ON CONFORMAL POWERS OF THE DIRAC OPERATOR ON EINSTEIN MANIFOLDS

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Abstract. We determine the structure of conformal powers of the Dirac operator on Einstein *Spin*-manifolds in terms of the product formula for shifted Dirac operators. The result is based on the techniques of higher variations for the Dirac operator on Einstein manifolds and spectral analysis of the Dirac operator on the associated Poincaré-Einstein metric, and relies on combinatorial recurrence identities related to the dual Hahn polynomials.

1. Introduction

Conformally covariant operators like the conformal Laplacian or the Dirac operator are of central interest in geometric analysis on manifolds. The conformal Laplacian is an example of conformally covariant operators termed GJMS operators, cf. [GJMS92, GZ03, GP03], and this is analogous for the Dirac operator, cf. [HS01, GMP12, Fis13]. Their original construction is based on the ambient metric, or, equivalently, on the associated Poincaré-Einstein metric introduced by Fefferman and Graham [FG85, FG11].

In the case of GJMS operators, it is shown in [GH04] that in even dimensions n there exists in general no conformal modification of the k-th power of the Laplace operator for $k > \frac{n}{2}$. In the case of conformal powers of the Dirac operator on general *Spin*-manifolds, all known constructions break down for even dimensions n if the order of the operator exceeds the dimension. The effect of non-existence for higher order conformal powers of the Laplace and Dirac operators does not apply for certain classes of manifolds, for example flat manifolds [Slo93] or Einstein manifolds [Gov06].

The proper understanding of the internal structure of conformal powers of the Laplace and Dirac operators is a difficult task, see the progress for GJMS operators in [Juh13]. In particular, there exists a sequence of second order differential operators such that all GJMS operators are polynomials in this collection of second order differential operators, and vice versa. Such a structure, in terms of first order differential operators, is also available for low order examples of conformal powers of the Dirac operator, cf. [Fis13, Chapter 6]. Explicit formulas are available on flat manifolds, where they are just powers of the Laplace or Dirac operators, on the spheres [Bra95, ES10], where they factor into a product of shifted Laplace or Dirac operators, or on Einstein manifolds, where the GJMS operators factor into a product of shifted Laplace operators [Gov06, FG11].

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Let us also mention that on Einstein manifolds there is a less uniform factorization formula [GŠ14] for the conformally covariant Branson-Gover operators [BG05] on differential forms.

The main aim of our article is to complete these results for conformal powers of the Dirac operator on Einstein *Spin*-manifolds.

Theorem. *Let (M,h) be a semi-Riemannian Einstein Spin-manifold of dimension n, normalized by $Ric(h) = \frac{2(n-1)J}{n}h$ for constant normalized scalar curvature $J \in \mathbb{R}$.*

The $(2N+1)$-st conformal power of the Dirac operator, $N \in \mathbb{N}_0$, satisfies

$$\mathcal{D}_{2N+1}\psi = \prod_{j=1}^{2N+1} \left(\not{D} - (N-j+1)\sqrt{\tfrac{2J}{n}} \right) \psi$$

for all $\psi \in \Gamma\big(S(M,h)\big)$. The empty product is regarded as 1.

The result is based on the proper understanding of higher variations of the Dirac operator on Einstein manifolds and the spectral analysis of the Dirac operator on the associated Poincaré-Einstein metric. The derivations of specific formulas rely on combinatorial recurrence identities related to dual Hahn polynomials.

Notice that our result implies the transfer of the spectral resolution for the Dirac operator $\not{D}$ to the conformal powers of the Dirac operator $\mathcal{D}_{2N+1}$. Moreover, the product structure (or, the factorization) of conformal powers of the Laplace and Dirac operators is applied in theoretical physics, cf. [Dow11, Dow13], to compute conformal and multiplicative anomalies of functional determinants in the context of the AdS/CFT correspondence.

The paper is organized as follows. In Section 2, we briefly review some basic notions related to *Spin*-geometry and Poincaré-Einstein metrics. In Section 3, we discuss variations to all orders of the Dirac operator on semi-Riemannian Einstein *Spin*-manifolds with respect to the 1-parameter family of metrics arising from the Poincaré-Einstein metric, cf. Theorem 3.5. In Section 4, we briefly recall the dual Hahn polynomials, which form a special class of generalized hypergeometric functions. The solution of certain recurrence relation, derived in Section 5, has an interpretation in terms of dual Hahn polynomials. Section 5 contains our main theorem, Theorem 5.2. Its proof is based on a recurrence relation, deduced from the construction of conformal powers on the Dirac operator of a semi-Riemannian Einstein *Spin*-manifold via the associated Poincaré-Einstein metric, cf. Proposition 4.1.

2. Semi-Riemannian *Spin*-geometry, Clifford algebras, and Poincaré-Einstein spaces

In the present section we review conventions and notation related to semi-Riemannian *Spin*-geometry and the Poincaré-Einstein metric construction used throughout the article.

Let (M,h) be a semi-Riemannian *Spin*-manifold of signature (p,q) and dimension $n = p+q$. Then any orthonormal frame $\{e_i\}_{i=1}^n$ fulfills $h(e_i,e_j) = \varepsilon_i\delta_{ij}$, where $\varepsilon_i = -1$ for $1 \le i \le p$ and $\varepsilon_i = 1$ for $p+1 \le i \le n$.

The Clifford algebra of $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{p,q})$, denoted by $Cl(\mathbb{R}^{p,q})$, is a quotient of the tensor algebra of $\mathbb{R}^n$ by the two sided ideal generated by the relations $x\otimes y + y\otimes x = -2\langle x,y\rangle_{p,q}$

for all $x, y \in \mathbb{R}^n$. In the even case $n = 2m$, the complexified Clifford algebra $Cl_{\mathbb{C}}(\mathbb{R}^{p,q})$ has a unique irreducible representation up to isomorphism, whereas in the odd case $n = 2m+1$ it has two non-equivalent irreducible representations on $\Delta_n := \mathbb{C}^{2^m}$, again unique up to isomorphism. The restriction of this representation to the spin group $Spin(p,q)$, regarded as a subgroup of the group of units $Cl^*(\mathbb{R}^{p,q})$, is denoted by κ_n.

The choice of a $Spin$-structure (Q, f) on (M, h) provides an associated spinor bundle $S(M,h) := Q \times_{(Spin_0(p,q),\kappa_n)} \Delta_n$, where $Spin_0(p,q)$ denotes the connected component of the spin group containing the identity element. (We could work with the full spin group as well, because we do not need the existence of a scalar product). Then the Levi-Civita connection ∇^h on (M,h) lifts to a covariant derivative $\nabla^{h,S}$ on the spinor bundle. The associated Dirac operator is denoted by $\not{D}$.

