ON THE NON-QUADRATICITY OF VALUES OF THE q-EXPONENTIAL FUNCTION AND RELATED q-SERIES

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To our great Peter Bundschuh on his 70th birthday

Abstract. We investigate arithmetic properties of values of the entire function

$$F(z)=F_q(z;\lambda)=\sum_{n=0}^{\infty}\frac{z^n}{\prod_{j=1}^{n}(q^j-\lambda)}, \qquad |q|>1, \quad \lambda\notin q^{\mathbb{Z}_{>0}},$$

that includes as special cases the Tschakaloff function ($\lambda=0$) and the q-exponential function ($\lambda=1$). In particular, we prove the non-quadraticity of the numbers $F_q(\alpha;\lambda)$ for integral q, rational λ and $\alpha\notin -\lambda q^{\mathbb{Z}_{>0}}$, $\alpha\ne 0$.

1. Introduction and main results

Consider the q-exponential function

$$E_q(z)=\sum_{n=0}^{\infty}\frac{z^n}{\prod_{j=1}^{n}(q^j-1)}, \tag{1.1}$$

which is an entire function in the complex z-plane for any $q\in\mathbb{C}$, $|q|>1$. It is not difficult to adopt the classical proof of the irrationality of

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}$$

to the case of the number $E_q(1)$ for an integer $q>1$. Indeed, assuming, by contradiction, that $E_q(1)=r/s$ for certain positive integers r and s, we see that the real

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number

$$r\prod_{j=1}^{k}(q^j-1)-s\sum_{n=0}^{k}\prod_{j=n+1}^{k}(q^j-1) = s\prod_{j=1}^{k}(q^j-1)\cdot\left(E_q(1)-\sum_{n=0}^{k}\frac{1}{\prod_{j=1}^{n}(q^j-1)}\right)=s\sum_{n=k+1}^{\infty}\frac{1}{\prod_{j=k+1}^{n}(q^j-1)} \tag{1.2}$$

is integral (according to the left-hand side representation) and positive (because of the right-hand side representation), hence it is at least 1, for any integer $k\ge 1$. On the other hand,

$$s\sum_{n=k+1}^{\infty}\frac{1}{\prod_{j=k+1}^{n}(q^j-1)}<\frac{s}{q^{k+1}-1}\sum_{n=0}^{\infty}\frac{1}{2^n} = \frac{2s}{q^{k+1}-1}\to 0 \quad \text{as } k\to\infty,$$

leading to a contradiction.

The above proof is based on the simple observation that truncations of the series defining $E_q(1)$ (see the intermediate term in (1.2)) provide rational approximations that are good enough to conclude the irrationality of the number in question. This argument has been generalized in various ways. For example, this truncation idea lies at the heart of Mahler's method [13] of proving the algebraic independence of values of the series satisfying certain, quite restrictive, functional equations. In the same paper [13], K. Mahler posed a transcendence problem for values of the series that form a solution to more general functional equations. This problem remains unsolved until today, with the sole exception of values of quasi-modular functions [14]. In particular, only irrationality and linear independence results are known so far for values of the q-exponential function.

Recently, J.-P. Bézivin [1] proposed a new approach for the study of arithmetic properties of values of certain q-series. Among other things, he managed to prove the non-quadraticity of values of the so-called Tschakaloff function

$$T_q(z)=\sum_{n=0}^{\infty}q^{-n(n+1)/2}z^n \tag{1.3}$$

at non-zero rational points if $q=\rho/\sigma\in\mathbb{Q}$ satisfies $\gamma:=\log|\rho|/\log|\sigma|>14$. Furthermore, he proved the irrationality of these values if $\gamma>28/15=1.866\dots$, and thus extended considerably the possible values of q in the earlier irrationality results [18], [2]–[5], where $\gamma>(3+\sqrt{5})/2=2.618\dots$. It is interesting that Bézivin's approach was also an implicit generalization of the truncation idea. The method of [1] was applied to the q-exponential function by R. Choulet [6], who could not prove the non-quadraticity of its values, but improved the bound $\gamma>7/3$ of the earlier irrationality result of Bundschuh [2] for $E_q(z)$ to $\gamma>2$. He also improved the above bound $\gamma>14$ in Bézivin's non-quadraticity result for $T_q(z)$ to $\gamma>14/3$ and the bound $\gamma>28/15$ in the irrationality result to $\gamma>28/17$.

The aim of this article is two-fold. First of all, we further generalize Bézivin's method [1] to prove non-quadraticity results for values of the q-series

$$F(z) = F_q(z;\lambda) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{n}(q^j - \lambda)}, \qquad |q| > 1, \tag{1.4}$$

that include the Tschakaloff function and the q-exponential function as special cases ($\lambda = 0$ and $\lambda = 1$, respectively), and we further extend the values of q giving irrational values for $F_q(z;\lambda)$. Secondly, in our proofs we use a more direct method than the p-adic approach used in [1] and [6]. This allows us to perceive the additional arithmetic information which can hardly be seen from the p-adic considerations.

We state our results in the following two theorems.

Theorem 1. *Let $q = \rho/\sigma \in \mathbb{Q}$ with $|q| > 1$, and let α and λ satisfy $\alpha \neq 0$, $\lambda \notin q^{\mathbb{Z}_{>0}}$ and $\alpha \notin -\lambda q^{\mathbb{Z}_{>0}}$. If*

$$\gamma = \frac{\log|\rho|}{\log|\sigma|} > \begin{cases} \dfrac{126\pi^2}{47\pi^2 - 72\sqrt{3}\,\mathrm{Im}\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}/3})} = 3.27694460\ldots & \text{if } \lambda = 0, \\ \dfrac{27\pi^2}{5\pi^2 - 18\sqrt{3}\,\mathrm{Im}\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}/3})} = 9.43194241\ldots & \text{if } \lambda \neq 0, \end{cases}$$

then α, λ, and $\mu = F_q(\alpha;\lambda)$ in (1.4) cannot all belong to a quadratic extension of $\mathbb{Q}$. In particular, if α and λ are rational then $F_q(\alpha;\lambda)$ is neither rational nor quadratic.

In the case that $\lambda \neq 0$, the above result is entirely new, while its special case $\lambda = 0$ improves Choulet's bound $\gamma > 14/3$ considerably.

The next theorem gives improvements for the above mentioned lower bounds of γ in the irrationality results.

