-C/q)(1-C)(1-G/q)(1-H/q)&\\+K_4(1-C/q)(1-C)(1-D/q)(1-D)(1-Gq^{k-1})(1-Hq^{k-1})&=0.\end{aligned}$$

If we are successful, then the above equation does indeed imply the contiguous relation (5.4). In the equation, we compare coefficients of powers of q^k. This yields a system of 3 linear equations in the 4 unknowns K_1, K_2, K_3, K_4. With the help of *Mathematica*, we obtain the solution

$$\begin{aligned}K_1&=\frac{(C-q)(D-q)(-GH-CDq+CGq+DGq+CHq+DHq-GHq-CDq^2)}{(G-q)(H-q)(Cq-D)(Dq-C)},\\K_2&=\frac{(C-1)(D-G)(D-H)(C-q)q}{(D-C)(G-q)(H-q)(Cq-D)},\\K_3&=\frac{(D-1)(C-G)(C-H)(D-q)q}{(C-D)(G-q)(H-q)(Dq-C)},\end{aligned}$$

where the free parameter K_4 was chosen to be 1 (see Appendix A for the *Mathematica* code). The required 4-term contiguous relation is then given by

$$\frac{(C-q)(D-q)(-GH-CDq+CGq+DGq+CHq+DHq-GHq-CDq^2)}{(G-q)(H-q)(Cq-D)(Dq-C)}\\\times\,{}_4\varphi_3\left(\begin{matrix}A,B,C,D\\F,G,H\end{matrix};q,\,q\right)+{}_4\varphi_3\left(\begin{matrix}A,B,C/q,D/q\\F,G/q,H/q\end{matrix};q,\,q\right)\\+\frac{(C-1)(D-G)(D-H)(C-q)q}{(D-C)(G-q)(H-q)(Cq-D)}\,{}_4\varphi_3\left(\begin{matrix}A,B,Cq,D/q\\F,G,H\end{matrix};q,\,q\right)\\+\frac{(D-1)(C-G)(C-H)(D-q)q}{(C-D)(G-q)(H-q)(Dq-C)}\,{}_4\varphi_3\left(\begin{matrix}A,B,C/q,Dq\\F,G,H\end{matrix};q,\,q\right)=0. \quad (5.5)$$

(This 4-term contiguous relation for the ${}_4\varphi_3$-functions can be extended to an arbitrary ${}_r\psi_s$-function, see Appendix A for more details.)

When $qABCD=FGH$, in view of the structure of the Askey–Wilson operator in (1.5), equation (5.5) should become

$$(L(x)+\lambda)\,{}_4\varphi_3\left(\begin{matrix}A,B,Cq,D/q\\F,G,H\end{matrix};q,\,q\right)$$

$$= \frac{\sigma(-s)}{\Delta x(s)\nabla x_1(s)}\, {}_4\varphi_3\left(\begin{matrix} A, B, Cq, D/q \\ F, G, H \end{matrix}; q, q\right) + \frac{\sigma(s)}{\nabla x(s)\nabla x_1(s)}\, {}_4\varphi_3\left(\begin{matrix} A, B, C/q, Dq \\ F, G, H \end{matrix}; q, q\right)$$
$$+ \frac{\lambda\Delta x(s)\nabla x(s)\nabla x_1(s) - \sigma(s)\Delta x(s) - \sigma(-s)\nabla x(s)}{\Delta x(s)\nabla x(s)\nabla x_1(s)}\, {}_4\varphi_3\left(\begin{matrix} A, B, C, D \\ F, G, H \end{matrix}; q, q\right).$$