Let $\widehat{h} = e^{2\sigma}h$ be a metric conformally related to h, $\sigma \in \mathcal{C}^\infty(M)$. The spinor bundles for $\widehat{h}$ and h can be identified by a vector bundle isomorphism $F_\sigma : S(M,h) \to S(M,\widehat{h})$, and the Dirac operator satisfies the conformal transformation law

$$\widehat{\not{D}}\big(e^{\frac{1-n}{2}\sigma}\widehat{\psi}\big) = e^{-\frac{1+n}{2}\sigma}\widehat{\not{D}\psi},$$

for all smooth sections $\psi \in \Gamma\big(S(M,h)\big)$, and $\widehat{\cdot}$ denotes the evaluation with respect to $\widehat{h}$. Conformal odd powers of the Dirac operator were constructed in [HS01, GMP12, Fis13], and are denoted by $\mathcal{D}_{2N+1} = \not{D}^{2N+1} + \mathrm{LOT}$, for $N \in \mathbb{N}_0$ ($N < \frac{n}{2}$ for even n). Here, LOT stands for "lower order terms." They satisfy

$$\widehat{\mathcal{D}}_{2N+1}\big(e^{\frac{2N+1-n}{2}\sigma}\widehat{\psi}\big) = e^{-\frac{2N+1+n}{2}\sigma}\widehat{\mathcal{D}_{2N+1}\psi}$$

for all smooth function $\sigma \in \mathcal{C}^\infty(M)$ and sections $\psi \in \Gamma\big(S(M,h)\big)$. Notice that even conformal powers of the Dirac operator do not exist, cf. [Slo93].

As for the Poincaré-Einstein metric construction we refer to [FG11]. The Poincaré-Einstein metric associated with an n-dimensional semi-Riemannian manifold (M,h), $n \geq 3$, is $X := M \times (0,\varepsilon)$, $\varepsilon \in \mathbb{R}_+$, equipped with the metric

$$g_+ = r^{-2}(dr^2 + h_r),$$

for a 1-parameter family of metrics h_r on M, $h_0 = h$. The requirement of the Einstein condition on g_+ for n odd,

$$Ric(g_+) + ng_+ = O(r^\infty),$$

uniquely determines the family h_r, while for n even the conditions

$$Ric(g_+) + ng_+ = O(r^{n-2}), \quad tr(Ric(g_+) + ng_+) = O(r^{n-1}),$$

uniquely determine the coefficients $h_{(2)}, \dots, h_{(n-2)}$, $\tilde{h}_{(n)}$ and the trace of $h_{(n)}$ in the formal power series

$$h_r = h + r^2 h_{(2)} + \cdots + r^{n-2}h_{(n-2)} + r^n(h_{(n)} + \tilde{h}_{(n)}\log r) + \cdots .$$

For example, we have

$$h_{(2)} = -P, \quad h_{(4)} = \tfrac{1}{4}\left(P^2 - \tfrac{B}{n-4}\right),$$

where P is the Schouten tensor and B is the Bach tensor associated with h.

All constructions in the present article, based on the Poincaré-Einstein metric, depend for even n on the coefficients $h_{(2)}, \dots, h_{(n-2)}$ and $tr(h_{(n)})$ only. Choosing different representatives $h, \widehat{h} \in [h]$ in the conformal class leads to Poincaré-Einstein metrics g_+^1

and g_+^2 related by a diffeomorphism $\Phi : U_1 \subset X \to U_2 \subset X$, where both U_i, $i = 1, 2$, contain $M \times \{0\}$, $\Phi|_M = Id_M$, and $g_+^1 = \Phi^* g_+^2$ (up to a finite order in r, for even n).

3. Variation of the Dirac operator induced by the Poincaré-Einstein metric

In this section we give a complete description of the variations of the Dirac operator, associated with the 1-parameter family h_r induced by the Poincaré-Einstein metric g_+, assuming that $(M, h_0 = h)$ is Einstein.

For a general 1-parameter family of metrics h_r on a Riemannian *Spin*-manifold, the first variation of the Dirac operator was discussed in [BG92], which we will adapt and make explicit for the 1-parameter family of metrics h_r induced by the Poincaré-Einstein metric.

Motivated by a proof of the fundamental theorem of hypersurface theory and a new way to identify spinors for different metrics, [BGM05] introduced the technique of generalized cylinders to derive the first order variation formula for the Dirac operator with respect to a deformation of the underlying metric.

In general, the topic of higher metric variations for the Dirac operator was not discussed in the literature. In case (M, h) is Einstein, the associated Poincaré-Einstein metric takes a very simple form, cf. Equation (3.1). This allows for a complete description of variation formulas of general order for the Dirac operator associated with h_r. The higher variation formulas are used in Section 5 to make the construction of conformal powers of the Dirac operator very explicit, ending in a product structure for $\mathcal{D}_{2N+1}$, $N \in \mathbb{N}_0$.

Throughout the article, we use the standard notation in semi-Riemannian geometry, e.g., Ric, τ are the Ricci tensor and its scalar curvature, respectively.

Let (M, h) be a semi-Riemannian Einstein manifold of dimension n with normalized Einstein metric h,

$$Ric(h) = 2\lambda(n-1)h, \quad \lambda \in \mathbb{R}.$$

This implies $P = \frac{J}{n}h$, where $J = \frac{\tau}{2(n-1)}$ is the normalized scalar curvature and $P = \frac{1}{n-2}(Ric - Jh)$ is the Schouten tensor. The associated Poincaré-Einstein metric $g_+ = r^{-2}(dr^2 + h_r)$ on X is determined by the 1-parameter family of metrics h_r on M,

$$h_r = h - r^2 \tfrac{J}{n} h + r^4 \left(\tfrac{J}{2n}\right)^2 h = \left(1 - \tfrac{J}{2n} r^2\right)^2 h, \quad h_0 = h. \tag{3.1}$$

For $r \in \mathbb{R}_+$ small enough we consider a point-wise isomorphism $f_r : T_xM \to T_xM$, $x \in M$, relating $h = h_0$ and h_r via

$$Y \mapsto f_r Y := \left(1 - \tfrac{J}{2n} r^2\right)^{-1} Y, \quad \text{for all } Y \in \Gamma(TM),$$

characterized by $h(Y, U) = h_r(f_r Y, f_r U)$ for all $Y, U \in \Gamma(TM)$ and $f_0 = Id_{TM}$.