Theorem 2. *Under the hypotheses of Theorem 1, if*

$$\gamma = \frac{\log|\rho|}{\log|\sigma|} > \begin{cases} \dfrac{252\pi^2}{173\pi^2 - 72\sqrt{3}\,\mathrm{Im}\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}/3})} = 1.53237645\ldots & \text{if } \lambda = 0, \\ \dfrac{27\pi^2}{16\pi^2 - 9\sqrt{3}\,\mathrm{Im}\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}/3})} = 1.80828115\ldots & \text{if } \lambda \neq 0, \end{cases}$$

then α, λ, and $\mu = F_q(\alpha;\lambda)$ in (1.4) cannot all be rational.

Since the function $F_q(z;\lambda)$ satisfies the functional equation

$$F(qz) = (z + \lambda)F(z) + (1 - \lambda),$$

the irrationality of the values of $F_q(z;\lambda)$ at non-zero rational points $\notin -\lambda q^{\mathbb{Z}_{>0}}$ follows from [17] if a rational number $\lambda \notin q^{\mathbb{Z}_{>0}}$ and a rational number q satisfies $\gamma > 7/3$ for $\lambda \in q^{\mathbb{Z}_{\le 0}}$ and $\gamma > 2 + \sqrt{2}$ otherwise.

Sections 2–4 prepare for the proofs of these theorems. In Section 2, we review Bézivin's construction, applied to our more general context. It involves in particular the introduction of a sequence $(v_n)_{n\in\mathbb{Z}}$, the Hankel determinant of which plays a fundamental role in the sequel. This determinant is a polynomial in q and two other variables. Propositions 1 and 2 in Section 3 address the power of q which appears in this Hankel determinant as a polynomial factor, while an asymptotic upper bound for the Hankel determinant is found in Proposition 3. Finally, Proposition 4 in Section 4 detects large amounts of cyclotomic factors (in q) in the Hankel determinant. All these ingredients are put together for the proofs of Theorems 1 and 2 in Section 5.

2. Review of Bézivin's construction

The general idea of Bézivin's method [1] refers to a function

$$F(z)=\sum_{n=0}^{\infty}a_n(q)z^n,\qquad a_0(q)=1,\quad \frac{a_{n-1}(q)}{a_n(q)}=b_n(q)=b(q^n)\ \text{ for } n=1,2,\dots, \tag{2.1}$$

where $b(\,\cdot\,)$ is a polynomial (in general, a rational function) over a number field. Let $\alpha\in\mathbb{C}$. One takes the coefficients v_n appearing in

$$\frac{F(\alpha z)-F(\alpha)}{z-1}=\sum_{n=0}^{\infty}v_na_n(q)z^n \tag{2.2}$$

and forms the Hankel determinant

$$V_n=\det_{0\le i,j\le n-1}(v_{i+j}). \tag{2.3}$$

Then one has to provide an analytic upper bound for $|V_n|$ and, under the assumption that both α and $\mu=F(\alpha)$ belong to a certain algebraic number field K, an arithmetic lower bound, in order to find them contradictory; this shows that the assumption on α and μ cannot be true.

Before going into the details of the construction, note that relation (2.2) may be written in the form

$$\sum_{n=0}^{\infty}a_n(q)\alpha^nz^n-\mu=(z-1)\sum_{n=0}^{\infty}v_na_n(q)z^n=-v_0+\sum_{n=1}^{\infty}\big(v_{n-1}a_{n-1}(q)-v_na_n(q)\big)z^n,$$

yielding

$$v_0=\mu-1\qquad\text{and}\qquad v_n=v_{n-1}b_n(q)-\alpha^n\quad\text{for } n=1,2,\dots. \tag{2.4}$$

Hence, by induction, we easily arrive at the formula

$$v_n=\mu\prod_{j=1}^{n}b_j(q)-\sum_{k=0}^{n}\alpha^k\prod_{j=k+1}^{n}b_j(q). \tag{2.5}$$

Remark 1. Since we shall make use of it later on, we point out that Formula (2.5) also holds for *negative* n (that is, if we extend the sequence (v_n) to all integers n by letting the recurrence (2.4) hold for all integers n) under the conventions

$$\sum_{k=m}^{n-1}\operatorname{Expr}(k)=\begin{cases}\sum_{k=m}^{n-1}\operatorname{Expr}(k) & n>m,\\ 0 & n=m\\ -\sum_{k=n}^{m-1}\operatorname{Expr}(k) & n<m,\end{cases}$$

and

$$\prod_{k=m}^{n-1} \operatorname{Expr}(k) = \begin{cases} \prod_{k=m}^{n-1} \operatorname{Expr}(k) & n > m, \\ 1 & n = m \\ 1 \Big/ \prod_{k=n}^{m-1} \operatorname{Expr}(k) & n < m. \end{cases}$$

Assuming that $b(\,\cdot\,)$ in (2.1) is a polynomial of degree s, Formula (2.5) shows that, for positive integers n, V_n is a polynomial in μ, α, and q of degree at most n in μ, $n(n-1)$ in α, and

$$s \sum_{i=0}^{n-1} \frac{2i(2i+1)}{2} = \frac{sn(n-1)(4n+1)}{6} \tag{2.6}$$

in q (cf. [1, Lemma 2.4]). Formula (2.5) may also be written as

$$\begin{aligned} v_n &= \prod_{j=1}^{n} b_j(q) \cdot \left(\mu - \sum_{k=0}^{n} \alpha^k \prod_{j=1}^{k} \frac{1}{b_j(q)} \right) \\ &= a_n(q)^{-1} \cdot \left(\sum_{k=0}^{\infty} a_k(q)\alpha^k - \sum_{k=0}^{n} a_k(q)\alpha^k \right) \\ &= a_n(q)^{-1} \cdot \sum_{k=n+1}^{\infty} a_k(q)\alpha^k = \sum_{k=n+1}^{\infty} \frac{\alpha^k}{\prod_{j=n+1}^{k} b_j(q)}, \end{aligned}$$

showing that the v_n's are nothing else but *tails* of the series $\mu = \sum_{k=0}^{\infty} a_k(q)\alpha^k$ (normalized by the factors $a_n(q)^{-1}$; cf. the intermediate part of (1.2)). This fact somehow explains why the determinant in (2.3) is expected to be 'small'.

Our basic example (1.4) corresponds to the choice $b_n(q) = q^n - \lambda$, for a fixed algebraic number λ. In this case, we have $a_n(q) = \prod_{k=1}^{n}(q^k - \lambda)^{-1}$, and the Hankel determinant V_n is also a polynomial in λ of degree at most $n(n-1)$. The choice $b_n(q) = q^n$ (that is, $\lambda = 0$), yielding the Tschakaloff function (1.3), was the illustrative example of the method in [1], while the choice $b_n(q) = q^n - 1$ (when $\lambda = 1$) results in the q-exponential function (1.1). In [6], Choulet treated both the Tschakaloff and q-exponential cases.

