q–Generalization of a ballot problem

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Abstract. n–dimensional lattice paths which do not touch the hyperplanes $x_i - x_{i+1} = -1$, $i = 1, 2, \dots, n-1$, and $x_n - x_1 = -1 - K$ are enumerated by certain statistics, one of which is MacMahon's major index, the others being variations of it. By a reflection–like proof, a formula involving determinants is obtained. It is a q–extension of Filaseta's (J. Combin. Theory A 39 (1985), 102–111) expression for the number of elections in a specific ballot problem.

1. Introduction. In [2] Filaseta considered the following ballot problem. Let $a_1, a_2, \dots, a_n$ and $t_1, t_2, \dots, t_n$ be positive integers such that

$$a_i < a_{i-1} + t_i \qquad \text{for } i = 1, 2, \dots, n\ ,$$

where we take $a_0 = a_n$ in the case $i = 1$. Suppose n candidates $A_1, A_2, \dots, A_n$ in an election in which candidate A_i received a_i votes, $i = 1, 2, \dots, n$. Let $A_i(m)$ denote the number of votes A_i received after the first m votes have been cast, $i = 1, 2, \dots, n$. How many possible elections are there such that

$$A_i(m) < A_{i-1}(m) + t_i \qquad \text{for } i = 1, 2, \dots, n \text{ and } m = 1, 2, \dots, \sum_{i=1}^{n} a_i\ ,$$

where again we take $A_0(m) = A_n(m)$ in the case $i = 1$. Using an analytical argument, Filaseta obtained the following expression for this number of elections,

$$(1.1) \qquad \sum_{\substack{u_1 \in \mathbb{Z}}} \cdots \sum_{\substack{u_n \in \mathbb{Z} \\ u_1 + \dots + u_n = 0}} (a_1 + \dots + a_s)! \det_{1 \le i,j \le n} (1/b_{ij}!)\ ,$$

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where

$$b_{ij} = a_j + u_i \sum_{\nu=1}^{n} t_\nu + \sum_{\nu=1}^{i} t_\nu - \sum_{\nu=1}^{j} t_\nu \ .$$

This ballot problem is equivalent to the problem of counting n–dimensional lattice paths from $\mathbf{0}$ to $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$, consisting of positive unit steps, which do not touch any one of the hyperplanes

$$x_i = x_{i-1} + t_i \qquad \text{for } i = 1, 2, \dots, n \ , \tag{1.2}$$

where again $x_0 := x_n$. This is seen by interpreting a path p from $\mathbf{0}$ to $\boldsymbol{a}$ by an election in which the m–th vote is a vote for candidate A_j if and only if the m–th step in the path p is a step in the x_j–direction.

In this paper we consider the problem of counting paths from $\mathbf{0}$ to $\boldsymbol{a}$ which do not touch any one of the hyperplanes in (1.2) by MacMahon's [5] greater index, which nowadays frequently is called major index. (To be precise, we consider the equivalent problem of counting lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ not touching any one of the hyperplanes in (2.2) and (2.3).) The counting formula (Theorem 1 in section 3) which we obtain is a q–analogue of (1.1). Besides, this result generalizes previous ones of one of the authors [4]. It is easily seen that, using the terminology of Gessel and Zeilberger [3], Filaseta's result solves the problem of counting lattice paths consisting of positive unit steps in a Weyl chamber corresponding to the affine root system $S(A_n)$. Hence, our results give a solution for the problem of q-counting lattice paths in a Weyl chamber of $S(A_n)$. The key for proving Theorem 1 lies in extending the proof method of [4] to this generalized path enumeration problem. Since this method heavily relies on Zeilberger's [6] reflection proof for the classical n–candidate ballot problem, the first thing to be done is to give a reflection proof of Filaseta's formula (1.1). This is the contents of section 4. (In fact, the arguments in section 4 are just the same as those in the proof of Theorem 1 in [3] but specialized to $S(A_n)$.) Section 2 contains all relevant definitions. Finally, in section 5, we give the bijections which replace the reflections of section 4, thus establishing a combinatorial proof of Theorem 1. Section 6 presents an extension of Theorem 1 to a family of major-like statistics (Theorem 2). This result does not only include the case of counting by MacMahon's greater index as special case, which is discussed in the preceding section, but also the corresponding result for counting lattice paths by MacMahon's lesser index [5, p. 136].

2. Definitions. We use the usual multidimensional notation. If $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$, then $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, $\boldsymbol{\lambda} - \boldsymbol{\mu} = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots, \lambda_n - \mu_n)$, and $\binom{|\boldsymbol{\lambda}|}{\boldsymbol{\lambda}} = |\boldsymbol{\lambda}|!/\lambda_1!\,\lambda_2!\cdots\lambda_n!$. If $\lambda_i \ge \mu_i$ for all $i = 1, 2, \dots, n$, we write $\boldsymbol{\lambda} \ge \boldsymbol{\mu}$.

Let q be an indeterminate. The q–notations we need are $[\alpha] = (1 - q^\alpha)$, $[m]! = [m]\cdot[m-1]\cdots[1]$, $[0]! = 1$, and $\begin{bmatrix} |\boldsymbol{\lambda}| \\ \boldsymbol{\lambda} \end{bmatrix} = [|\boldsymbol{\lambda}|]!/[\lambda_1]!\,[\lambda_2]!\cdots[\lambda_n]!$.

In the sequel we assume K to be an arbitrary fixed positive integer. Let K_n stand for the set of all n–tuples $\boldsymbol{\lambda}$ of integers with

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge \lambda_1 - K \ .$$

Let S_n denote the symmetric group of order n. Given $\sigma \in S_n$ we write $\boldsymbol{\lambda}_\sigma$ for $(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \lambda_{\sigma(n)})$. For the set of all permutations of the multiset $\{1^{\mu_1}, 2^{\mu_2}, \dots, n^{\mu_n}\}$ (the multiset consisting of μ_i copies of i, $i = 1, 2, \dots, n$) we write $S(\boldsymbol{\mu})$.