Let us introduce the Levi-Civita covariant derivatives on TM corresponding to h and h_r:

$$\nabla^h, \nabla^{h_r} : \Gamma(TM) \to \Gamma(T^*M \otimes TM),$$

and

$$\begin{aligned}\nabla^{h,h_r} : \Gamma(TM) &\to \Gamma(T^*M \otimes TM)\\ Y &\mapsto \big\{U \to \nabla^{h,h_r}_Y U := (f_r^{-1} \circ \nabla^{h_r}_Y \circ f_r)U\big\}.\end{aligned} \tag{3.2}$$

The covariant derivatives $\nabla^h, \nabla^{h_r}, \nabla^{h,h_r}$ extend by the Leibniz rule and the spin representation to tensor-spinor fields. For example one has

$$(\nabla^h_U f)(Y) = \nabla^h_U (fY) - f(\nabla^h_U Y),$$

for all $f \in End(TM)$, and $U, Y \in \Gamma(TM)$.

Lemma 3.1. *The covariant derivative ∇^{h,h_r} is metric for h, and its torsion T^r satisfies*

$$T^r(U,Y) = f_r^{-1}\big((\nabla^{h_r}_U f_r)(Y) - (\nabla^{h_r}_Y f_r)(U)\big), \quad \textit{for all } U, Y \in \Gamma(TM). \tag{3.3}$$

Proof. Let $Y, U, Z \in \Gamma(TM)$. First we show the h-metricity of ∇^{h,h_r}:

$$\begin{aligned}(\nabla^{h,h_r}_Y h)(U,Z) &= Y\big(h(U,Z)\big) - h(\nabla^{h,h_r}_Y U, Z) - h(\nabla^{h,h_r}_Y Z, U)\\ &= Y\big(h_r(f_rU, f_rZ)\big) - h(f_r^{-1}\nabla^{h_r}_Y(f_rU), Z) - h(f_r^{-1}\nabla^{h_r}_Y(f_rZ), U)\\ &= Y\big(h_r(f_rU, f_rZ)\big) - h_r(\nabla^{h_r}_Y(f_rU), f_rZ) - h(\nabla^{h_r}_Y(f_rZ), f_rU)\\ &= \nabla^{h_r}_Y h_r(f_rU, f_rZ) = 0.\end{aligned}$$

The second statement follows from

$$\begin{aligned}T^r(U,Y) &= \nabla^{h,h_r}_U Y - \nabla^{h,h_r}_Y U - [U,Y]\\ &= f_r^{-1}\big(\nabla^{h_r}_U f_rY - \nabla^{h_r}_Y f_rU\big) - \nabla^{h_r}_U Y + \nabla^{h_r}_Y U\\ &= f_r^{-1}\big(\nabla^{h_r}_U f_rY - f_r(\nabla^{h_r}_U Y) - \nabla^{h_r}_Y f_rU + f_r(\nabla^{h_r}_Y U)\big)\\ &= f_r^{-1}\big((\nabla^{h_r}_U f_r)(Y) - (\nabla^{h_r}_Y f_r)(U)\big).\end{aligned}$$

□

It is well known that two h-metric covariant derivatives on (M,h) differ by their torsions.

Lemma 3.2. *We have*

$$h(\nabla^{h,h_r}_U Y, Z) = h(\nabla^h_U Y, Z) + \tfrac{1}{2}\big[h(T^r(Z,Y),U) - h(T^r(Y,U),Z) - h(T^r(U,Z),Y)\big]$$

for all $U, Y, Z \in \Gamma(TM)$.

Proof. Let $U, Y, Z \in \Gamma(TM)$. Any two covariant derivatives differ by a tensor field $\omega \in \Gamma(T^*M \otimes TM \otimes T^*M)$, i.e., $\nabla^{h,h_r}_U Y = \nabla^h_U Y + \omega(U)Y$. Since both covariant derivatives are metric with respect to h, we get

$$0 = \nabla^{h,h_r}_U h(Y,Z) - \nabla^h_U h(Y,Z) = -\big[h(\omega(U)Y, Z) + h(\omega(U)Z, Y)\big]. \tag{3.4}$$

Since ∇^h is torsion-free, we have

$$\begin{aligned}T^r(U,Y) &= \nabla^{h,h_r}_U Y - \nabla^{h,h_r}_Y U - [U,Y]\\ &= \nabla^{h,h_r}_U Y - \nabla^{h,h_r}_Y U - \nabla^h_U Y + \nabla^h_Y U\\ &= \omega(U)Y - \omega(Y)U.\end{aligned}$$

Using Equation (3.4), we see that this implies

$$\begin{aligned} h(T^r(Z,Y),U) - h(T^r(Y,U),Z) - h(T^r(U,Z),Y) \\ = 2h(\omega(U)Y,Z) = 2h(\nabla^{h,h_r}_U Y - \nabla^h_U Y, Z), \end{aligned}$$

and the proof is complete. □

Any h-metric covariant derivative induces a covariant derivative on the spinor bundle $S(M,h)$, hence ∇^h, ∇^{h_r} and ∇^{h,h_r} induce

$$\begin{aligned} \nabla^{h,S}:\quad & \Gamma\big(S(M,h)\big) \to \Gamma\big(T^*M \otimes S(M,h)\big), \\ \nabla^{h_r,S}:\quad & \Gamma\big(S(M,h_r)\big) \to \Gamma\big(T^*M \otimes S(M,h_r)\big), \\ \nabla^{r,S}:\quad & \Gamma\big(S(M,h)\big) \to \Gamma\big(T^*M \otimes S(M,h)\big). \end{aligned} \tag{3.5}$$

Note that the last of the above covariant derivatives equals $\nabla^{h,h_r,S}$, but we will use the abbreviation $\nabla^{r,S}$. It follows from Lemma 3.2 that locally

$$\begin{aligned} \nabla^{r,S}_{s_i}\psi &= s_i(\psi) + \tfrac{1}{2}\sum_{j<k} \varepsilon_j\varepsilon_k h(\nabla^{h,h_r}_{s_i} s_j, s_k) s_j \cdot s_k \cdot \psi \\ &= \nabla^{h,S}_{s_i}\psi + \tfrac{1}{8}\sum_{j\neq k} \varepsilon_j\varepsilon_k T^{r,\sigma}_{ijk} s_j \cdot s_k \cdot \psi, \end{aligned} \tag{3.6}$$

where $\psi \in \Gamma\big(S(M,h)\big)$, $\{s_i\}_{i=1}^n$ is an h-orthonormal frame, $T^{r,\sigma}_{ijk} := (1-\sigma-\sigma^2)T^r_{kji}$ with $\sigma T^r_{ijk} := T^r_{jki}$ and $T^r(U,Y,Z) := h(T^r(U,Y),Z)$ for $U,Y,Z \in \Gamma(TM)$. The isometry $f_r : T_xM \to T_xM$, $x \in M$, pulls-back h to h_r and lifts to a spinor bundle isomorphism

$$\beta_r : S(M,h) \to S(M,h_r).$$

It preserves the base points on M and satisfies $\beta_r(U\cdot\psi) = f_r(U)\cdot\beta_r(\psi)$ for all $U \in \Gamma(TM), \psi \in \Gamma\big(S(M,h)\big)$.

