AUTOMATIC GENERATION OF HYPERGEOMETRIC IDENTITIES BY THE BETA INTEGRAL METHOD

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Abstract. In this article hypergeometric identities (or transformations) for ${}_{p+1}F_p$-series and for Kampé de Fériet series of unit arguments are derived systematically from known transformations of hypergeometric series and products of hypergeometric series, respectively, using the beta integral method in an automated manner, based on the *Mathematica* package HYP. As a result we obtain some known and some identities which seem to not have been recorded before in literature.

1. Introduction

Euler's beta integral evaluation

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.1}$$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, is at the heart of many identities in the theory of hypergeometric series. An example is the well known integral representation (see e.g. [11, (11), Sec. 3.6]) of a hypergeometric series

$$\begin{aligned}{}_{p+1}F_p&\left[\begin{matrix}\alpha,\alpha_1,\dots,\alpha_p\\ \gamma,\beta_1,\dots,\beta_{p-1}\end{matrix};t\right]\\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 z^{\alpha-1}(1-z)^{\gamma-\alpha-1}{}_pF_{p-1}\left[\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_{p-1}\end{matrix};zt\right]dz. \end{aligned}\tag{1.2}$$

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Here we use the standard hypergeometric notation

$$ {}_rF_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{k!\,(b_1)_k \cdots (b_s)_k} z^k \ , $$

where the Pochhammer symbol $(\alpha)_k$ is defined by $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$, $k > 0$, $(\alpha)_0 = 1$. (In order to prove (1.2), one would interchange integration and summation on the right-hand side, and then use (1.1) to evaluate the integral inside the summation.)

In this paper we propose the exploitation of (what we call) the "*beta integral method*," a method of deriving "new" hypergeometric identities from "old" ones, using the beta integral evaluation, which is folklore in the hypergeometric literature, although appearances can be only found sporadically (see, for example, [1, Ch. 3, Ex. 5, 14 and 16]). It seems that it was never exploited systematically, probably because of the effort it takes to do all the computations. However, with the help of a computer, these computations become completely painless. This is what we propose here: the *completely automatic* application of the beta integral method.

To convey the idea, we briefly recall an early occurrence of this method (but, very likely, not the first) in [18, Sec. 3]. We start with the well known transformation formula [14, (1.8.10), with a, b, c replaced by $-n, \alpha, \gamma$, respectively]:

$$ {}_2F_1\left[\begin{matrix} -n, \alpha \\ \gamma \end{matrix}; z\right] = \frac{(\gamma-\alpha)_n}{(\gamma)_n} {}_2F_1\left[\begin{matrix} -n, \alpha \\ \alpha - n - \gamma + 1 \end{matrix}; 1-z\right], \tag{1.3} $$

multiply both sides by $z^{\beta-1}(1-z)^{\delta-\beta-1}$, integrate both sides with respect to z, $0 \le z \le 1$,[1] interchange integration and summation on both sides, then use (1.1), and finally convert the result back to hypergeometric notation, to get the transformation formula:

$$ {}_3F_2\left[\begin{matrix} -n, \alpha, \beta \\ \gamma, \delta \end{matrix}; 1\right] = \frac{(\gamma-\alpha)_n}{(\gamma)_n} {}_3F_2\left[\begin{matrix} -n, \alpha, \delta-\beta \\ 1+\alpha-\gamma-n, \delta \end{matrix}; 1\right], \tag{1.4} $$

where n is a nonnegative integer. In this case we obtain an already known transformation formula, namely one of Thomae's ${}_3F_2$-transformation formulas. (The formula (1.4) can for example be extracted from Tables II_A and II_B ($Fp(0;4,5) = Fn(4;0,1)$) in Bailey's tract [3] which summarizes and groups the equivalent numerator and denominator functions obtained by Thomae [17] in the notation of Whipple [20].)

However, not only can this method be applied to a large variety of identities, its application can be *completely automatized* (as we already announced) with the help of the *Mathematica* program HYP created by one of us [9]. Thus, this method is similar in spirit to the idea of "dual identities" in "WZ-theory" [19] (though it is less sophisticated), and the idea of "parameter augmentation" in [6, 7] (which, however, introduces just one additional parameter instead of two as in the "beta integral method"). Also there, the idea is to start with a known identity, and then apply a certain procedure to automatically produce a (possibly) new identity. (In WZ-theory, the procedure consists

[1] In fact, we have to temporarily restrict the parameters β and δ to $\Re(\beta) > 0$ and $\Re(\delta-\beta) > 0$, because otherwise the beta integral (1.1) would not converge. However, these restrictions can in the end be removed by analytic continuation. We shall, if necessary, make similar temporary assumptions without mentioning in subsequent derivations.

in finding the "WZ-pair" which is behind the original identity, and then use summation over the "other" variable to obtain a different identity. "Parameter augmentation" applies a certain (q-)differential operator to a known identity.)

The algorithm has also been applied to transformation formulae for products of hypergeometric functions known in literature. As a result formulae for the Kampé de Fériet series of unit arguments, which transform them into single-sum hypergeometric series, are obtained. Again, some of the results obtained are known ones but some of the results obtained seem to be new and interesting.

In Section 2, the algorithm and its implementation is presented. A section of the *Mathematica* session is reproduced to give a feel for the methodology adopted. In Section 3, we list the results if we apply our algorithm to known hypergeometric transformation formulae between single-sum series. In Section 4, we list a few of the results for Kampé de Fériet that we obtain when we apply our machinery to identities involving products of hypergeometric series. Concluding remarks are made in Section 5.

