EXPLICIT FORMULAS FOR ENUMERATION OF LATTICE PATHS: BASKETBALL AND THE KERNEL METHOD

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Dedicated to Sri Gopal Mohanty,
a pioneer in the field of lattice paths combinatorics,
on the occasion of his 84th birthday.

Abstract. This article deals with the enumeration of directed lattice walks on the integers with any finite set of steps, starting at a given altitude j and ending at a given altitude k, with additional constraints such as, for example, to never attain altitude 0 in-between. We first discuss the case of walks on the integers with steps $-h, \dots, -1, +1, \dots, +h$. The case $h = 1$ is equivalent to the classical Dyck paths, for which many ways of getting explicit formulas involving Catalan-like numbers are known. The case $h = 2$ corresponds to "basketball" walks, which we treat in full detail. Then we move on to the more general case of walks with any finite set of steps, also allowing some weights/probabilities associated with each step. We show how a method of wide applicability, the so-called "kernel method", leads to explicit formulas for the number of walks of length n, for any h, in terms of nested sums of binomials. We finally relate some special cases to other combinatorial problems, or to problems arising in queuing theory.

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Contents

1. Introduction

While analysing permutations sortable by a stack, Knuth [34, Ex. 1–4 in Sec. 2.2.1] showed they were counted by Catalan numbers, and were therefore in bijection with Dyck paths (lattice paths with steps $(1, 1)$ and $(1, -1)$ in the plane integer lattice, from the origin to some point on the x-axis, and never running below the x-axis in-between). He used a method to derive the corresponding generating function (see [34, p. 536ff]) which Flajolet coined *"kernel method"*. That name stuck among combinatorialists, although the method already existed in the folklore of statistics and statistical physics — without a name. The method was later generalized to enumeration and asymptotic analysis of directed lattice paths with any set of steps, and many other combinatorial structures enumerated by bivariate or trivariate functional equations (see, e.g., [6, 8, 19, 20, 26, 27]). We refer to the introduction of [11] for a more detailed history of the kernel method.

The emphasis in [8] is on asymptotic analysis, for which the derived (exact) enumeration results serve as a starting point. The latter are in a sense *implicit*, since they involve solutions to certain algebraic equations. They are nevertheless perfect for carrying out singularity analysis, which in the end leads to very precise asymptotic results.

In general, it is not possible to simplify the exact enumeration results from [8]. However, for models involving special choices of step sets, this is possible. These potential simplifications are the main focus of our paper.

Such models appear frequently in queuing theory. Indeed, birth and death processes and queues, like the one shown in Figure 1, are naturally encoded by lattice paths (see [16, 29, 30, 35, 36, 41]). In this article, we solve a problem raised during the 2015 International Conference on "Lattice Path Combinatorics and Its Applications": to find closed-form formulas for the number of walks of length n from 0 to k for a full family of models similar to Figure 1. As it turns out, the essential tool to achieve this goal is indeed the kernel method.

Our paper is organized as follows. We begin with some preliminaries in Section 2. In particular, we introduce the directed lattice paths that we are going to discuss here, we provide a first glimpse of the kernel method, and we briefly review the Lagrange–Bürmann inversion formula for the computation of the coefficients of implicitly defined power series. Section 3 is devoted to (old-time) "basketball walks", which, by definition, are directed lattice walks with steps from the set $\{(1,-2),(1,-1),(1,1),(1,2)\}$ which always stay above the x-axis. (They may be seen as the evolution of — pre 1984 — basketball games; see the beginning of that section for a more detailed explanation of the terminology.) We provide exact formulas (often several, not obviously equivalent ones) for generating functions and for the numbers of walks under various constraints. At the end of Section 3, we also briefly address the asymptotic analysis of the number of these walks. Section 4 then considers the more general problem of enumerating directed walks where the allowed steps are of the form $(1,i)$ with $-h \leq i \leq h$ (including $i = 0$ or not). Again, we provide exact formulas for generating functions — in terms of roots of the so-called kernel equation — and for numbers of walks — in terms of nested sums of binomials. All these results are obtained by appropriate combinations of the kernel method with variants of the Lagrange–Bürmann inversion formula. In the concluding Section 5, we relate basketball walks with other combinatorial objects, namely

- with certain trees coming from option pricing,
- with increasing unary-binary trees which avoid a certain pattern which arose in work of Riehl [39],
- and with certain Boolean bracketings which appeared in work of Bender and Williamson [13].

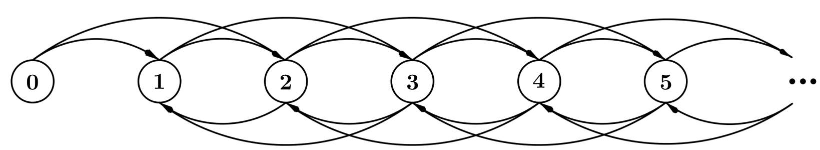


FIGURE 1. A queue corresponding to the basketball walk model.

2. The general setup, and some preliminaries

In this section, we describe the general setup that we consider in this article. We use (subclasses of) so-called *Łukasiewicz paths* as main example(s) which serve to illustrate this setup. We recall here as well the main tools that we shall use in this article: the *kernel method* and the *Lagrange–Bürmann inversion formula*.

We start with the definition of the lattice paths under consideration.

Definition 2.1. A *step set* $\mathcal{S} \subset \mathbb{Z}^2$ is a finite set of vectors

$$\{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)\}.$$

An n-step *lattice path* or *walk* is a sequence of vectors $v = (v_1, v_2, \dots, v_n)$, such that v_j is in $\mathcal{S}$. Geometrically, it may be interpreted as a sequence of points $\omega = (\omega_0, \omega_1, \dots, \omega_n)$, where $\omega_i \in \mathbb{Z}^2, \omega_0 = (0, 0)$ (or another starting point), and $\omega_i - \omega_{i-1} = v_i$ for $i = 1, 2, \dots, n$. The elements of $\mathcal{S}$ are called *steps*. The *length* $|\omega|$ of a lattice path is its number n of steps.

The lattice paths can have different additional constraints shown in Table 1.

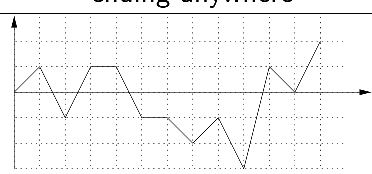
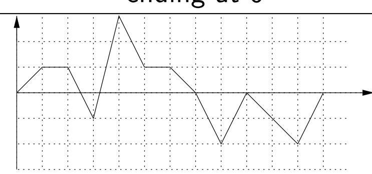
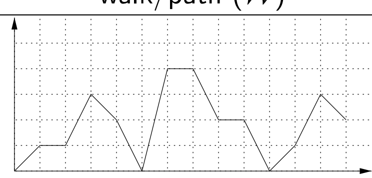
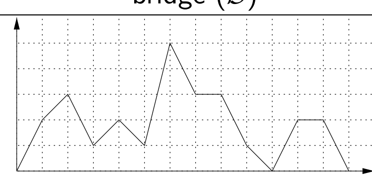
	ending anywhere	ending at 0
unconstrained (on $\mathbb{Z}$)	 walk/path ($\mathcal{W}$)	 bridge ($\mathcal{B}$)
constrained (on $\mathbb{N}$)	 meander ($\mathcal{M}$)	 excursion ($\mathcal{E}$)

Table 1. The four types of walks: unconstrained walks, bridges, meanders, and excursions.

We restrict our attention to *directed paths*, which are defined by the fact that, for each step $(x, y) \in \mathcal{S}$, one has $x \geq 0$. Moreover, we will focus only on the subclass of *simple paths*, where every element in the step set $\mathcal{S}$ is of the form $(1, b)$. In other words, these paths constantly move one step to the right. Thus, they are essentially one-dimensional objects and can be seen as walks on the integers. We introduce the abbreviation $\mathcal{S} = \{b_1, b_2, \dots, b_n\}$ in this case. A *Łukasiewicz path* is a simple path where its associated step set $\mathcal{S}$ is a subset of $\{-1, 0, 1, \dots\}$ and $-1 \in \mathcal{S}$.

Example 2.2 (DYCK PATHS)**.** *A Dyck path is a path constructed from the step set $\mathcal{S} = \{-1, +1\}$, which starts at the origin, never passes below the x-axis, and ends on the x-axis. In other words, Dyck paths are excursions with step set $\mathcal{S} = \{-1, +1\}$.*

The next definition allows to merge the probabilistic point of view (*random walks*) and the combinatorial point of view (*lattice paths*).

Definition 2.3. For a given step set $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$, we define the corresponding *system of weights* as $\{p_1, p_2, \ldots, p_m\}$, where $p_j > 0$ is the weight associated with step s_j for $j = 1, 2, \ldots, m$. The *weight of a path* is defined as the product of the weights of its individual steps.

Next we introduce the algebraic structures associated with the previous definitions. The *step polynomial* of a given step set $\mathcal{S}$ is defined as the Laurent polynomial[1]

$$P(u) := \sum_{j=1}^{m} p_j u^{s_j}.$$

Let

$$c = -\min_j s_j \quad \text{and} \quad d = \max_j s_j \tag{2.1}$$

be the two extreme step sizes, and assume throughout that $c, d > 0$. Note that for Łukasiewicz paths we have $c = 1$.

We start with the easy case of unconstrained paths. We define their bivariate generating function as

$$W(z,u) = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} W_{n,k} z^n u^k,$$

where $W_{n,k}$ is the number of unconstrained paths ending after n steps at altitude k.

It is well-known and straightforward to derive that

$$W(z,u) = \frac{1}{1 - zP(u)}. \tag{2.2}$$

We continue with the generating function of meanders:

$$F(z,u) := \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_{n,k} z^n u^k,$$

where $F_{n,k}$ is the number of paths ending after n steps at altitude k, and constrained to be always at altitude ≥ 0 in-between. Note that we are mainly interested in solving the counting problem, i.e., determining the numbers $F_{n,k}$ for specific families of paths (see Table 1). The generating function encodes all information we are interested in.

[1] By a *Laurent polynomial* in u we mean a polynomial in u and u^{-1}.

We decompose $F(z,u)$ in two ways, namely

$$F(z,u) = \sum_{k\geq 0} F_k(z) u^k = \sum_{n\geq 0} f_n(u) z^n.$$

Here, the generating functions $F_k(z)$ enumerate paths ending at altitude k, i.e., $F_k(z) = \sum_{n\geq 0} F_{n,k} z^n$. In particular, the generating function for excursions is equal to $F_0(z)$. On the other hand, the polynomials $f_n(u)$ enumerate paths of length n. The power of u encodes their final altitude. We will use this decomposition for a step-by-step approach, similar to the one in the case of unconstrained paths.

For the sake of illustration, we show below how the kernel method can be used to find a closed form for the generating function of a given class of Łukasiewicz paths.

Theorem 2.4. *Let $\mathcal{S}$ be the step set of a class of Łukasiewicz paths, and let $P(u)$ be the associated step polynomial. Then the bivariate generating function of meanders (where z marks length, and u marks final altitude) and excursions are*

$$F(z,u) = \frac{1 - zF_0(z)/u}{1 - zP(u)} \qquad \text{and} \qquad F_0(z) = \frac{u_1(z)}{z}, \tag{2.3}$$

respectively, where $u_1(z)$ is the unique small solution of the implicit equation

$$1 - zP(u) = 0,$$

that is, the unique solution satisfying $\lim_{z\to 0} u_1(z) = 0$.

Proof. A meander of length n is either empty, or it is constructed from a meander of length $n-1$ by appending a possible step from $\mathcal{S}$. However, a meander is not allowed to pass below the x-axis, thus at altitude 0 it is not allowed to use the step -1. This translates into the relations

$$f_0(u) = 1, \qquad\qquad f_{n+1}(u) = \{u^{\geq 0}\}\left(P(u) f_n(u)\right),$$

where $\{u^{\geq 0}\}$ is the linear operator extracting all terms in the power series representation containing non-negative powers of u. Multiplying both sides of the above equation by z^{n+1} and subsequently summing over all $n \geq 0$, we obtain the functional equation

$$F(z,u) = 1 + zP(u)F(z,u) - \frac{z}{u} F_0(z).$$

Equivalently,

$$(1 - zP(u))F(z,u) = 1 - \frac{z}{u} F_0(z)\,. \tag{2.4}$$

We write $K(z,u) := 1 - zP(u)$ and call this factor $K(z,u)$ the *kernel*. The above functional equation looks like an underdetermined equation as there are two unknown functions, namely $F(z,u)$ and $F_0(z)$. However, the special structure on the left-hand side will resolve this problem and leads us to the *kernel method*.

Using the theory of Newton polygons and Puiseux expansions (cf. [24, Appendix of Sec. 3]), we know that the *kernel equation*

$$1 - zP(u) = 0,$$

has $d+1$ distinct solutions in u (recall that $c = 1$, see Equation (2.1)). One of them, say $u_1(z)$, maps 0 to 0. We call this solution the "small branch" of the kernel equation. It is in modulus smaller than the other d branches. These in turn grow to infinity in modulus while z approaches 0. Consequently, we call the latter the "large branches" and denote them by $v_1(z), v_2(z), \ldots, v_d(z)$. Inserting the small branch into (2.4) (this is legitimate as we stay in the integral domain of Puiseux power series: substitution of the small branch always leads to series having a finite number of terms with negative exponents, even for intermediate computations), we get $F_0(z) = u_1(z)/z$. This proves our second claim. Using this result, we can solve (2.4) for $F(z,u)$ to get the first claim. ☐

The formula (2.3) in the previous theorem implies that the number m_n of meanders of length n is directly related to the number e_n of excursions of length n via

$$m_n = P(1)^n - \sum_{k=0}^{n-1} P(1)^k e_{n-k-1}.$$

In the sequel, we therefore focus on giving explicit expressions for e_n.

A key tool for finding a formula for the coefficients of power series satisfying implicit equations is the Lagrange inversion formula [37], independently discovered in a slightly extended form by Bürmann [22] (see also [38]). In the statement of the theorem and also later, we use the *coefficient extractor* $[z^n]F(z) := f_n$ for a power series $F(z) = \sum f_n z^n$.