We replace the argument of Bézivin and Choulet by a more direct approach (see Sections 3–5 below); in particular, we do not require the non-trivial p-adic techniques used in [1] and [6], thus making our proofs more 'concrete' and elementary. An essential gain, which allows us to succeed in proving the non-quadraticity of the values of (1.4), is due to extraction of cyclotomic factors in the factorization of the Hankel determinant (2.3); this is explained in Section 4.

3. Determinant calculus

Define the (q-)*order* of a Laurent series $f(q) = \sum_{n\in\mathbb{Z}} c_n q^n$ as

$$\operatorname{ord} f(q) = \operatorname{ord}_q f(q) = \min\{n : c_n \neq 0\}.$$

The *q-binomial coefficient* $\left[\begin{smallmatrix} m \\ k \end{smallmatrix} \right]_q$ is defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)} & \text{if } k \ge 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Moreover, we adopt the usual notation for shifted q-factorials, given by $(a;q)_m := (1-a)(1-aq)\cdots(1-aq^{m-1})$ if $m > 0$, and $(a;q)_0 := 1$.

Specializing $b_j(q) = q^j - \lambda$ in (2.4), where $\lambda \ne q^{\mathbb{Z}_{>0}}$, we consider the sequence defined by

$$v_0 = \mu - 1, \qquad v_n = (q^n - \lambda)v_{n-1} - \alpha^n, \tag{3.1}$$

where

$$\mu = \sum_{n=0}^{\infty} \frac{\alpha^n}{\prod_{k=1}^n (q^k - \lambda)}. \tag{3.2}$$

We follow Remark 1 in requiring the recursive relation to be valid for all $n \in \mathbb{Z}$. This does, in fact, not work if $q^n - \lambda = 0$ for some integer $n \le 0$. However, since the only places where we take recourse on the extension of (3.1) to negative integers is in Remark 2 and in the proof of Proposition 2, in a context where $\lambda = 0$, we do not have to worry about these exceptional cases.

Let $\mathcal{N}$ denote the backward shift operator acting (solely) on the index of the sequence $(v_n)_{n\in\mathbb{Z}}$, that is $\mathcal{N} v_n = v_{n-1}$. Introduce the difference operator

$$\mathcal{D}_l = (-\lambda\mathcal{N};q)_l\,(\alpha\mathcal{N};q)_l = \prod_{k=0}^{l-1}(\mathcal{I} + (\lambda-\alpha)q^k\mathcal{N} - \lambda\alpha q^{2k}\mathcal{N}^2), \tag{3.3}$$

where $\mathcal{I}$ is the identity operator.

Lemma 1. *For $n \in \mathbb{Z}$ and $l \ge 0$ we have*

$$\mathcal{D}_l v_n = q^{l(n-l)} \sum_{s=0}^{l} \begin{bmatrix} l \\ s \end{bmatrix}_q q^{\binom{l-s+1}{2}} (-\alpha)^s v_{n-l-s}. \tag{3.4}$$

Proof. By the q-binomial theorem (cf. [9, Ex. 1.2(vi)])

$$(1+z)(1+qz)\cdots(1+q^{m-1}z) = \sum_{\ell=0}^{m} q^{\binom{\ell}{2}} \begin{bmatrix} m \\ \ell \end{bmatrix}_q z^\ell, \tag{3.5}$$

we can write

$$\mathcal{D}_l = \sum_{k_1=0}^{l}\sum_{k_2=0}^{l} q^{\binom{k_1}{2}+\binom{k_2}{2}} \begin{bmatrix} l \\ k_1 \end{bmatrix}_q \begin{bmatrix} l \\ k_2 \end{bmatrix}_q \lambda^{k_1}(-\alpha)^{k_2}\mathcal{N}^{k_1+k_2}.$$

Hence, what we want to prove is

$$\sum_{k_1=0}^{l}\sum_{k_2=0}^{l} q^{\binom{k_1}{2}+\binom{k_2}{2}} \begin{bmatrix} l \\ k_1 \end{bmatrix}_q \begin{bmatrix} l \\ k_2 \end{bmatrix}_q \lambda^{k_1}(-\alpha)^{k_2} v_{n-k_1-k_2} = q^{l(n-l)} \sum_{s=0}^{l} \begin{bmatrix} l \\ s \end{bmatrix}_q q^{\binom{l-s+1}{2}} (-\alpha)^s v_{n-l-s} \tag{3.6}$$

for $n \in \mathbb{Z}$ and $l \in \mathbb{N}_0$. For $l = 0, 1$, this equality can be readily verified.

We now assume that (3.6) is valid for some $l \geq 1$ and all n. Substituting $n-1$ and $n-2$ instead of n, we get

$$\sum_{k_1=0}^{l}\sum_{k_2=0}^{l} q^{\binom{k_1}{2}+\binom{k_2}{2}} \begin{bmatrix} l \\ k_1 \end{bmatrix}_q \begin{bmatrix} l \\ k_2 \end{bmatrix}_q \lambda^{k_1}(-\alpha)^{k_2} v_{n-k_1-k_2-1} = q^{l(n-l-1)} \sum_{s=0}^{l} \begin{bmatrix} l \\ s \end{bmatrix}_q q^{\binom{l-s+1}{2}} (-\alpha)^s v_{n-l-s-1}, \tag{3.7}$$

respectively

$$\sum_{k_1=0}^{l}\sum_{k_2=0}^{l} q^{\binom{k_1}{2}+\binom{k_2}{2}} \begin{bmatrix} l \\ k_1 \end{bmatrix}_q \begin{bmatrix} l \\ k_2 \end{bmatrix}_q \lambda^{k_1}(-\alpha)^{k_2} v_{n-k_1-k_2-2} = q^{l(n-l-2)} \sum_{s=0}^{l} \begin{bmatrix} l \\ s \end{bmatrix}_q q^{\binom{l-s+1}{2}} (-\alpha)^s v_{n-l-s-2}. \tag{3.8}$$

Next we form the linear combination

$$(3.6) + (\lambda-\alpha)q^l \cdot (3.7) - \lambda\alpha q^{2l} \cdot (3.8). \tag{3.9}$$