Lemma 3.3. *Let $\psi \in \Gamma\big(S(M,h)\big), Y \in \Gamma(TM)$. Then*

$$\nabla^{r,S}_Y \psi = (\beta_r^{-1} \circ \nabla^{h_r,S}_Y \circ \beta_r)\psi. \tag{3.7}$$

Proof. For $\psi \in \Gamma\big(S(M,h)\big)$, we have

$$\begin{aligned} \nabla^{r,S}_Y\psi &= Y(\psi) + \tfrac{1}{2}\sum_{j<k} \varepsilon_j\varepsilon_k h(\nabla^{h,h_r}_Y s_j, s_k) s_j\cdot s_k\cdot\psi \\ &= Y(\psi) + \tfrac{1}{2}\sum_{j<k}\varepsilon_j\varepsilon_k h_r(\nabla^{h_r}_Y f_r s_j, f_r s_k)\beta_r^{-1}\big(f_r s_j \cdot f_r s_k \cdot \beta_r\psi\big) \\ &= (\beta_r^{-1}\circ\nabla^{h_r,S}_Y \circ \beta_r)\psi, \end{aligned}$$

which completes the proof. □

Let us introduce the notation

$$f^{(l)} := \tfrac{d^l}{dr^l}(f_r)|_{r=0}, \quad T^{(l)}(U,Y) := \tfrac{d^l}{dr^l}\big(T^r(U,Y)\big)|_{r=0}, \tag{3.8}$$

for $l \in \mathbb{N}_0$.

Lemma 3.4. (1) *Let $l \in \mathbb{N}_0$. Then*

$$f^{(2l+1)}U = 0, \quad f^{(2l)}U = (2l)!\left(\tfrac{J}{2n}\right)^l U, \quad U \in \Gamma(TM). \tag{3.9}$$

(2) *Let $l \in \mathbb{N}$. Then the torsion T^r of ∇^{h,h_r} fulfills*

$$T^{(l)}(U,Y) = 0 \tag{3.10}$$

for all $U, Y \in \Gamma(TM)$.

Proof. Expansion of f_r into a formal power series

$$f_r = \left(1 - \tfrac{J}{2n} r^2\right)^{-1} Id_{TM} = \sum_{l \geq 0} \tfrac{r^{2l}}{(2l)!}(2l)!\left(\tfrac{J}{2n}\right)^l Id_{TM}$$

gives the first statements. It follows from Lemma 3.1 that the l-th derivative of T^r at $r = 0$ is given by a sum, where each contribution contains derivatives of f_r, f_r^{-1} and ∇^{h_r} at $r = 0$. Using $f^{(2l+1)} = 0$ and $f^{(2l)} = (2l)!(\frac{J}{2n})^l Id_{TM}$, for all $l \in \mathbb{N}_0$, we just have to show that $\frac{d^l}{dr^l}(\nabla^{h_r})|_{r=0}$ acts as a covariant derivative, hence annihilating the identity map. But this is obvious, since $\frac{d^l}{dr^l}$ does not effect the property of ∇^{h_r} being a covariant derivative. □

In what follows, we use two Dirac operators induced by $\nabla^{h,S}$ and $\nabla^{h_r,S}$:

$$\begin{aligned} \not{D}^h : \quad & \Gamma\big(S(M,h)\big) \to \Gamma\big(S(M,h)\big), \\ \not{D}^{h_r} : \quad & \Gamma\big(S(M,h_r)\big) \to \Gamma\big(S(M,h_r)\big). \end{aligned} \tag{3.11}$$

Furthermore, we define

$$\begin{aligned} \not{D}^{h,h_r} : \Gamma\big(S(M,h)\big) &\to \Gamma\big(S(M,h)\big) \\ \psi &\mapsto \beta_r^{-1} \circ \not{D}^{h_r} \circ \beta_r(\psi). \end{aligned} \tag{3.12}$$

Lemma 3.3 and $\beta_r(U \cdot \psi) = f_r(U) \cdot \beta_r(\psi)$ for $U \in \Gamma(TM), \psi \in \Gamma\big(S(M,h)\big)$ imply

$$\not{D}^{h,h_r}\psi = \sum_{i=1}^n \varepsilon_i s_i \cdot \nabla^{r,S}_{f_r(s_i)}\psi, \tag{3.13}$$

and the r-derivatives of $\not{D}^{h,h_r}$ at $r = 0$ yield the variation formulas for $\not{D}^h$ with respect to the 1-parameter deformation h_r of h. Note that $\not{D}^{h,h_r}$ is not the Dirac operator induced by $\nabla^{r,S}$.

Theorem 3.5. *Let (M,h) be a semi-Riemannian Einstein Spin-manifold with* $\dim(M) = n$, *i.e., $Ric = 2(n-1)\lambda h$. Let g_+ be the associated Poincaré-Einstein metric on X with $g_+ = r^{-2}(dr^2 + h_r)$, $h_r = (1 - \frac{J}{2n} r^2)^2 h$. Then, for all $l \in \mathbb{N}, \psi \in \Gamma\big(S(M,h)\big)$, we have*

$$\begin{aligned} \tfrac{d^{(2l)}}{dr^{(2l)}}(\not{D}^{h,h_r}\psi)|_{r=0} &= (2l)!\left(\tfrac{J}{2n}\right)^l \not{D}^h\psi, \\ \tfrac{d^{(2l+1)}}{dr^{(2l+1)}}(\not{D}^{h,h_r}\psi)|_{r=0} &= 0. \end{aligned} \tag{3.14}$$

Proof. The r-derivatives of $\not{D}^{h,h_r}$ at $r = 0$ are

$$\tfrac{d^l}{dr^l}(\not{D}^{h,h_r}\psi)|_{r=0} = \sum_{k=0}^l \binom{l}{k} \sum_{i=1}^n \varepsilon_i s_i \cdot \nabla^{r,S;(k)}_{f^{(l-k)}s_i}\psi,$$

where $\nabla^{r,S;(k)} := \frac{d^k}{dr^k}(\nabla^{r,S})|_{r=0}$, for $k \in \mathbb{N}_0$. From Equation (3.6), we obtain for $l \in \mathbb{N}$

$$\tfrac{d^l}{dr^l}(\nabla^{r,S}_{s_i}\psi) = \tfrac{1}{8}\sum_{j\neq k}\varepsilon_j\varepsilon_k \tfrac{d^l}{dr^l}(T^{r,\sigma}_{ijk})s_j \cdot s_k \cdot \psi,$$

which vanishes at $r = 0$ due to Lemma 3.4 and the linearity of $\frac{d^l}{dr^l}$. Thus we get

$$\tfrac{d^l}{dr^l}(\not{D}^{h,h_r}\psi)|_{r=0} = \sum_{i=1}^{n}\varepsilon_i s_i \cdot \nabla^{0,S}_{f^{(l)}s_i}\psi.$$

