ANALYTIC PROOF OF THE PARTITION IDENTITY $A_{5,3,3}(n) = B^0_{5,3,3}(n)$

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Abstract. In this paper we give an analytic proof of the identity $A_{5,3,3}(n) = B^0_{5,3,3}(n)$, where $A_{5,3,3}(n)$ counts the number of partitions of n subject to certain restrictions on their parts, and $B^0_{5,3,3}(n)$ counts the number of partitions of n subject to certain other restrictions on their parts, both too long to be stated in the abstract. Our proof establishes actually a refinement of that partition identity. The original identity was first discovered by the first author jointly with M. Ruby Salestina and S. R. Sudarshan in ["A new theorem on partitions," Proc. Int. Conference on "Analytic Number Theory with Special Emphasis on L-functions," Ramanujan Math. Soc., Mysore, 2005, pp. 57–70] where it was also given a combinatorial proof, thus answering a question of Andrews.

1. Introduction

For an even integer λ, let $A_{\lambda,k,a}(n)$ denote the number of partitions of n such that

- no part $\not\equiv 0 \pmod{\lambda+1}$ may be repeated, and
- no part is $\equiv 0, \pm\left(a - \frac{\lambda}{2}\right)(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}$.

For an odd integer λ, let $A_{\lambda,k,a}(n)$ denote the number of partitions of n such that

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- no part $\not\equiv 0 \pmod{\frac{\lambda+1}{2}}$ may be repeated,
- no part is $\equiv \lambda+1 \pmod{2\lambda+2}$, and
- no part is $\equiv 0, \pm(2a-\lambda)\left(\frac{\lambda+1}{2}\right) \pmod{(2k-\lambda+1)(\lambda+1)}$.

Let $B_{\lambda,k,a}(n)$ denote the number of partitions of n of the form $b_1+\cdots+b_s$ with $b_i \ge b_{i+1}$, such that

- no part $\not\equiv 0 \pmod{\lambda+1}$ is repeated,
- $b_i - b_{i+k-1} \ge \lambda+1$, with strict inequality if b_i is a multiple of $\lambda+1$, and
- $\sum_{i=j}^{\lambda-j+1} f_i \le a-j$ for $1 \le j \le \frac{\lambda+1}{2}$ and $f_1+\cdots+f_{\lambda+1} \le a-1$, where f_j is the number of appearances of j in the partition.

In 1969, Andrews [1] proved the following theorem.

Theorem 1 ([1, Th. 2]). *If λ, k, and a are positive integers with $\frac{\lambda}{2} \le a \le k$ and $k \ge 2\lambda-1$, then, for every positive integer n, we have*

$$A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n).$$

Schur's theorem [6] addresses the case $\lambda = k = a = 2$. Hence, it is *not* a particular case of Theorem 1 as $k \ge 2\lambda - 1$ is not satisfied. Motivated by this observation, Andrews [1] first conjectured, and later proved in [2], that Theorem 1 is still true for $k \ge \lambda$.

In the same paper [2], Andrews raised the following question: Is it possible to modify the conditions on the partitions enumerated by $B_{\lambda,k,a}(n)$ so that values of $k < \lambda$ would be admissible? In fact, Schur [6] had proved that $A_{3,2,2}(n) = B^0_{3,2,2}(n)$, where $B^0_{3,2,2}(n)$ denotes the number of partitions enumerated by $B_{3,2,2}(n)$ with the added condition that no parts are $\equiv 2 \pmod 4$.

This led Andrews [2] to state the following conjecture.

Conejecture 2. *There holds the identity $A_{4,3,3}(n) = B^0_{4,3,3}(n)$ for all positive integers n, where $B^0_{4,3,3}(n)$ denotes the number of partitions of n enumerated by $B_{4,3,3}(n)$ with the added restrictions:*

$$\begin{aligned} f_{5j+2} + f_{5j+3} &\le 1 \quad \textit{for} \quad j \ge 0, \\ f_{5j+4} + f_{5j+6} &\le 1 \quad \textit{for} \quad j \ge 0, \\ f_{5j-1} + f_{5j} + f_{5j+5} + f_{5j+6} &\le 3 \quad \textit{for} \quad j \ge 1, \end{aligned}$$

where, as before, f_j denotes the number of appearances of j in the partition.