We claim that the left-hand side of (3.9) is equal to the left-hand side of (3.6) with l replaced by $l+1$. To see this, we rewrite the left-hand side of $\lambda q^l \cdot (3.7)$ in the form

$$\lambda q^l \sum_{k_1=0}^{l}\sum_{k_2=0}^{l} q^{\binom{k_1}{2}+\binom{k_2}{2}} \begin{bmatrix} l \\ k_1 \end{bmatrix}_q \begin{bmatrix} l \\ k_2 \end{bmatrix}_q \lambda^{k_1}(-\alpha)^{k_2} v_{n-k_1-k_2-1} = \sum_{k_1=0}^{l+1}\sum_{k_2=0}^{l+1} q^{l-k_1+1+\binom{k_1}{2}+\binom{k_2}{2}} \begin{bmatrix} l \\ k_1-1 \end{bmatrix}_q \begin{bmatrix} l \\ k_2 \end{bmatrix}_q \lambda^{k_1}(-\alpha)^{k_2} v_{n-k_1-k_2}, \tag{3.10}$$

we rewrite the left-hand side of $-\alpha q^l \cdot (3.7)$ in the form

$$-\alpha q^l \sum_{k_1=0}^{l}\sum_{k_2=0}^{l} q^{\binom{k_1}{2}+\binom{k_2}{2}} \begin{bmatrix} l \\ k_1 \end{bmatrix}_q \begin{bmatrix} l \\ k_2 \end{bmatrix}_q \lambda^{k_1}(-\alpha)^{k_2} v_{n-k_1-k_2-1} = \sum_{k_1=0}^{l+1}\sum_{k_2=0}^{l+1} q^{l-k_2+1+\binom{k_1}{2}+\binom{k_2}{2}} \begin{bmatrix} l \\ k_1 \end{bmatrix}_q \begin{bmatrix} l \\ k_2-1 \end{bmatrix}_q \lambda^{k_1}(-\alpha)^{k_2} v_{n-k_1-k_2}, \tag{3.11}$$

and we rewrite the left-hand side of $-\lambda\alpha q^{2l}\cdot$(3.8) in the form

$$-\lambda\alpha q^{2l}\sum_{k_1=0}^{l}\sum_{k_2=0}^{l} q^{\binom{k_1}{2}+\binom{k_2}{2}}\begin{bmatrix}l\\k_1\end{bmatrix}_q\begin{bmatrix}l\\k_2\end{bmatrix}_q\lambda^{k_1}(-\alpha)^{k_2}v_{n-k_1-k_2-2}$$
$$=\sum_{k_1=0}^{l+1}\sum_{k_2=0}^{l+1} q^{(l-k_1+1)+(l-k_2+1)+\binom{k_1}{2}+\binom{k_2}{2}}\begin{bmatrix}l\\k_1-1\end{bmatrix}_q\begin{bmatrix}l\\k_2-1\end{bmatrix}_q\lambda^{k_1}(-\alpha)^{k_2}v_{n-k_1-k_2}. \tag{3.12}$$

By summing the left-hand side of (3.6) and the right-hand sides of (3.10), (3.11), and (3.12), we obtain indeed the left-hand side of (3.6) with l replaced by $l+1$, after little simplification.

We now turn our attention to the right-hand side of (3.9), that is, to

$$q^{l(n-l)}\sum_{s=0}^{l}\begin{bmatrix}l\\s\end{bmatrix}_q q^{\binom{l-s+1}{2}}(-\alpha)^s(v_{n-l-s}+(\lambda-\alpha)v_{n-l-s-1}-\lambda\alpha v_{n-l-s-2}).$$

By (3.6) with $l=1$ and n replaced by $n-l-s$, this is equal to

$$q^{l(n-l)}\sum_{s=0}^{l}\begin{bmatrix}l\\s\end{bmatrix}_q q^{\binom{l-s+1}{2}}(-\alpha)^s q^{n-l-s-1}(qv_{n-l-s-1}-\alpha v_{n-l-s-2}).$$

It is not difficult to transform this into the right-hand side of (3.6) with l replaced by $l+1$. □

As a corollary, we get for $l\ge 0$ and $n\ge 2l-1$

$$\operatorname{ord}_q \mathcal{D}_l v_n = l(n-l). \tag{3.13}$$

Moreover, we have

$$q^{-l(n-l)}\mathcal{D}_l v_n\big|_{q=0}=(-\alpha)^l v_{n-2l}\big|_{q=0}.$$

Remark 2. In the proof of Proposition 2 below, under the hypothesis $\lambda=0$, we require the estimate

$$\operatorname{ord}_q \mathcal{D}_l v_n > l(n-l) \quad \text{if } n<2l-1,$$

which complements (3.13) and also follows from Lemma 1. This estimate is valid for *negative* indices n as well: recall that the definition (3.1) of our sequence $(v_n)_{n\in\mathbb{Z}}$ in the case $\lambda=0$ and $\alpha\ne 0$ results in

$$v_{-n-1}=q^n(v_{-n}+\alpha^{-n}),$$

whence $\operatorname{ord}_q v_{-n}=n-1$ for $n\ge 1$, implying the desired estimate for negative indices.

Proposition 1. *Let $\lambda\ne 0$, and let the sequence v_n be given by* (3.1)*. Then the Hankel determinant $V_n=\det_{0\le i,j\le n-1}(v_{i+j})$, viewed as an analytic function* (*in fact, a polynomial*) *in q, α, λ, and μ, admits the representation*

$$V_n=\begin{cases}\alpha^{n(n-1)/2}\lambda^{n(n-2)/4}(\lambda-(\lambda+\alpha)\mu)^{n/2}\cdot q^{e_0(n)}+O(q^{e_0(n)+1}) & \text{if } n \text{ is even,}\\ \alpha^{n(n-1)/2}\lambda^{(n-1)^2/4}(\mu-1)(\lambda-(\lambda+\alpha)\mu)^{(n-1)/2}\cdot q^{e_0(n)}+O(q^{e_0(n)+1}) & \\ & \text{if } n \text{ is odd,}\end{cases}$$

where

$$e_0(n) = \frac{n(n-1)(n-2)}{6} = \binom{n}{3}. \tag{3.14}$$

In particular, its q-order under any specialization of α, λ, and μ is at least $e_0(n)$.