2. The algorithm

Let us recall the basic steps of the algorithm that produced (1.4) from (1.3):

(i) convert the hypergeometric series on both sides of a given transformation into sums;
(ii) multiply both sides of the equation by the factors $z^{\beta-1}(1-z)^{\delta-\beta-1}$;
(iii) integrate term-by-term with respect to z for $0 \le z \le 1$;
(iv) interchange integration and summation;
(v) use the beta integral to evaluate the integrals inside the summations;
(vi) convert the sums back into hypergeometric notation.

We give below (see `In[2]` in the *Mathematica* session below) an implementation of this algorithm in *Mathematica*. There, a function `T` is defined, which is applied to some equation. To briefly explain the code: Step (i) is performed in the first line (`X=Ers[Equ,FSUM,{1,2}]`), where the variable `X` is set to the equation `Equ` where on both sides the hypergeometric series is converted into a sum (which is achieved by the application of `FSUM`). Then, in the next line, both sides of the equation are multiplied by $z^{A-1}(1-z)^{B-A-1}$, thus performing Step (ii). The subsequent line has the purpose of bringing everything inside the summations (which is achieved by `SUMSammle`). Then Steps (iii)-(vi) are performed, in the following way. First, the variable `Y` is set to the integral over z, $0 \le z \le 1$, of the *summand* (represented by `X[[1,1]]`) of the series on the left hand side of the equation. Subsequently, the result is summed over k, and then converted to hypergeometric notation (the latter being achieved by `SUMF`). Then the same is done for the right hand side, and the result is stored in the variable `X`. Finally, in the last line (`Y==X`) the results on both sides are equated.

To illustrate the use of this algorithm, we adjoin a segment of the *Mathematica* output where the algorithm is applied to the transformation formula [14, (1.3.15)]. Within HYP, the transformation formula is input as Tgl2103 (see `In[3]`). Then the algorithm is invoked, by applying the function `T` to the transformation formula. In the interactive mode the questions raised by *Mathematica* have to be answered appropriately. The result is displayed in `Out[4]`.

```
Mathematica 2.2 for DOS 387
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In[1]:= <<hyp.m

In[2]:= T[Equ_]:=
            Module[{X,Y},
              X=Ers[Equ,FSUM,{1,2}];
              Mal[z^(A-1)*(1-z)^(B-A-1)];
              X=(Gleichung/.SUMSammle);
              Y=Integrate[X[[1,1]],{z,0,1}];
              Y=(SUM[Y,{k,0,Infinity}]/.SUMF);
              X=Integrate[X[[2,1]],{z,0,1}];
              X=(SUM[X,{k,0,Infinity}]/.SUMF);
              Y==X
             ]

In[3]:= Tgl2103
Do you want to set values for the equation?  [y|n]:  n
```

$$\text{Out[3]=}\quad {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};z\right] == (1-z)^{-a-b+c}\,{}_2F_1\left[\begin{matrix}-a+c,\ -b+c\\ c\end{matrix};z\right]$$

```
In[4]:= T[%]
Is -a a nonnegative integer?
[y|n]:    n
Is -b a nonnegative integer?
[y|n]:    n
A hypergeometric series is converted into a sum.
Enter a variable for the summation index:  k
Is a - c a nonnegative integer?
[y|n]:    n
Is b - c a nonnegative integer?
[y|n]:    n
A hypergeometric series is converted into a sum.
Enter a variable for the summation index:  k
```

$$\text{Out[4]=}\quad \frac{{}_3F_2\left[\begin{matrix}A,\ a,\ b\\ B,\ c\end{matrix};1\right]\Gamma(A)\,\Gamma(-A+B)}{\Gamma(B)} ==$$

```
        [ A, -a + c, -b + c              ]
      F [                           ; 1  ] Γ(A) Γ(-a - A - b + B + c)
    3 2 [ -a - b + B + c, c              ]
⟩   -------------------------------------------------------
                         Γ(-a - b + B + c)
```

Notice that $\Gamma(A)$ is occurring on both sides of the output equation. Simplification of the result can be done either by hand or using the various commands available in HYP. After the following replacements, in succession: $c \to e$, $A \to c$, $B \to d$, the final result obtained is:

$$ {}_3F_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};1\right] = \frac{\Gamma(d)\,\Gamma(s)}{\Gamma(-c+d)\,\Gamma(s+c)}\,{}_3F_2\left[\begin{matrix} c,-a+e,-b+e \\ s+c,d \end{matrix};1\right] \tag{2.1} $$

where $s = d+e-a-b-c$, is the parameter excess. This identity belongs to the set of ten non-terminating ${}_3F_2[1]$ series (see [3, Ex. 7, p. 98] and [15, (III)]). Within HYP, this output can then be converted into any of the commonly used forms of TeX (LaTeX, $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX, $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX or Plain-TeX). In the case of this article, conversion into $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX code was applied, which is reproduced here.