Theorem 2.5 (Lagrange–Bürmann inversion formula). *Let $F(z)$ be a formal power series which satisfies $F(z) = z\phi(F(z))$, where $\phi(z)$ is a power series with $\phi(0) \neq 0$. Then, for any Laurent*[2] *series $H(z)$ and for all non-zero integers n, we have*

$$[z^n]H(F(z)) = \frac{1}{n}[z^{n-1}]H'(z)\phi^n(z)\,.$$

Proof. See [28, Chapter A.6] or [48, Theorem 5.4.2]. ☐

[2] Here, by Laurent series we mean a series of the form $H(z) = \sum_{n\geq a} H_n\, z^n$ for some (possibly negative) integer a.

name and the associated step polynomial $P(u)$	number e_n of excursions of length n
Dyck paths $P(u) = \frac{1}{u} + u$	$e_{2n} = \frac{1}{n+1}\binom{2n}{n}$
Motzkin paths $P(u) = \frac{1}{u} + 1 + u$	$e_n = \frac{1}{n+1}\sum_{k=0}^{\lceil \frac{n+1}{2} \rceil}\binom{n+1}{k}\binom{n+1-k}{k-1}$
weighted Motzkin paths $P(u) = \frac{p_{-1}}{u} + p_0 + p_1 u$	$e_n = \frac{1}{n+1}\sum_{k=0}^{\lceil \frac{n+1}{2} \rceil}\binom{n+1}{k}\binom{n+1-k}{k-1}(p_1p_{-1})^{k-1}p_0^{n+2-2k}$
bicoloured Motzkin paths $P(u) = \frac{1}{u} + 2 + u$	$e_{n+1} = \frac{1}{n+1}\binom{2n}{n}$
Łukasiewicz paths $P(u) = \frac{1}{u} + 1 + u + u^2 + \cdots$	$e_n = \frac{1}{n+1}\binom{2n}{n}$
d-ary trees $P(u) = \frac{1}{u} + u^{d-1}$	$e_{dn+1} = \frac{1}{(d-1)n+1}\binom{dn}{n}$
$\{1,2,\dots,d\}$-ary trees $P(u) = \frac{1}{u} + 1 + \cdots + u^{d-1}$	$e_n = \frac{1}{n}\sum_{j=0}^{\lfloor \frac{n-1}{d+1} \rfloor}(-1)^j\binom{n}{j}\binom{2n-2-j(d+1)}{n-1}$
$\{d,d+1\}$-ary trees $P(u) = \frac{1}{u} + u^{d-1} + u^d$	$e_n = \frac{1}{n}\sum_{k=0}^{\lfloor \frac{n-1}{d} \rfloor}\binom{n}{k}\binom{k}{n-1-dk}$

TABLE 2. Closed-form formulas for some famous families of lattice paths.

Table 2 presents several applications of this Lagrange inversion formula to lattice path enumeration. It leads to the Catalan numbers for Dyck paths, and to the Motzkin numbers for the Motzkin paths, i.e., excursions associated with the step set $\mathcal{S} = \{-1, 0, +1\}$. They are two of the most ubiquitous number sequences in combinatorics, see [48, Ex. 6.19, 6.25, and 6.38] for more information. Table 2 also contains an example of weighted paths (namely weighted Motzkin paths and the special case of bicoloured Motzkin paths), as well as an example with an infinite set of steps (namely the Łukasiewicz paths with all possible steps allowed).

All of the examples in Table 2 are intimately related to families of trees (as suggested by some of the namings in the table). In order to explain this, we recall that an *ordered tree* is a rooted tree for which an ordering of the children is specified for each vertex, and for which its arity (i.e., the outdegree, the number of children of each node) is restricted to be in a subset $\mathcal{A}$ of $\mathbb{N}$.[3] If $\mathcal{A} = \{0, 2\}$, this leads to the classical binary trees counted by the Catalan numbers; if $\mathcal{A} = \{0, 1, 2\}$, this leads to the unary-binary trees counted by Motzkin numbers, and if $\mathcal{A} = \mathbb{N}$, this gives the ordered trees (also called planted plane trees), which are also counted by Catalan numbers. Any ordered tree can be traversed starting from the root in *prefix order*: one starts from the root and proceeds depth-first and left-to-right. The listing of the outdegrees of nodes in prefix order is called the *preorder degree sequence*. This characterizes a tree unambiguously, see Figure 2, and it is best summarized by the following folklore proposition.

Proposition 2.6 (Łukasiewicz correspondence)**.** *Ordered trees are in bijection with Łukasiewicz excursions.*

Proof. Given an ordered tree with n nodes, the preorder sequence can be interpreted as a lattice path. Let $(\sigma_j)_{j=1}^n$ be a preorder degree sequence. With each σ_j we associate a step $(1, \sigma_j - 1) \in \mathbb{N} \times \mathbb{Z}$. Note that, as the minimal degree is 0, our smallest step is -1. Starting at the origin, we concatenate these steps for $j = 1, 2, \ldots, n-1$, ignoring the last step. In this way, we obtain a Łukasiewicz excursion of length $n - 1$. $\square$

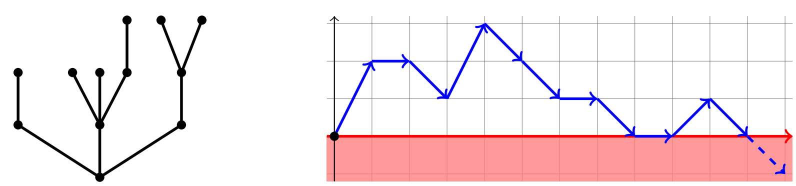


Figure 2. The bijection between trees and Łukasiewicz paths. The preorder degree sequence $(3, 1, 0, 3, 0, 0, 1, 0, 1, 2, 0, 0)$ uniquely characterizes the tree, and gives the corresponding Łukasiewicz path with step sequence $(2, 0, -1, 2, -1, -1, -1, 0, -1, 0, 1, -1, -1)$. Dropping the last -1 step yields an excursion.

As one can see, the combinatorics of the Łukasiewicz paths is well understood (see e.g. [28, 49]), and the true challenge is to analyse lattice paths with other negative steps than just -1. The smallest non-Łukasiewicz cases are the Duchon lattice paths (steps $\mathcal{S} = \{-2, +3\}$), and the Knuth lattice paths (steps $\mathcal{S} = \{-2, +5\}$). Their enumerative and asymptotic properties are the subject of another article in this volume [11]. For these two families of lattice paths, the asymptotics are tricky, because the generating functions involve several dominant singularities. In the next sections, we concentrate on closed formulas which appear for many other non-Łukasiewicz cases.

[3] In this article, by convention $0 \in \mathbb{N}$.

3. (Old-time) Basketball walks: steps $\mathcal{S} = \{-2, -1, +1, +2\}$

FIGURE 3. Since its creation in 1892 by James Naismith (November 6, 1861 – November 28, 1939), the rules of basketball evolved. For example, since 1896, field goals and free throws were counted as two and one points, respectively. The international rules were changed in 1984 so that a "far" field goal was now rewarded by 3 points, while "ordinary" field goals remained at 2 points, a free throw still being worth one point.

We now turn our attention to a class of lattice paths (lattice walks) with rich combinatorial properties: *the basketball walks*. They are constructed from the step set $\mathcal{S} = \{-2, -1, +1, +2\}$. This terminology was introduced by Arvind Ayyer and Doron Zeilberger [5], and these walks were later also considered by Mireille Bousquet-Mélou [18]. They can be seen as the evolution of the score during a(n old-time) basketball game (see Figure 3).

Ayyer, Zeilberger, and Bousquet-Mélou found interesting results on the shape of the algebraic equations satisfied by the excursion generating function, and similar properties when the height of the excursion is bounded. In this article, we analyse a generalization in which the starting point and the end point of the walks do not necessarily have altitude 0. Since, in that case, we lose a natural factorization happening for excursions, we are led to variations of certain parts in the kernel method. In addition, we are interested in closed-form expressions for the number of walks of length n. This is complementary to the results in [8] and in [11]. Moreover, contrary to the previous section, these walks are not Łukasiewicz paths any more. This makes them harder to analyse (the easy bijection with trees is lost, for example). Despite all that, the kernel method will strike again, thus illustrating our main motto:

"The kernel method is the method of choice for problems on directed lattice paths!"

3.1. **Generating functions for positive (old-time) basketball walks: the kernel method.** We define *positive walks* as walks staying strictly above the x-axis, possibly touching it at the first or last step. Returning to the basketball interpretation, these correspond to the evolution of basketball scores where one team (the stronger team, the richer team?) is always ahead of the other team.

Let $G_{j,n,k}$ be the number of such walks running from $(0, j)$ to (n, k), and define by $G_j(z, u)$ the generating function of positive walks starting at $(0, j)$. We write

$$G_j(z,u) := \sum_{n,k\geq 0} G_{j,n,k} z^n u^k = \sum_{n=0}^{\infty} g_{j,n}(u) z^n = \sum_{k=0}^{\infty} G_{j,k}(z) u^k.$$

Similar to Section 2, we shall need the polynomial $g_{j,n}(u)$, the generating function for all walks with n steps, and the series $G_{j,k}(z)$, the generating function for all walks ending at altitude k. The bivariate generating function $G_j(z, u)$ is analytic for $|z| < 1/P(1)$ and $|u| \leq 1$.

A walk is either the single initial point at altitude j, or a walk followed by a step not reaching altitude 0 or below. This leads to the functional equation

$$(1 - zP(u))G_j(z,u) = u^j - z\Big(G_{j,1}(z) + G_{j,2}(z) + G_{j,1}(z)/u\Big), \qquad j > 0, \tag{3.1}$$

where the *step polynomial* $P(u)$ is given by

$$P(u) := u^{-2} + u^{-1} + u + u^2.$$

Again, we call the factor $1 - zP(u)$ on the left-hand side of (3.1) the *kernel* of the equation, and denote it by $K(z, u)$.

We refer to (3.1) as the *fundamental functional equation* for $G_j(z, u)$. The equation has a small problem though: this is *one* equation with *three* unknowns, namely $G_j(z, u), G_{j,1}(z)$, and $G_{j,2}(z)$! The idea of the so-called *'kernel method'* is to equate the kernel $K(z, u)$ to 0, thus binding u and z in such a way that the left-hand side of (3.1) vanishes. This produces two extra equations.

To equate $K(z, u)$ to zero means to put

$$1 - zP(u) = 0 \quad \text{or equivalently} \quad u^2 - zu^2P(u) = 0. \tag{3.2}$$

We call this equation the *kernel equation*. As an equation of degree 4 in u, it has four roots. We call the two small roots (that is, the roots which tend to 0 when z approaches 0) $u_1(z)$ and $u_2(z)$.

Then, on the complex plane slit along the negative real axis, we can identify the small roots $u_1(z)$ and $u_2(z)$ as

$$\begin{aligned}
u_1(z) &= -\frac{1}{4}\left(\frac{z-\sqrt{4z+9z^2}}{z}+\sqrt{\frac{4-6z-2\sqrt{4z+9z^2}}{z}}\right)\\
&= \sqrt{z}+\frac{1}{2}z+\frac{1}{8}z^{3/2}+\frac{1}{2}z^2+\frac{159}{128}z^{5/2}+O(z^3),\\
u_2(z) &= -\frac{1}{4}\left(\frac{z+\sqrt{4z+9z^2}}{z}-\sqrt{\frac{4-6z+2\sqrt{4z+9z^2}}{z}}\right)\\
&= -\sqrt{z}+\frac{1}{2}z-\frac{1}{8}z^{3/2}+\frac{1}{2}z^2-\frac{159}{128}z^{5/2}+O(z^3).
\end{aligned}$$

Moreover, their Puiseux expansions are related via the following proposition.

Proposition 3.1 (Conjugation principle for two small roots)**.** *The small roots u_1 and u_2 of $1 - zP(u) = 0$ satisfy*

$$u_1(z) = \sum_{n\geq 1} a_n z^{n/2} \text{ and } u_2(z) = \sum_{n\geq 1} (-1)^n a_n z^{n/2}.$$

Proof. The kernel equation yields

$$u = X(1+u+u^3+u^4)^{1/2},$$

with $X = z^{1/2}$ or $X = -z^{1/2}$. Since the above equation possesses a unique formal power series solution $u(X)$, the claim follows. □

By substituting the small roots $u_1(z)$ and $u_2(z)$ of the kernel equation (3.2) into the fundamental functional equation (3.1), we see that the left-hand side vanishes. Subsequently, we solve for $G_{j,1}(z)$ and $G_{j,2}(z)$ and get[4]

$$G_{j,1}(z) = -\frac{u_1u_2(u_1^j-u_2^j)}{z(u_1-u_2)}, \qquad j>0, \tag{3.3}$$

$$G_{j,2}(z) = \frac{u_1u_2(u_1^j-u_2^j)+u_1^{j+1}-u_2^{j+1}}{z(u_1-u_2)}, \qquad j>0. \tag{3.4}$$

Substitution in the fundamental functional equation (3.1) then yields

$$G_j(z,u) = \frac{u^j - z(G_{j,1}(z)+G_{j,2}(z)+G_{j,1}(z)/u)}{1-zP(u)}, \qquad j>0. \tag{3.5}$$

[4]In this article, whenever we thought it could ease the reading, without harming the understanding, we write u_1 for $u_1(z)$, or F for $F(z)$, etc.

By means of the kernel method, we have thus derived an explicit expression for the bivariate generating function $G_j(z,u)$ for walks starting at altitude $j > 0$.

In the following proposition, we summarize our findings so far. In addition, we express the generating function for walks from altitude j to altitude k (with $j, k > 0$) explicitly in terms of the small roots $u_1(z)$ and $u_2(z)$, and we also cover the special case $j = 0$, which offers some nice simplifications.

Proposition 3.2. *As before, let $G_{j,k}(z)$ be the generating function for positive basketball walks with steps $-2, -1, +1, +2$ starting at altitude j and ending at altitude k. Furthermore, let $u_1(z)$ and $u_2(z)$ be the small roots of the kernel equation $1 - zP(u) = 0$, with $P(u) = u^{-2} + u^{-1} + u + u^2$. Then, for $j, k > 0$, we have*

$$G_{0,k}(z) = \frac{u_1^{k+1}(z) - u_2^{k+1}(z)}{u_1(z) - u_2(z)}, \tag{3.6}$$

$$G_{j,k}(z) = -\frac{u_1(z)u_2(z)}{z} \sum_{i=0}^{j} \frac{u_1^{j-i+1}(z) - u_2^{j-i+1}(z)}{u_1(z) - u_2(z)} \frac{u_1^{k-i+1}(z) - u_2^{k-i+1}(z)}{u_1(z) - u_2(z)}, \tag{3.7}$$

Proof. We start with the proof of (3.6). The first step of a walk can only be a step of size $+1$ or $+2$. Thus, removing this first step and shifting the origin, we have

$$G_{0,k}(z) = z\left(G_{1,k}(z) + G_{2,k}(z)\right),$$

where $G_{1,k}(z)$ and $G_{2,k}(z)$ are the generating functions for positive walks running from altitude 1 to altitude k, respectively from altitude 2 to altitude k. This decomposition is illustrated in Figure 4.