Throughout this paper, a lattice path p will be a finite sequence $(p(0), p(1), \dots, p(L))$ of points $p(i)$ in the integer lattice $\mathbb{Z}^n$, where $p(i) \in \mathbb{Z}^n$ and $p(i+1) - p(i) = E_j$ for some j, $1 \le j \le n$, $i = 0, 1, 2, \dots$. E_j stands for the vector with the j–th coordinate being equal to 1, all other being equal to 0. $[p(i)]_k$ will denote the k–th coordinate of the i–th point on the path p. A lattice path p from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ can be symbolized by a pair $(\boldsymbol{\mu}, \pi)$, where $\boldsymbol{\mu} = p(0)$ and $\pi \in S(\boldsymbol{\lambda} - \boldsymbol{\mu})$. π is obtained by proceeding along the path, starting at its first point $\boldsymbol{\mu}$, and successively writing a k for each step in the x_k–direction. To be precise, $\pi = \pi_1 \dots \pi_L$, where $L = |\boldsymbol{\lambda} - \boldsymbol{\mu}|$ and $\pi_i = k$ if and only if $p(i) - p(i-1) = E_k$. Conversely, if a path p is given by $(\boldsymbol{\mu}, \pi)$, the points of p are determined by

$$p(i) = \boldsymbol{\mu} + \sum_{k=1}^{n} E_k \cdot |\{r : 1 \le r \le i \text{ and } \pi_r = k\}| \ . \tag{2.1}$$

To give an example, take $n = 3$ and $p_0 = ((1,2,1), (1,2,2), (1,2,3), (2,2,3), (2,3,3), (3,3,3))$. Then p_0 is symbolized by the pair $((1,2,1), 33121)$.

By $N(\boldsymbol{\mu} \to \boldsymbol{\lambda})$ we denote the set of all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$. We write $N(\boldsymbol{\mu} \to \boldsymbol{\lambda})^+$ for the set of all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ whose points all lie in K_n. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in K_n$ the set $N(\boldsymbol{\mu} \to \boldsymbol{\lambda})^+$ is equal to the set of all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ which do not touch any one of the hyperplanes

$$x_i - x_{i+1} = -1, \qquad i = 1, 2, \dots, n-1 \ , \tag{2.2}$$

and

$$x_n - x_1 = -1 - K \ . \tag{2.3}$$

Let H_i denote the hyperplane $x_i - x_{i+1} = -1$, $i = 1, 2, \dots, n-1$, let H_n denote the hyperplane in (2.3), and let R_i be the reflection with respect to H_i, $i = 1, 2, \dots, n$. $N(\boldsymbol{\mu} \to \boldsymbol{\lambda})^-$ stands for the set of all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ which touch at least one of the hyperplanes in (2.2) and (2.3).

Next, for a given multiset permutation $\pi = \pi_1 \pi_2 \dots \pi_L$ we define the major statistics by

$$\operatorname{maj} \pi = \sum_{i=1}^{L-1} i \cdot \chi(\pi_i > \pi_{i+1}) \ .$$

($\chi(A) = 1$ if A is true, and $\chi(A) = 0$ otherwise.) Having in mind the representation of a lattice path p by a pair $(\boldsymbol{\mu}, \pi)$, we extend maj to lattice paths by defining $\operatorname{maj} p = \operatorname{maj} \pi$, if $p = (\boldsymbol{\mu}, \pi)$. Finally we introduce the generating function

$$G(\boldsymbol{\mu} \to \boldsymbol{\lambda}; q) = \sum_{p} q^{\operatorname{maj} p} \ ,$$

where the sum is taken over all $p \in N(\boldsymbol{\mu} \to \boldsymbol{\lambda})$. Analogously,

$$G(\boldsymbol{\mu} \to \boldsymbol{\lambda}; q)^+ = \sum_p q^{\operatorname{maj} p} ,$$

where the sum is over all $p \in N(\boldsymbol{\mu} \to \boldsymbol{\lambda})^+$, and

$$G(\boldsymbol{\mu} \to \boldsymbol{\lambda}; q)^- = \sum_p q^{\operatorname{maj} p} ,$$

where the sum is over all $p \in N(\boldsymbol{\mu} \to \boldsymbol{\lambda})^-$.

3. Formulation of the Theorem. In section 1 it was shown that Filaseta's ballot problem is equivalent with the problem of counting all lattice paths from $\mathbf{0}$ to $\boldsymbol{a}$ which do not touch any one of the hyperplanes in (1.2). Let $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ be an element of $\mathbb{Z}_n$. By performing the transformation $\boldsymbol{y} \to \boldsymbol{z}$, where

$$y_i = z_i + \sum_{\nu=1}^{i} t_\nu - i \quad \text{for} \quad i = 1, 2, \dots, n ,$$

it is seen that the above lattice path counting problem is equivalent with the problem of counting all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ which do not touch any one of the hyperplanes in (2.2) and (2.3), where

$$\mu_i = i - \sum_{\nu=1}^{i} t_\nu ,$$

$$\lambda_i = a_i + i - \sum_{\nu=1}^{i} t_\nu ,$$

and

$$K = \sum_{\nu=1}^{n} t_\nu - n .$$

The major counting extension of this counting problem, which is proved in section 5, is the following.

Theorem 1. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in K_n$ and $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$. The generating function $\sum q^{\operatorname{maj} p}$, where the sum is over all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ which do not touch any one of the hyperplanes in* (2.2) *and* (2.3)*, is given by*

$$G(\boldsymbol{\mu} \to \boldsymbol{\lambda}; q)^+ = \sum_{\boldsymbol{u} \in U_n} [|\boldsymbol{\lambda} - \boldsymbol{\mu}|]! \det_{1 \le s,t \le n} \left(q^{T(s,t,\boldsymbol{u},\boldsymbol{\mu})} / [\lambda_s - s - \mu_t + t - (K+n) u_s]! \right) ,$$

where U_n is the set of all n–tuples $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$ of integers with $u_1 + \cdots + u_n = 0$, and

$$T(s,t,\boldsymbol{u},\boldsymbol{\mu}) = (\mu_t - t)(s-t) + (n-1)u_s(\mu_t - t) + (K+n)((n-1)\frac{u_s^2}{2} + su_s) \ .$$

The special case $K > |\boldsymbol{\lambda} - \boldsymbol{\mu}|$ (this is the corresponding result for counting lattice paths which do not touch any of the hyperplanes in (2.2)) has previously appeared in [4, Theorem 1].