3. "New" single sum hypergeometric identities from old ones

In this section, we apply the algorithm of the previous section systematically to known hypergeometric series transformations. Whenever we have been able to trace the obtained identity in the literature, we provide the respective reference. In any case, we have already seen two such examples: If we start with the transformation formula (1.3), then the algorithm results in (1.4), and if we start with the transformation formula displayed as `Out[3]` of the *Mathematica* session, then the algorithm results in (2.1). The transformation formula [14, (1.7.1.3)],

$$ {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix};z\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix} a,c-b \\ c \end{matrix};-\frac{z}{1-z}\right], $$

will also result in one of the 18 terminating ${}_3F_2[1]$ transformations (see [16, (IX), p. 91]), when one of the numerator parameters is a negative integer.

1. If we start with the quadratic transformation formula [12, (3.2)]

$$ {}_2F_1\left[\begin{matrix} a,b \\ 1+a-b \end{matrix};-z\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix} \frac{a}{2},\frac{1}{2}+\frac{a}{2} \\ 1+a-b \end{matrix};-\frac{4\,z}{(1-z)^2}\right] \tag{3.1} $$

and assume that a is a nonpositive integer (so that the ${}_2F_1$-series terminate), then we obtain

$$
{}_3F_2\left[\begin{matrix}a,b,d\\1+a-b,e\end{matrix};-1\right]
= \frac{\Gamma(e)\,\Gamma(e-a-d)}{\Gamma(e-a)\,\Gamma(e-d)}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}+\frac{a}{2},\frac{a}{2},d,1+a-e\\1+a-b,\frac{1}{2}+\frac{a}{2}+\frac{d}{2}-\frac{e}{2},1+\frac{a}{2}+\frac{d}{2}-\frac{e}{2}\end{matrix};1\right]. \tag{3.2}
$$

This transformation relating a nearly-poised ${}_3F_2[-1]$ to a ${}_4F_3[1]$ is valid provided a is a nonpositive integer. However, by a standard polynomial trick, it can be seen that it is also true if a is arbitrary but d is a nonpositive integer: let d by a fixed nonpositive integer. By multiplying both sides of (3.2) by

$$(e-a)_d\,(1+a-b)_d\,(1+a+d-e)_{2d},$$

both sides become polynomials in a of degree at most $5d$. These two polynomials agree for all nonpositive a, since we know that (3.2) is true for nonpositive a. These are infinitely many values of a, whence the polynomials must be identical.

2. If we start with the transformation formula [12, (5.10)]

$$
{}_2F_1\left[\begin{matrix}a,\frac{1}{2}+a\\\frac{1}{2}+b\end{matrix};z^2\right] = (1-z)^{-2\,a}\,{}_2F_1\left[\begin{matrix}2\,a,b\\2\,b\end{matrix};\frac{2\,z}{z-1}\right] \tag{3.3}
$$

and assume that a is a nonpositive integer, then we obtain

$$
{}_4F_3\left[\begin{matrix}a,\frac{1}{2}+a,\frac{1}{2}+\frac{d}{2},\frac{d}{2}\\\frac{1}{2}+b,\frac{1}{2}+\frac{e}{2},\frac{e}{2}\end{matrix};1\right] = \frac{\Gamma(e)\,\Gamma(e-2\,a-d)}{\Gamma(e-2\,a)\,\Gamma(e-d)}\,{}_3F_2\left[\begin{matrix}2\,a,b,d\\2\,b,1+2\,a+d-e\end{matrix};2\right]. \tag{3.4}
$$

This identity is true provided a is a nonpositive integer or d is a nonpositive integer, the latter because of the same arguments as sketched in item 1. It can be found in the literature as a special case of a more general transformation for basic hypergeometric series (see [8, Ex. 3.4, $q\to 1$, reversed]; it is available as `T3235` within HYP, see [10]).

3. If we start with the quadratic transformation formula of Gauß [3, Ex. 4.(iii), p. 97, with $\alpha\to a/2$, $\beta\to b/2$, $x\to z$]

$$
{}_2F_1\left[\begin{matrix}a,b\\\frac{1}{2}+\frac{a}{2}+\frac{b}{2}\end{matrix};z\right] = {}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{b}{2}\\\frac{1}{2}+\frac{a}{2}+\frac{b}{2}\end{matrix};4\,(1-z)\,z\right] \tag{3.5}
$$

and assume that a is a nonpositive integer, then we obtain

$$
{}_3F_2\left[\begin{matrix}a,b,d\\\frac{1}{2}+\frac{a}{2}+\frac{b}{2},e\end{matrix};1\right] = {}_4F_3\left[\begin{matrix}\frac{a}{2},\frac{b}{2},d,-d+e\\\frac{1}{2}+\frac{a}{2}+\frac{b}{2},\frac{1}{2}+\frac{e}{2},\frac{e}{2}\end{matrix};1\right]. \tag{3.6}
$$

This identity is true provided both hypergeometric series terminate. It is the main theorem in [13] (actually, there even a q-analogue of (3.6) can be found; see also [8, (3.10.13); Appendix (III.21)]). This same result (3.6) will be obtained when $z\to 1-z$ in (3.5).