Proof. We act on the i-th row of the matrix $(v_{i+j})_{0\le i,j\le n-1}$ by the operator $\mathcal{D}_{\lfloor i/2\rfloor}$; doing this for $i = n-1, n-2, \dots, 1, 0$ (in this order!). By definition (3.3) we have a sequence of elementary row operations, hence the new matrix with entries $a_{ij} = \mathcal{D}_{\lfloor i/2\rfloor} v_{i+j}$, $0 \le i, j \le n-1$, has the same determinant V_n. According to (3.4),

$$e_{ij} := \operatorname{ord}_q a_{ij} = \left\lfloor \frac{i}{2} \right\rfloor \left(i + j - \left\lfloor \frac{i}{2} \right\rfloor \right) = \left\lfloor \frac{i}{2} \right\rfloor \left(\left\lceil \frac{i}{2} \right\rceil + j \right),$$

and for a permutation τ of $\{0, 1, \dots, n-1\}$ we have

$$\sum_{i=0}^{n-1} e_{i,\tau(i)} = \sum_{i=0}^{n-1} \left\lfloor \frac{i}{2} \right\rfloor \left\lceil \frac{i}{2} \right\rceil + \sum_{i=0}^{n-1} \left\lfloor \frac{i}{2} \right\rfloor \tau(i).$$

We claim that the minimal value of the latter expression is equal to $n(n-1)(n-2)/6$ and is attained, e.g., for $\tau(0) > \tau(1) > \dots > \tau(n-1)$. To see this, first observe that, if $\lfloor i_1/2\rfloor > \lfloor i_2/2\rfloor$ and $j_1 > j_2$, then

$$\left\lfloor \frac{i_1}{2} \right\rfloor j_1 + \left\lfloor \frac{i_2}{2} \right\rfloor j_2 > \left\lfloor \frac{i_1}{2} \right\rfloor j_2 + \left\lfloor \frac{i_2}{2} \right\rfloor j_1.$$

Hence we necessarily have $\tau(0) > \tau(i)$ and $\tau(1) > \tau(i)$ for all $i \ge 2$. In other words, the 2-element set $\{\tau(0), \tau(1)\}$ is $\{n-1, n-2\}$. Continuing in this manner, we obtain $\{\tau(2), \tau(3)\} = \{n-3, n-4\}$, $\{\tau(4), \tau(5)\} = \{n-5, n-6\}$, and so on. It follows that indeed, for any permutation τ, we have

$$\sum_{i=0}^{n-1} e_{i,\tau(i)} \ge \sum_{i=0}^{n-1} \left\lfloor \frac{i}{2} \right\rfloor \left\lceil \frac{i}{2} \right\rceil + \sum_{i=0}^{\lfloor n/2\rfloor - 1} i\big(n-1-2i+n-2-2i\big) = \binom{n}{3}.$$

Moreover, the coefficient of the minimal power $q^{n(n-1)(n-2)/6}$ is equal to the determinant of the 'anti-diagonal' matrix

$$\begin{pmatrix} 0 & & & \boxed{\begin{matrix} v_{n-2} & v_{n-1} \\ v_{n-1} & v_n \end{matrix}} \\ & & \iddots & \\ & \boxed{\begin{matrix} (-\alpha)^l v_{n-2l-2} & (-\alpha)^l v_{n-2l-1} \\ (-\alpha)^l v_{n-2l-1} & (-\alpha)^l v_{n-2l} \end{matrix}} & & \\ \iddots & & & 0 \end{pmatrix} \tag{3.15}$$

evaluated at $q = 0$. Here, if n is odd, the left lower angle of the matrix contains just the 1×1-matrix $(-\alpha)^{(n-1)/2} v_0$. Let us compute the determinant of a 2×2-box in (3.15) assuming $q = 0$ throughout. For $n \ge 0$, the explicit expression (2.5) for v_n with $b_j(q) = q^j - \lambda = -\lambda$ yields

$$v_n = \mu(-\lambda)^n - (-\lambda)^n \frac{1 - (-\frac{\alpha}{\lambda})^{n+1}}{1 + \frac{\alpha}{\lambda}}.$$

Hence,

$$\det \begin{pmatrix} (-\alpha)^l v_{n-2l-2} & (-\alpha)^l v_{n-2l-1} \\ (-\alpha)^l v_{n-2l-1} & (-\alpha)^l v_{n-2l} \end{pmatrix} = \alpha^{n-1}(-\lambda)^{n-2l-1}\left(\mu\left(1+\frac{\alpha}{\lambda}\right)-1\right).$$

Therefore the desired coefficient of $q^{n(n-1)(n-2)/6}$ in V_n is equal to

$$\prod_{l=0}^{n/2-1}\left(\alpha^{n-1}(-\lambda)^{n-2l-2}(\lambda-(\lambda+\alpha)\mu)\right)$$

if n is even, and it is equal to

$$(-\alpha)^{(n-1)/2}(\mu-1)\prod_{l=0}^{(n-1)/2-1}\left(\alpha^{n-1}(-\lambda)^{n-2l-2}(\lambda-(\lambda+\alpha)\mu)\right)$$

if n is odd. □

Proposition 2. *Let $\lambda = 0$, and let the sequence v_n be given by (3.1). Then the Hankel determinant $V_n = \det_{0\le i,j\le n-1}(v_{i+j})$, viewed as an analytic function (in fact, a polynomial) in q, α, and μ, admits the representation*

(3.16)
$$V_n = (-1)^{n(n+2)/8}\alpha^{n(5n-2)/8}\mu^{n/2}\cdot q^{e_0(n)} + O(q^{e_0(n)+1}) \qquad \text{if } n \text{ is even,}$$

(3.17)
$$V_n = (-1)^{(n-1)(n-3)/8}\alpha^{(n-1)(5n+1)/8}K_{(n-1)/2}\cdot q^{e_0(n)} + O(q^{e_0(n)+1}) \qquad \text{if } n \text{ is odd,}$$

where the sequence $K_n = K_n(\alpha,\mu)$ is defined in (3.20) below, and

$$e_0(n) = \begin{cases} \dfrac{n(n-2)(5n-2)}{24} & \text{if } n \text{ is even,} \\ \dfrac{n(n-1)(5n-7)}{24} & \text{if } n \text{ is odd.} \end{cases} \tag{3.18}$$

In particular, the q-order of V_n under any specialization of α and μ is at least $e_0(n)$.

Proof. This time we act on the i-th row of the matrix $(v_{i+j})_{0\le i,j\le n-1}$, for $i = n-1$, $n-2, \dots, 1, 0$, by the operator $\mathcal{D}_{l_i}$, where $l_i = \min\{i, \lfloor n/2 \rfloor\}$. Again, these are elementary row transformations because $\mathcal{D}_l = (\alpha\mathcal{N}; q)_l$ in (3.3) for $\lambda = 0$. For the entries $a_{ij} = \mathcal{D}_{l_i} v_{i+j}$ of the resulting matrix, whose determinant is V_n, we have $e_{ij} := \operatorname{ord}_q a_{ij} \ge l_i(i+j-l_i)$, with equality occurring when $i+j \ge 2l_i - 1$ (cf. Remark 2). This fact and the fact that the sequence (l_i) is non-decreasing imply, for any permutation τ of $\{0, 1, \dots, n-1\}$, that

$$\sum_{i=0}^{n-1} e_{i,\tau(i)} \ge \sum_{i=0}^{n-1} l_i(i+\tau(i)-l_i) \ge \sum_{i=0}^{n-1} l_i(n-1-l_i) = e_0(n), \tag{3.19}$$

for $e_0(n)$ defined in (3.18). To get equality in (3.19), the following two conditions should be satisfied: (a) for each i we have $i+\tau(i)\ge 2l_i-1$, implying $\tau(i)\ge 2\lfloor n/2\rfloor - i - 1$ for $i\ge\lfloor n/2\rfloor$, and (b) for $i<\lfloor n/2\rfloor$ we have $\tau(i)=n-i-1$.