To give a concrete example, take $n = 3$, $K = 2$, $\boldsymbol{\mu} = (1,-1,-1)$ and $\boldsymbol{\lambda} = (3,2,1)$. We want to count all paths p from $(1,-1,-1)$ to $(3,2,1)$ with respect to the major index which do not touch $x_1 - x_2 = -1$, $x_2 - x_3 = -1$, and $x_3 - x_1 = -3$. Let such a path be represented by $p = ((1,-1,-1),\pi)$ as was explained above. The following table lists all possibilities of π together with the corresponding major–values.

π	$\operatorname{maj}\pi = \operatorname{maj}p$
2312321	13
2312312	7
2312231	8
2321231	11
2321312	10
2321321	16
2323112	6
2323121	12
2231231	9
2231312	8
2231321	14
2233112	4
2233121	10

Hence,

$$G((1,-1,-1) \to (3,2,1))^+ = q^4 + q^6 + q^7 + 2q^8 + q^9 + 2q^{10} + q^{11} + q^{12} + q^{13} + q^{14} + q^{16},$$

which agrees with the claimed expression in Theorem 1 as can be easily checked.

4. The reflection proof. Let $\boldsymbol{\lambda},\boldsymbol{\mu} \in K_n$ and $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$. For the number of all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$, $G(\boldsymbol{\mu} \to \boldsymbol{\lambda}; 1)$, we have

$$G(\boldsymbol{\mu} \to \boldsymbol{\lambda}; 1) = \binom{|\boldsymbol{\lambda} - \boldsymbol{\mu}|}{\boldsymbol{\lambda} - \boldsymbol{\mu}} \ . \tag{4.1}$$

Following Zeilberger [6], for $\sigma \in S_n$ we define

$$\boldsymbol{e}_\sigma = (1 - \sigma(1), 2 - \sigma(2), \dots, n - \sigma(n)) \ .$$

We are going to construct a bijection between

$$\bigcup_{\substack{\sigma \text{ even} \\ \boldsymbol{u} \in U_n}} N(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u} \to \boldsymbol{\lambda})^- \tag{4.2}$$

and

$$\bigcup_{\substack{\sigma \text{ odd} \\ \boldsymbol{u}\in U_n}} N(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u} \to \boldsymbol{\lambda})^- \ , \tag{4.3}$$

which proves

$$\sum_{\substack{\sigma \text{ even} \\ \boldsymbol{u}\in U_n}} G(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u} \to \boldsymbol{\lambda}; 1)^- = \sum_{\substack{\sigma \text{ odd} \\ \boldsymbol{u}\in U_n}} G(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u} \to \boldsymbol{\lambda}; 1)^- \ . \tag{4.4}$$

This is done by application of the reflection principle. Consider a lattice path p of $N(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{\mu} \to \boldsymbol{\lambda})^-$. Let H_j be the last hyperplane of those in (2.2) and (2.3) which the path p meets. If $j \le n-1$ reflection with respect to H_j has the effect

$$R_j(x_1, \dots, x_j, x_{j+1}, \dots, x_n) = (x_1, \dots, x_{j+1} - 1, x_j + 1, \dots, x_n) \ , \tag{4.5}$$

while reflection with respect to H_n yields

$$R_n(x_1, x_2, \dots, x_{n-1}, x_n) = (x_n + K + 1, x_2, \dots, x_{n-1}, x_1 - K - 1) \ . \tag{4.6}$$

Hence, for $j \le n-1$, we have

$$R_j(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u}) = \boldsymbol{\mu}_{\sigma(j,j+1)} + \boldsymbol{e}_{\sigma(j,j+1)} + (K+n)\boldsymbol{u}^{(j)} \ , \tag{4.7}$$

where $\boldsymbol{u}^{(j)} = (u_1, \dots, u_{j+1}, u_j, \dots, u_n)$, and if $j = n$,

$$R_n(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u}) = \boldsymbol{\mu}_{\sigma(1,n)} + \boldsymbol{e}_{\sigma(1,n)} + (K+n)\boldsymbol{u}^{(n)} \ , \tag{4.8}$$

where $\boldsymbol{u}^{(n)} = (u_n+1, u_2, \dots, u_{n-1}, u_1-1)$. ($(g,h)$ is the transposition which exchanges g and h. An expression of the form $\sigma(g,h)$ means the composition $\sigma \circ (g,h)$ of the permutations σ and (g,h).) Therefore, replacing the portion of the path p until the last meeting with H_j by its reflection with respect to H_j, turns p into a path of $N(\boldsymbol{\mu}_{\sigma(j,j+1)} + \boldsymbol{e}_{\sigma(j,j+1)} + (K+n)\boldsymbol{u}^{(j)} \to \boldsymbol{\lambda})^-$ if $j \le n-1$, in the case $j = n$ by this reflection we obtain a path of $N(\boldsymbol{\mu}_{\sigma(1,n)} + \boldsymbol{e}_{\sigma(1,n)} + (K+n)\boldsymbol{u}^{(n)} \to \boldsymbol{\lambda})^-$. Since the reflected path again meets H_j as last hyperplane, and its last meeting point is the same, renewed application of this reflection procedure would give the original path. Therefore this mapping in fact is an involution which consequently defines the desired bijection between the sets (4.2) and (4.3). This implies that (4.4) holds.