Since $\nabla^{0,S}$ agrees with the spinor covariant derivative $\nabla^{h,S}$ on $S(M,h)$, we may conclude by Lemma 3.4 that

$$\tfrac{d^{2l}}{dr^{2l}}(\not{D}^{h,h_r}\psi)|_{r=0} = (2l)!\left(\tfrac{J}{2n}\right)^l \not{D}^h\psi, \quad \tfrac{d^{2l+1}}{dr^{2l+1}}(\not{D}^{h,h_r}\psi)|_{r=0} = 0,$$

hence completing the proof. □

4. Generalized hypergeometric functions and dual Hahn polynomials

The aim of the present section is to introduce a certain class of polynomials, to prove some of their combinatorial properties, and to give their interpretation in terms of dual Hahn polynomials. These polynomials will be responsible for the product structure of conformal powers of the Dirac operator.

The Pochhammer symbol of a complex number $a \in \mathbb{C}$ is denoted by $(a)_l$, and it is defined by $(a)_l := a(a+1)\cdots(a+l-1)$ for $l \in \mathbb{N}$, and $(a)_0 := 1$. The generalized hypergeometric function ${}_pF_q$, for $p, q \in \mathbb{N}$, with p upper parameters, q lower parameters, and argument z, is defined by

$${}_pF_q\left[\begin{matrix}a_1, \dots, a_p\\ b_1, \dots, b_q\end{matrix}; z\right] := \sum_{l=0}^{\infty}\frac{(a_1)_l\cdots(a_p)_l}{(b_1)_l\cdots(b_q)_l}\frac{z^l}{l!}, \tag{4.1}$$

for $a_i \in \mathbb{C}$ $(1 \le i \le q)$, $b_j \in \mathbb{C}\setminus\{-\mathbb{N}_0\}$ $(1 \le j \le q)$, and $z \in \mathbb{C}$.

For later purposes, we introduce the polynomials

$$\tilde{q}_m(y) := \sum_{l=0}^{m}(-1)^{m-l}(\tfrac{n}{2}+1+l)_{m-l}\,(k-m)_{m-l}\binom{m}{l}\prod_{j=1}^{l}(y-j^2), \tag{4.2}$$

for $m \in \mathbb{N}_0$, $k, n \in \mathbb{N}$ and an abstract variable y.

Proposition 4.1. *The polynomials $\tilde{q}_m(y)$, $m \in \mathbb{N}_0$, satisfy the recurrence relation*

$$\begin{aligned}\tilde{q}_{m+1}(y) = \big(y - 2m(k-m-\tfrac{n}{2}-\tfrac{1}{2}) - \tfrac{n}{2}(k-1) - k\big)\,\tilde{q}_m(y)\\ - m(m-k)(m+\tfrac{n}{2})(m-k+\tfrac{n}{2}-1)\tilde{q}_{m-1}(y),\end{aligned} \tag{4.3}$$

with $\tilde{q}_{-1}(y) := 0$ and $\tilde{q}_0(y) := 1$.

Proof. We prove the statement by comparing the coefficients of $\prod_{j=1}^{l}(y-j^2)$ on both sides of (4.3). The only detail to observe is that one must replace y in the coefficient of $\tilde{q}_m(y)$ on the right-hand side of (4.3) by $(y-(l+1)^2)+(l+1)^2$, with l being the summation index of the sum in the definition of $\tilde{q}_m(y)$. The term $(y-(l+1)^2)$ then combines with the product $\prod_{j=1}^{l}(y-j^2)$ to become $\prod_{j=1}^{l+1}(y-j^2)$. The verification that the coefficients of $\prod_{j=1}^{l}(y-j^2)$ do indeed agree is then a routine matter. □

Remark. Our considerations are motivated by [FG11], where the analogous recurrence relation

$$\begin{aligned} q_{m+1}(y) = \big(y - 2m(k-m-\tfrac{n}{2}) - \tfrac{n}{2}(k-1)\big)\, q_m(y) \\ - m(m-k)(m-1+\tfrac{n}{2})(m-1+\tfrac{n}{2}-k)q_{m-1}(y), \end{aligned}$$

for $k, n \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $q_{-1}(y) := 0$, $q_0(y) := 1$, appears. Its solution is given by

$$q_m(y) := \sum_{l=0}^{m} (-1)^{m-l} (\tfrac{n}{2}+l)_{m-l}\, (k-m)_{m-l} \binom{m}{l} \prod_{j=1}^{l} \big(y - j(j-1)\big). \tag{4.4}$$

In the rest of the section, we discuss interpretations of $\tilde{q}_m(y)$ and $q_m(y)$ in terms of dual Hahn polynomials, cf. [KM61, KLS10]. The Hahn polynomial $Q_n(x) := Q_n(x; \alpha, \beta, N)$ is defined by

$$Q_n(x) := {}_3F_2 \left[\begin{matrix} -n\ ,\ -x\ ,\ n+\alpha+\beta+1 \\ \alpha+1\ ,\ -N+1 \end{matrix} ; 1 \right], \tag{4.5}$$

for $\Re(\alpha), \Re(\beta) > -1$, $N \in \mathbb{N}$, and $n = 0, \dots, N-1$. It is known that, beside recurrence relations, Hahn polynomials satisfy a difference relation, cf. [KM61, Equation (1.3)]. The dual Hahn polynomials can be defined by recurrence relations with the same coefficients as the Hahn polynomials have in their difference relations, cf. [KM61, Equation (1.18)].

For $\lambda(n) := n(n+\alpha+\beta+1)$, the relation between Hahn polynomials $Q_n(x)$ and dual Hahn polynomials $R_k(\lambda) := R_k(\lambda; \alpha, \beta, N)$ is given by

$$R_k\big(\lambda(n)\big) = Q_n(k). \tag{4.6}$$

Notice that

$$\begin{aligned} (-y)_l\, (1+y)_l &= (-1)^l \prod_{j=1}^{l} \big(y(y+1) - j(j-1)\big), \\ (1-y)_l\, (y+1)_l &= (-1)^l \prod_{j=1}^{l} \big(y^2 - j^2\big), \end{aligned}$$

for $l \in \mathbb{N}$. Furthermore, by using the identities for Pochhammer symbols

$$\begin{aligned} (\tfrac{n}{2}+l)_{m-l} = (\tfrac{n}{2})_m \tfrac{1}{(\frac{n}{2})_l}, &\quad (\tfrac{n}{2}+1+l)_{m-l} = (\tfrac{n}{2}+1)_m \tfrac{1}{(\frac{n}{2}+1)_l}, \\ (k-m)_{m-l} = (k-m)_m \tfrac{(-1)^l}{(1-k)_l}, &\quad \binom{m}{l} = (-1)^l \tfrac{(-m)_l}{l!}, \end{aligned}$$

one obtains the following precise relations.