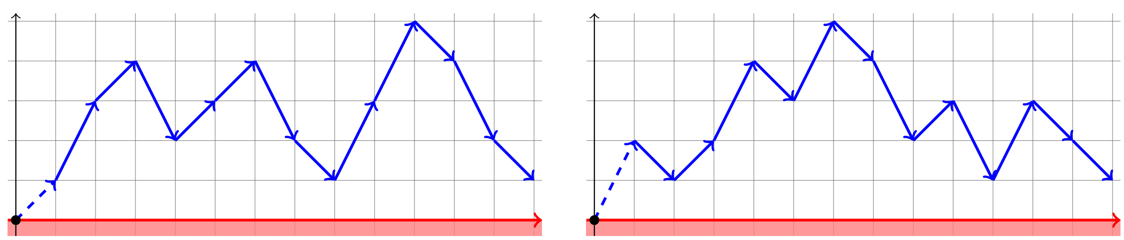


FIGURE 4. Two different instances of walks counted by $G_{0,1}(z)$ showing the two possible first steps $+1$ and $+2$.

By "time reversal" (due to the symmetry of our step set, i.e., $P(u) = P(u^{-1})$), we also have

$$G_{1,k}(z) = G_{k,1}(z), \quad \text{and} \quad G_{2,k}(z) = G_{k,2}(z),$$

where $G_{k,1}(z)$ and $G_{k,2}(z)$ are known from Equations (3.3) and (3.4). Now notice that

$$\begin{aligned} G_{k,2}(z) &= \frac{u_1u_2(u_1^k - u_2^k) + u_1^{k+1} - u_2^{k+1}}{z(u_1 - u_2)} = \frac{u_1u_2(u_1^k - u_2^k)}{z(u_1 - u_2)} + \frac{u_1^{k+1} - u_2^{k+1}}{z(u_1 - u_2)} \\ &= \frac{u_1^{k+1} - u_2^{k+1}}{z(u_1 - u_2)} - G_{k,1}(z). \end{aligned}$$

This leads directly to (3.6).

For computing $G_{j,k}(z)$ with $j, k > 0$, we use a first passage decomposition with respect to minimal altitude of the walk. Combining (3.6) with time reversal, we see that $h_m(z) := \frac{u_1^{m+1} - u_2^{m+1}}{u_1 - u_2}$ is the generating function for basketball walks starting at altitude m, staying always above the x-axis, but ending on the x-axis. Furthermore, by (3.3) with $j = 1$, the series $E(z) = -\frac{u_1u_2}{z}$ is the generating function for excursions (allowed to touch the x-axis). Then the walks from altitude j to altitude k can be decomposed into three sets, as illustrated by Figure 5:

(1) The walk starts at altitude j, and continues until it hits for the first time altitude i (the lowest altitude of the walk, so $1 \leq i \leq j$). This part is counted by $h_{j-i}(z)$.
(2) The second part is the one from that point to the last time reaching altitude i. In other words, this part is an excursion on level i counted by $E(z)$.
(3) The last part runs from altitude i to altitude j without ever returning to altitude i. By time reversal one sees that this is counted by $h_{k-i}(z)$.

Summing over all possible i's, we get (3.7). $\square$

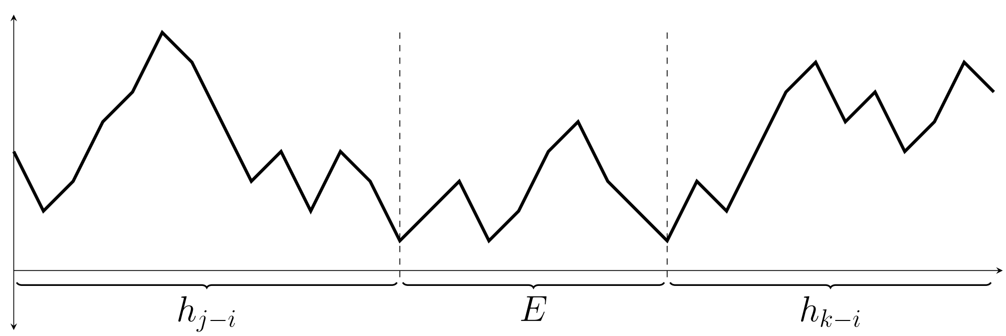


FIGURE 5. The decomposition for $G_{j,k}$

There is an alternative expression for the generating function $G_{j,k}(z)$, which we present in the next proposition.

Proposition 3.3 (Formula for walks from altitude j to altitude k). *Let $u_1(z)$ and $u_2(z)$ be the small roots of the kernel equation $1-zP(u)=0$, with $P(u)=u^{-2}+u^{-1}+u+u^2$, and let $G_{j,k}(z)$ be the generating function for positive basketball walks starting at altitude j and ending at altitude k. Then*

$$G_{j,k}(z)=W_{j-k}+h_j(u_1,u_2)W_{-k}+u_1u_2h_{j-1}(u_1,u_2)W_{-k+1}, \tag{3.8}$$

where

$$W_i(z)=z\left(\frac{u_1'}{u_1^{i+1}}+\frac{u_2'}{u_2^{i+1}}\right)$$

is the generating function of unconstrained walks starting at the origin and ending at altitude i, and

$$h_i(x_1,x_2)=\frac{x_1^{i+1}-x_2^{i+1}}{x_1-x_2}$$

is the complete homogeneous symmetric polynomial of degree i in x_1 and x_2.

Proof. Since $G_{j,k}(z)=G_{k,j}(z)$, without loss of generality we may assume that $j\leq k$. We start with (3.5). Extraction of the coefficient of u^k on the left-hand side gives $G_{j,k}(z)$. As coefficient extraction is linear, we need to find expressions for

$$[u^i]\frac{1}{1-zP(u)}.$$

By (2.2), these are the generating functions $W_i(z)$ for unconstrained walks starting at the origin and ending at altitude i. For basketball walks, we have $P(u)=P(u^{-1})$, hence $W_i(z)=W_{-i}(z)$. Using a straightforward contour integral argument, using Cauchy's integral formula and the residue theorem, we have

$$W_i(z)=[u^i]\frac{1}{1-zP(u)}=\frac{1}{2\pi\sqrt{-1}}\int_C\frac{du}{u^{i+1}(1-zP(u))}=z\left(\frac{u_1'(z)}{u_1^{i+1}(z)}+\frac{u_2'(z)}{u_2^{i+1}(z)}\right).$$

Thus, we obtain the claimed expression for $W_i(z)$ in terms of the small branches. Finally, the remaining factors in (3.8) are obtained by simplifications in (3.5). □

Thus, by (3.6), walks starting at the origin are given by complete homogeneous symmetric polynomials in the small branches. In particular, we have

$$\begin{aligned}G_{0,1}(z)&=u_1(z)+u_2(z),\\ G_{0,2}(z)&=u_1^2(z)+u_1(z)u_2(z)+u_2^2(z).\end{aligned} \tag{3.9}$$

We now derive an explicit expression for $G_{0,1}(z)$ and $G_{0,2}(z)$. Note that, as (3.9) is not defined on the negative real axis, we apply analytic continuation in order to derive an expression which is defined for every $|z|<\frac{1}{4}$, which is the radius of convergence of $G_{0,1}(z)$. The function $G_{0,1}(z)$ is an algebraic function since it is the sum of two algebraic functions (namely, $u_1(z)$ and $u_2(z)$). Using a computer algebra package, it is easy to derive an algebraic equation for $G_{0,1}(z)$. For example, the following *Maple* commands (see [46] for more on these aspects) gives the desired equation:

```
> AllRoots:=allvalues(solve(1-z*P(u),u)):
> u1:=AllRoots[2]:  u2:=AllRoots[3]:
> algeq:=algfuntoalgeq(u1+u2,u(z));
```

$$zu^4 + 2zu^3 + (3z-1)u^2 + (2z-1)u + z. \tag{3.10}$$

In particular, $G_{0,1}(z)$ is uniquely determined by the previous equation and the fact that its expansion at $z = 0$ is a power series with non-negative coefficients. Solving this equation, we arrive at an analytic expression for $G_{0,1}(z)$ for $|z| < 1/4$:

$$\begin{aligned} G_{0,1}(z) &= -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{2-3z-2\sqrt{1-4z}}{z}} \\ &= z + z^2 + 3z^3 + 7z^4 + 22z^5 + 65z^6 + 213z^7 + \cdots . \end{aligned} \tag{3.11}$$

Using a computer algebra package again, we find that $G_{0,2}(z)$ satisfies

$$z^3u^4 - 3z^2u^3 - (z^2-3z)u^2 + (z-1)u + z = 0. \tag{3.12}$$

Among its four branches, only one is a power series at $z = 0$ with non-negative coefficients, namely

$$\begin{aligned} G_{0,2}(z) &= \frac{3-\sqrt{1-4z}-\sqrt{2+12z+2\sqrt{1-4z}}}{4z} \\ &= z + z^2 + 4z^3 + 9z^4 + 31z^5 + 91z^6 + 309z^7 + \cdots . \end{aligned} \tag{3.13}$$

In order to undertake a small digression on complexity of computation: these explicit forms are not the fastest way to access the coefficients. A better way is to take advantage of the theory of holonomic functions (as, e.g., implemented in the *gfun Maple* package, see [46]). To begin with, the kernel method gave us an algebraic equation. Applying the derivative to both sides of this equation and using the obtained new relations, we are led to a linear differential equation satisfied by the function $G(z)$ (where we write $G(z)$ instead of $G_{0,1}(z)$ for short):

```
> diffeq:=algeqtodiffeq(subs(u=G,algeq),G(z),G(0)=0):
```

$$\begin{cases} G(0) = 0, \\ 6\,z + 6 + 12\,(z+1)\,G(z) + 2\,(162\,z^3 + 66\,z^2 + z - 3)\,\frac{d}{dz}G(z) \\ +z\,(9\,z+4)\,(4\,z-1)\,(6\,z+1)\,\frac{d^2}{dz^2}G(z) = 0 \end{cases}$$

Then, extraction of $[z^n]$ on both sides of the differential equation yields a linear recurrence satisfied by the coefficients $g(n)$ of G, namely

```
> rec:=diffeqtorec(diffeq,G(z),g(n)):
```

$$\begin{cases} g(0) = 0, g(1) = 1, g(2) = 1\,, \\ 0 = 108\,n\,(2\,n+1)\,g\,(n) + 6\,\left(13\,n^2 + 35\,n + 24\right)g\,(n+1) \\ -\left(17\,n^2 + 49\,n + 18\right)g\,(n+2) - 2\,(2\,n+7)\,(n+3)\,g\,(n+3)\,. \end{cases}$$

From this recurrence, a binary splitting approach introduced by the Chudnovskys gives a procedure which surprisingly computes $g(n)$ in only $O(\sqrt{n})$ operations (and $O(n \ln n \ln(\ln n))$ bit complexity):

```
> g:=rectoproc(rec,g(n)):
> g(10^5):    #a 6014-digits number computed in only 2 seconds!
```

The same approach applies to all our directed lattice path models. This approach is much faster than the naive approach by means of dynamic programming (which would compute the bivariate generating function, and would then extract the desired $G(z)$ from it: this would cost $O(n^2)$ in time and $O(n^3)$ in memory).

We just saw how to efficiently compute $g(n)$, for any given value of n, but is there a closed-form formula holding for all n at once? We now further investigate this question.

3.2. **How to get a closed form for coefficients: Lagrange–Bürmann inversion.** In Section 4, we present a closed form for the numbers of lattice walks with step polynomial $P(u) = u^{-h} + u^{-h+1} + \cdots + u^{h-1} + u^h$, for any h. In the case $h = 2$ that we are dealing with in the current section, a nice miracle occurs: a more ad hoc approach allows one to derive simpler expressions.

3.2.1. *Closed form for coefficients of $G_{0,1}(z)$.* The generating function $G_{0,1}(z)$ of walks starting at the origin, ending at altitude 1, and never touching the x-axis, satisfies the algebraic equation (3.10). We rewrite it in the form

$$G_{0,1}(z) + G_{0,1}^2(z) = z(1 + G_{0,1}(z) + G_{0,1}^2(z))^2.$$

Here, substitution of $G_{0,1}(z) + G_{0,1}^2(z)$ by $C(z) - 1$ gives the striking equation

$$1 + G_{0,1}(z) + G_{0,1}^2(z) = C(z), \tag{3.14}$$

where $C(z) = 1 + zC(z)^2$ is the generating function for Catalan numbers. A recursive bijection for this identity was found by Axel Bacher and (independently) by Jérémie Bettinelli and Éric Fusy (personal communication, see also [14]). It remains a challenge to find a more direct simple bijection. This identity is the key to get nice closed-form expressions for the coefficients, via the following variant of Lagrange inversion.