Since, because of $\boldsymbol{\mu} \in K_n$, for $\sigma \ne id$ or $\boldsymbol{u} \ne \boldsymbol{0}$ we have

$$(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u}) \notin K_n \ ,$$

we get

$$N(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda})^- = N(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda}) \quad \text{for } \sigma\neq id \text{ or } \boldsymbol{u}\neq\mathbf{0}\,. \tag{4.9}$$

Therefore, by consecutive use of (4.4), (4.9), and (4.1), we have

$$\begin{aligned} G(\boldsymbol{\mu}\to\boldsymbol{\lambda};1)^+ &= G(\boldsymbol{\mu}\to\boldsymbol{\lambda};1)-G(\boldsymbol{\mu}\to\boldsymbol{\lambda};1)^- \\ &= G(\boldsymbol{\mu}\to\boldsymbol{\lambda};1)-G(\boldsymbol{\mu}\to\boldsymbol{\lambda};1)^- \\ &\qquad+\sum_{\substack{\sigma\in S_n\\ \boldsymbol{u}\in U_n}}\operatorname{sgn}\sigma\, G(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda};1)^- \\ &= \sum_{\substack{\sigma\in S_n\\ \boldsymbol{u}\in U_n}}\operatorname{sgn}\sigma\, G(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda};1) \\ &= \sum_{\substack{\sigma\in S_n\\ \boldsymbol{u}\in U_n}}\operatorname{sgn}\sigma\binom{|\boldsymbol{\lambda}-\boldsymbol{\mu}|}{\boldsymbol{\lambda}-\boldsymbol{\mu}_\sigma-\boldsymbol{e}_\sigma-(K+n)\boldsymbol{u}} \\ &= \sum_{\boldsymbol{u}\in U_n}|\boldsymbol{\lambda}-\boldsymbol{\mu}|!\det(1/(\lambda_s-s-\mu_t+t-(K+n)u_s)!)\,. \end{aligned} \tag{4.10}$$

5. Proof of the Theorem. To prove Theorem 1, the reflection procedure of section 4 must be modified because, when reflecting a part of a lattice path, nothing can be said about the major of its image. Therefore, in order to extend the reflection proof to $q\neq 1$, we construct a map φ, which sends the initial point of a path to the same point as the reflection procedure would do, but is (in a certain sense) also maj–preserving. Let the weight $w(p)$ of a path $p\in N(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda})$ be defined by

$$w(p)=q^{\sum_{s=1}^n T(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu})}q^{\operatorname{maj}p}\,, \tag{5.1}$$

where $T(s,t,\boldsymbol{u},\boldsymbol{\mu})$ is the number given in Theorem 1. What we are going to do, is to construct a weight–preserving bijection φ between the two sets (4.2) and (4.3), which sends the initial point of a path to the same point as the reflection procedure would do.

Now, let $p\in N(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda})^-$. First, suppose that the path p meets H_j as the last hyperplane of those in (2.2) and (2.3), and $j\le n-1$. In [4] a mapping Φ was given satisfying the following properties (cf. [4, Def. 3, p. 145 (3),(4), (5.25)]):

(1) Φ acts on paths which meet one of the hyperplanes H_j, $1\le j\le n-1$.

(2) If the path $\overline{p}\in N(\overline{\boldsymbol{\mu}}\to\overline{\boldsymbol{\lambda}})^-$ meets H_j as the last hyperplane, $j\le n-1$, then $\Phi(\overline{p})\in N(R_j(\overline{\boldsymbol{\mu}})\to\overline{\boldsymbol{\lambda}})^-$ and

$$\operatorname{maj}\Phi(\overline{p})=\operatorname{maj}\overline{p}-\overline{\mu}_j+\overline{\mu}_{j+1}-1\,. \tag{5.2}$$

(3) Φ is an involution.

For p satisfying the above properties we define

$$\varphi(p) := \Phi(p) \ .$$

By (2) this immediately implies

$$p \in N(\boldsymbol{\mu}_{\sigma(j,j+1)} + \boldsymbol{e}_{\sigma(j,j+1)} + (K+n)\boldsymbol{u}^{(j)} \to \boldsymbol{\lambda})^- \ ,$$

with the $\boldsymbol{u}^{(j)}$ of section 4. Moreover, using the weight property (5.2), we obtain

$$\begin{aligned} w(\varphi(p)) &= q^{\sum_{s=1}^n T(s,(\sigma(j,j+1))(s),\boldsymbol{u}^{(j)},\boldsymbol{\mu})} q^{\operatorname{maj}(\Phi(p))} \\ &= q^{\sum_{s=1}^n T(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu}) - \mu_{\sigma(j+1)} + \sigma(j+1) + \mu_{\sigma(j)} - \sigma(j) + (K+n)(u_j - u_{j+1})} \\ &\quad \cdot q^{\operatorname{maj} p - \mu_{\sigma(j)} - j + \sigma(j) - (K+n)u_j + \mu_{\sigma(j+1)} + j + 1 - \sigma(j+1) + (K+n)u_{j+1} - 1} \\ &= w(p) \ . \end{aligned} \tag{5.3}$$

Now suppose that p meets H_n as last hyperplane. To define $\varphi(p)$ in this case, we proceed in a manner analogous to [4, sec. 5]. Given a multiset permutation $\pi = \pi_1\pi_2\dots\pi_L$, $\pi_i \in \{1,2,\dots,n\}$, we define the difference function $d_n(\pi,i)$ by

$$d_n(\pi,i) = |\{r : r \le i \text{ and } \pi_r = n\}| - |\{r : r \le i \text{ and } \pi_r = 1\}| \ .$$

$d_n(\pi,i)$ is the difference between the number of n's and the number of 1's in $\pi_1\pi_2\dots\pi_i$. In the following two definitions two auxiliary maps, $\mathcal{L}_n$ and $\mathcal{R}_n$, are introduced. $\mathcal{L}_n$ has the effect of changing a single step in the x_1–direction into a step in x_n–direction, $\mathcal{R}_n$ has the reverse effect.