In the case of even n, condition (a) gives us $\tau(i)\ge n-i-1$ for $i\ge n/2$, which in view of condition (b) is possible if and only if $\tau(i)=n-i-1$ for each $i=0,1,\dots,n-1$. Therefore, the unique anti-diagonal product $(-1)^{n(n-1)/2}\prod_{i=0}^{n-1}a_{i,n-1-i}$ provides the lowest power $q^{n(n-2)(5n-2)/24}$ in the determinant $\det(a_{ij})_{0\le i,j\le n-1}$, implying (3.16).

If n is odd, conditions (a) and (b) take the form

$$\begin{aligned} \tau(i) &= n-i-1 \qquad \text{for } i=0,1,\dots,\tfrac{n-3}{2},\\ \tau(i) &\ge n-i-2 \qquad \text{for } i=\tfrac{n-1}{2},\dots,n-1. \end{aligned}$$

In this case the coefficient of the lowest power $q^{n(n-1)(5n-7)/24}$ in V_n is equal to the determinant of the matrix

$$\left(\begin{array}{c|c} \mathbf{0} & \begin{matrix} & & & (-\alpha)^0 v_{n-1}\\ & & (-\alpha)^1 v_{n-3} & \\ & \iddots & & \\ (-\alpha)^{(n-3)/2}v_2 & & & \end{matrix} \\ \boxed{\begin{matrix} 0 & & (-\alpha)^{\lfloor n/2\rfloor}v_{-1} & (-\alpha)^{\lfloor n/2\rfloor}v_0\\ & \iddots & (-\alpha)^{\lfloor n/2\rfloor}v_0 & (-\alpha)^{\lfloor n/2\rfloor}v_1\\ (-\alpha)^{\lfloor n/2\rfloor}v_{-1} & \iddots & & \vdots\\ (-\alpha)^{\lfloor n/2\rfloor}v_0 & \dots & \dots & (-\alpha)^{\lfloor n/2\rfloor}v_{\lfloor n/2\rfloor} \end{matrix}} & \mathbf{0} \end{array}\right)$$

evaluated at $q=0$. It is clear that the non-vanishing of the coefficient will follow from the non-vanishing of the determinant

$$\begin{aligned} \det\begin{pmatrix} 0 & & v_{-1} & v_0\\ & \iddots & v_0 & v_1\\ v_{-1} & \iddots & & \vdots\\ v_0 & \dots & \dots & v_{(n-1)/2} \end{pmatrix}\Bigg|_{q=0} &= \det\begin{pmatrix} 0 & & \mu & \mu-1\\ & \iddots & \mu-1 & -\alpha\\ \mu & \iddots & & \vdots\\ \mu-1 & \dots & \dots & -\alpha^{(n-1)/2} \end{pmatrix}\\ &= (-1)^{(n-1)(n-3)/8}K_{(n-1)/2}, \end{aligned}$$

where

$$K_n=K_n(\alpha,\mu)=\det\begin{pmatrix} \mu-1 & -\alpha & -\alpha^2 & \dots & -\alpha^n\\ \mu & \mu-1 & -\alpha & \dots & -\alpha^{n-1}\\ & \mu & \mu-1 & \dots & -\alpha^{n-2}\\ & & \ddots & \ddots & \vdots\\ \mathbf{0} & & & \mu & \mu-1 \end{pmatrix}. \tag{3.20}$$

Summarizing we have (3.17), and the proposition follows. $\square$

Remark 3. Generically, the power $e_0(n)$ in (3.18) is exact. Indeed, the determinant K_n is not identically zero, since it is trivially non-zero for $\alpha=0$ and $\mu\ne 1$. In fact, we can also write down explicit formulas for K_n, since the sequence satisfies the linear recurrence

$$K_{n+2} = (\mu - 1 - \alpha\mu)K_{n+1} + \alpha\mu^2 K_n \quad \text{for } n = 0, 1, 2, \dots,$$
$$K_0 = \mu - 1, \quad K_1 = (\mu - 1)^2 + \alpha\mu.$$

To find an (asymptotic) upper bound for our Hankel determinant

$$\det_{0\le i,j\le n-1}(v_{i+j}),$$

where

$$v_n = \sum_{k=n+1}^{\infty} \frac{\alpha^k}{\prod_{j=n+1}^{k}(q^j - \lambda)} \quad \text{for } n = 0, 1, 2, \dots, \tag{3.21}$$

we will use the difference operator

$$\widetilde{\mathcal{D}}_l = (\alpha q^{-1}\mathcal{N}; q^{-1})_l = \prod_{k=1}^{l}(\mathcal{I} - \alpha q^{-k}\mathcal{N}). \tag{3.22}$$

Remark 4. The operators (3.3) and (3.22) are directly related by

$$\mathcal{D}_l v_n = q^{ln-\binom{l}{2}} \widetilde{\mathcal{D}}_l v_{n-l},$$

where $l \ge 0$ and $n \in \mathbb{Z}$. This is seen by applying the q-binomial theorem (3.5) to (3.22), and by comparing the result with (3.4). Equivalently,

$$\widetilde{\mathcal{D}}_l v_n = q^{\binom{l}{2}-l(n+l)} \mathcal{D}_l v_{n+l}.$$

Lemma 2. *Let q, α, λ be complex numbers with $|q| > 1$, $\alpha \ne 0$, and λ not a (positive) power of q. Then, for $0 \le l \le n$, we have*

$$|\widetilde{\mathcal{D}}_l v_n| \le \begin{cases} |q|^{\binom{l}{2}-nl} C_1^{n+1} & \text{if } \lambda \ne 0, \\ |q|^{-nl} C_2^{n+1} & \text{if } \lambda = 0, \end{cases}$$

where C_1 and C_2 are positive real numbers not depending on n and l.