Proposition 4.2. *For all $m \in \mathbb{N}_0, k, n \in \mathbb{N}$, we have*

$$\begin{aligned} \tilde{q}_m\big(\lambda(y-1)\big) &= (-1)^m (\tfrac{n}{2}+1)_m (k-m)_m\ {}_3F_2 \left[\begin{matrix} -(y-1)\ ,\ -m\ ,\ 1+y \\ \tfrac{n}{2}+1\ ,\ 1-k \end{matrix} ; 1 \right] \\ &= (-1)^m (\tfrac{n}{2}+1)_m (k-m)_m\ Q_{y-1}(m; \tfrac{n}{2}, 1-\tfrac{n}{2}, k) \\ &= (-1)^m (\tfrac{n}{2}+1)_m (k-m)_m\ R_m\big(\lambda(y-1); \tfrac{n}{2}, 1-\tfrac{n}{2}, k\big) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} q_m\big(\lambda(y)\big) &= (\tfrac{n}{2})_m(k-m)_m(-1)^m \ {}_3F_2\left[\begin{matrix} -y\ ,\ -m\ ,\ 1+y \\ \frac{n}{2}\ ,\ 1-k \end{matrix};1\right] \\ &= (\tfrac{n}{2})_m(k-m)_m(-1)^m \ Q_y(m;\tfrac{n}{2}-1,1-\tfrac{n}{2},k) \\ &= (\tfrac{n}{2})_m(k-m)_m(-1)^m \ R_m\big(\lambda(y);\tfrac{n}{2}-1,1-\tfrac{n}{2},k\big). \end{aligned} \tag{4.8}$$

Hence, up to a multiplicative factor, both $\tilde{q}_m(y)$ and $q_m(y)$ can be realized as dual Hahn polynomials.

5. Product structure (factorization) of conformal powers of the Dirac operator

In the present section, we show that conformal powers of the Dirac operator on Einstein manifolds obey a product structure, in the sense that they factor into linear factors based on shifted Dirac operators. This result is parallel to the case of conformal powers of the Laplace operator on Einstein manifolds, cf. [Gov06, Theorem 1.2].

Let us denote the Dirac operator on (M,h) by $\not{D}$. (Notice that in Section 3 we used $\not{D}^h$ instead of $\not{D}$.) The proof of our main result, Theorem 5.2, relies on the construction of conformal powers of the Dirac operator.

Theorem 5.1 ([GMP12])**.** *Let (M,h) be a semi-Riemannian Spin-manifold of dimension n. For every $N\in\mathbb{N}_0$ ($N\le\frac{n}{2}$ for even n) there exists a linear differential operator, called conformal power of the Dirac operator,*

$$\mathcal{D}_{2N+1}:\Gamma\big(S(M,h)\big)\to\Gamma\big(S(M,h)\big), \tag{5.1}$$

satisfying

1. *$\mathcal{D}_{2N+1}$ is of order $2N+1$ and $\mathcal{D}_{2N+1}=\not{D}^{2N+1}+LOT$, where, as before, LOT denotes lower order terms;*
2. *$\mathcal{D}_{2N+1}$ is conformally covariant, that is,*
$$\widehat{\mathcal{D}}_{2N+1}\left(e^{\frac{2N+1-n}{2}\sigma}\widehat{\psi}\right)=e^{-\frac{2N+1+n}{2}\sigma}\widehat{\mathcal{D}_{2N+1}\psi} \tag{5.2}$$
for every $\psi\in\Gamma\big(S(M,h)\big)$ and $\sigma\in\mathcal{C}^\infty(M)$. Note that $\widehat{\cdot}$ denotes evaluation with respect to $\widehat{h}=e^{2\sigma}h$.

We briefly outline the main point of the proof, which will be then analyzed in detail on Einstein manifolds. Let g_+ be the associated Poincaré-Einstein metric on X with conformal infinity $(M,[h])$. The conformal compactification of $\big(X=M\times(0,\varepsilon),g_+\big)$ is

$$\big(M\times[0,\varepsilon),\bar{g}:=r^2g_+=dr^2+h_r\big),$$

where $\bar{g}$ smoothly extends to $r=0$. Corresponding spinor bundles are denoted by

$$S(M,h),\quad S(X,g_+),\quad S(X,\bar{g}),$$

respectively. The spinor bundle $S(X,\bar{g})|_{r=0}$ is isomorphic to $S(M,h)$ if n is even, and it is isomorphic to $S(M,h)\oplus S(M,h)$ if n is odd. The proof of Theorem 5.1 is based on the extension of a boundary spinor $\psi\in\Gamma(S(X,\bar{g})|_{r=0})$ to the interior $\theta\in\Gamma\big(S(X,\bar{g})\big)$: one requires θ to be a formal solution of

$$D(\bar{g})\theta=i\lambda\theta,\quad \lambda\in\mathbb{C}. \tag{5.3}$$

Here, $D(\bar{g})$ arises by applying the vector bundle isomorphism $F_r : S(X, g_+) \to S(X, \bar{g})$, which exists since g_+ and $\bar{g}$ are conformally equivalent, to the equation $\not{D}^{g_+}\varphi = i\lambda\varphi$, $\lambda \in \mathbb{C}$ and $\varphi \in \Gamma\big(S(X, g_+)\big)$. The solution of Equation (5.3) is obstructed for $\lambda = -\frac{2N+1}{2}$, and the obstruction induces a conformally covariant linear differential operator $\mathcal{D}_{2N+1} = \not{D}^{2N+1} + LOT$.