Definition 1. *Let $\pi = \pi_1\pi_2\dots\pi_L$ be a multiset permutation where the function $i \to d_n(\pi,i)$ does not reach its minimal value at $i=0$ or $i=L$. Let k be the smallest integer, $1 \le k \le L-1$, where the function $i \to d_n(\pi,i)$ reaches its minimal value, in other words*

$$\begin{aligned} i < k \quad &\text{implies} \quad d_n(\pi,i) > d_n(\pi,k) \ , \\ i \ge k \quad &\text{implies} \quad d_n(\pi,i) \ge d_n(\pi,k) \ . \end{aligned}$$

Obviously, $\pi_k = 1$. This given, we define $\mathcal{L}_n$ by:

(A) *If π satisfies*

$$\pi_e > \pi_{e+1} \le \dots \le \pi_{k-1} \le 1 < \pi_{k+1} \le \dots \le \pi_f > \pi_{f+1} \ ,$$

where $1 \le e+1 < f \le L$, and

$$\begin{aligned} \textit{either} \quad (a) \quad & e+1 < k < f \\ \textit{or} \quad (b) \quad & k = e+1 \text{ and } \pi_e > \pi_{e+2} \ , \end{aligned}$$

then

$$\mathcal{L}_n(\pi) = \pi_1 \dots \pi_e \pi_{e+1} \dots \pi_{k-1}\pi_{k+1} \dots \pi_f \, n \, \pi_{f+1} \dots \pi_L \ .$$

(B) *If π satisfies*

$$\pi_e \leq \pi_{e+1} > \cdots > \pi_{k-1} > 1 < \pi_{k+1}$$

where $1 \leq e+1 < k \leq L$, and $\pi_{k-1} \leq \pi_{k+1}$, then

$$\mathcal{L}_n(\pi) = \pi_1 \dots \pi_e \, n \, \pi_{e+1} \dots \pi_{k-1}\pi_{k+1} \dots \pi_L \ .$$

(By convention, if $i > j$, the symbol $\pi_i \dots \pi_j$ means the empty word. For the detailed conventions of how to read the inequality chains, we refer the reader to [4, sec. 5].)

Defintion 2. *Let $\pi_1\pi_2 \dots \pi_L$ be a multiset permutation where $d_n(\pi, L)$ exceeds the minimal value of the function $i \to d_n(\pi, i)$ by at least two. Let $(r-1)$ be the largest integer, $0 \leq r-1 \leq L-1$, where $i \to d_n(\pi, i)$ reaches its minimal value, meaning*

$$\begin{aligned} i \leq r-1 \quad &\textit{implies} \quad d_n(\pi, i) \geq d_n(\pi, r-1) \ , \\ i > r-1 \quad &\textit{implies} \quad d_n(\pi, i) > d_n(\pi, r-1) \ . \end{aligned}$$

Then $\pi_r = n$. This given we define $\mathcal{R}_n$ by:

(A) *If π satisfies*

$$\pi_e > \pi_{e+1} \leq \cdots \leq \pi_{r-1} < n \leq \pi_{r+1} \leq \cdots \leq \pi_f > \pi_{f+1} \ ,$$

where $1 \leq e+1 < f \leq L$ and

$$\begin{aligned} \textit{either} \quad (a) &\quad e+1 < r < f \\ \text{or} \quad (b) &\quad r = f \text{ and } \pi_{f-1} > \pi_{f+1} \ , \end{aligned}$$

then

$$\mathcal{R}_n(\pi) = \pi_1 \dots \pi_e \, 1 \, \pi_{e+1} \dots \pi_{r-1}\pi_{r+1} \dots \pi_f\pi_{f+1} \dots \pi_L \ .$$

(B) *If π satisfies*

$$\pi_{r-1} < n > \pi_{r+1} > \cdots > \pi_f \leq \pi_{f+1} \ ,$$

where $1 \leq r < f \leq L$, and $\pi_{r-1} \leq \pi_{r+1}$, then

$$\mathcal{R}_n(\pi) = \pi_1 \dots \pi_{r-1}\pi_{r+1} \dots \pi_f \, 1 \, \pi_{f+1} \dots \pi_L \ .$$

Following the lines of [4, sec. 5], it can be shown that $\mathcal{L}_n$ and $\mathcal{R}_n$ are well–defined, and that

$$\mathcal{L}_n \circ \mathcal{R}_n = \mathcal{R}_n \circ \mathcal{L}_n = id \ .$$

Besides, from the definition of $\mathcal{L}_n$ and $\mathcal{R}_n$ it is easily seen that for any multiset permutation π there holds

$$\operatorname{maj} \mathcal{L}_n(\pi) = \operatorname{maj} \mathcal{R}_n(\pi) = \operatorname{maj} \pi \ . \tag{5.4}$$

Now we are able to define $\varphi(p)$ for p meeting H_n as last hyperplane. Let $p = (\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u}, \pi)$, then

$$\varphi(p) = (R_n(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u}), \mathcal{L}_n^{E(\sigma,K,\boldsymbol{\mu},\boldsymbol{u})}(\pi)) \ , \tag{5.5}$$

where

$$\begin{aligned} E(\sigma, K, \boldsymbol{\mu}, \boldsymbol{u}) &= [\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u}]_n - [\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u}]_1 + K + 1 \\ &= \mu_{\sigma(n)} + n - \sigma(n) - \mu_{\sigma(1)} - 1 + \sigma(1) \\ &\qquad + (K+n)(u_n - u_1) + K + 1 \ . \end{aligned}$$

If the exponent $E(\sigma, K, \boldsymbol{\mu}, \boldsymbol{u})$ in (5.5) is negative, by $\mathcal{L}_n^{E(\sigma,K,\boldsymbol{\mu},\boldsymbol{u})}$ we mean $\mathcal{R}_n^{-E(\sigma,K,\boldsymbol{\mu},\boldsymbol{u})}$.