In the year 1994, Andrews et al. [3] gave an analytic proof of the above conjecture. The first author and Ruby Salestina, M. gave a combinatorial proof in [4]. In [5], these two authors and Sudarshan, S.R. first conjectured, and then proved combinatorially, the following result, which is analogous to Conjecture 2.

Theorem 3. *There holds the identity $A_{5,3,3}(n) = B^0_{5,3,3}(n)$ for all positive integers n, where $B^0_{5,3,3}(n)$ denotes the number of partitions of n enumerated by $B_{5,3,3}(n)$ with* *the added restrictions:*

$$\begin{aligned} f_{6j+3} &= 0 \quad \textit{for} \quad j \geq 0, \\ f_{6j+2} + f_{6j+4} &\leq 1 \quad \textit{for} \quad j \geq 0, \\ f_{6j+5} + f_{6j+7} &\leq 1 \quad \textit{for} \quad j \geq 0, \\ f_{6j-1} + f_{6j} + f_{6j+6} + f_{6j+7} &\leq 3 \quad \textit{for} \quad j \geq 1. \end{aligned}$$

The object of this paper is to give an analytic proof of the partition identity stated in Theorem 3. Actually, we are going to prove a new refinement of that partition identity, which we state in the next section. The method of our proof in Section 3 is similar to that of Andrews et al. in [3].

2. A refinement of the partition identity in Theorem 3

Before being able to state the announced refinement of Theorem 3, we need to make two definitions.

Definition 1. Let $A(\mu, \nu, N)$ denote the number of partitions of N into distinct non-multiples of 6 of which μ are congruent to 1 or 2 mod 6 and ν are congruent to 4 or 5 mod 6.

Clearly, we have

$$\sum_{\mu,\nu,N\geq 0} A(\mu,\nu,N) a^{\mu} b^{\nu} q^{N} = \prod_{n=0}^{\infty} (1+aq^{6n+1})(1+aq^{6n+2})(1+bq^{6n+4})(1+bq^{6n+5}). \tag{2.1}$$

Definition 2. Let $B(\mu, \nu, N)$ denote the number of partitions $\lambda = b_1 + \cdots + b_s$ of N satisfying the following conditions:

(i) Only multiples of 6 may be repeated.
(ii) $b_i - b_{i+2} \geq 6$ with strict inequality if b_i is a multiple of 6.
(iii) The multiplicities f_i, $1 \leq i \leq s$, satisfy

$$\begin{aligned} f_{6j+3} &= 0 \quad \text{for all} \quad j \geq 0, \\ f_{6j+2} + f_{6j+4} &\leq 1 \quad \text{for all} \quad j \geq 0, \\ f_{6j+5} + f_{6j+7} &\leq 1 \quad \text{for all} \quad j \geq 0, \\ f_{6j-1} + f_{6j} + f_{6j+6} + f_{6j+7} &\leq 3 \quad \text{for all} \quad j \geq 1. \end{aligned}$$

(iv) There are μ parts of the partition $\equiv 0$, 1 or 2 (mod 6).
(v) There are ν parts of the partition $\equiv 0$, 4 or 5 (mod 6).

The following theorem is the announced refinement of Theorem 3.

Theorem 4. *For each $\mu, \nu, N \geq 0$ we have*

$$A(\mu, \nu, N) = B(\mu, \nu, N). \tag{2.2}$$

It is obvious that Theorem 3 follows immediately from Theorem 4 by summing both sides of (2.2) over all μ and ν. The proof of Theorem 4 is given in the next section.

3. Proof of Theorem 4

We first observe that for any partition which satisfies (i)–(iii) of Definition 2 there are exactly 16 possibilities (numbered 0–15 in Table 1) for the subset of summands of the partition that lie in the interval $[6i+1, 6i+6]$.

0 :	$\emptyset$: empty
1 :	$6i+1$
2 :	$6i+2$
3 :	$6i+2,\ 6i+1$
4 :	$6i+4$
5 :	$6i+4,\ 6i+1$
6 :	$6i+5$
7 :	$6i+5,\ 6i+1$
8 :	$6i+5,\ 6i+2$
9 :	$6i+5,\ 6i+4$
10 :	$6i+6$
11 :	$6i+6,\ 6i+1$
12 :	$6i+6,\ 6i+2$
13 :	$6i+6,\ 6i+4$
14 :	$6i+6,\ 6i+5$
15 :	$6i+6,\ 6i+6$

Table 1

We now refine the partitions from Definition 2 further, using the classification given in Table 1.