4. If we start with the quadratic transformation formula [12, (5.12), reversed]

$$
{}_2F_1\left[\begin{matrix}a,b\\2\,b\end{matrix};z\right] = (1-z)^{-a/2}\,{}_2F_1\left[\begin{matrix}\frac{a}{2},-\frac{a}{2}+b\\\frac{1}{2}+b\end{matrix};\frac{z^2}{4\,(z-1)}\right] \tag{3.7}
$$

and assume that a is an even nonpositive integer, then we obtain

$$ {}_3F_2\left[\begin{matrix}a,b,d\\ 2\,b,e\end{matrix};1\right]=\frac{\Gamma(e)\,\Gamma(e-\frac{a}{2}-d)}{\Gamma(e-\frac{a}{2})\,\Gamma(e-d)}\,{}_4F_3\left[\begin{matrix}\frac{a}{2},-\frac{a}{2}+b,\frac{1}{2}+\frac{d}{2},\frac{d}{2}\\ \frac{1}{2}+b,1+\frac{a}{2}+d-e,-\frac{a}{2}+e\end{matrix};1\right], \tag{3.8}$$

provided a is an even nonpositive integer or d is any nonpositive integer. This identity can also be obtained in a different (but more complicated) way: In the transformation formula listed as `T4391` in HYP (see [10]; it is Eq. (3.5.7) from [8] with $q\to 1$) let $e\to\infty$. Then on the right-hand side the second term vanishes, while the first is a very-well-poised ${}_7F_6$-series to which Whipple's ${}_7F_6$-to-${}_4F_3$ transformation (see [14, (2.4.1.1)]) can be applied. The result is the above transformation formula (3.8).

5. The transformation formula [12, (3.31), reversed]

$$ {}_2F_1\left[\begin{matrix}a,1-a\\ c\end{matrix};z\right]=(1-z)^{c-1}\,{}_2F_1\left[\begin{matrix}-\frac{a}{2}+\frac{c}{2},-\frac{1}{2}+\frac{a}{2}+\frac{c}{2}\\ c\end{matrix};4\,(1-z)\,z\right] \tag{3.9}$$

results in

$$ {}_3F_2\left[\begin{matrix}1-a,a,d\\ c,e\end{matrix};1\right] \\ =\frac{\Gamma(e)\,\Gamma(c-d+e-1)}{\Gamma(c+e-1)\,\Gamma(e-d)}\,{}_4F_3\left[\begin{matrix}-\frac{a}{2}+\frac{c}{2},-\frac{1}{2}+\frac{a}{2}+\frac{c}{2},d,c-d+e-1\\ c,-\frac{1}{2}+\frac{c}{2}+\frac{e}{2},\frac{c}{2}+\frac{e}{2}\end{matrix};1\right], \tag{3.10}$$

provided both hypergeometric series terminate.

6. If we start with the transformation formula [4, (4.10), with $x\to -z$]

$$ {}_2F_1\left[\begin{matrix}a,b\\ 2\,b\end{matrix};-\frac{4\,z}{(1-z)^2}\right]=(1-z)^{2\,a}\,{}_2F_1\left[\begin{matrix}a,\frac{1}{2}+a-b\\ \frac{1}{2}+b\end{matrix};z^2\right], \tag{3.11}$$

which is a combination of [12, (5.10)] and [12, (6.2), reversed], and assume that a is a nonpositive integer, then we obtain

$$ {}_4F_3\left[\begin{matrix}a,b,d,e\\ 2\,b,\frac{d}{2}+\frac{e}{2},\frac{1}{2}+\frac{d}{2}+\frac{e}{2}\end{matrix};1\right] \\ =\frac{\Gamma(1-e)\,\Gamma(1+2\,a-d-e)}{\Gamma(1+2\,a-e)\,\Gamma(1-d-e)}\,{}_4F_3\left[\begin{matrix}a,\frac{1}{2}+a-b,\frac{1}{2}+\frac{d}{2},\frac{d}{2}\\ \frac{1}{2}+b,\frac{1}{2}+a-\frac{e}{2},1+a-\frac{e}{2}\end{matrix};1\right], \tag{3.12}$$

provided a or d is a nonpositive integer.

7. If we start with the transformation formula ([4, (4.22), with $\alpha\to a$, $\beta\to b$, $x\to z$] and [12, (5.12), with $z\to z/(2-z)$]):

$$ {}_2F_1\left[\begin{matrix}a,b\\ \frac{1}{2}+a+b\end{matrix};\frac{z^2}{4\,(z-1)}\right]=(1-z)^{a}\,{}_2F_1\left[\begin{matrix}2\,a,a+b\\ 2\,a+2\,b\end{matrix};z\right] \tag{3.13}$$

and assume that a is a nonpositive integer, then we obtain

$$ {}_4F_3\left[\begin{matrix}a,b,\frac{1}{2}+\frac{d}{2},\frac{d}{2}\\ \frac{1}{2}+a+b,1+d-e,e\end{matrix};1\right]=\frac{\Gamma(e)\,\Gamma(a-d+e)}{\Gamma(a+e)\,\Gamma(e-d)}\,{}_3F_2\left[\begin{matrix}2\,a,a+b,d\\ 2\,a+2\,b,a+e\end{matrix};1\right], \tag{3.14}$$

provided a or d is a nonpositive integer.