It can be shown that φ is well-defined, that the endpoint of $\varphi(p)$ is the same as that of p, that the last meeting point with H_n is the same as that of p, and that the portion of the path after this last meeting point remains unchanged. Hence, by (4.8),

$$\varphi(p) \in N(\boldsymbol{\mu}_{\sigma(1,n)} + \boldsymbol{e}_{\sigma(1,n)} + (K+n)\boldsymbol{u}^{(n)} \to \boldsymbol{\lambda})^- \ ,$$

with the $\boldsymbol{u}^{(n)}$ of section 4. Moreover, by (5.4) obviously we have

$$\operatorname{maj} \varphi(p) = \operatorname{maj} p \ . \tag{5.6}$$

It is easy to see that this definition again establishes an involution. Therefore, using (5.6) we obtain

$$\begin{aligned} w(\varphi(p)) &= q^{\sum_{s=1}^n T(s,(\sigma(1,n))(s),\boldsymbol{u}^{(n)},\boldsymbol{\mu})} q^{\operatorname{maj}(\varphi(p))} \\ &= q^{\sum_{s=1}^n T(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu}) + (n-1)(\mu_{\sigma(1)} - \sigma(1))} \\ &\qquad \cdot q^{-(n-1)(\mu_{\sigma(n)} - \sigma(n)) + (n-1)(\mu_{\sigma(n)} - \sigma(n)) - (n-1)(\mu_{\sigma(1)} - \sigma(1))} \\ &\qquad \cdot q^{(K+n)\left((n-1)\frac{1}{2}(2u_n + 1 - 2u_1 + 1) - (n-1)u_n + 1 + (n-1)u_1 - n\right)} \\ &\qquad \cdot q^{\operatorname{maj} p} \\ &= w(p) \ . \end{aligned} \tag{5.7}$$

To give an example, take $n = 3$, $K = 2$, $\boldsymbol{\mu} = (1, -1, -1)$, $\boldsymbol{\lambda} = (3, 2, 1)$, $\sigma = (123)$ (in cycle notation), and $\boldsymbol{u} = (1, 0, -1)$. Then $\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u} = (3, -2, -2)$. Consider the path $p_0 = \big((3,-2,-2), 2233223\big)$. This path only touches the hyperplane H_3, which is given by $x_n - x_1 = -3$ (cf. (2.3)). The meeting points are $(3,0,0)$, $(3,1,0)$, and $(3,2,0)$, the last meeting point being $(3,2,0)$. To apply the mapping φ, first we observe that

$$E\big((123), 2, (1,-1,-1), (1,0,-1)\big) = -2 \ .$$

Hence, we have

$$\varphi(p_0) = \big(R_3(3,-2,-2), \mathcal{R}_3^2(2233223)\big) = \big((1,-2,0), \mathcal{R}_3^2(2233223)\big) \ .$$

Using Definition 2 (A) ($e = 0$, $r = 3$, $f = 4$), we obtain $\mathcal{R}_3(2233223) = 1223223$, and by Definition 2 (B) ($r = 4$, $f = 5$) $\mathcal{R}_3(1223223) = 1222123$. Hence, $\varphi(p_0) = \big((1,-2,0), 1222123\big)$. Obviously, the final point of p_0 again is $(3,2,1)$. $\varphi(p_0)$ meets H_3 as the last hyperplane, just as p_0 did. Again $(3,2,0)$ is the last meeting point with H_3 and after this point nothing was changed. Moreover, we have $\operatorname{maj} p_0 = \operatorname{maj} \varphi(p_0) = 4$. Similarly, use of Definition 1 (B) ($e = 3$, $k = 5$) would give $\mathcal{L}_3(1222123) = 1223223$, and by Definition 1 (A) ($e = 0$, $k = 1$, $f = 4$) we would get $\mathcal{L}_3(1223223) = 2233223$. This establishes $\varphi(\varphi(p_0)) = p_0$, which must be satisfied, since φ is an involution.

Summarizing, the map φ maps elements of the set in (4.2) onto elements of the set in (4.3) and vice versa. Since it is an involution it is a bijection. Moreover it is weight–preserving with respect to w. Therefore the equation which generalizes (4.4) reads

$$\begin{aligned}(5.8)\qquad \sum_{\substack{\sigma \text{ even}\\ \boldsymbol{u}\in U_n}} & q^{\sum_{s=1}^n T(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu})} G(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u} \to \boldsymbol{\lambda}; q)^- \\ &= \sum_{\substack{\sigma \text{ odd}\\ \boldsymbol{u}\in U_n}} q^{\sum_{s=1}^n T(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu})} G(\boldsymbol{\mu}_\sigma + \boldsymbol{e}_\sigma + (K+n)\boldsymbol{u} \to \boldsymbol{\lambda}; q)^- \ .\end{aligned}$$

The q–analogue of (4.1) is MacMahon's well–known result [1, Theorem 3.7, p. 42], which in our notation is

$$(5.9)\qquad G(\boldsymbol{\mu} \to \boldsymbol{\lambda}; q) = \begin{bmatrix} |\boldsymbol{\lambda}-\boldsymbol{\mu}| \\ \boldsymbol{\lambda}-\boldsymbol{\mu} \end{bmatrix} \ .$$

Successive use of (5.8), (4.9), and (5.9) gives

$$\begin{aligned} G(\boldsymbol{\mu}\to\boldsymbol{\lambda};q)^+ &= G(\boldsymbol{\mu}\to\boldsymbol{\lambda};q)-G(\boldsymbol{\mu}\to\boldsymbol{\lambda};q)^- \\ &= G(\boldsymbol{\mu}\to\boldsymbol{\lambda};q)-G(\boldsymbol{\mu}\to\boldsymbol{\lambda};q)^- \\ &\qquad + \sum_{\substack{\sigma\in S_n\\ \boldsymbol{u}\in U_n}} \operatorname{sgn}\sigma\, q^{\sum_{s=1}^n T(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu})} G(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda};q)^- \\ &= \sum_{\substack{\sigma\in S_n\\ \boldsymbol{u}\in U_n}} \operatorname{sgn}\sigma\, q^{\sum_{s=1}^n T(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu})} G(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda};q) \\ &= \sum_{\substack{\sigma\in S_n\\ \boldsymbol{u}\in U_n}} \operatorname{sgn}\sigma\, q^{\sum_{s=1}^n T(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu})} \begin{bmatrix} |\boldsymbol{\lambda}-\boldsymbol{\mu}| \\ \boldsymbol{\lambda}-\boldsymbol{\mu}_\sigma-\boldsymbol{e}_\sigma-(K+n)\boldsymbol{u}\end{bmatrix} \\ &= \sum_{\boldsymbol{u}\in U_n} [|\boldsymbol{\lambda}-\boldsymbol{\mu}|]!\det\left(q^{T(s,t,\boldsymbol{u},\boldsymbol{\mu})}/[\lambda_s-\mu_t-s+t-(K+n)u_s]!\right) , \end{aligned} \tag{5.10}$$

which exactly is the expression in Theorem 1.