Definition 3. Let $S_n(j, a, b, q)$ denote the generating function

$$\sum B_n(\mu, \nu, N) a^\mu b^\nu q^N,$$

where $B_n(\mu, \nu, N)$ is the number of all partitions considered in Definition 2, which in addition satisfy the two conditions

(vi) all parts are $\le 6n+6$, and

(vii) the subset of summands that lie in the interval $[6n+1, 6n+6]$ must have number $\le j$ in Table 1.

When $n = -1$, we define $S_{-1}(j, a, b, q) = 1$ and for $n < -1$, we define $S_n(j, a, b, q) = 0$.

For example,

$$S_0(9, a, b, q) = 1 + aq + aq^2 + a^2q^3 + bq^4 + abq^5 + bq^5 + abq^6 + abq^7$$

and

$$S_0(15, a, b, q) = 1 + aq + aq^2 + a^2q^3 + bq^4 + abq^5 + bq^5 + 2abq^6 + a^2bq^7 + abq^7 \\ + a^2bq^8 + b^2q^9 + ab^2q^{10} + ab^2q^{11} + a^2b^2q^{12}.$$

It is easy to verify that

$$S_0(15, a, b, q) = (1 + aq)(1 + aq^2)(1 + bq^4)(1 + bq^5).$$

For convenience, we write $S_n(j)$ for $S_n(j,a,b,q)$. Along the lines of [3], we obtain the following recurrence relations for $S_n(j)$:

$$S_n(0) = S_{n-1}(15), \tag{3.1}$$

$$S_n(1) = S_n(0) + aq^{6n+1}[S_{n-1}(11) - S_{n-1}(9) + S_{n-1}(5)] - a^3b^3q^{24n-12}S_{n-3}(9), \tag{3.2}$$

$$S_n(2) = S_n(1) + aq^{6n+2}[S_{n-1}(12) - S_{n-1}(9) + S_{n-1}(8)], \tag{3.3}$$

$$S_n(3) = S_n(2) + a^2q^{12n+3}S_{n-1}(3), \tag{3.4}$$

$$S_n(4) = S_n(3) + bq^{6n+4}S_{n-1}(13), \tag{3.5}$$

$$S_n(5) = S_n(4) + abq^{12n+5}S_{n-1}(5), \tag{3.6}$$

$$S_n(6) = S_n(5) + bq^{6n+5}S_{n-1}(14), \tag{3.7}$$

$$S_n(7) = S_n(6) + abq^{12n+6}S_{n-1}(5), \tag{3.8}$$

$$S_n(8) = S_n(7) + abq^{12n+7}S_{n-1}(8), \tag{3.9}$$

$$S_n(9) = S_n(8) + b^2q^{12n+9}S_{n-1}(9), \tag{3.10}$$

$$S_n(10) = S_n(9) + abq^{6n+6}S_{n-1}(14), \tag{3.11}$$

$$S_n(11) = S_n(10) + a^2bq^{12n+7}S_{n-1}(5), \tag{3.12}$$

$$S_n(12) = S_n(11) + a^2bq^{12n+8}S_{n-1}(8), \tag{3.13}$$

$$S_n(13) = S_n(12) + ab^2q^{12n+10}S_{n-1}(9), \tag{3.14}$$

$$S_n(14) = S_n(13) + ab^2q^{12n+11}S_{n-1}(9), \tag{3.15}$$

$$S_n(15) = S_n(14) + a^2b^2q^{12n+12}S_{n-1}(9). \tag{3.16}$$

We now define two linear combinations of the $S_n(9)$'s and the $S_n(15)$'s,

(3.17)
$$\begin{aligned} J(n) := S_n(9) &- (1-q^{6n})(1+aq^{6n+1}+aq^{6n+2}+bq^{6n+4}+bq^{6n+5})S_{n-1}(15) \\ &- q^{6n}(1+aq^{6n+1}+aq^{6n+2}+a^2q^{6n+3}+bq^{6n+4}+bq^{6n+5}+abq^{6n+5} \\ &\qquad + abq^{6n+6}+abq^{6n+7}+b^2q^{6n+9})S_{n-1}(9) \\ &+ (1-q^{6n})abq^{18n-3}(a^2+abq^2+abq^3+abq^4+a^2bq^4+a^2bq^5+b^2q^6 \\ &\qquad + ab^2q^7+ab^2q^8)S_{n-2}(9) \\ &+ a^3b^3q^{24n-12}(1-q^{6n})(1-q^{6n-6})S_{n-3}(9), \end{aligned}$$