8. The transformation formula [12, (3.31), with $z \to 4z/(1-z)$, $a \to a+c+\frac{1}{2}$, $c \to b$]:

$$ {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+a+b \end{matrix}; 4\,(1-z)\,z\right] = (1-z)^{\frac{1}{2}-a-b}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}+a-b, \frac{1}{2}-a+b \\ \frac{1}{2}+a+b \end{matrix}; z\right] \tag{3.15}$$

and assume that a is a nonpositive integer, then we obtain

$$ {}_4F_3\left[\begin{matrix} a, b, d, e-d \\ \frac{1}{2}+a+b, \frac{1}{2}+\frac{e}{2}, \frac{e}{2} \end{matrix}; 1\right] \\ = \frac{\Gamma(e)\,\Gamma(\frac{1}{2}-a-b-d+e)}{\Gamma(\frac{1}{2}-a-b+e)\,\Gamma(e-d)} {}_3F_2\left[\begin{matrix} \frac{1}{2}+a-b, \frac{1}{2}-a+b, d \\ \frac{1}{2}+a+b, \frac{1}{2}-a-b+e \end{matrix}; 1\right], \tag{3.16}$$

provided both hypergeometric series terminate. This transformation can also be obtained by combining the ${}_4F_3$-to-${}_3F_2$ transformation from [13] that occurred already in item 3 with the ${}_3F_2$-transformation (2.1).

9. If we start with the transformation formula [8, (3.4.8), $q \to 1$, reversed]

$$ (1+z)\, {}_2F_1\left[\begin{matrix} a, \frac{1}{2}+a \\ b \end{matrix}; -\frac{4\,z}{(1-z)^2}\right] = (1-z)^{2\,a}\, {}_3F_2\left[\begin{matrix} 2\,a-1, \frac{1}{2}+a, 2\,a-b \\ -\frac{1}{2}+a, b \end{matrix}; -z\right] \tag{3.17}$$

and assume that a is a nonpositive integer, then we obtain

$$ (d+e)\, {}_4F_3\left[\begin{matrix} a, \frac{1}{2}+a, d, -e \\ b, \frac{1}{2}+\frac{d}{2}-\frac{e}{2}, 1+\frac{d}{2}-\frac{e}{2} \end{matrix}; 1\right] \\ = \frac{\Gamma(1+e)\,\Gamma(2\,a-d+e)}{\Gamma(2\,a+e)\,\Gamma(e-d)} {}_4F_3\left[\begin{matrix} 2\,a-1, \frac{1}{2}+a, 2\,a-b, d \\ -\frac{1}{2}+a, b, 2\,a+e \end{matrix}; -1\right], \tag{3.18}$$

provided a or d is a nonpositive integer. (In fact, when applying the beta integral method, we have to deal with a sum of two series on the left-hand side, which generates the factor $(d+e)$ in the result.)

10. If we start with the transformation formula [3, p. 97, Ex. 4.(iv), with $x \to z$]

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}+\frac{a}{2}, 1+a-b-c \\ 1+a-b, 1+a-c \end{matrix}; -\frac{4\,z}{(1-z)^2}\right] = (1-z)^a\, {}_3F_2\left[\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}; z\right] \tag{3.19}$$

and assume that a is a nonpositive integer, then we obtain

$$ {}_5F_4\left[\begin{matrix} \frac{1}{2}+\frac{a}{2}, \frac{a}{2}, 1+a-b-c, d, 1-e \\ 1+a-b, 1+a-c, \frac{1}{2}+\frac{d}{2}-\frac{e}{2}, 1+\frac{d}{2}-\frac{e}{2} \end{matrix}; 1\right] \\ = \frac{\Gamma(e)\,\Gamma(a-d+e)}{\Gamma(a+e)\,\Gamma(e-d)} {}_4F_3\left[\begin{matrix} a, b, c, d \\ 1+a-b, 1+a-c, a+e \end{matrix}; 1\right], \tag{3.20}$$

provided a or d is a nonpositive integer. This is a known transformation between a nearly-poised ${}_4F_3[1]$ series and a Saalschützian ${}_5F_4[1]$ series ([14, (2.4.2.3)]; to see this do the replacements $a \to f$, $b \to 1+f-h$, $c \to h-a$, $e \to g-f$, in (3.20)).

11. If we start with the transformation formula [3, p. 97, Ex. 6, with $b \to 1+a-b$, $c \to 1+a-c$, $x \to z$]

$$(1+z)\ {}_3F_2\left[\begin{matrix}\frac{1}{2}+\frac{a}{2}, 1+\frac{a}{2}, -a+b+c-1\\ b, c\end{matrix}; -\frac{4\,z}{(z-1)^2}\right]$$
$$= (1-z)^{1+a}\ {}_4F_3\left[\begin{matrix}a, 1+\frac{a}{2}, 1+a-b, 1+a-c\\ \frac{a}{2}, b, c\end{matrix}; z\right] \quad (3.21)$$

and assume that a is a nonpositive integer, then we obtain

$${}_5F_4\left[\begin{matrix}\frac{1}{2}+\frac{a}{2}, 1+\frac{a}{2}, -a+b+c-1, d, -e\\ b, c, \frac{1}{2}+\frac{d}{2}-\frac{e}{2}, 1+\frac{d}{2}-\frac{e}{2}\end{matrix}; 1\right]$$
$$= \frac{1}{d+e}\frac{\Gamma(1+e)\,\Gamma(1+a-d+e)}{\Gamma(1+a+e)\,\Gamma(e-d)}\,{}_5F_4\left[\begin{matrix}a, 1+\frac{a}{2}, 1+a-b, 1+a-c, d\\ \frac{a}{2}, b, c, 1+a+e\end{matrix}; 1\right], \quad (3.22)$$

provided a or d is a nonpositive integer. On replacing $b \to 1+a-b, c \to 1+a-c, d \to -m, e \to -1-a+w$, this identity corresponds to [3, (4.5.2)]. (The parenthetical remark in item 9 applies also here. However, in the result, we moved the factor $(d+e)$, which is generated by the sum of two series on the left-hand side of (3.21), to the right-hand side.)