6. An extension of Theorem 1. In this section we extend Theorem 1, which is a theorem about major counting, to other, major-like statistics. To motivate the following considerations, recall that MacMahon not only dealt with the major (i.e. greater) index but also with what he called the “lesser index” [5, p. 136]. This lesser index, which we denote by maj’, of a multiset permutation $\pi=\pi_1\pi_2\dots\pi_L$ is defined by

$$\operatorname{maj'}\pi=\sum_{i=1}^{L-1} i\cdot\chi(\pi_i<\pi_{i+1}) .$$

While for the major index each j with $\pi_j>\pi_{j+1}$ makes a contribution, for the lesser index it is each j with $\pi_j<\pi_{j+1}$ which makes a contribution. One also could roughly say that taking the lesser index is nothing but taking the major index where the order $1<2<\dots<n$ of the numbers $\{1,2,\dots,n\}$ has been changed into $1>2>\dots>n$. Having this idea in mind, it is also reasonable to consider other total orders of $\{1,2,\dots,n\}$ and to take major index with respect to these orders. Thus, for each n one obtains a set of $n!$ different major variants, the original major index and the lesser index being only two of them.

To be more precise, we proceed as follows. Given a specific total order $i_1<i_2<\dots<i_n$ of the set $\{1,2,\dots,n\}$, let the permutation τ be defined by $\tau(i_j):=j$, $j=1,2,\dots,n$. Then taking the major index of a multiset permutation $\pi=\pi_1\pi_2\dots\pi_L$ with respect to the order $i_1<i_2<\dots<i_n$ is just the same as $\operatorname{maj}(\tau(\pi_1)\tau(\pi_2)\dots\tau(\pi_L))$. This leads to the following definition.

Let τ be a fixed permutation, $\tau\in S_n$. We extend τ to multiset permutations $\pi=\pi_1\pi_2\dots\pi_L$ which satisfy $\pi_i\in\{1,2,\dots,n\}$ by

$$\tau(\pi)=\tau(\pi_1)\tau(\pi_2)\dots\tau(\pi_L) .$$

The action of τ on a path $p=(\boldsymbol{\mu},\pi)$ is defined by $\tau(p)=(\boldsymbol{\mu},\tau(\pi))$. Now we introduce the following permutation-indexed statistics:

$$\operatorname{maj}_\tau p := \operatorname{maj}\tau(p)\ . \tag{6.1}$$

Obviously, maj_{id} is identical with the major index, while for τ_0 given by $\tau_0(i)=n+1-i$ the statistics $\operatorname{maj}_{\tau_0}$ coincides with MacMahon's lesser index.

To give another example, take $n=3$, $\tau_1=\binom{1\ 2\ 3}{2\ 3\ 1}$, $\pi_1=1223213$. This choice of τ_1 corresponds to the total order $3<1<2$. We have $\operatorname{maj}_{\tau_1}\pi_1=3+5+6=14$.

The same idea as that which was used in the proof of Theorem 1 also serves to establish an analogous result for counting $N(\boldsymbol{\mu}\to\boldsymbol{\lambda})^+$ by maj_τ. In fact, the result for the generating function $\sum q^{\operatorname{maj}_\tau p}$ almost looks to be identical, except that the powers of q are changed according to the choice of τ.

Theorem 2. *Let $\boldsymbol{\lambda},\boldsymbol{\mu}\in K_n$ and $\boldsymbol{\lambda}\ge\boldsymbol{\mu}$. The generating function $\sum q^{\operatorname{maj}_\tau p}$, where the sum is over all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ which do not touch any one of the hyperplanes in* (2.2) *and* (2.3)*, is given by*

$$\sum_{\boldsymbol{u}\in U_n}[|\boldsymbol{\lambda}-\boldsymbol{\mu}|]!\det_{1\le s,t\le n}\left(q^{T_\tau(s,t,\boldsymbol{u},\boldsymbol{\mu})}/[\lambda_s-s-\mu_t+t-(K+n)u_s]!\right)\ .$$

The exponents $T_\tau(s,t,\boldsymbol{u},\boldsymbol{\mu})$ are given by

$$T_\tau(s,t,\boldsymbol{u},\boldsymbol{\mu})=(\mu_t-t)\sum_{j=t}^{s-1}c(j)+u_s(\mu_t-t)\sum_{j=1}^{n}c(j)$$
$$+(K+n)\left(\frac{u_s^2}{2}\sum_{j=1}^{n}c(j)+u_s\sum_{j=1}^{s-1}c(j)\right)\ ,$$

where $c(j)=\chi(\tau(j)<\tau(j+1))$, and where we adopt the convention that $\sum_{j=l}^{l-1}a_j=0$, and $\sum_{j=l}^{k-1}a_j=-\sum_{j=k}^{l-1}a_j$ whenever $k<l$. By $\tau(n+1)$ here and in the sequel we mean $\tau(1)$. Analogous conventions will be assumed for $n+1$ appearing as a subscript: $\mu_{n+1}:=\mu_1$, etc.