Equating coefficients, one obtains

$$\begin{aligned}(1 - C/q)(1 - D/q)(D - C)(GH + CDq - CGq - DGq - CHq - DHq + GHq + CDq^2)& \\ = \frac{2qa^3}{1-q}\left(\sigma(-s)\nabla x(s) + \sigma(s)\Delta x(s) - \lambda\Delta x(s)\nabla x(s)\nabla x_1(s)\right)&\end{aligned}$$

and

$$\begin{aligned}
(D - C)(D - C/q)(-C + D/q) &= \frac{2aq^{1/2}}{1-q}\nabla x_1(s)\frac{2a}{1-q}\Delta x(s)\frac{2a}{1-q}\nabla x(s),\\
(C - 1)(G - D)(H - D)(q - C) &= -qa^2\sigma(-s),\\
(D - C)(-C + D/q) &= \frac{2aq^{1/2}}{1-q}\nabla x_1(s)\frac{2a}{1-q}\Delta x(s),\\
(D - 1)(G - C)(H - C)(q - D) &= -qa^2\sigma(s),\\
(D - C)(D - C/q) &= \frac{2aq^{1/2}}{1-q}\nabla x_1(s)\frac{2a}{1-q}\nabla x(s),\\
(G - q)(H - q) &= q^2\frac{(1-q)^2}{4q^{3/2}}\lambda.
\end{aligned}$$

This gives the required formula (3.5) for the Askey–Wilson operator with

$$\sigma(s) = q^{-2s}\,(q^s - a)\,(q^s - a/q)\,(q^s - c)\,(q^s - d)\,, \quad \lambda = \frac{4q^{3/2}}{(1-q)^2}\,(1 - ac/q)\,(1 - ad/q)\,.$$

The proof of the lemma is complete.

Appendix A. 4-Term Contiguous Relations

In order to derive the contiguous relation (5.5), one can use the following *Mathematica* program:[1]

[1]A corresponding *Mathematica* notebook is available on the article's website `http://www.mat.univie.ac.at/~kratt/artikel/AssAWPols.html`.

```
In[1]:= X1 = K1*(1 - C) (1 - D) (1 - C*K/q) (1 - D*K/q) (1 - G/q) (1 - H/q) +
             K2*(1 - C*K) (1 - C*K/q) (1 - D/q) (1 - D) (1 - G/q) (1 - H/q) +
             K3*(1 - D*K) (1 - D*K/q) (1 - C/q) (1 - C) (1 - G/q) (1 - H/q) +
             K4*(1 - C/q) (1 - C) (1 - D/q) (1 - D) (1 - G*K/q) (1 - H*K/q) ;
        X1 = Table[Coefficient[X1, K, i] == 0, i, 0, 2];
        X1 = Solve[X1, K1, K2, K3, K4];
        X1 = {K1 -> Factor[K1/.X1[[1]]], K2 -> Factor[K2/.X1[[1]]],
              K3 -> Factor[K3/.X1[[1]]], K4 -> Factor[K4/.X1[[1]]]}

Out[1]= {K1 -> (K4 (C - q) (D - q)
>            (G H + C D q - C G q - D G q - C H q - D H q + G H q + C D q^2)) /
>            ((G - q) (H - q) (-D + C q) (C - D q)),
>         K2 -> - ((-1 + C) (D - G) (D - H) K4 (C - q) q) /
>            ((C - D) (G - q) (H - q) (-D + C q)),
>         K3 -> - ((-1 + D) (C - G) (C - H) K4 (D - q) q) /
>            ((C - D) (G - q) (H - q) (C - D q)),
>         K4 -> K4}
```

It is evident from the proof of (5.5) that, actually, an extension for bilateral series (see [9, equation (5.1.1)] for the definition) with an arbitrary number of parameters holds, namely:

$$\begin{aligned}\frac{(c-q)\,(d-q)\,(-gh-cdq+cgq+dgq+chq+dhq-ghq-cdq^2)}{(g-q)\,(h-q)\,(cq-d)\,(dq-c)}&\\ \times\ {}_r\psi_s\!\left(\begin{matrix}a_1,\dots,a_i,\ c,d\\ b_0,\dots,b_k,\ g,h\end{matrix};q,\,t\right)&\\ +\frac{(c-1)\,(d-g)\,(d-h)\,(c-q)\,q}{(d-c)\,(g-q)\,(h-q)\,(cq-d)}\ {}_r\psi_s\!\left(\begin{matrix}a_1,\dots,a_i,\ cq,d/q\\ b_0,\dots,b_k,\ g,h\end{matrix};q,\,t\right)&\\ +\frac{(d-1)\,(c-g)\,(c-h)\,(d-q)\,q}{(c-d)\,(g-q)\,(h-q)\,(dq-c)}\ {}_r\psi_s\!\left(\begin{matrix}a_1,\dots,a_i,\ c/q,dq\\ b_0,\dots,b_k,\ g,h\end{matrix};q,\,t\right)&\\ +\ {}_r\psi_s\!\left(\begin{matrix}a_1,\dots,a_i,\ c/q,d/q\\ b_0,\dots,b_k,\ g/q,h/q\end{matrix};q,\,t\right)=0.&\end{aligned}\tag{A.1}$$