and

$$\begin{aligned} K(n) := S_n(9) - S_n(15) &+ abq^{6n+6}(1-q^{6n})S_{n-1}(15) \\ &+ abq^{12n+6}(1+aq+aq^2+bq^4+bq^5+abq^6)S_{n-1}(9) \\ &- a^3b^3q^{18n+6}(1-q^{6n})S_{n-2}(9). \end{aligned} \tag{3.18}$$

Along lines similar to those in [3], we are able to obtain a recurrence for $S_n(9)$ (see Lemma 6).

Lemma 5. *For $n \geq 0$, $J(n) = 0 = K(n)$.*

Sketch of proof. We prove the lemma by using the identities (3.1)–(3.16). The 14 sequences $S_n(j)$, where j is different from 9 and 15, can be expressed as linear combinations of the $S_m(9)$'s and the $S_m(15)$'s in the following way: from (3.16), we find that $S_n(14)$ is given by

$$S_n(14) = S_n(15) - a^2b^2q^{12n+12}S_{n-1}(9).$$

Using the above equation in (3.15), $S_n(13)$ becomes such a linear combination. Similarly for $S_n(12)$ if we use (3.14). Equation (3.10) yields such a linear combination for $S_n(8)$. Subsequently, (3.13) yields a linear combination for $S_n(11)$, and (3.11) yields a linear combination for $S_n(10)$. Replacing n by $n+1$ in (3.12), we get

$$S_n(5) = a^{-2}b^{-1}q^{-12n-19}[S_{n+1}(11) - S_{n+1}(10)],$$

which in turn yields an expression for $S_n(5)$ in terms of the $S_m(9)$'s and the $S_m(15)$'s. Equations (3.9), (3.8), (3.6), (3.5), (3.4) and (3.3) yield respectively linear combinations in terms of the $S_m(9)$'s and the $S_m(15)$'s for $S_n(7)$, $S_n(6)$, $S_n(4)$, $S_n(3)$, $S_n(2)$ and $S_n(1)$. Finally, we know already from (3.1) that $S_n(0) = S_{n-1}(15)$.

We are now in the position to prove $K(n) = 0$. Let us consider (3.7), that is

$$S_n(6) = S_n(5) + bq^{6n+5}S_{n-10}(14).$$

Substituting the expression in terms of the $S_m(9)$'s and $S_m(15)$'s obtained earlier for $S_n(5)$ and the respective one for $S_{n-1}(14)$ in the equation above, we get a certain identity, (A) say.

On the other hand, from (3.8), we have

$$S_n(6) = S_n(7) - abq^{12n+6}S_{n-1}(5).$$

Substituting the expression in terms of the $S_m(9)$'s and $S_m(15)$'s obtained earlier for $S_n(7)$ and the respective one for $S_{n-1}(5)$, we obtain another identity, (B) say. It can now be verified that (A)$-$(B), when multiplied by a^2bq^{12n+19}, is exactly the equation $K(n+1) = 0$.

Now we prove $J(n) = 0$. Substituting the expressions obtained earlier for $S_n(1)$, $S_n(0)$, $S_{n-1}(11)$ and $S_{n-1}(5)$ into (3.2), we obtain

$$0 = a^2bq^{12n+19}J(n) - K(n+1) + aq^{6n+13}(1 + aq^{6n+2} + bq^{6n+4} + bq^{6n+5})K(n).$$

Since $K(n) = 0$ for all $n \geq 0$, we conclude that $J(n) = 0$ for all $n \geq 0$. This proves the lemma. $\square$