Let us be more specific. Let (M, h) be a semi-Riemannian Einstein *Spin*-manifold, normalized by $Ric(h) = \frac{2(n-1)J}{n} h$ for constant normalized scalar curvature $J \in \mathbb{R}$. Consider the embedding $\iota_r : M \to X$ given by $\iota_r(m) := (r, m)$. Then $(M, \iota_r^*(\bar{g}) = h_r)$ is a hypersurface in $(X, \bar{g})$ with trivial space-like normal bundle. It follows from [BGM05] that the Dirac operator $\not{D}^{\bar{g}}$ of $(X, \bar{g})$ and the leaf-wise (or, hypersurface) Dirac operator

$$\widetilde{\not{D}}^{h_r} := \partial_r \cdot \sum_{i=1}^{n} \varepsilon_i s_i \cdot \widetilde{\nabla}^{h_r, S}_{s_i} : \Gamma\big(S(X, \bar{g})\big) \to \Gamma\big(S(X, \bar{g})\big)$$

for an h_r-orthonormal frame $\{s_i\}_i$ on M are related by

$$\iota_r^* \partial_r \cdot \not{D}^{\bar{g}} = \widetilde{\not{D}}^{h_r} \iota_r^* + \tfrac{n}{2} \iota_r^* H_r - \iota_r^* \nabla^{\bar{g}, S}_{\partial_r}, \tag{5.4}$$

where $H_r := \frac{1}{n} tr_{h_r}(W_r)$ is the h_r-trace of the Weingarten map associated with the embedding ι_r. We used a swung dash (on $\widetilde{\not{D}}^{h_r}$) in order to emphasize the action on the spinor bundle on $(X, \bar{g})$. At $r = 0$, we have the identification

$$\widetilde{\not{D}} := \widetilde{\not{D}}^{h_0} \simeq \begin{cases} \not{D}, & \text{if } n \text{ is even}, \\ \begin{pmatrix} \not{D} & 0 \\ 0 & -\not{D} \end{pmatrix}, & \text{if } n \text{ is odd}. \end{cases}$$

The equation $\not{D}^{g_+}\varphi = i\lambda\varphi$, $\lambda \in \mathbb{C}$ and $\varphi \in \Gamma\big(S(X, g_+)\big)$, is equivalent to Equation (5.3) by combination of conformal covariance, Equation (5.4), and the isomorphism F_r, where the linear differential operator $D(\bar{g}) : \Gamma\big(S(X, \bar{g})\big) \to \Gamma\big(S(X, \bar{g})\big)$ is given by

$$D(\bar{g})\theta = -r\partial_r \cdot \widetilde{\not{D}}^{h_r} \theta - \tfrac{n}{2} r H_r \partial_r \cdot \theta + r\partial_r \cdot \nabla^{\bar{g}, S}_{\partial_r} \theta - \tfrac{n}{2} \partial_r \cdot \theta$$

for $\theta = F_r(\varphi)$. Using Theorem 3.5, we find the explicit formulas

$$\widetilde{\not{D}}^{h_r} = \left(1 - \tfrac{J}{2n} r^2\right)^{-1} \widetilde{\not{D}}, \quad H_r = \tfrac{J}{n} r \left(1 - \tfrac{J}{2n} r^2\right)^{-1}. \tag{5.5}$$

This is a consequence of the Einstein assumption on M. In general, there is no explicit formula analogous to Equation (5.5). We decompose the spinor bundle $S(X, \bar{g})$ into the $\pm i$-eigenspaces $S^{\pm \partial_r}(X, \bar{g})$ with respect to the linear map $\partial_r \cdot : S(X, \bar{g}) \to S(X, \bar{g})$ satisfying $\partial_r^2 = -1$. The formal solution of Equation (5.3) is constructed inside

$$\mathcal{A} := \Big\{\theta = \sum_{j \geq 0} r^j \theta_j \mid \theta_j \in \Gamma\big(S(X, \bar{g})\big),\ \nabla^{\bar{g}, S}_{\partial_r} \theta_j = 0\Big\},$$

and

$$\bar{\theta} := r^{\frac{n}{2} + \lambda} \theta = \sum_{j \geq 0} r^{\frac{n}{2} + \lambda + j} (\theta_j^+ + \theta_j^-) \in \mathcal{A}$$

for $\theta_j^\pm \in \Gamma\big(S^{\pm\partial_r}(X,\bar g)\big)$, $j\in\mathbb{N}_0$, is a solution of Equation (5.3) provided the coupled system of recurrence relations

$$\begin{aligned} j\theta_j^+ &= \widetilde{\not{D}}\theta_{j-1}^- + \tfrac{n+j-2}{2n}J\theta_{j-2}^+,\\ (2\lambda+j)\theta_j^- &= \widetilde{\not{D}}\theta_{j-1}^+ + \tfrac{2\lambda+n+j-2}{2n}J\theta_{j-2}^-, \end{aligned} \tag{5.6}$$

holds for all $j\in\mathbb{N}_0$. Note that we only consider restrictions to $r=0$ and then extend $\theta_j^\pm$, $j\geq 0$, by parallel transport with respect to $\nabla^{\bar g,S}$ along the geodesic induced by the r-coordinate. The initial data are given by $\theta_0^+ := \psi^+$ for some $\psi^+\in\Gamma(S^{+\partial_r}(X,\bar g)|_{r=0})$, and $\theta_0^-=0$. Provided $\lambda\notin -\mathbb{N}+\frac12$, the system can be solved uniquely for all $j\in\mathbb{N}$ if n is odd, and for all $j\in\mathbb{N}$ such that $j\leq n$ if n is even. The obstruction at $\lambda=-\frac{2N+1}{2}$, for $N\in\mathbb{N}_0$ ($N\leq\frac n2$ for even n), is given by $\mathcal{D}_{2N+1}$ for $N\in\mathbb{N}$.

The application of $\widetilde{\not{D}}$ to the system (5.6) together with the shift of j to $j-1$, respectively $j-3$, implies

$$\begin{aligned} (j-1)\widetilde{\not{D}}\theta_{j-1}^+ &= \widetilde{\not{D}}^2\theta_{j-2}^- + \tfrac{n+j-3}{2n}J\widetilde{\not{D}}\theta_{j-3}^+,\\ (2\lambda+j-1)\widetilde{\not{D}}\theta_{j-1}^- &= \widetilde{\not{D}}^2\theta_{j-2}^+ + \tfrac{2\lambda+n+j-3}{2n}J\widetilde{\not{D}}\theta_{j-3}^-,\\ \widetilde{\not{D}}\theta_{j-3}^- &= (j-2)\theta_{j-2}^+ - \tfrac{n+j-4}{2n}J\theta_{j-4}^+,\\ \widetilde{\not{D}}\theta_{j-3}^+ &= (2\lambda+j-2)\theta_{j-2}^- - \tfrac{2\lambda+n+j-4}{2n}J\theta_{j-4}^-. \end{aligned}$$