Furthermore, in the same way, the following variation can be obtained:[2]

$$\begin{aligned}\frac{(g-1)\,(h-1)\,(-gh-cdq+cgq+dgq+chq+dhq-ghq-cdq^2)}{(c-1)\,(d-1)\,(gq-h)\,(hq-g)}&\\ \times\ {}_r\psi_s\!\left(\begin{matrix}a_1,\dots,a_i,\ c,d\\ b_0,\dots,b_k,\ g,h\end{matrix};q,\,t\right)&\end{aligned}$$

[2]Again, a corresponding *Mathematica* notebook is available on the article's website `http://www.mat.univie.ac.at/~kratt/artikel/AssAWPols.html`.

$$+\frac{(c-g)\,(d-g)\,(h-1)\,(h-q)}{(c-1)\,(d-1)\,(h-g)\,(gq-h)}\ {}_r\psi_s\left(\begin{matrix}a_1,\dots,a_i,\ c,d\\ b_0,\dots,b_k,\ gq,h/q\end{matrix};q,\,t\right)$$
$$+\frac{(c-h)\,(d-h)\,(g-1)\,(g-q)}{(c-1)\,(d-1)\,(g-h)\,(hq-g)}\ {}_r\psi_s\left(\begin{matrix}a_1,\dots,a_i,\ c,d\\ b_0,\dots,b_k,\ g/q,hq\end{matrix};q,\,t\right)$$
$$+\ {}_r\psi_s\left(\begin{matrix}a_1,\dots,a_i,\ cq,dq\\ b_0,\dots,b_k,\ gq,hq\end{matrix};q,\,t\right)=0. \tag{A.2}$$

Appendix B. An Inverse of the Askey–Wilson Operator

The Askey–Wilson divided difference operator on the left-hand side of equation (3.5) can be inverted by the method of Ref. [5]. The end result is

$$\frac{(q,q^2;q)_\infty}{2\pi}\int_{-1}^{1}L\,(x,y)\ \ p_n\,(x;a,b,c,d)\,\rho\,(x;a,b,c,d)\ \ dx=p_n\,(x;aq,b/q,c,d)\,, \tag{B.1}$$

where $\rho\,(x;a,b,c,d)$ is the weight function of the Askey–Wilson polynomials (3.2) and the kernel is given by

$$L\,(x,y)=\left(ac,ad,qce^{i\varphi},qde^{-i\varphi};q\right)_1\ \frac{\left(be^{i\theta},be^{-i\theta},qde^{i\theta},qde^{-i\theta},qae^{i\varphi},qae^{-i\varphi},qce^{i\varphi},qce^{-i\varphi};q\right)_\infty}{\left(qe^{i\theta+i\varphi},qe^{i\theta-i\varphi},qe^{i\varphi-i\theta},qe^{-i\theta-i\varphi};q\right)_\infty}$$
$$\times\ {}_8\varphi_7\left(\begin{matrix}qde^{-i\varphi},q\sqrt{qde^{-i\varphi}},-q\sqrt{qde^{-i\varphi}},qe^{i\theta-i\varphi},qe^{-i\theta-i\varphi},qd/c,q\\ \sqrt{qde^{-i\varphi}},\sqrt{qde^{-i\varphi}},qde^{-i\theta},qde^{i\theta},q^2,qce^{-i\varphi},qde^{-i\varphi}\end{matrix};q,\,ce^{i\varphi}\right).$$

Here, $x=\cos\theta$ and $y=\cos\varphi$. Computational details are left to the reader.

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