Lemma 6. *For $n \geq 0$,*

$$\begin{aligned}(3.19)\qquad &(1 + aq^{6n-5} + aq^{6n-4} + bq^{6n-2} + bq^{6n-1})S_n(9)\\ &= p_1(n,a,b,q)S_{n-1}(9) + (1-q^{6n})p_2(n,a,b,q)S_{n-2}(9)\\ &\quad + p_3(n,a,b,q)(1-q^{6n})(1-q^{6n-6})S_{n-3}(9)\\ &\quad + a^4b^4q^{30n-36}(1-q^{6n})(1-q^{6n-6})(1-q^{6n-12})\\ &\qquad \times [(1 + aq^{6n+1} + aq^{6n+2} + bq^{6n+4} + bq^{6n+5})S_{n-4}(9)],\end{aligned}$$

where

$$\begin{aligned}
p_1(n,a,b,q) = {} & 1 + aq^{6n-5} + aq^{6n-4} + bq^{6n-2} + bq^{6n-1} + abq^{6n} + 2abq^{12n-1} + 3abq^{12n} \\
& + aq^{6n+2} + 2a^2q^{12n-3} + a^2q^{12n-2} + 2abq^{12n+1} + a^2bq^{12n+2} + bq^{6n+4} \\
& + b^2q^{12n+2} + 2b^2q^{12n+3} + ab^2q^{12n+4} + bq^{6n+5} + b^2q^{12n+4} + ab^2q^{12n+5} \\
& + a^2q^{12n+3} + 2a^2bq^{18n+1} + 2a^2bq^{18n+2} + abq^{12n+5} + a^2bq^{18n} \\
& + ab^2q^{18n+3} + a^2q^{12n-4} + 2ab^2q^{18n+4} + abq^{12n+6} + 2ab^2q^{18n+5} \\
& + abq^{12n+7} + a^2bq^{18n+3} + ab^2q^{18n+6} + b^2q^{12n+9} + a^3q^{18n-2} \\
& + a^3q^{18n-1} + b^3q^{18n+7} + b^3q^{18n+8} + a^2bq^{12n+1} + aq^{6n+1},
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
p_2(n,a,b,q) = {} & a^2bq^{12n-5} + a^2bq^{12n-4} + ab^2q^{12n-2} + ab^2q^{12n-1} + a^2b^2q^{12n} + a^3bq^{18n-4} \\
& + a^3bq^{18n-3} + a^3bq^{18n-2} + 3a^2b^2q^{18n} + a^2b^2q^{18n-1} + ab^3q^{18n+2} \\
& + ab^3q^{18n+3} + a^2b^2q^{18n+1} + ab^3q^{18n+4} + a^3bq^{18n-10} + a^2b^2q^{18n-7} \\
& + 3a^2b^2q^{18n-6} + a^3b^2q^{18n-5} + a^4bq^{24n-9} + a^4bq^{24n-8} + a^4bq^{24n-7} \\
& + 3a^3b^2q^{24n-5} + a^3b^2q^{24n-6} + 3a^2b^3q^{24n-2} + 3a^3b^2q^{24n-4} + 3a^2b^3q^{24n-1} \\
& + a^3bq^{18n-9} + a^3bq^{18n-8} + a^2b^2q^{18n-5} + a^3b^2q^{18n-4} + a^2b^3q^{24n} \\
& + ab^3q^{18n-4} + ab^4q^{24n} + a^3b^2q^{24n-3} + ab^4q^{24n+2} + ab^3q^{18n-2} \\
& + ab^4q^{24n+1} + ab^4q^{24n+3} + a^4bq^{24n-6} + ab^3q^{18n-3} + a^2b^3q^{18n-2} \\
& + a^2b^3q^{18n-1} + a^2b^3q^{24n-3},
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
p_3(n,a,b,q) = {} & -a^3b^3q^{24n-12} - a^3b^3q^{18n-12} - a^4b^3q^{24n-11} - a^4b^3q^{24n-10} - a^3b^4q^{24n-8} \\
& - a^3b^4q^{24n-7} + 2a^4b^3q^{30n-17} + 2a^4b^3q^{30n-16} + 2a^3b^4q^{30n-14} \\
& + 2a^3b^4q^{30n-13} + a^4b^2q^{24n-21} + a^5b^2q^{30n-20} + a^3b^3q^{24n-19} \\
& + a^5b^2q^{30n-19} + a^4b^3q^{30n-18} + a^3b^4q^{30n-15} + a^3b^3q^{24n-18} + a^3b^3q^{24n-17} \\
& + a^4b^3q^{30n-15} + a^3b^4q^{30n-12} + a^2b^5q^{30n-11} + a^2b^5q^{30n-10} + a^2b^4q^{24n-15}.
\end{aligned} \tag{3.22}$$