12. If we start with the transformation formula [2, (5.8)]:

$$\left(1-\frac{z}{2}\right)\ {}_4F_3\left[\begin{matrix}1+a, a-2\,b, 1-a+2\,b, \frac{4}{3}+\frac{a}{3}\\ \frac{3}{2}+b, 1+a-b, \frac{1}{3}+\frac{a}{3}\end{matrix}; -\frac{z^2}{4\ (1-z)}\right]$$
$$= (1-2\,z)\ (1-z)^{1+a}\ {}_4F_3\left[\begin{matrix}1+a, 1+b, \frac{1}{2}+a-b, \frac{5}{3}+\frac{2\,a}{3}\\ 2+2\,b, 1+2\,a-2\,b, \frac{2}{3}+\frac{2\,a}{3}\end{matrix}; 4\ (1-z)\ z\right]. \quad (3.23)$$

and assume that a is a negative integer, then we obtain

$${}_6F_5\left[\begin{matrix}\frac{4}{3}+\frac{a}{3}, 1+a, a-2\,b, 1-a+2\,b, \frac{1}{2}+\frac{d}{2}, \frac{d}{2}\\ \frac{1}{3}+\frac{a}{3}, 1+a-b, \frac{3}{2}+b, 1+d-e, 1+e\end{matrix}; 1\right]$$
$$= \frac{2\ (1+a-2\,d+e)}{2\,e-d}\ \frac{\Gamma(1+e)\,\Gamma(1+a-d+e)}{\Gamma(2+a+e)\,\Gamma(e-d)}$$
$$\times\ {}_6F_5\left[\begin{matrix}\frac{5}{3}+\frac{2\,a}{3}, 1+a, \frac{1}{2}+a-b, 1+b, d, 1+a-d+e\\ \frac{2}{3}+\frac{2\,a}{3}, 1+2\,a-2\,b, 2+2\,b, 1+\frac{a}{2}+\frac{e}{2}, \frac{3}{2}+\frac{a}{2}+\frac{e}{2}\end{matrix}; 1\right], \quad (3.24)$$

provided a is a negative integer or d is a nonpositive integer. (Again, the parenthetical remark of item 9 applies, this time on both sides. The factors generated appear in the first term on the right-hand side.)

13. If we start with the cubic transformation formula [4, (4.05), with $\rho_1 \to b$, $\rho_2 \to 3a-b+\frac{3}{2}$, $x \to z$]

$$(1-z)^{-3\,a}\,{}_3F_2\left[\begin{matrix}a, \frac{1}{3}+a, \frac{2}{3}+a\\ b, \frac{3}{2}+3\,a-b\end{matrix}; -\frac{27\,z}{4\,(1-z)^3}\right] = {}_3F_2\left[\begin{matrix}3\,a, -3\,a+2\,b-1, 2+3\,a-2\,b\\ b, \frac{3}{2}+3\,a-b\end{matrix}; \frac{z}{4}\right] \quad (3.25)$$

and assume that a is a nonpositive integer, then we obtain

$$
{}_6F_5\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a,d,\frac{1}{2}+\frac{3\,a}{2}-\frac{e}{2},1+\frac{3\,a}{2}-\frac{e}{2}\\ \frac{3}{2}+3\,a-b,b,\frac{1}{3}+a+\frac{d}{3}-\frac{e}{3},\frac{2}{3}+a+\frac{d}{3}-\frac{e}{3},1+a+\frac{d}{3}-\frac{e}{3}\end{matrix};1\right]
= \frac{\Gamma(-3\,a+e)\,\Gamma(e-d)}{\Gamma(e)\,\Gamma(-3\,a-d+e)}\,{}_4F_3\left[\begin{matrix}3\,a,2+3\,a-2\,b,-3\,a+2\,b-1,d\\ \frac{3}{2}+3\,a-b,b,e\end{matrix};\frac{1}{4}\right], \tag{3.26}
$$

provided a or d is a nonpositive integer. This is an unusual identity featuring a transformation between a ${}_6F_5[1]$-series and a ${}_4F_3[\frac{1}{4}]$-series.

14. If we start with the second cubic transformation formula of Bailey [4, (4.06), with $\rho_1 \to b$, $\rho_2 \to \frac{3}{2}+3a-b$, $x \to z$]:

$$
{}_3F_2\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a\\ b,\frac{3}{2}+3\,a-b\end{matrix};\frac{27\,z^2}{4\,(1-z)^3}\right] = (1-z)^{3\,a}\,{}_3F_2\left[\begin{matrix}3\,a,-\frac{1}{2}+b,1+3\,a-b\\ 2\,b-1,2+6\,a-2\,b\end{matrix};4\,z\right] \tag{3.27}
$$

and assume that a is a nonpositive integer, then we obtain

$$
{}_6F_5\left[\begin{matrix}a,\frac{1}{3}+a,\frac{2}{3}+a,\frac{1}{2}+\frac{d}{2},\frac{d}{2},e\\ \frac{3}{2}+3\,a-b,b,\frac{d}{3}+\frac{e}{3},\frac{1}{3}+\frac{d}{3}+\frac{e}{3},\frac{2}{3}+\frac{d}{3}+\frac{e}{3}\end{matrix};1\right]
= \frac{\Gamma(1-e)\,\Gamma(1+3\,a-d-e)}{\Gamma(1+3\,a-e)\,\Gamma(1-d-e)}\,{}_4F_3\left[\begin{matrix}3\,a,1+3\,a-b,-\frac{1}{2}+b,d\\ 2+6\,a-2\,b,2\,b-1,1+3\,a-e\end{matrix};4\right], \tag{3.28}
$$

provided a or d is a nonpositive integer.