Lemma 3.4 (LAGRANGE–BÜRMANN INVERSION VARIANT)**.** *Let $F(z)$ and $H(z)$ be two formal power series satisfying the equations*

$$F(z) = z\phi(F(z)), \qquad\qquad H(z) = z\psi(H(z)),$$

where $\phi(z)$ and $\psi(z)$ are formal power series such that $\phi(0) \neq 0$ and $\psi(0) \neq 0$. Then,

$$[z^n]H(F(z)) = \frac{1}{n}\sum_{k=1}^{n}\left([z^{k-1}]\psi^k(z)\right)\left([z^{n-k}]\phi^n(z)\right). \tag{3.15}$$

Proof. By the Lagrange–Bürmann inversion (Theorem 2.5), we have

$$[z^n]H(F(z)) = \frac{1}{n}[z^{n-1}]H'(z)\phi^n(z).$$

Now we apply the Cauchy product formula $[z^m]A(z)B(z) = \sum_{k=0}^{m} a_k b_{m-k}$ with $m = n-1$, $A(z) = H'(z)$, and $B(z) = \phi^n(z)$. This leads to

$$\begin{aligned}[z^n]H(F(z)) &= \frac{1}{n}\sum_{k=0}^{n-1}\left([z^k]H'(z)\right)\left([z^{n-1-k}]\phi^n(z)\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}\left([z^{k-1}]H'(z)\right)\left([z^{n-k}]\phi^n(z)\right).\end{aligned}$$

This gives Formula (3.15), after observing $[z^{k-1}]H'(z) = k[z^k]H(z) = [z^{k-1}]\psi^k(z)$, where we used Lagrange–Bürmann inversion again. ☐

Proposition 3.5. *The number of basketball walks of length n from the origin to altitude 1 with steps in $\mathcal{S} = \{-2,-1,+1,+2\}$ and never returning to the x-axis equals*

$$\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\binom{2k-2}{k-1}\binom{2n}{n-k} = \frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{2n+1-3i}.$$

Proof. Equation (3.14) implies that $G_{0,1}(z) = H(C(z)-1)$, where $H(z)$ is the functional inverse of the polynomial x^2+x. Thus $H(z) = z\psi(H(z))$, with $\psi(z) = \frac{1}{1+z}$. Furthermore, it is well-known that $C_0(z) := C(z)-1$ satisfies $C_0(z) = z\phi(C_0(z))$ with $\phi(z) = (1+z)^2$. Hence, Equation (3.15) yields

$$\begin{aligned}[z^n]G_{0,1}(z) &= \frac{1}{n}\sum_{k=1}^{n}\left([z^{k-1}]\frac{1}{(1+z)^k}\right)\left([z^{n-k}](1+z)^{2n}\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\binom{2k-2}{k-1}\binom{2n}{n-k}.\end{aligned}$$

The alternative expression without the $(-1)^{k+1}$ factors comes from Formula (3.9), to which we apply the Lagrange–Bürmann inversion formula for u_1, remembering that u_1 satisfies $u^2 = zu^2P(u)$, and that the conjugation property of the small roots from Proposition 3.1 holds:

$$[z^n]G_{0,1}(z) = [z^n](u_1(z) + u_2(z)) = 2[z^n]u_1(z) = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{2n+1-3k}. \qquad \square$$

The last closed-form expression can also be explained via the so-called cycle lemma (cf. [48, Ex. 5.3.8]). Namely, by (2.2) combined with the factorization $u^{-2}+u^{-1}+u+u^2 = u^{-2}(1+u^3)(1+u)$, the number of unrestricted walks from 0 to 1 in n steps is given by

$$[u^1z^n]W(z,u) = [u^1]P(u)^n = [u^1]\left(\frac{(1+u^3)(1+u)}{u^2}\right)^n = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{2n+1-3i}.$$

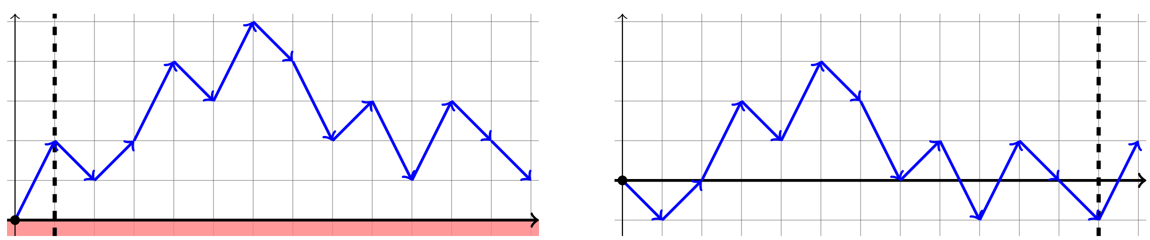


FIGURE 6. Transforming a walk counted by $G_{0,1}(z)$ into a walk counted by $W_{0,1}(z)$.

From the formulas, we see that $[z^n]G_{0,1}(z) = \frac{1}{n}[z^n]W_{0,1}(z)$. There exists indeed a 1-to-n correspondence between walks counted by $G_{0,1}(z)$ and those counted by $W_{0,1}(z)$. For each walk ω counted by $G_{0,1}(z)$, decompose ω into $\omega = \omega_\ell B \omega_r$ where B is any point in the walk. A new walk ω' counted by $W_{0,1}(z)$ is constructed by putting B at the origin and adjoining ω_ℓ at the end of ω_r, i.e., $\omega' = B\omega_r\omega_\ell$, see Figure 6. If ω is of length n, then there are n choices for B. All these walks are different because there are no walks from altitude 0 to altitude 1 which are the concatenation of several copies of one and the same walk. (This is not true for walks from altitude 0 to altitude 2. For example, the walk $(0,2,1,3,2)$ is the concatenation of two copies of the walk $(0,2,1)$.)

Conversely, given a walk τ of length n counted by $W_{0,1}(z)$, we decompose τ into $\tau = \tau_\ell B \tau_r$, where B is the right-most minimum of τ. Then, $\tau' = B\tau_r\tau_\ell$ is a walk of length n counted by $G_{0,1}(z)$.

3.2.2. *Closed form for the coefficients of $G_{0,2}(z)$.* Recall that, by means of the kernel method, we derived a closed form expression for the generating function $G_{0,2}(z)$ in (3.13).

Proposition 3.6. *The number of basketball walks of length n from the origin to altitude 2 with steps in $\mathcal{S} = \{-2, -1, +1, +2\}$ and never returning to the x-axis equals*

$$\frac{1}{2n+1} \sum_{k=0}^{n+1} (-1)^{n+k+1} \binom{2n+1}{n+k} \binom{n+2k-1}{k}.$$

Proof. We define the series $F(z)$ by

$$-\frac{1}{F(z)} = G_{0,2}(z) - \frac{1}{z}. \tag{3.16}$$

It is straightforward to see from this equation that $F(z) = z + z^3 + \cdots$. The equation (3.12) translates into the equation

$$(F^3(z) - zF(z))(1 + F(z)) + z^2 = 0$$

for $F(z)$. We may rewrite this equation in the form

$$\left(F^2 - \frac{z}{2}\right)^2 = \frac{z^2}{4} \cdot \frac{1 - 3F(z)}{1 + F(z)}.$$

Next we take the square root on both sides. In order to decide the sign, we have to observe that $F^2(z) = z^2 + \cdots$, hence

$$F^2(z) - \frac{z}{2} = -\frac{z}{2}\sqrt{\frac{1 - 3F(z)}{1 + F(z)}},$$

or, equivalently, $F(z)$ satisfies $F^2(z) = zB(F(z))$, where

$$B(z) = \frac{1}{2}\left(1 - \sqrt{\frac{1-3z}{1+z}}\right).$$

It is straightforward to verify that $B(z)$ satisfies the equation $B(z) = zA(B(z))$ with $A(z) = \frac{1}{1-z} - z$, and it is the only power series solution of this equation. Hence, for $n \geq 1$, by (3.16), Lagrange–Bürmann inversion (Theorem 2.5) with $H(z) = z^{-1}$, we have

$$[z^n]G_{0,2}(z) = -[z^n]\frac{1}{F(z)} = \frac{1}{n}[z^{n-1}]z^{-2}\left(\frac{B(z)}{z}\right)^n = \frac{1}{n}[z^{2n+1}]B^n(z).$$

Now we apply Lagrange–Bürmann inversion again, this time with $F(z)$ replaced by $B(z)$, n replaced by $2n+1$, and $H(z) = z^n$. This yields

$$\begin{aligned}
[z^n]G_{0,2}(z) &= \frac{1}{n(2n+1)}[z^{2n}]nz^{n-1}A^{2n+1}(z) \\
&= \frac{1}{2n+1}[z^{n+1}]\left(\frac{1}{1-z} - z\right)^{2n+1}.
\end{aligned}$$

By applying the binomial theorem, we then obtain

$$[z^n]G_{0,2}(z) = \frac{1}{2n+1}[z^{n+1}]\sum_{k=0}^{2n+1}(-1)^{k+1}\binom{2n+1}{k}z^{2n+1-k}\left(\frac{1}{1-z}\right)^k.$$

Since

$$\left(\frac{1}{1-z}\right)^k = \sum_{\ell\geq 0}\binom{k+\ell-1}{\ell}z^\ell,$$

we get

$$\begin{aligned}
[z^n]G_{0,2}(z) &= \frac{1}{2n+1}[z^{n+1}]\sum_{\ell\geq 0}\sum_{k=0}^{2n+1}(-1)^{k+1}\binom{2n+1}{k}\binom{k+\ell-1}{\ell}z^{2n+1-k+\ell} \\
&= \frac{1}{2n+1}\sum_{k=n}^{2n+1}(-1)^{k+1}\binom{2n+1}{k}\binom{2k-n-1}{k-n} \\
&= \frac{1}{2n+1}\sum_{k=0}^{n+1}(-1)^{n+k+1}\binom{2n+1}{n+k}\binom{n+2k-1}{k},
\end{aligned}$$

as desired. □

The idea of the above proof was to "build up" a chain of dependencies between the actual series of interest, $G_{0,2}(z)$, and several auxiliary series, namely the series $F(z)$, $B(z)$, and $A(z)$, so that repeated application of Lagrange–Bürmann inversion could be applied to provide an explicit expression for the coefficients of the series of interest. This raises the question whether this example is just a coincidence, or whether there exists a general method to transform a power series into a Laurent series with the same positive part, and a "nice" algebraic expression, allowing multiple Lagrange–Bürmann inversions to get "nice" closed forms for the coefficients. We have no answer to this question and therefore leave this to future research.

3.2.3. *Closed form for the coefficients of basketball excursions.* Here, we enumerate basketball *excursions*, that is, basketball walks which start at the origin, return to altitude 0, and in between do not pass below the x-axis. A main difference to the previously considered *positive* basketball walks is that the excursions are allowed to touch the x-axis anywhere.

Proposition 3.7 (Enumeration of basketball excursions). *The number of basketball walks with steps in $\mathcal{S} = \{-2, -1, +1, +2\}$ of length n from the origin to altitude 0 never passing below the x-axis is*

$$e_n := \frac{1}{n+1} \sum_{k=0}^{n} (-1)^{n+k} \binom{2n+2}{n-k} \binom{n+2k+1}{k} = \frac{1}{n+1} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{2n+2}{i} \binom{n-i-1}{n-2i}. \tag{3.17}$$

Remark. The first few values of the sequence defined by (3.17) are

$$1, 0, 2, 2, 11, 24, 93, 272, 971, 3194, 11293, 39148, 139687, 497756, \dots$$

Proof of Proposition 3.7. By the kernel method, we know that the generating function for excursions, $E(z)$ say, is given by $E(z) = -\frac{u_1 u_2}{z}$, and that it satisfies the algebraic equation

$$z^4 E^4 - (2z^3 + z^2) E^3 + (3z^2 + 2z) E^2 - (2z+1) E + 1 = 0\,.$$

Among the branches of this algebraic equation, only one has a power series expansion. The equation may be rewritten in the form

$$zE(z) = z \left(\frac{1}{(1 - zE(z))^2} - \frac{2zE(z)}{1 - zE(z)} + z^2 E^2(z) \right) = z \left(\frac{1}{1 - zE(z)} - zE(z) \right)^2.$$

This shows that we may apply Lagrange–Bürmann inversion (Theorem 2.5) with $\phi(z) = (\frac{1}{1-z} - z)^2$. So we have

$$\begin{aligned} [z^n] E(z) &= \frac{1}{n+1} [z^n] \phi^{n+1}(z) = \frac{1}{n+1} [z^n] \left(\frac{1}{1-z} - z \right)^{2n+2} \\ &= \frac{1}{n+1} \sum_{k=0}^{n} (-1)^{n+k} \binom{2n+2}{n-k} \binom{n+2k+1}{k}. \end{aligned}$$

It is possible to get an expression involving only positive summands by making use of the rewriting $\phi(z) = (1 + \frac{z^2}{1-z})^2$. This leads to (3.17). $\square$

The trick used in this proof can in fact be translated into an algorithm of wider use:

`The "Lagrangean scheme" algorithm`

`input:` an algebraic power series (given in terms of its algebraic equation $P(z, F) = 0$, plus the first terms of the expansion of F, so that we can uniquely identify the correct branch of the equation)

`output:` a "Lagrangean equation" satisfied by F
(i.e., $H(z^a F) = z\phi(z^a F)$, where $z^a F$ has valuation[a] 1.)

`way to process:` if we assume that $H = H_1/H_2$ and $\phi = \phi_1/\phi_2$ are rational functions, then we identify them via an indeterminate coefficient approach, by substituting the *polynomials* H_1, H_2, ϕ_1, ϕ_2 in the equation $P(z, F) = 0$.

[a]The valuation of a power series $\sum_{n\geq 0} f_n z^n$ is the least n such that $f_n \neq 0$.

This algorithm therefore provides a way to get multiple-binomial-sum representations. See [17,25,50] for other approaches not relying on the algebraic nature of F, but designed for the class of functions which can be written as diagonals of rational functions (these two classes coincide in the bivariate case). For example, Formula (3.17) for e_n has the following alternative representation:

$$(n+1)e_n = [t^n]\,\mathrm{diagonal}\left(\frac{(1+u)^6 u t^2}{1-(u(u+1)^2 t + u(1+u)^4 t^2)} + (u+1)^2\right).$$

The rational function on the right-hand side has the striking feature that its bivariate series expansion has only non-negative coefficients. In fact, it is even a bivariate $\mathbb{N}$-rational function (i.e., a function obtained as iteration of addition, multiplication, and quasi-inverse,[5] starting from polynomials in u and t with positive integer coefficients). Given a multivariate rational function, it is a hard task to write it as an $\mathbb{N}$-rational expression (an algorithm is known in the univariate case), so some human computations were needed here to get the above expression.

In fact (and we believe that it was not observed before), these multivariate rational functions appearing in the computation of diagonals related to nested sums of binomials are always $\mathbb{N}$-rational: this follows from the closure properties of $\mathbb{N}$-rational functions. It is an open question to give a combinatorial interpretation (in terms of the initial structure counted by the diagonal) of the other diagonals of this rational function. It is also not easy to extrapolate from this rational function a general pattern which could appear for more general sets of steps: we shall see in Section 4 which type of formulas generalize the rich combinatorics that we had for $P(u) = u^{-2} + u^{-1} + u + u^2$.