These formulas can be used to decouple the system (5.6) into

$$\begin{aligned} j\theta_j^+ &= \left(\tfrac{1}{2\lambda+j-1}\widetilde{\not{D}}^2 + \tfrac{(2\lambda+n+j-3)(j-2)}{2n(2\lambda+j-1)}J + \tfrac{(2\lambda+j-1)(n+j-2)}{2n(2\lambda+j-1)}J\right)\theta_{j-2}^+\\ &\qquad - \tfrac{(2\lambda+n+j-3)(n+j-4)}{4n^2(2\lambda+j-1)}J^2\theta_{j-4}^+,\\ (2\lambda+j)\theta_j^- &= \left(\tfrac{1}{j-1}\widetilde{\not{D}}^2 + \tfrac{(2\lambda+j-2)(n+j-3)}{2n(j-1)}J + \tfrac{(2\lambda+n+j-2)(j-1)}{2n(j-1)}J\right)\theta_{j-2}^-\\ &\qquad - \tfrac{(2\lambda+n+j-4)(n+j-3)}{4n^2(j-1)}J^2\theta_{j-4}^-, \end{aligned} \tag{5.7}$$

for all $j\geq 2$, with the initial data $\theta_0^+=\psi^+$, $\theta_0^-=0$, $\theta_1^+=0$, and $\theta_1^-=\frac{1}{2\lambda+1}\widetilde{\not{D}}\psi^+$. Introducing $\phi_l := \theta_{2l+1}^-$ for $l\in\mathbb{N}_0$, Equation (5.7) is equivalent to

$$\begin{aligned} 2l(2\lambda+2l+1)\phi_l &= \left(\widetilde{\not{D}}^2 + \tfrac{(2\lambda+2l-1)(n+2l-2)}{2n}J + \tfrac{2l(2\lambda+n+2l-1)}{2n}J\right)\phi_{l-1},\\ &\qquad - \tfrac{(2\lambda+n+2l-3)(n+2l-2)}{4n^2}J^2\phi_{l-2}, \end{aligned} \tag{5.8}$$

for $l\in\mathbb{N}$ and $\phi_0 := \frac{1}{2\lambda+1}\widetilde{\not{D}}\psi^+$. We define the solution operators $\tilde q_l(y)$ by

$$4^l l!\,(\tfrac{n}{2J})^l(\lambda+\tfrac32)_l\,\phi_l = \tilde q_l(y)\phi_0, \tag{5.9}$$

where $y := \frac{n}{2J}\widetilde{\not{D}}^2$. Then Equation (5.8) yields a recurrence relation for $\tilde q_l(y)$, namely

$$\begin{aligned} \tilde q_l(y) &= \big(y + (\lambda+l-\tfrac12)(l+\tfrac n2-1) + l(\lambda+l+\tfrac n2-\tfrac12)\big)\tilde q_{l-1}(y)\\ &\qquad - (l-1)(l+\tfrac n2-1)(l+\lambda-\tfrac12)(l+\lambda+\tfrac n2-\tfrac32)\tilde q_{l-2}(y), \end{aligned}$$

$l \in \mathbb{N}$, $\tilde{q}_{-1}(y) := 0$ and $\tilde{q}_0(y) := 1$. Changing l to $(m+1)$ and substituting $\lambda = -\frac{2N+1}{2}$ for $N \in \mathbb{N}_0$, we obtain

$$\begin{aligned} \tilde{q}_{m+1}(y) = \big(y - 2m(N - m - \tfrac{n}{2} - \tfrac{1}{2}) - \tfrac{n}{2}(N-1) - N\big)\tilde{q}_m(y) \\ - m(m-N)(m+\tfrac{n}{2})(m-N+\tfrac{n}{2}-1)\tilde{q}_{m-1}(y). \end{aligned} \tag{5.10}$$

The unique solution of the recurrence relation (5.10) is given by

$$\tilde{q}_m(y) := \sum_{l=0}^{m} (-1)^{m-l} (\tfrac{n}{2}+1+l)_{m-l}\,(N-m)_{m-l} \binom{m}{l} \prod_{j=1}^{l} (y - j^2), \tag{5.11}$$

cf. Proposition 4.1, and it specializes for $m = N$ to

$$\tilde{q}_N(y) = \prod_{j=1}^{N} (y - j^2) = \prod_{j=1}^{N} \left(\sqrt{\tfrac{n}{2J}}\widetilde{\not{D}} - j\right)\left(\sqrt{\tfrac{n}{2J}}\widetilde{\not{D}} + j\right).$$

The solution ϕ_l, cf. Equation (5.9), multiplied by $(2\lambda+1)$ is obstructed at $\lambda = -\frac{2N+1}{2}$, $N \in \mathbb{N}_0$, and we get

$$4^N N!(-N)_N \phi_N = \left(\tfrac{n}{2J}\right)^{-N} \tilde{q}_N(y)\widetilde{\not{D}}\psi^+.$$

Repeating all the previous steps with eigen-equation (5.3) for the eigenvalue $-\lambda$ and initial data $\psi^- \in \Gamma(S^{-\partial_r}(X,\bar{g})|_{r=0})$, the obstruction at $\lambda = -\frac{2N+1}{2}$, for $N \in \mathbb{N}_0$, induces

$$\begin{aligned} \mathcal{D}_{2N+1} &= (\frac{n}{2J})^{-N} \not{D} \prod_{j=1}^{N} \left(\sqrt{\tfrac{n}{2J}}\not{D} - j\right)\left(\sqrt{\tfrac{n}{2J}}\not{D} + j\right) \\ &= \not{D} \prod_{j=1}^{N} \left(\not{D} - j\sqrt{\tfrac{2J}{n}}\right)\left(\not{D} + j\sqrt{\tfrac{2J}{n}}\right), \end{aligned} \tag{5.12}$$

the conformal power of the Dirac operator in the factorized form. Note that there is no restriction on $N \in \mathbb{N}_0$ in the case of even n. Thus we have the following result.

Theorem 5.2. *Let (M,h) be a semi-Riemannian Einstein Spin-manifold of dimension n, normalized by $Ric(h) = \frac{2(n-1)J}{n}h$ for constant normalized scalar curvature $J \in \mathbb{R}$.*

The $(2N+1)$-th conformal power of the Dirac operator, $N \in \mathbb{N}_0$, satisfies

$$\begin{aligned} \mathcal{D}_{2N+1}\psi &= \prod_{j=1}^{2N+1} \left(\not{D} - (N-j+1)\sqrt{\tfrac{2J}{n}}\right)\psi \\ &= \not{D} \prod_{j=1}^{N} \left(\not{D}^2 - j^2\left(\tfrac{2J}{n}\right)\right)\psi \end{aligned} \tag{5.13}$$

for all $\psi \in \Gamma\big(S(M,h)\big)$. The empty product is regarded as 1.

In particular, Theorem 5.2 applies to the standard round sphere (S^n, h) of radius 1. We get

$$\mathcal{D}_{2N+1} = (\not{D} - N)\cdots(\not{D} - 1)\not{D}(\not{D}+1)\cdots(\not{D}+N) \tag{5.14}$$

for all $N \in \mathbb{N}_0$, since the scalar curvature is $\tau = n(n-1)$ and so $J = \frac{n}{2}$. This agrees with the results in [Bra95, ES10].

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