Sketch of proof. Using the identity $J(n) = 0$, we find that $S_{n-1}(15)$ is a linear combination of $S_n(9)$, $S_{n-1}(9)$, $S_{n-2}(9)$ and $S_{n-3}(9)$. By Lemma 5, we have $K(n) = 0$. We substitute that linear combination for $S_{n-1}(15)$ and the corresponding one for $S_n(15)$ in (3.18). After some simplification, and after replacing n by $n-1$, we arrive exactly at (3.19). □

We are now able to prove a recurrence for $S_n(15)$.

Lemma 7. *For $n \geq 0$, we have*

$$
\begin{aligned}
(1 + aq^{6n-5} &+ aq^{6n-4} + bq^{6n-2} + bq^{6n-1})S_n(15) \\
&= p_1(n-1, aq^6, bq^6, q)S_{n-1}(15) + (1-q^{6n-6})p_2(n-1, aq^6, bq^6, q)S_{n-2}(15) \\
&\quad + (1-q^{6n-6})(1-q^{6n-12})p_3(n-1, aq^6, bq^6, q)S_{n-3}(15) \\
&\quad + a^4b^4q^{30n-18}(1 + aq^{6n+1} + aq^{6n+2} + bq^{6n+4} + bq^{6n+5}) \\
&\qquad \times (1-q^{6n-6})(1-q^{6n-12})(1-q^{6n-18})S_{n-4}(15).
\end{aligned} \tag{3.23}
$$

Proof. Since $J(n) = 0$, we can express $S_n(15)$ in terms of the $S_n(9)'s$. Substituting these expressions in (3.23), we get an equation involving $S_{n+1}(9), S_n(9), \dots, S_{n-6}(9)$. In that equation we apply Lemma 6 to $S_{n+1}(9)$. In the result thus obtained, we apply Lemma 6 to $S_n(9)$. In the subsequent result obtained, we apply Lemma 6 to $S_{n-1}(9)$. In the result obtained, we again apply Lemma 6, this time to $S_{n-2}(9)$. The result is zero. All these calculations have been performed using *Mathematica*. □

Lemma 8. *For $n \geq 0$,*

$$S_n(15, a, b, q) = (1 + aq)(1 + aq^2)(1 + bq^4)(1 + bq^5)S_{n-1}(9, aq^6, bq^6, q).$$

Proof. Comparing Lemma 7 and Lemma 6, we find that both sides of Lemma 8 satisfy the same fourth order recurrence valid for $n \geq 1$. Hence we have only to verify Lemma 8 for $n = 0, 1, 2, 3$. This is a routine verification, and can therefore be left to the reader. □

Proof of Theorem 4. For $0 \leq j \leq 15$, we have

$$\lim_{n\to\infty} S_n(j, a, b, q) = \sum_{\mu,\nu,N\geq 0} B(\mu, \nu, N)a^\mu b^\nu q^N \equiv S(a, b, q).$$

Letting $n \to \infty$ in Lemma 8, we find that

$$S(a, b, q) = (1 + aq)(1 + aq^2)(1 + bq^4)(1 + bq^5)S(aq^6, bq^6, q).$$

Iterating the above equation, we obtain

$$
\begin{aligned}
S(a, b, q) &= \prod_{n=0}^{\infty}(1 + aq^{6n+1})(1 + aq^{6n+2})(1 + bq^{6n+4})(1 + bq^{6n+5}) \\
&= \sum_{\mu,\nu,N\geq 0} A(\mu, \nu, N)a^\mu b^\nu q^N,
\end{aligned}
$$

the latter equality being due to (2.1). Thus we get

$$\sum_{\mu,\nu,N\geq 0} B(\mu, \nu, N)a^\mu b^\nu q^N = \sum_{\mu,\nu,N\geq 0} A(\mu, \nu, N)a^\mu b^\nu q^N,$$

which, upon comparison of coefficients of $a^\mu b^\nu q^N$, implies

$$A(\mu, \nu, N) = B(\mu, \nu, N)$$

for all non-negative μ, ν and N. This is exactly the claim in Theorem 4. □

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