15. If we start with the transformation formula [5, Entry 4 of Ramanujan, Ch. 11, p. 50]:

$$
{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1}{2}+\frac{a}{2}\\ \frac{1}{2}+b\end{matrix};-\frac{4\,z}{(1-z)^2}\right] = (1-z)^{a}\,{}_2F_1\left[\begin{matrix}a,\frac{1}{2}+a-b\\ \frac{1}{2}+b\end{matrix};-z\right] \tag{3.29}
$$

and assume that a is a nonpositive integer, then we obtain

$$
{}_4F_3\left[\begin{matrix}\frac{1}{2}+\frac{a}{2},\frac{a}{2},d,e\\ \frac{1}{2}+b,\frac{d}{2}+\frac{e}{2},\frac{1}{2}+\frac{d}{2}+\frac{e}{2}\end{matrix};1\right] = \frac{\Gamma(1-e)\,\Gamma(1+a-d-e)}{\Gamma(1+a-e)\,\Gamma(1-d-e)}\,{}_3F_2\left[\begin{matrix}a,\frac{1}{2}+a-b,d\\ \frac{1}{2}+b,1+a-e\end{matrix};-1\right], \tag{3.30}
$$

provided a or d is a nonpositive integer.

16. The transformation formula [14, (1.8.10), with c replaced by $1+a+b-c$ and z replaced by $1-z$] expressing the Gauß solution valid for $|z|<1$, in terms of the Gauß functions valid for $|z-1|<1$ is

$$
{}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right] = (1-z)^{c-a-b}\,\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\,{}_2F_1\left[\begin{matrix}-b+c,-a+c\\ 1-a-b+c\end{matrix};1-z\right]
+ \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-b)\,\Gamma(c-a)}\,{}_2F_1\left[\begin{matrix}a,b\\ 1+a+b-c\end{matrix};1-z\right]. \tag{3.31}
$$

If our algorithm is applied, it results in

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};1\right] &= \frac{\Gamma(d)\,\Gamma(d-a-b)}{\Gamma(d-a)\,\Gamma(d-b)}\,{}_3F_2\left[\begin{matrix}a,b,-c+e\\ 1+a+b-d,e\end{matrix};1\right]\\
&+\frac{\Gamma(a+b-d)\,\Gamma(d)\,\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(e-c)\,\Gamma(d+e-a-b)}\,{}_3F_2\left[\begin{matrix}d-a,d-b,d+e-a-b-c\\ 1+d-a-b,d+e-a-b\end{matrix};1\right]
\end{aligned}
\tag{3.32}
$$

This three-term ${}_3F_2$-transformation formula can for example be found in [14, (4.3.4.2)].

17. The transformation formula [5, Entry 21 of Ramanujan]

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}a,b\\ \frac12+\frac a2+\frac b2\end{matrix};\frac{1-z}{2}\right] &= z\,\frac{\Gamma(-\frac12)\,\Gamma(\frac12+\frac a2+\frac b2)}{\Gamma(\frac a2)\,\Gamma(\frac b2)}\,{}_2F_1\left[\begin{matrix}\frac12+\frac a2,\frac12+\frac b2\\ \frac32\end{matrix};z^2\right]\\
&+\frac{\Gamma(\frac12)\,\Gamma(\frac12+\frac a2+\frac b2)}{\Gamma(\frac12+\frac a2)\,\Gamma(\frac12+\frac b2)}\,{}_2F_1\left[\begin{matrix}\frac a2,\frac b2\\ \frac12\end{matrix};z^2\right]
\end{aligned}
\tag{3.33}
$$

results in

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}a,b,e-d\\ \frac12+\frac a2+\frac b2,e\end{matrix};\frac12\right] &= \frac{\Gamma(\frac12)\,\Gamma(\frac12+\frac a2+\frac b2)}{\Gamma(\frac12+\frac a2)\,\Gamma(\frac12+\frac b2)}\,{}_4F_3\left[\begin{matrix}\frac a2,\frac b2,\frac12+\frac d2,\frac d2\\ \frac12,\frac12+\frac e2,\frac e2\end{matrix};1\right]\\
&+\frac{\Gamma(-\frac12)\,\Gamma(\frac12+\frac a2+\frac b2)\,\Gamma(1+d)\,\Gamma(e)}{\Gamma(\frac a2)\,\Gamma(\frac b2)\,\Gamma(d)\,\Gamma(1+e)}\,{}_4F_3\left[\begin{matrix}\frac12+\frac a2,\frac12+\frac b2,\frac12+\frac d2,1+\frac d2\\ \frac32,\frac12+\frac e2,1+\frac e2\end{matrix};1\right].
\end{aligned}
\tag{3.34}
$$