[5]The *quasi-inverse* of a power series $f(z)$ of positive valuation is $1/(1 - f(z))$.

3.3. **How to derive the corresponding asymptotics: singularity analysis.** We close this section by briefly addressing how to find the asymptotics of numbers of basketball walks. Indeed, standard techniques from singularity analysis suffice to get the asymptotic growth of the coefficients of z^n in the generating functions that we consider here for $n \to \infty$. The interested reader is referred to [28] for more details on this subject (see Figure VI.7 therein for an illustration of singularity analysis).

Theorem 3.8. *Let $G_{0,1}(z)$ and $G_{0,2}(z)$ be the generating functions for positive basketball walks with steps $-2,-1,+1,+2$ starting at the origin and ending at altitude 1, respectively at 2. Then, as $n \to \infty$, the coefficients are asymptotically equal to*

$$[z^n]G_{0,1}(z) = \frac{1}{\sqrt{5\pi}} \frac{4^n}{n^{3/2}} \left(1 - \frac{81}{200}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right),$$
$$[z^n]G_{0,2}(z) = \frac{5+\sqrt{5}}{10\sqrt{\pi}} \frac{4^n}{n^{3/2}} \left(1 - \frac{201+24\sqrt{5}}{200}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Proof. The asymptotic growth of the coefficients is governed by the location of the dominant singularity (the singularity closest to the origin). The dominant singularity of (3.11) and (3.13) is given by $1/4$, since the square root becomes singular at this point.

Next, we compute the singular expansion for $z \to 1/4$, which is a Puiseux series:

$$G_{0,1}(z) = -\frac{1-\sqrt{5}}{2} - \frac{2}{\sqrt{5}}\sqrt{1-4z} + O\left(1-4z\right),$$
$$G_{0,2}(z) = \left(3-\sqrt{5}\right) - \frac{5+\sqrt{5}}{5}\sqrt{1-4z} + O\left(1-4z\right).$$

Finally, we apply the standard function scale from [28, Theorem VI.1] and the transfer for the error term [28, Theorem VI.3] to get the asymptotics. □

More generally, asymptotics for the number of walks from altitude i to altitude j in n steps can be obtained via singularity analysis of the small roots, similarly to what was done in [8]. Note that it is easy derive as many terms as needed in the asymptotic expansion of the coefficients by including more terms in the Puiseux expansion. We also want to point out that this process was implemented in *SageMath* (see [31]) or in *Maple* by Bruno Salvy (as a part of the `algolib` package). There, the `equivalent` command directly gives the above result:

```
> equivalent(G01,z,n,3);
```

$$\frac{1}{5}\frac{\sqrt{5}\,4^n}{\sqrt{\pi}\,n^{3/2}} - \frac{81}{1000}\frac{\sqrt{5}\,4^n}{\sqrt{\pi}\,n^{5/2}} + O\left(\frac{4^n}{n^{7/2}}\right).$$

4. General case: Lattice walks with arbitrary steps

We first prove a theorem which holds for any symmetric set of steps, i.e., when the step polynomial satisfies $P(u) = P(1/u)$.

Theorem 4.1 (Positive walk enumeration). *Consider walks with a symmetric step polynomial $P(u)$. Let $G_{0,k}(z)$ be the generating function for positive walks, i.e., walks starting at the origin, ending at altitude k, and always staying strictly above the x-axis in-between, and let $M_{>0}(z)$ be the generating function of positive meanders, i.e., positive walks ending at any altitude > 0. Then*

$$M_{>0}(z) = \sum_{k>0} G_{0,k}(z) = \prod_{i=1}^{h} \frac{1}{1-u_i(z)},$$
$$G_{0,k}(z) = h_k\left(u_1(z), u_2(z), \ldots, u_h(z)\right),$$

where $u_1(z), u_2(z), \ldots, u_h(z)$ are the small roots of the kernel equation $1 - zP(u) = 0$, and

$$h_k(x_1, x_2, \ldots, x_h) = \sum_{\substack{i_1,\ldots,i_h \geq 0 \\ i_1+\cdots+i_h=k}} x_1^{i_1} x_2^{i_2} \cdots x_h^{i_h}$$

is the complete homogeneous symmetric polynomial of degree k in the variables $x_1, x_2, \ldots, x_h$.

Proof. The formula for positive meanders follows from the expression for meanders (which are allowed to touch the x-axis!) in [8, Corollary 1],

$$M_{\geq 0}(z) = -\frac{1}{z} \prod_{i=1}^{h} \frac{1}{1-v_i(z)},$$

where $v_1(z), v_2(z), \ldots, v_h(z)$ are the large roots of $1 - zP(u) = 0$, i.e., those roots $v(z)$ for which $\lim_{z\to 0} |v(z)| = \infty$. Every meander starts with an initial excursion, and later never returns to the x-axis any more. This simple fact implies the generating function equation $M_{\geq 0}(z) = E(z)M_{>0}(z)$. Hence, we need to divide the above expression for $M_{\geq 0}(z)$ by the generating function for excursions — which, by [8, Theorem 2], is given by

$$E(z) = \frac{(-1)^{h-1}}{z} \prod_{i=1}^{h} u_i(z).$$

Finally, due to $P(u) = P(u^{-1})$, we have $u_i(z) = 1/v_i(z)$, which gives the final expression for $M_{>0}$, while the formula for $G_{0,k}(z)$ is proven in [10]. □

This proof shows, in particular, that generating functions for strictly positive walks, respectively for weakly positive walks, are intimately related, and are therefore given by similar expressions. (The price of positivity is a division by $E(z)$, which encodes the excursion prefactor.) The proof also extends to non-symmetric steps, but then the formulas involve one more factor. It is possible to deal with them exactly in the way we proceed for symmetric steps, but this leads to slightly less nice formulas.

In the sequel, we focus on positive walks with symmetric steps. We show in which way we can use the obtained expressions for the generating functions in order to get nice closed-form expressions for their coefficients.

4.1. **Counting walks with steps in $\mathcal{S} = \{0, \pm 1, \dots, \pm h\}$.** In Section 3 on basketball walks, we had a taste of what the kernel method could do for us when combined with Lagrange–Bürmann inversion. This was, however, only for the case $\mathcal{S} = \{\pm 1, \pm 2\}$. In this section, we illustrate again the power of the kernel method, when applied to more general step sets $\mathcal{S}$. We first start with a generalization of Section 2 to $\mathcal{S} = \{0, \pm 1, \dots, \pm h\}$. In order to have a convenient notation, we introduce m*-nomial coefficients* by defining

$$\binom{n}{k}_m := [u^k](1 + u + \cdots + u^{m-1})^n\,,$$

where k is between 0 and $(m-1)n$.

Proposition 4.2. *The m-nomial coefficient equals*

$$\binom{n}{k}_m = \sum_{i=1}^{n} (-1)^i \binom{n}{i} \binom{n+k-mi-1}{n-1}. \tag{4.1}$$

Proof. Coefficient extraction in the defining expression for $\binom{n}{k}_m$ yields

$$\begin{aligned}
\binom{n}{k}_m &= [u^k](1 + u + \cdots + u^{m-1})^n = [u^k](1-u^m)^n \frac{1}{(1-u)^n} \\
&= [u^k] \left(\sum_{i=0}^{n} \binom{n}{i} (-1)^i u^{mi} \right) \left(\sum_{j \ge 0} \binom{n+j-1}{n-1} u^j \right) \\
&= \sum_{i=0}^{\lfloor (n+k-1)/m \rfloor} (-1)^i \binom{n}{i} \binom{n+k-mi-1}{n-1}.
\end{aligned}$$

The upper bound in the sum can be taken more naturally to be $i = n$, using the convention that binomials $\binom{n}{k}$ are 0 for $n < 0$ or $k > n$ (the reader should be warned that this is not the convention of *Maple* or *Mathematica*). This gives Formula (4.1). $\square$

Historical remark. These m-nomial coefficients appear in more than fifty articles (many of them focusing on trinomial coefficients) dealing with their rich combinatorial aspects (see e.g. [2, 4, 12, 15]). We use the notation $\binom{n}{k}_m$ promoted by George Andrews [3]. It should be noted that they were previously called *polynomial coefficients* by Louis Comtet [23, p. 78], who is mentioning early work of Désiré André (with a typo in the date) and Paul Montel [1, 43], and who was himself using another notation for these numbers, namely $\binom{n,m}{k}$.

These coefficients have a direct combinatorial interpretation in terms of lattice walk enumeration.

Theorem 4.3 (Unconstrained walk enumeration). *The number of unconstrained[6] walks running from the origin to altitude k in n steps taken from $\{0, \pm 1, \pm 2, \ldots, \pm h\}$ equals $\binom{n}{k+hn}_{2h+1}$.*

Proof. By (2.2), the generating function for unconstrained walks is

$$W(z,u) = \frac{1}{1 - zP(u)} = \sum_{n=0}^{\infty} P^n(u) z^n .$$

Then a simple factorization shows that

$$[u^k]P^n(u) = [u^k] \left(\sum_{i=-h}^{h} u^i \right)^n = [u^k] u^{-hn} \left(\sum_{i=0}^{2h} u^i \right)^n = \binom{n}{k+hn}_{2h+1}. \qquad \square$$

Now we will see how to link these coefficients with *constrained* lattice walks. To this end, we first state the general version of the conjugation principle that we encountered in Proposition 3.1.

Proposition 4.4 (Conjugation principle for small roots). *Let*

$$P(u) = \sum_{i=-c}^{d} p_i u^i$$

be the step polynomial, and let $\omega = e^{2\pi i/c}$ be a c-th root of unity. The small roots $u_i(z)$, $i = 1, 2, \ldots, c$, of $1 - zP(u) = 0$ satisfy

$$u_i(z) = \sum_{n \geq 1} \omega^{n(i-1)} a_n z^{n/c}$$

for certain "universal" coefficients a_n, $n = 1, 2, \ldots$.

[6] Unconstrained means that the walks are allowed to have both positive and negative altitudes.

Proof. The kernel equation yields

$$u = X\left(p_{-c} + p_{-c+1}u + p_{-c+2}u^2 + \cdots + p_{d-1}u^{c+d-1} + p_d u^{c+d}\right)^{1/c},$$

with $X = \omega^j z^{1/c}$ for $j = 0, 1, \ldots, c-1$. Since the above equation possesses a unique formal power series solution $u(X)$, the claim follows. $\square$

Next, we apply Lagrange–Bürmann inversion to the small roots given by the kernel method, and combine it with the conjugation principle.

Proposition 4.5 (Explicit expansion of the roots u_i). *For lattice walks with step polynomial given by $P(u) = u^{-h} + u^{-h+1} + \cdots + u^{h-1} + u^h$, let $U(z)$ be the root of $1 - z^h P(U) = 0$ whose Taylor expansion at 0 starts $U(z) = z + \cdots$. The series $U(z)$ is a power series, not a genuine Puiseux series. Then all small and large roots can be expressed in terms of $U(z)$, namely we have*

$$u_i(z) = U(\omega^{i-1} z^{1/h}) \quad \text{and} \quad v_i(z) = 1/U(\omega^{i-1} z^{1/h}), \quad i = 1, 2, \ldots, h,$$

where $\omega = e^{2\pi i/h}$ is a primitive h-th root of unity. The expansion of a power of the series $U(z)$ is explicitly given by

$$U^m(z) = \sum_{n=m}^{\infty} \frac{m}{n} \binom{n/h}{n-m}_{2h+1} z^n.$$

Proof. We want to solve $1 - zP(u) = 0$ for u. We may rewrite this equation as

$$z = \frac{u^h}{1 + u + \cdots + u^{2h}}.$$

Taking the h-th root, we get

$$\omega^{i-1} z^{1/h} = \frac{u}{(1 + u + \cdots + u^{2h})^{1/h}},$$

for some i with $1 \leq i \leq h$.

Since an equation of the form $Z = u\phi(u)$, where $\phi(u)$ is a power series in u, has a unique power series solution $u(Z)$, the above equation has a unique solution $u_i(z)$, which turns out to have exactly the form described in the proposition. The equation for v_i follows from $u_i = 1/v_i$ as we have $P(u) = P(1/u)$.

The equation for U^m comes from Lagrange–Bürmann inversion:

$$\begin{aligned}[z^n]U^m(z) &= \frac{1}{n}[z^{-1}](z^m)'P^{n/h}(z)\\ &= \frac{m}{n}[z^{-m}]\sum_k z^k\binom{n/h}{k+n}_{2h+1}\\ &= \frac{m}{n}\binom{n/h}{n-m}_{2h+1}.\end{aligned}$$ ☐

Theorem 4.6 (Closed-form expression for walks with $\mathcal{S}=\{0,\pm1,\dots,\pm h\}$). *The numbers of positive walks and meanders from the origin to altitude k in n steps from $\mathcal{S}=\{0,\pm1,\dots,\pm h\}$ admit the closed-form expressions*

$$\begin{aligned}[z^n]G_{0,k}(z) = \sum_{n_1+\dots+n_h=nh}\ \sum_{i_1+\dots+i_h=k} &\frac{i_1}{n_1}\binom{n_1/h}{n_1-i_1}_{2h+1}\\ &\cdots\frac{i_h}{n_h}\binom{n_h/h}{n_h-i_h}_{2h+1}\omega^{\sum_{j=1}^h (j-1)n_j},\end{aligned}$$

$$[z^n]M_{>0}(z) = \sum_{n_1+\dots+n_h=nh}\ \sum_{i_1,\dots,i_h\geq 0}\frac{i_1}{n_1}\binom{n_1/h}{n_1-i_1}_{2h+1}\cdots\frac{i_h}{n_h}\binom{n_h/h}{n_h-i_h}_{2h+1}\omega^{\sum_{j=1}^h (j-1)n_j}.$$

Proof. We use the expansions from Proposition 4.5 in the generating function formulas from Theorem 4.1. ☐

Here are some sequences of numbers of positive walks with steps $\mathcal{S}=\{0,\pm1,\dots,\pm h\}$, starting at the origin, and ending at altitude 1, for different values of h:

$h=1$ (A168049)[7] : $0,1,1,2,4,9,21,51,127,323,835,\dots$
$h=2$ (A104632) : $0,1,2,6,20,73,281,1125,4635,19525,83710,\dots$
$h=3$ (A276902) : $0,1,3,12,56,284,1526,8530,49106,289149,1733347,\dots$
$h=4$ (A277920) : $0,1,4,20,120,780,5382,38638,285762,2162033,16655167,\dots$

[7]Axxxxxx refers to the corresponding sequence in the On-Line Encyclopedia of Integer Sequences, available electronically at https://oeis.org.