Proof. Let first $\lambda \ne 0$. From (3.21), it is easy to see that we can find a real number $C > 1$ (depending on q, α and λ, but not on n), such that $|v_n| \le C^{n+1}$. Making use of the q-binomial theorem (3.5), of the fact that

$$\left| \begin{bmatrix} m \\ k \end{bmatrix}_q \right| \le \begin{bmatrix} m \\ k \end{bmatrix}_{|q|}$$

(following from the polynomiality in q of $\left[{m \atop k}\right]_q$, the coefficients being non-negative integers), and of our observation in Remark 4, we have

$$\begin{aligned}
|\widetilde{\mathcal{D}}_l v_n| &= |q|^{\binom{l}{2}-l(n+l)}|\mathcal{D}_l v_{n+l}| \\
&\le |q|^{-\binom{l+1}{2}-ln}\sum_{k_1=0}^{l}|q|^{\binom{k_1}{2}}\begin{bmatrix} l \\ k_1\end{bmatrix}_{|q|}|\lambda|^{k_1}\sum_{k_2=0}^{l}|q|^{\binom{k_2}{2}}\begin{bmatrix} l \\ k_2\end{bmatrix}_{|q|}|\alpha|^{k_2}|v_{n+l-k_1-k_2}| \\
&\le |q|^{-\binom{l+1}{2}-ln}C^{n+l+1}\prod_{j=0}^{l-1}(1+|\lambda|\,|q|^j)(1+|\alpha|\,|q|^j) \\
&\le |q|^{\binom{l}{2}-l-ln}C^{n+l+1}(1+|\lambda|)^l(1+|\alpha|)^l,
\end{aligned}$$

which, in view of $l \le n$, is exactly in line with the first assertion of the lemma.

In the case $\lambda = 0$, we can proceed in the same way. The only difference is that the above sum over k_1 reduces to just the summand for $k_1 = 0$. Hence, we obtain

$$\begin{aligned}
|\widetilde{\mathcal{D}}_l v_n| &\le |q|^{-\binom{l+1}{2}-ln}\sum_{k_2=0}^{l}|q|^{\binom{k_2}{2}}\begin{bmatrix} l \\ k_2\end{bmatrix}_{|q|}|\alpha|^{k_2}|v_{n+l-k_2}| \\
&\le |q|^{-\binom{l+1}{2}-ln}C^{n+l+1}\prod_{j=0}^{l-1}(1+|\alpha|\,|q|^j) \\
&\le |q|^{-l-ln}C^{n+l+1}(1+|\alpha|)^l,
\end{aligned}$$

which is exactly in line with the second assertion of the lemma. □

Proposition 3. *Let the sequence v_n be given by (3.21). Then, as n tends to ∞, the Hankel determinant $V_n = \det_{0\le i,j\le n-1}(v_{i+j})$ is asymptotically*

$$|V_n| \le \begin{cases} |q|^{-n^3/3}\exp(O(n^2)) & \text{if } \lambda \ne 0, \\ |q|^{-n^3/2}\exp(O(n^2)) & \text{if } \lambda = 0. \end{cases} \tag{3.23}$$

Proof. Acting by the operator $\widetilde{\mathcal{D}}_i$ on the i-th row of the matrix $(v_{i+j})_{0\le i,j\le n-1}$ (this results in elementary row operations according to (3.22)) we get the matrix $(a_{ij})_{0\le i,j\le n-1}$ with entries $a_{ij} = \widetilde{\mathcal{D}}_i v_{i+j}$, whose determinant is equal to V_n.

Let now $\lambda \ne 0$. Writing $\mathfrak{S}_n$ for the symmetric group on $\{0,1,2,\dots,n-1\}$, we have

$$\begin{aligned}
|V_n| &\le n!\max_{\tau\in\mathfrak{S}_n}\prod_{i=0}^{n-1}|\widetilde{\mathcal{D}}_i v_{i+\tau(i)}| \\
&\le n!\max_{\tau\in\mathfrak{S}_n}\prod_{i=0}^{n-1}|q|^{\binom{i}{2}-(i+\tau(i))i}C_1^{i+\tau(i)+1} \\
&\le \exp(O(n^2))\prod_{i=0}^{n-1}|q|^{\binom{i}{2}-(n-1)i},
\end{aligned}$$

where, to go from the next-to-last to the last line, we used again the fact that the permutation achieving the maximum is the permutation sending i to $n-i-1$, $i=0,1,\dots,n-1$. This implies the first claim of the proposition.

On the other hand, if $\lambda=0$, then we have

$$\begin{aligned}|V_n| &\le n!\max_{\tau\in\mathfrak{S}_n}\prod_{i=0}^{n-1}|\widetilde{\mathcal{D}}_i v_{i+\tau(i)}|\\ &\le n!\max_{\tau\in\mathfrak{S}_n}\prod_{i=0}^{n-1}|q|^{-(i+\tau(i))i}C_2^{i+\tau(i)+1}\\ &\le \exp(O(n^2))\prod_{i=0}^{n-1}|q|^{-(n-1)i},\end{aligned}$$

implying the second claim of the proposition. □

Remark 5. Using an analytic method for the (entire) generating series $\sum_{n=0}^{\infty} v_n z^n$, Choulet [6, Lemmas 3.3 and 3.4] proves estimates that may be informally summarized in our settings as follows:

$$\operatorname{ord}_q V_n \ge \begin{cases}\frac{5}{24}n^3+O(n^2) & \text{if } \lambda=0,\\ \frac{1}{6}n^3+O(n^2) & \text{if } \lambda\ne 0,\end{cases}\qquad \text{as}\quad n\to\infty, \tag{3.24}$$

and

$$|V_n| \le \begin{cases}|q|^{-\frac{1}{2}n^3+O(n^2)} & \text{if } \lambda=0,\\ |q|^{-\frac{1}{3}n^3+O(n^2)} & \text{if } \lambda\ne 0,\end{cases}\qquad \text{as}\quad n\to\infty. \tag{3.25}$$

Therefore, our Propositions 1–2 sharpen the estimate (3.24) of Choulet, while our Proposition 3 provides a different proof of (3.25). (Strictly speaking, Choulet did not arrive at the better estimate in (3.25) for the case $\lambda=0$ himself; this had been done earlier by Bézivin [1] using elementary considerations.) On the other hand, it is Choulet's method that suggested to us the form of the difference operators (3.3) and (3.22).

4. Cyclotomic factorization

We now turn to the general Hankel determinant (2.3) with the sequence (v_n) defined in (2.4) and (2.5). Let us fix the notation

$$\Phi_l(q)=\prod_{\substack{1\le j\le l\\ \gcd(j,l)=1}}(q-e^{2\pi j\sqrt{-1}/l}),\qquad l=1,2,\dots,$$

for the cyclotomic polynomials.