4. Products of hypergeometric series and identities for Kampé de Fériet series

There exist many relations connecting products of hypergeometric series, including the ones called the theorems of Cayley and Orr [3, Sec. 10.1]. In this section we apply our machinery to identities of the form

$$\text{“product of two hypergeometric series = hypergeometric series.”} \tag{4.1}$$

As before, on the right hand side we will obtain a new hypergeometric series. However, we will see that on the left hand side we obtain Kampé de Fériet series. These are hypergeometric double series. The standard notation, which will be used in the sequel, for these is as follows:

$$
F^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b');\\ (c):(d);(d');\end{matrix}\ z_1,z_2\right] = \sum_{m,n\ge0}\frac{\prod_{j=1}^{A}(a_j)_{m+n}\prod_{j=1}^{B}(b_j)_m\prod_{j=1}^{B'}(b'_j)_n}{\prod_{j=1}^{C}(c_j)_{m+n}\prod_{j=1}^{D}(d_j)_m\prod_{j=1}^{D'}(d'_j)_n}\frac{z_1^m z_2^n}{m!\,n!}.
$$

We report below the interesting results we obtain from the identities of the form (4.1) found in [14, Sec. 2.5].

1. “The” standard theorem of the type (4.1) is the celebrated Clausen's theorem. It concerns the square of a Gauß series and it is

$$
{}_2F_1\left[\begin{matrix}a,b\\ \frac12+a+b\end{matrix};z\right]^2 = {}_3F_2\left[\begin{matrix}2a,2b,a+b\\ 2a+2b,\frac12+a+b\end{matrix};z\right].
\tag{4.2}
$$

The *Mathematica* program given in Section 2 above has been modified to take care of cases like this and it generated the following result from (4.2):

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} d: & a,b & ; & a,b & ; \\ e: & \frac{1}{2}+a+b & ; & \frac{1}{2}+a+b & ; \end{matrix}\; x,x\right] = {}_4F_3\left[\begin{matrix} 2\,a, 2\,b, a+b, d \\ \frac{1}{2}+a+b, 2\,a+2\,b, e \end{matrix}; x\right], \tag{4.3}$$

(The reader should note that the above result is obtained by first replacing z by xz in (4.2) and subsequently applying our method.)

2. The transformation which follows from Theorem VII of Bailey's [14, (2.5.31)]

$${}_2F_1\left[\begin{matrix} a,b \\ 1+a+b-c \end{matrix}; z\right] {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix}; z\right] = {}_4F_3\left[\begin{matrix} a, b, \frac{a}{2}+\frac{b}{2}, \frac{1}{2}+\frac{a}{2}+\frac{b}{2} \\ a+b, c, 1+a+b-c \end{matrix}; 4\,(1-z)\,z\right] \tag{4.4}$$

results in

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} d: & a,b & ; & a,b & ; \\ e: & 1+a+b-c & ; & c & ; \end{matrix}\; 1,1\right] = {}_6F_5\left[\begin{matrix} a, \frac{a}{2}+\frac{b}{2}, \frac{1}{2}+\frac{a}{2}+\frac{b}{2}, b, d, e-d \\ a+b, 1+a+b-c, c, \frac{1}{2}+\frac{e}{2}, \frac{e}{2} \end{matrix}; 1\right]. \tag{4.5}$$

3. If we start with the transformation which follows from Theorem VIII of Bailey's [14, (2.5.32)]

$${}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix}; z\right] {}_2F_1\left[\begin{matrix} a,-b+c \\ c \end{matrix}; z\right] = (1-z)^{-a}\, {}_4F_3\left[\begin{matrix} a, b, -a+c, -b+c \\ c, \frac{c}{2}, \frac{1}{2}+\frac{c}{2} \end{matrix}; -\frac{z^2}{4\,(1-z)}\right] \tag{4.6}$$

and assume that a or b is a nonpositive integer, then we obtain

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} d: & a,b & ; & a,-b+c & ; \\ e: & c & ; & c & ; \end{matrix}\; 1,1\right] \\ = \frac{\Gamma(e)\,\Gamma(e-a-d)}{\Gamma(e-a)\,\Gamma(e-d)}\, {}_6F_5\left[\begin{matrix} a, b, -a+c, -b+c, \frac{1}{2}+\frac{d}{2}, \frac{d}{2} \\ \frac{1}{2}+\frac{c}{2}, \frac{c}{2}, c, 1+a+d-e, -a+e \end{matrix}; 1\right], \tag{4.7}$$

provided a, b, or d is a nonpositive integer.

5. Concluding Remarks

We have considered in this article a method, which we called the *beta integral method*, to obtain transformations and summation theorems from known transformations of hypergeometric and products of hypergeometric series. While being itself folklore, this method was automated with the help of programs written in *Mathematica* using the software package HYP developed by one of us [9]. We obtained some known identities (which we displayed only partially), and some which we were not able to trace in the literature, among them some very interesting ones, such as (3.10), (3.24) (3.26), or (3.28).

Clearly, variations (that can be again automated) may also be considered. For example, instead of using the beta integral evaluation, we may replace z by zt at the beginning and then use the Euler integral representation (1.2) (with $p=1$) of a ${}_2F_1[t]$-series. This yields identities which equate single sums to sums of ${}_2F_1$-series (aside from some explicit multiplicative factors). Another direction would be to use other extensions of the beta integral evaluation, most notably q-analogues thereof. However, it seems that, in particular in the case of q-analogues, the repertory of identities which can be used as a starting point, is very limited.

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