Furthermore, here are some sequences of numbers of positive walks with steps $\mathcal{S} = \{0, \pm 1, \ldots, \pm h\}$, starting at the origin, and ending at altitude 2, for small values of h:

$h = 1$ (A105695) : $0, 0, 1, 2, 5, 12, 30, 76, 196, 512, 1353, \ldots$

$h = 2$ (A276903) : $0, 1, 2, 7, 25, 96, 382, 1567, 6575, 28096, 121847, 534953, \ldots$

$h = 3$ (A276904) : $0, 1, 3, 14, 68, 358, 1966, 11172, 65104, 387029, 2337919, \ldots$

$h = 4$ (A277921) : $0, 1, 4, 23, 142, 950, 6662, 48420, 361378, 2753687, 21334313, \ldots$

Here are the corresponding sequences for positive meanders:

$h = 1$ (A005773) : $1, 1, 2, 5, 13, 35, 96, 267, 750, 2123, 6046, 17303, \ldots$

$h = 2$ (A278391) : $1, 2, 7, 29, 126, 565, 2583, 11971, 56038, 264345, \ldots$

$h = 3$ (A278392) : $1, 3, 15, 87, 530, 3329, 21316, 138345, 906853, \ldots$

$h = 4$ (A278393) : $1, 4, 26, 194, 1521, 12289, 101205, 844711, 7120398, \ldots$

Here are the corresponding sequences for meanders (allowed to touch 0):

$h = 1$ (A005773) : $1, 2, 5, 13, 35, 96, 267, 750, 2123, 6046, 17303, 49721, \ldots$

$h = 2$ (A180898) : $1, 3, 12, 51, 226, 1025, 4724, 22022, 103550, 490191, \ldots$

$h = 3$ (A180899) : $1, 4, 22, 130, 803, 5085, 32747, 213419, 1403399, \ldots$

$h = 4$ (A180900) : $1, 5, 35, 265, 2100, 17075, 141246, 1182719, 9994086, \ldots$

Here are the corresponding sequences for excursions:

$h = 1$ (A001006) : $1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, \ldots$

$h = 2$ (A104184) : $1, 1, 3, 9, 32, 120, 473, 1925, 8034, 34188, 147787, 647141, \ldots$

$h = 3$ (A204208) : $1, 1, 4, 16, 78, 404, 2208, 12492, 72589, 430569, 2596471, \ldots$

$h = 4$ (A204209) : $1, 1, 5, 25, 155, 1025, 7167, 51945, 387000, 2944860, \ldots$

Remark. Most of the above sequences for $h \geq 3$ were not contained in the On-Line Encyclopedia of Integer Sequences (OEIS) before we added them. In Section 5, we discuss the combinatorial structures related to the sequences which were already in the OEIS.

4.2. **Counting walks with steps in $\mathcal{S} = \{\pm 1, \ldots, \pm h\}$.** In this Section 4.2, we consider the same steps as in the previous one, except that we drop the 0-step.

Certainly, for any type of walks consisting of k steps with 0-step included, enumerated by f_k say, the number of walks of the same type consisting of n steps, all of which different from the 0-step, can be obtained by the inclusion-exclusion principle. The result is $\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_k$.

Here, our way to derive the corresponding formulas is more ad hoc and relies on the shape of the considered steps in $\mathcal{S}$. This offers the advantage of leading to positive sum formulas, as opposed to the alternating sums produced by inclusion-exclusion. For convenience, we introduce the mock-m-nomial coefficients by

$$\binom{n}{k}^*_{2m} := [u^k](1+\cdots+u^{m-1}+u^{m+1}+\cdots+u^{2m})^n\,.$$

Proposition 4.7. *The mock-m-nomial coefficients can be expressed in terms of the (ordinary) m-nomial coefficients in the form*[8]

$$\binom{n}{k}^*_{2m} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{k-(m+1)i}_m.$$

Proof. Factoring the expression and extracting coefficients, we obtain

$$\begin{aligned}\binom{n}{k}^*_{2m} &= [u^k](1+\cdots+u^{m-1}+u^{m+1}+\cdots+u^{2m})^n\\ &= [u^k](1+u^{m+1})^n(1+u+\cdots+u^{m-1})^n\\ &= [u^k]\left(\sum_{i\geq 0}\binom{n}{i}u^{(m+1)i}\right)\left(\sum_{j\geq 0}\binom{n}{j}_m u^j\right)\\ &= \sum_{i=0}^{n}\binom{n}{i}\binom{n}{k-(m+1)i}_m.\end{aligned}$$

□

These mock-m-nomial coefficients have also a direct combinatorial interpretation in terms of lattice walk enumeration.

Theorem 4.8 (Unconstrained walk enumeration)**.** *The mock-m-nomial coefficient $\binom{n}{k+hn}^*_{2h}$ is the number of unconstrained walks running from 0 to k in n steps taken from $\{\pm 1, \pm 2, \ldots, \pm h\}$.*

Proof. We have

$$\begin{aligned}[u^k]P^n(u) &= [u^k]\left(\sum_{i=-h}^{-1}u^i+\sum_{i=1}^{h}u^i\right)^n\\ &= [u^k]u^{-hn}\left(\sum_{i=0}^{h-1}u^i+\sum_{i=h+1}^{2h}u^i\right)^n = \binom{n}{k+hn}^*_{2h}.\end{aligned}$$

□

[8] Here, the * is a mnemonic to remind us that we do not have the 0 step.

Proposition 4.9 (Explicit expansion of the roots u_i)**.** *For lattice walks with step polynomial given by $P(u) = u^{-h} + \cdots + u^{-1} + u^1 + \cdots + u^h$, let $U(z)$ be the root of $1 - z^h P(U) = 0$ whose Taylor expansion at 0 starts $U(z) = z + \cdots$. Again, $U(z)$ is a power series, not a genuine Puiseux series. Then $U(z)$ satisfies*

$$U^m(z) = \sum_{n=1}^{\infty} \frac{m}{n} \binom{n/h}{n-m}_{2h}^{*} z^n,$$

and all small and large roots are expressed in terms of $U(z)$ as

$$u_i(z) = U(\omega^{i-1} z^{1/h}) \quad \text{and} \quad v_i(z) = 1/U(\omega^{i-1} z^{1/h}), \quad \text{for } i = 1, 2, \ldots, h,$$

where $\omega = e^{2\pi i/h}$ is a primitive h-th root of unity.

Proof. We apply Lagrange–Bürmann inversion to get

$$[z^n]U^m(z) = \frac{1}{n}[z^{-1}](z^m)' P^{n/h}(z) = \frac{m}{n}[z^{-m}] \sum_k u^k \binom{n/h}{k+n}_{2h}^{*} = \frac{m}{n} \binom{n/h}{n-m}_{2h}^{*}. \qquad \square$$

Theorem 4.10 (Closed-form expression for walks with $\mathcal{S} = \{\pm 1, \ldots, \pm h\}$)**.** *The numbers of positive walks and meanders from the origin to altitude k in n steps from $\mathcal{S} = \{\pm 1, \ldots, \pm h\}$ admit the closed-form expressions*

$$[z^n]G_{0,k}(z) = \sum_{n_1 + \cdots + n_h = nh} \; \sum_{i_1 + \cdots + i_h = k} \frac{i_1}{n_1} \binom{n_1/h}{n_1 - i_1}_{2h}^{*} \cdots \frac{i_h}{n_h} \binom{n_h/h}{n_h - i_h}_{2h}^{*} \omega^{\sum_{j=1}^{h} (j-1) n_j},$$

$$[z^n]M_{>0}(z) = \sum_{n_1 + \cdots + n_h = nh} \; \sum_{i_1, \ldots, i_h \geq 0} \frac{i_1}{n_1} \binom{n_1/h}{n_1 - i_1}_{2h}^{*} \cdots \frac{i_h}{n_h} \binom{n_h/h}{n_h - i_h}_{2h}^{*} \omega^{\sum_{j=1}^{h} (j-1) n_j}.$$

Proof. We use the expansions from Proposition 4.9 in the generating function formulas from Theorem 4.1. $\square$

Here are some sequences of numbers of walks with steps in $\mathcal{S} = \{\pm 1, \pm 2, \ldots, \pm h\}$, starting at the origin, and ending at altitude 1, for different values of h:

$h = 1$ (A000108) : $0, 1, 0, 1, 0, 2, 0, 5, 0, 14, 0, \ldots$

$h = 2$ (A166135) : $0, 1, 1, 3, 7, 22, 65, 213, 693, 2352, 8034, \ldots$

$h = 3$ (A276852) : $0, 1, 2, 7, 28, 121, 560, 2677, 13230, 66742, 343092, \ldots$

$h = 4$ (A277922) : $0, 1, 3, 13, 71, 405, 2501, 15923, 104825, 704818, 4827957, \ldots$

Furthermore, here are some sequences of numbers of walks with steps in $\mathcal{S} = \{\pm 1, \pm 2, \ldots, \pm h\}$, starting at the origin, and ending at altitude 2, for different values of h:

$$
\begin{aligned}
h=1 \quad & \texttt{(A000108)}: \quad 0,0,1,0,2,0,5,0,14,0,42,\ldots\\
h=2 \quad & \texttt{(A111160)}: \quad 0,1,1,4,9,31,91,309,1009,3481,11956,\ldots\\
h=3 \quad & \texttt{(A276901)}: \quad 0,1,2,9,34,159,730,3579,17762,90538,467796,\ldots\\
h=4 \quad & \texttt{(A277923)}: \quad 0,1,3,16,84,505,3121,20180,133604,904512,6224305,\ldots
\end{aligned}
$$

Here are the corresponding sequences for positive meanders:

$$
\begin{aligned}
h=1 \quad & \texttt{(A001405)}: \quad 1,1,1,2,3,6,10,20,35,70,126,252,462,924,\ldots\\
h=2 \quad & \texttt{(A278394)}: \quad 1,2,5,17,58,209,761,2823,10557,39833,151147\ldots\\
h=3 \quad & \texttt{(A278395)}: \quad 1,3,12,60,311,1674,9173,51002,286384,1620776,\ldots\\
h=4 \quad & \texttt{(A278396)}: \quad 1,4,22,146,1013,7269,53156,394154,2951950,\ldots
\end{aligned}
$$

Here are the corresponding sequences for meanders (allowed to touch 0):

$$
\begin{aligned}
h=1 \quad & \texttt{(A001405)}: \quad 1,1,2,3,6,10,20,35,70,126,252,462,924,1716,3432,\ldots\\
h=2 \quad & \texttt{(A047002)}: \quad 1,2,7,23,83,299,1107,4122,15523,58769,223848,\ldots\\
h=3 \quad & \texttt{(A278398)}: \quad 1,3,15,75,400,2169,11989,66985,377718,2144290,\ldots\\
h=4 \quad & \texttt{(A278416)}: \quad 1,4,26,174,1231,8899,65492,487646,3664123,\ldots
\end{aligned}
$$

Here are the corresponding sequences for excursions:

$$
\begin{aligned}
h=1 \quad & \texttt{(A126120)}: \quad 1,0,1,0,2,0,5,0,14,0,42,0,132,0,429,0\ldots\\
h=2 \quad & \texttt{(A187430)}: \quad 1,0,2,2,11,24,93,272,971,3194,11293,39148,139687\ldots\\
h=3 \quad & \texttt{(A205336)}: \quad 1,0,3,6,35,138,689,3272,16522,83792,434749,\ldots\\
h=4 \quad & \texttt{(A205337)}: \quad 1,0,4,12,82,454,2912,18652,124299,841400,\ldots
\end{aligned}
$$

Remark. The cases with $h = 1$ lead to famous sequences, having many links with the combinatorics of trees, via the Łukasiewicz correspondence (see Section 2). It is surprising that the cases with $h = 2$ also offer many links with trees, as we show in the next section.

5. Some links with other combinatorial problems

In this section, we establish some links between our lattice walks and other combinatorial problems. Thereby we prove several conjectures issued in the On-Line Encyclopedia of Integer Sequences.

5.1. **Trees and basketball walks from 0 to 1.**

First, we prove that the sequence A166135 from the On-Line Encyclopedia of Integer Sequences, coming from the enumeration of certain tree structures used in financial mathematics, is in fact related to basketball walks, and corresponds more precisely to the coefficients of $G_{0,1}(z)$.

The m-nomial tree is a lattice-based computational model used in financial mathematics to price options. It was developed by Phelim Boyle [21] in 1986. For example, for $m = 3$, the underlying stock price is modelled as a recombining tree, where, at each node, the price has three possible paths: an up, down, or stable path. The case $m = 2$ has a long history going back to one of the founding problems of financial mathematics and probability theory, the "ruin problem", analysed in the XVIIIth and XIXth century by de Moivre, Laplace, Huygens, Ampère, Rouché, before to be revisited by combinatorialists like Catalan, Whitworth, Bertrand, André, Delannoy (see [9] for more on these aspects). Figure 7 illustrates a 4-nomial tree.

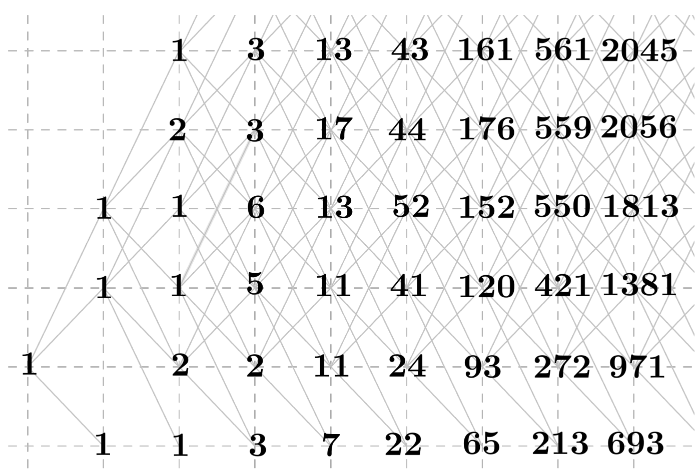


Figure 7. Cutting a 4-nomial tree at one unit from its root gives the above picture, which thus naturally corresponds to the lattice supporting our lattice basketball walks. The numbers near each node indicate the number of walks from the root to this node.