Sketch of proof. In analogy with (5.1) let the weight $w_\tau(p)$ of a path $p\in N(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda})$ be defined by

$$w_\tau(p)=q^{\sum_{s=1}^n T_\tau(s,\sigma(s),\boldsymbol{u},\boldsymbol{\mu})}q^{\operatorname{maj}_\tau p}\ . \tag{6.2}$$

Let $p\in N(\boldsymbol{\mu}_\sigma+\boldsymbol{e}_\sigma+(K+n)\boldsymbol{u}\to\boldsymbol{\lambda})^-$ be a path which meets H_j as last hyperplane of those in (2.2) and (2.3). Suppose that there are mappings ψ_j which satisfy the following properties:

(1) ψ_j acts on paths which meet H_j as last hyperplane.
(2) If $\overline{p}\in N(\overline{\boldsymbol{\mu}}\to\overline{\boldsymbol{\lambda}})^-$ meets H_j as last hyperplane, then $\psi_j(\overline{p})\in N(R_j(\overline{\boldsymbol{\mu}})\to\overline{\boldsymbol{\lambda}})^-$ and $\psi_j(\overline{p})$ also meets H_j as last hyperplane.

(3) The action of ψ_j changes maj_τ according to

$$\operatorname{maj}_\tau \psi_j(\overline{p}) = \begin{cases} \operatorname{maj}_\tau \overline{p} - c(j)(\overline{\mu}_j - \overline{\mu}_{j+1} + 1) & 1 \le j \le n-1 \\ \operatorname{maj}_\tau \overline{p} - c(n)(\overline{\mu}_n - \overline{\mu}_1 + 1 + K) & j = n \end{cases} , \tag{6.3}$$

with $c(j) = \chi(\tau(j) < \tau(j+1))$.

(4) ψ_j is an involution.

Then simple calculations show that $\psi_j(p) \in N(\boldsymbol{\mu}_{\sigma(j,j+1)} + \boldsymbol{e}_{\sigma(j,j+1)} + (K+n)\boldsymbol{u}^{(j)} \to \boldsymbol{\lambda})^-$ with $\boldsymbol{u}^{(j)}$ being defined in section 4. Moreover, $w_\tau(\psi_j(p)) = w_\tau(p)$. Let the mapping φ_τ, which acts on elements of the union of the sets in (4.2) and (4.3), be defined as follows: If an element of the union of the sets in (4.2) and (4.3) meets H_j as last hyperplane of those in (2.2) and (2.3), then the image of this path under application of φ_τ is given by applying ψ_j to this path. φ_τ is seen to be an involution, and hence a bijection between the sets in (4.2) and (4.3). Besides, it is weight-preserving with respect to w_τ. Computations which proceed analogously to those in (5.8)–(5.10) would complete the proof of Theorem 2.

It remains to construct those mappings ψ_j. The idea is as follows:

If $\tau(j) < \tau(j+1)$ modify Definitions 1,2 of [4] such that $\mathbf{L}_j$ changes a single $\tau(j+1)$ into $\tau(j)$, $\mathbf{R}_j$ changes a single $\tau(j)$ into a $\tau(j+1)$, and

$$\begin{aligned} \mathbf{R}_j \circ \mathbf{L}_j = \mathbf{L}_j \circ \mathbf{R}_j &= id , \\ \operatorname{maj} \mathbf{L}_j(\pi) &= \operatorname{maj} \pi - 1 , \\ \operatorname{maj} \mathbf{R}_j(\pi) &= \operatorname{maj} \pi + 1 . \end{aligned}$$

Then for a path $p = (\boldsymbol{\mu}, \pi)$ which meets H_j as last hyperplane, ψ_j is given by

$$\psi_j(p) = (R_j(\boldsymbol{\mu}),\ \mathbf{L}_j^{\mu_j - \mu_{j+1} + 1 + K\cdot\chi(j=n)}(\tau(\pi))) .$$

If N is negative, by $\mathbf{L}_j^N$ we mean $\mathbf{R}_j^{-N}$. Similarly, if $\tau(j) > \tau(j+1)$, modify Definitions 3,4 of [4] such that $\overline{\mathbf{L}}_j$ changes a $\tau(j+1)$ into a $\tau(j)$, $\overline{\mathbf{R}}_j$ changes a $\tau(j)$ into a $\tau(j+1)$, and

$$\begin{aligned} \overline{\mathbf{R}}_j \circ \overline{\mathbf{L}}_j = \overline{\mathbf{L}}_j \circ \overline{\mathbf{R}}_j &= id , \\ \operatorname{maj} \overline{\mathbf{L}}_j(\pi) = \operatorname{maj} \overline{\mathbf{R}}_j(\pi) &= \operatorname{maj} \pi . \end{aligned}$$

In this case, ψ_j is given by

$$\psi_j(p) = (R_j(\boldsymbol{\mu}),\ \overline{\mathbf{L}}_j^{\mu_j - \mu_{j+1} + 1 + K\cdot\chi(j=n)}(\tau(\pi))) ,$$

with the analogous convention concerning negative exponents of $\overline{\mathbf{L}}_j$. $\square$

Setting $\tau = \tau_0$ in Theorem 2 yields the corresponding result for counting $N(\boldsymbol{\mu} \to \boldsymbol{\lambda})^+$ by the lesser index.

Corollary 1. *Let $\boldsymbol{\lambda},\boldsymbol{\mu} \in K_n$ and $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$. The generating function $\sum q^{\mathrm{maj}' P}$, where the sum is over all lattice paths from $\boldsymbol{\mu}$ to $\boldsymbol{\lambda}$ which do not touch any of the hyperplanes in (2.2) and (2.3), is given by*

$$\sum_{\boldsymbol{u}\in U_n} [|\boldsymbol{\lambda}-\boldsymbol{\mu}|]! \det_{1\le s,t\le n} \left(q^{T'(s,t,\boldsymbol{u},\boldsymbol{\mu})}/[\lambda_s - s - \mu_t + t - (K+n)u_s]!\right) ,$$

where

$$T'(s,t,\boldsymbol{u},\boldsymbol{\mu},) = u_s(\mu_t - t) + (K+n)\frac{u_s^2}{2} . \quad \square$$

The special case $K > |\boldsymbol{\lambda}-\boldsymbol{\mu}|$ has been proved previously in [4, Theorem 4].

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