The following proposition gives the exact link between these trees and a generalization of basketball walks.

Proposition 5.1 (LINK BETWEEN LATTICE WALKS AND m-NOMIAL TREES). *Consider the step sets*

$$\mathcal{S}_{2n} = \{-n, \ldots, -1, 1, \ldots, n\} \text{ and } \mathcal{S}_{2n+1} = \mathcal{S}_{2n} \cup \{0\}.$$

For each step set S_m, define $T_m(z)$ to be the generating function for walks using steps from $\mathcal{S}_m$, starting at the origin and getting absorbed at -1. (By this, we mean that the walks may never touch $y = -1$ except, possibly, at the very last step.) Then the coefficients of $T_2(z)$ are the Catalan numbers, the coefficients of $T_3(z)$ are the Motzkin numbers, while the coefficients of $T_4(z)$ count our basketball walks from 0 to 1 (walks with steps $\pm 2, \pm 1$, starting at the origin and ending at altitude 1, and never touching 0 in-between).

Proof. While the correspondence is direct for $m \leq 3$, it follows for $m = 4$ from a time reversal, as each walk from T_4 can then be obtained from $G_{0,1}$ and vice versa (see Table 3). Thus, $T_4(z) = G_{0,1}(z)$. □

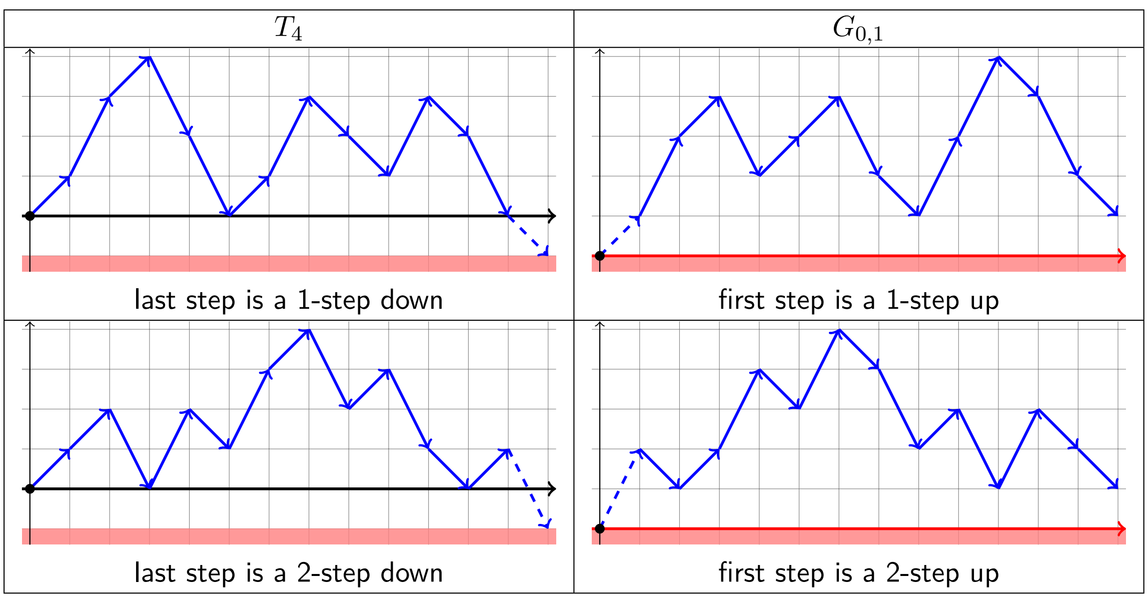


TABLE 3. By time reversal, $T_4(z) = G_{0,1}(z)$.

5.2. **Increasing trees and basketball walks.** A *unary-binary tree* is an ordered tree such that each node has $0, 1$, or 2 children. An *increasing unary-binary tree on n vertices* is a unary-binary tree with n vertices labelled $1, 2, \ldots, n$ such that the labels along each walk from the root are increasing (cf. [49, p. 51]). Given an increasing unary-binary tree T, we associate with T the *permutation* σ_T constructed by reading the tree left to right, level by level, starting at the root. A permutation σ is said to *contain the pattern* π if there exists a subsequence of σ that has the same relative order as π. Otherwise, σ is said to *avoid the pattern* π. For example, the permutation $\sigma = 14235$ contains the pattern 213 because σ contains the subsequence 425, in which the numbers have the same relative order as in 213, while the permutation 12453 avoids 213.

Manda Riehl initiated studies of increasing trees for which the associated permutation avoids a given pattern (see also [39]). By a computer program, she obtained the first terms of the corresponding sequences for patterns of length 3. She observed that "the number of increasing unary-binary trees with associated permutation avoiding 213" seems to coincide with sequence `A166135`, which we proved to count basketballs walks from altitude 0 to altitude 1. Figure 8 shows a verification of this claim for $n = 5$: there are 39 increasing unary-binary trees on 5 vertices, among them, 22 correspond to permutations avoiding the pattern 213. (The forbidden subsequences are highlighted in red. The trees in black all avoid 213. The trees are grouped according to their associated permutations. Tree labels are read left to right.)

Here is the reformulation of Riehl's conjecture which takes into account our findings.

Conjecture 5.2. *The number of basketball walks of length n starting at the origin and ending at altitude 1 that never touch or pass below the x-axis equals the number of increasing unary-binary trees on n vertices with associated permutation avoiding 213.*

After the first version of this article was circulated via the `arXiv`, Bettinelli, Fusy, Mailler, and Randazzo [14] found a nice bijective proof of this conjecture.

How strong is the constraint of avoiding the pattern 213? For this, we need to compute the probability that an increasing unary-binary tree avoids the pattern 213. Due to Conjecture 5.2, proved in [14], we know the number of increasing unary-binary trees which avoid 213. Hence, the question is to compute the total number t_n of increasing unary-binary trees, which can be done via the so-called boxed product.

The boxed product (written $^{\square}\!\times$) is the combinatorial construction corresponding to a labelled product, in which the minimal label is forced to be in the first component of this product (see [28]). This leads the following recursive decomposition for binary-ternary increasing trees $\mathcal{T}$:

$$\mathcal{T} = leaf + root\,^{\square}\!\times \mathcal{T} + root\,^{\square}\!\times \mathcal{T} \times \mathcal{T}\,,$$

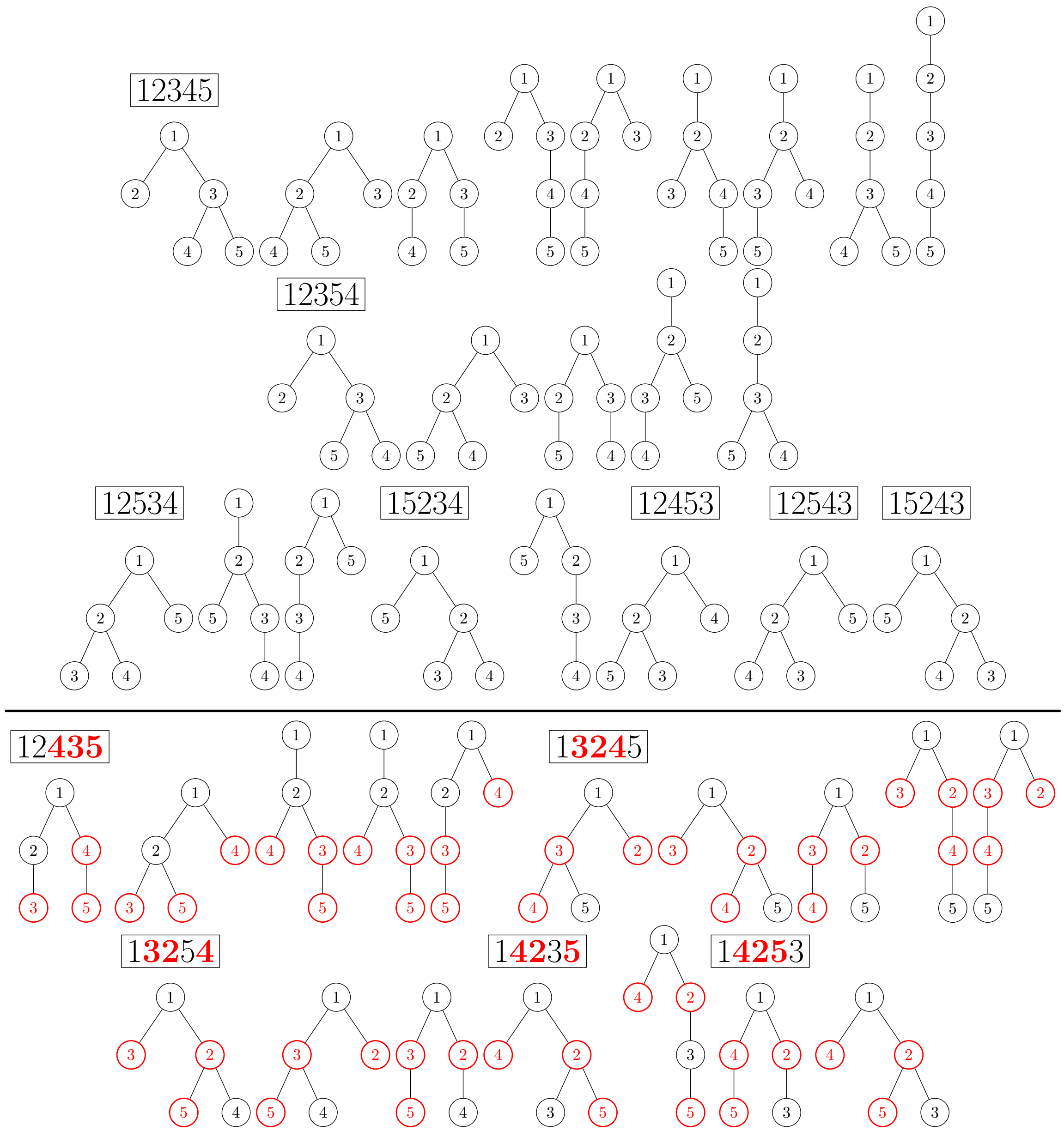


FIGURE 8. All increasing unary-binary trees with 5 nodes, where patterns 213 are marked in red. There are 22 trees (drawn in black) for which the associated permutation avoids this pattern.

which translates into the following functional equation for the corresponding exponential generating function:

$$T(z) = z + \int_0^z T(t)dt + \int_0^z T^2(t)dt\,.$$

By solving the associated differential equation $T'(z) = 1 + T(z) + T^2(z)$, we obtain

$$T(z) = \frac{\sqrt{3}}{2}\tan\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}z\right) - \frac{1}{2}\,.$$

The corresponding Taylor expansion is

$$T(z) = \sum_{n\geq 1} t_n \frac{z^n}{n!} = z + \frac{z^2}{2!} + 3\frac{z^3}{3!} + 9\frac{z^4}{4!} + 39\frac{z^5}{5!} + 189\frac{z^6}{6!} + 1107\frac{z^7}{7!} + O(z^8)\,.$$

Singularity analysis on the dominant poles of the $\tan$ function implies that

$$t_n \sim 3\sqrt{\frac{3}{2\pi}}\left(\frac{3^{3/2}}{2e\pi}\right)^n \sqrt{n}\, n^n\,.$$

In conclusion, increasing unary-binary trees grow like $n^{1/2}A^n n^n$, while the same trees avoiding the pattern 213 grow like $n^{-3/2}4^n$. This observation suggests the following natural conjecture.

Conjecture 5.3 (A Stanley–Wilf-like conjecture for pattern avoidance in increasing trees)**.** *Let $\mathcal{T}$ be a class of increasing trees of prescribed arity encoded by a power series ϕ, i.e., one has $\mathcal{T}' = z\phi(\mathcal{T})$. Then the number a_n of such trees avoiding a given pattern satisfies $a_n = O(C^n)$, for some C depending on the pattern and on ϕ.*

This conjecture shares the spirit of the Stanley–Wilf conjecture (proven by a combination of [33] and [40]), which asserted that any class of pattern-avoiding permutations has an exponential growth rate.

5.3. **Boolean trees and basketball walks from 0 to 2.** In [13], Bender and Williamson considered the problem of bracketing some binary operations (objects that are in bijection with the Boolean trees that we present in Figure 9). It turns out that this problem is doubly related to our basketball walks (walks with steps $\pm 1, \pm 2$, always positive). This is what we address in the next two propositions.

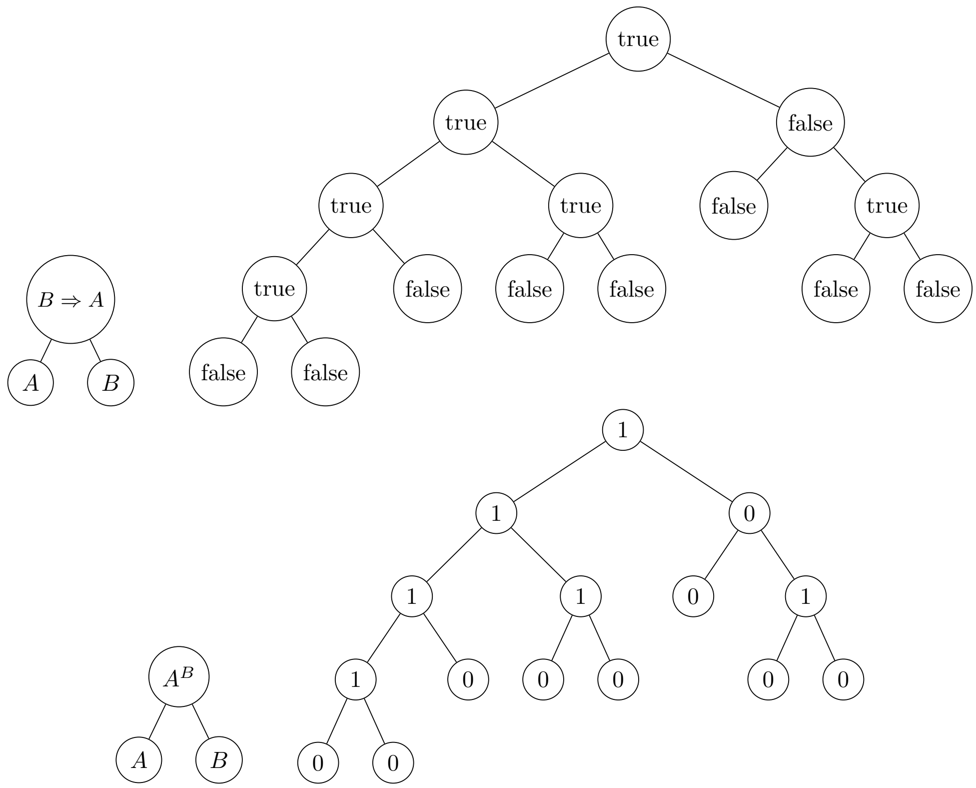


FIGURE 9. Boolean trees (i.e., binary trees where each node is labelled either "false" or "true") such that a node having children with Boolean value A and B will have the Boolean value "$B \Rightarrow A$".

Proposition 5.4. *Under the conventions $1^1 = 1^0 = 0^0 = 1$ and $0^1 = 0$, the number of bracketings of $n+1$ zeroes $0\,\hat{}\cdots\hat{}\,0$ giving result 1 is equal to the number of basketball walks from altitude 0 to altitude 2 of length n.*

Proof. Let $W(z)$ (respectively $Z(z)$) be the generating function for the number of bracketings of n zeroes 0^0^$\cdots$^0 producing result 1 (respectively 0). The objects that are counted by $W(z)$ are of the form ("1")^("1"), ("1")^("0"), or ("0")^("0"), where "1" stands for a bracketing producing the result 1, and "0" stands for a bracketing producing the result 0. This observation translates into the generating function equation

$$W(z) = W^2(z) + Z(z)W(z) + Z^2(z). \tag{5.1}$$

Similarly, a bracketing producing 0 may either be a single 0 or a bracketing of the form ("0")^("1"). This yields the equation

$$Z(z) = z + Z(z)W(z). \tag{5.2}$$

Let $C(z) := Z(z) + W(z)$. Equations (5.1) and (5.2) imply $C(z) = 1 + zC^2(z)$, i.e., $C(z) = \frac{1}{2z} - \frac{1}{2z}\sqrt{1-4z}$. This is not a surprise because $W + Z$ corresponds to well parenthesized words, known to be counted by Catalan numbers.

We may "replace" $W(z)$ by $C(z)$ in Equation (5.2). This leads to

$$Z(z) = z + Z(z)(C(z) - Z(z)).$$

Solving for $Z(z)$, we obtain

$$\begin{aligned} Z(z) &= \frac{C(z) - 1 + \sqrt{(C(z)-1)^2 + 4z}}{2} \\ &= -\frac{1}{4} - \frac{1}{4}\sqrt{1-4z} + \frac{1}{4}\sqrt{2 + 12z + 2\sqrt{1-4z}}. \end{aligned} \tag{5.3}$$

Therefore, we get

$$W(z) = C(z) - Z(z) = \frac{3}{4} - \frac{1}{4}\sqrt{1-4z} - \frac{1}{4}\sqrt{2 + 12z + 2\sqrt{1-4z}}.$$

Comparison of this expression with Expression (3.13) for $G_{0,2}(z)$ shows that $W(z) = zG_{0,2}(z)$. □

We leave it to the reader to find a bijective proof between bracketings of 0^...^0 having value 1 and basketball walks from altitude 0 to altitude 2.

Proposition 5.5. *The number of basketball walks of length n starting at the origin, ending at altitude 1, never running below the x-axis in-between, is equal to the number of bracketings of $n+2$ zeroes $0\,\hat{}\,0\,\hat{}\cdots\hat{}\,0$ producing result 0.*

Proof. The generating function $F_1(z)$ for walks ending at 1 is given by (3.4) in the form

$$F_1(z) = G_{1,2}(z) = \frac{u_1(z)u_2(z) + u_1(z) + u_2(z)}{z}.$$

The generating function $Z(z)$ for the number of bracketings of n zeroes 0^⋯^0 having value 0 is given by (5.3). Substitution of the closed-form expressions for the small roots into $F_1(z)$ yields $z^2F_1(z) = Z(z)$. This establishes the claim. □

6. Conclusion

In this article, we show how to derive closed-form expressions for the enumeration of lattice walks satisfying various constraints (starting point, ending point, positivity, allowed steps, ...). The key is a proper use of the Lagrange–Bürmann inversion in combination with the expressions given by the kernel method. This technique admits many extensions, which will work in a similar way: it is possible to extend it to walks in which we want to keep track of some parameters (marking a specific step, pattern, altitude, ...), allowing an infinite set of steps, or unbounded steps (this would encode what is called catastrophes in queuing theory language). It is also possible to consider other constraints, such as to force the walk to live in some cone or to have some forbidden patterns. In all these cases, the kernel method will give a closed-form expression for the generating function, in terms of the roots of the kernel, and thus, our mix of kernel method and Lagrange–Bürmann inversion will lead in these situations also to some closed-form expression for the coefficients of the generating function (in terms of nested sums of binomials).

In several cases, these nested sums of binomials provide the nice challenge of finding bijective proofs. It is satisfying to find *some* formula for the enumeration of certain lattice paths which is efficient (in terms of algorithmic complexity), but the fact that many of these sums involve only positive terms is an indication that combinatorics has still its word to say on these formulas.

The holonomic approach, as well illustrated by the book of Petkovšek, Wilf, and Zeilberger [45], or Kauers and Paule [32], is a way to prove that different binomial expressions correspond in fact to the same sequence. It remains an open question to know which methods can lead to the most concise formula: the platypus algorithms and the Flajolet–Soria formula [7, 8], or the cycle lemma, and extraction of diagonals of rational functions seem to indicate that we could in fact need an arbitrarily large amount of nested sums. In some cases, one can reduce the number of nested sums with techniques from symbolic summation theory (e.g., by $\Sigma\Pi$ extension theory [47], or geometric simplifications in diagonal extractions of rational functions [17]), but it is still unknown if, for the directed lattice path models we considered, there is a miraculous simple formula (with just one or two nested sums).

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References

[1] Désiré André. Mémoire sur les combinaisons régulières et leurs appliquations. *Annales scientifiques de l'École Normale Supérieure,* 2^e *série*, 5:155–198, 1876. Available on Numdam: [online pdf].

[2] George Andrews, Henrik Eriksson, Fedor Petrov, and Dan Romik. Integrals, partitions and MacMahon's theorem. *J. Combin. Theory Ser. A*, 114(3):545–554, 2007. [online pdf].

[3] George E. Andrews. Euler's "exemplum memorabile inductionis fallacis" and q-trinomial coefficients. *J. Amer. Math. Soc.*, 3(3):653–669, 1990. [online pdf].

[4] George E. Andrews and R. J. Baxter. Lattice gas generalization of the hard hexagon model. III. q-trinomial coefficients. *J. Statist. Phys.*, 47(3-4):297–330, 1987. [online pdf].

[5] Arvind Ayyer and Doron Zeilberger. The number of [old-time] basketball games with final score $n\,n$ where the home team was never losing but also never ahead by more than w points. *Electron. J. Combin.*, 14(1):Research Paper 19, 8, 2007. [online pdf].

[6] Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy, and Dominique Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246(1-3):29–55, 2002. Formal power series and algebraic combinatorics (Barcelona, 1999).

[7] Cyril Banderier and Michael Drmota. Formulae and asymptotics for coefficients of algebraic functions. *Combin. Probab. Comput.*, 24(1):1–53, 2015. [online pdf].

[8] Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.*, 281(1-2):37–80, 2002. Selected papers in honour of Maurice Nivat. [online pdf].

[9] Cyril Banderier and Sylviane Schwer. Why Delannoy numbers? *J. Statist. Plann. Inference*, 135(1):40–54, 2005.

[10] Cyril Banderier and Michael Wallner. The reflection-absorption model for directed lattice paths. *In preparation*, 2016.

[11] Cyril Banderier and Michael Wallner. The kernel method for lattice paths below a rational slope. In *Lattice path combinatorics and applications*, Developments in Mathematics Series. Springer, 2017. To appear.

[12] E. Barcucci, R. Pinzani, and R. Sprugnoli. The Motzkin family. *Pure Math. Appl. Ser. A*, 2(3-4):249–279, 1992.

[13] Edward A. Bender and S. Gill Williamson. *Foundations of Combinatorics with Applications*. Dover, 2006. New version of "Foundations of applied combinatorics", published by Addison-Wesley in 1991. [online pdf].

[14] Jérémie Bettinelli, Éric Fusy, Cécile Mailler, and Lucas Randazzo. A bijective study of basketball walks. *arXiv*, pages 1–20, 2016. [online pdf].

[15] P. Blasiak, G. Dattoli, A. Horzela, K. A. Penson, and K. Zhukovsky. Motzkin numbers, central trinomial coefficients and hybrid polynomials. *J. Integer Seq.*, 11(1):Article 08.1.1, 11, 2008. [online pdf].

[16] Walter Böhm. Lattice path counting and the theory of queues. *J. Statist. Plann. Inference*, 140(8):2168–2183, 2010.

[17] Alin Bostan, Pierre Lairez, and Bruno Salvy. Multiple binomial sums. *J. Symbolic Comput.*, 2016. [online pdf].

[18] Mireille Bousquet-Mélou. Discrete excursions. *Sém. Lothar. Combin.*, 57:23 pp., 2008. [online pdf].

[19] Mireille Bousquet-Mélou and Arnaud Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96(5):623–672, 2006.

[20] Mireille Bousquet-Mélou and Marko Petkovšek. Linear recurrences with constant coefficients: the multivariate case. *Discrete Math.*, 225(1-3):51–75, 2000. Formal power series and algebraic combinatorics.
[21] Phelim Boyle. Option valuation using a three-jump process. *International Options Journal*, 3:7–12, 1986.
[22] Hans Heinrich Bürmann. Formules du développement, de retour et d'integration. *Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, 1798. submitted to the Institut National de France; the manuscript survives in the archives of the École Nationale des Ponts et Chaussées in Paris.
[23] Louis Comtet. *Advanced combinatorics*. D. Reidel Publishing Co., 1974. Enlarged edition of the 2 volumes "Analyse combinatoire" published in French in 1970, by Presses Universitaires de France.
[24] Jean Dieudonné. *Infinitesimal calculus*. Hermann, Paris; Houghton Mifflin Co., 1971. Translated from the French.
[25] Georgiĭ P. Egorychev. *Integral representation and the computation of combinatorial sums*, volume 59 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1984.
[26] Bertrand Eynard. *Counting surfaces*, volume 70 of *Progress in Mathematical Physics*. Birkhäuser/Springer, [Cham], 2016. CRM Aisenstadt chair lectures.
[27] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev. *Random walks in the quarter-plane*, volume 40 of *Applications of Mathematics*. Springer-Verlag, 1999.
[28] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, 2009. [online pdf].
[29] Donald Gross, John F. Shortle, James M. Thompson, and Carl M. Harris. *Fundamentals of queueing theory*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., Hoboken, NJ, fourth edition, 2008. 1st ed.: 1974.
[30] Joti Lal Jain, Sri Gopal Mohanty, and Walter Böhm. *A course on queueing models*. Statistics: Textbooks and Monographs. Chapman & Hall/CRC, Boca Raton, FL, 2007.
[31] Manuel Kauers, Fredrik Johansson, and Maximilian Jaroschek. Ore Polynomials in Sage. In *Computer Algebra and Polynomials*, Lecture Notes in Computer Science, pages 105–125. Springer, 2015. [online pdf].
[32] Manuel Kauers and Peter Paule. *The concrete tetrahedron*. Texts and Monographs in Symbolic Computation. Springer, 2011. Symbolic sums, recurrence equations, generating functions, asymptotic estimates.
[33] Martin Klazar. The Füredi–Hajnal conjecture implies the Stanley–Wilf conjecture. In *Formal Power Series and Algebraic Combinatorics (Moscow, 2000)*, pages 250–255. Springer, 2000.
[34] Donald E. Knuth. *The art of computer programming. Vol. 2: Seminumerical algorithms.* Addison-Wesley, 1998. (3rd edition, 2nd ed.: 1981, 1st ed.:1969).
[35] Alan Krinik and Sri Gopal Mohanty. On batch queueing systems: a combinatorial approach. *J. Statist. Plann. Inference*, 140(8):2271–2284, 2010.
[36] Alan C. Krinik and Ken Shun. Markov processes with constant transition rates of size one & two. *J. Stat. Theory Pract.*, 5(3):475–495, 2011.
[37] Joseph-Louis Lagrange. Nouvelle méthode pour résoudre les équations littérales par le moyen des séries. *Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, 24:251–326, 1770. Reprinted in Œuvres de Lagrange, tome 2. Paris: Gauthier-Villars, pp. 655-726, 1868.

[38] Joseph-Louis Lagrange and Adrien-Marie Legendre. Rapport sur deux mémoires d'analyse du professeur burmann. *Mémoires de l'Institut National des Sciences et Arts: Sciences Mathématiques et Physiques*, 2:13–17, 1799.
[39] Derek Levin, Lara K. Pudwell, Manda Riehl, and Andrew Sandberg. Pattern avoidance in k-ary heaps. *Australas. J. Combin.*, 64:120–139, 2016. [online pdf].
[40] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A*, 107(1):153–160, 2004.
[41] Barbara Margolius. Asymptotic estimates for queueing systems with time-varying periodic transition rates. In *Lattice path combinatorics and applications*, Developments in Mathematics Series. Springer, 2017. To appear.
[42] Sri Gopal Mohanty. *Lattice path counting and applications*. Academic Press, 1979. Probability and Mathematical Statistics.
[43] Paul Montel. Sur les combinaisons avec répétitions limitées. *Bull. Sci. Math. (2)*, 66:86–103, 1942.
[44] Tadepalli Venkata Narayana. *Lattice path combinatorics with statistical applications*, volume 23 of *Mathematical Expositions*. University of Toronto Press, 1979.
[45] Marko Petkovšek, Herbert Saul Wilf, and Doron Zeilberger. $A = B$. A. K. Peters, 1996. [online pdf].
[46] Bruno Salvy and Paul Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994. [online pdf].
[47] Carsten Schneider. Symbolic summation assists combinatorics. *Sém. Lothar. Combin.*, 56:Art. B56b, 36, 2007. [online pdf].
[48] Richard Peter Stanley. *Enumerative combinatorics. Volume 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.
[49] Richard Peter Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. Second ed. (1st ed. in 1986).
[50] Guoce Xin. The ring of Malcev–Neumann series and the residue theorem. *Ph.D. thesis, Brandeis University*, 2004.

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