Proposition 4. *For any integer l in the range $1\le l<n/2$, the Hankel determinant $V_n=\det_{0\le i,j\le n-1}(v_{i+j})$, where the v_n's are given in (2.5) with $b_j(q)=b(q^j)$ for a*

polynomial $b(\,\cdot\,)$ *of arbitrary degree, is divisible by* $\Phi_l(q)^{e_l(n)}$*, where*

$$e_l(n)=\sum_{i=0}^{n-1}\left(\left\lfloor\frac{i+l}{3l}\right\rfloor+\left\lfloor\frac{i}{3l}\right\rfloor\right). \tag{4.1}$$

Remark 6. From (4.1), it is straightforward to compute a compact formula for $e_l(n)$, namely,

$$e_l(n)=\frac{(n-j)(n+j-2l)}{3l}+\begin{cases}0&\text{if } 0\le j<2l,\\ j-2l&\text{if } 2l\le j<3l,\end{cases}\quad\text{where } n\equiv j \pmod{3l}.$$

(In other words, $j/(3l)$ is the fractional part of $n/(3l)$.) In particular,

$$e_1(n)=\left\lfloor\frac{(n-1)^2}{3}\right\rfloor=\begin{cases}\dfrac{(n-1)^2}{3}&\text{if } n\equiv 1 \pmod 3,\\[2mm] \dfrac{n(n-2)}{3}&\text{otherwise,}\end{cases}\qquad e_2(n)=\left\lfloor\frac{(n-2)^2}{6}\right\rfloor.$$

Proof. We have to do some preparatory work first, before we are in the position to embark on the "actual" proof of the divisibility assertion in the proposition. The central part of this preparatory work is the identity (4.10).

Changing notation slightly, recall that, for $n\ge 1$, we have

$$v_n=b(q^n)v_{n-1}-\alpha^n,$$

with $b(x)=x-\lambda$. (In fact, the subsequent arguments hold for *any* function $b(x)$. It is therefore that we shall write $b(\,.\,)$ in the sequel instead of its explicit form which is of relevance in our context.) Hence, for $n\ge l\ge 1$, we have

$$v_n=v_{n-l}\prod_{k=0}^{l-1}b(q^{n-k})-\sum_{j=0}^{l-1}\alpha^{n-j}\prod_{k=0}^{j-1}b(q^{n-k}). \tag{4.2}$$

Writing

$$P_j(n,q)=\prod_{k=0}^{j-1}b(q^{n-k})$$

for non-negative integers j, the recurrence (4.2) takes the form

$$v_n=v_{n-l}P_l(n,q)-\sum_{j=0}^{l-1}\alpha^{n-j}P_j(n,q). \tag{4.3}$$

Fix a positive integer l and a primitive l-th root of unity ζ. For integers j,t,n with $j,t\ge 0$, set

$$P_j^{(t)}(n)=\frac{d^t}{dq^t}P_j(n,q)\Big|_{q=\zeta}. \tag{4.4}$$

In particular, we have

$$P_0^{(t)}(n)=\begin{cases}1&\text{if } t=0,\\ 0&\text{if } t\ge 1.\end{cases} \tag{4.5}$$

For $j \ge 1$, we have

$$\frac{d^t}{dq^t} P_j(n,q) = \sum_{t_0+\dots+t_{j-1}=t} \frac{t!}{t_0! \cdots t_{j-1}!} \prod_{k=0}^{j-1} \frac{d^{t_k}}{dq^{t_k}} b(q^{n-k}). \tag{4.6}$$

Applying the Faà di Bruno formula (cf. [7, Sec. 3.4]; but see also [8, 11]) we get

$$\frac{d^t}{dq^t} b(q^{n-k}) = \sum_{m_1+2m_2+\dots+tm_t=t} \frac{t!}{m_1! \cdots m_t!} b^{(m)}(q^{n-k}) \prod_{\nu=1}^{t} \left(\binom{n-k}{\nu} q^{n-k-\nu} \right)^{m_\nu}, \tag{4.7}$$

where $m = m_1 + \dots + m_k$.

It is straightforward to see that Equations (4.4)–(4.7) imply that, for any non-negative integers j and t, the quantity $P_j^{(t)}(n)$, as a function in n, is an l-quasi-polynomial of degree at most t, where an *l-quasi-polynomial of degree at most t* is a sequence of complex numbers $(Q(n))_{n\in\mathbb{Z}}$ of the form

$$Q(n) = \sum_{\nu=0}^{t} a_\nu(n) n^\nu,$$

the sequence $\{a_\nu(n)\}_{n\in\mathbb{Z}}$ being l-periodic for each ν (cf. [16, Sec. 4.4]). We denote the set of all l-quasi-polynomials of degree at most t by $\mathcal{Q}_t = \mathcal{Q}_t(l)$. Furthermore note that if $Q(n) \in \mathcal{Q}_t$, with $t > 0$, then $Q(n) - Q(n-l) \in \mathcal{Q}_{t-1}$. These facts will be used repeatedly.

Our next observation is that the sequence $(P_l^{(0)}(n))_{n\in\mathbb{Z}}$ is constant, where $P_l^{(0)}(n) = P_l^{(0)}(0) = \prod_{k=0}^{l-1} b(\zeta^k)$. We let

$$B = B_l = \prod_{k=0}^{l-1} b(\zeta^k).$$

Clearly, B is independent of the particular choice of the primitive l-th root of unity ζ.

Now we introduce the difference operators

$$\mathcal{F} = \mathcal{I} - B\mathcal{N}^l$$

and

$$\mathcal{G} = \mathcal{I} - \alpha^l \mathcal{N}^l,$$

where $\mathcal{I}$ and $\mathcal{N}$ have the same meaning as earlier. Clearly, the operators $\mathcal{F}$ and $\mathcal{G}$ commute.

It is easy to check, using elementary facts of difference calculus that, for any $Q(n) \in \mathcal{Q}_t$, we have

$$\mathcal{G}^{t+1}\big(Q(n)\alpha^n\big) = 0. \tag{4.8}$$

For non-negative integers m and n, let

$$v_n^{(m)} = \frac{\partial^m v_n}{\partial q^m}\bigg|_{q=\zeta}.$$

By differentiating both sides of (4.3) m times, and by subsequently substituting $q=\zeta$, we obtain for $n \ge l \ge 1$ the equation

$$v_n^{(m)} = \sum_{\nu=0}^{m} \binom{m}{\nu} P_l^{(m-\nu)}(n) v_{n-l}^{(\nu)} - \sum_{j=0}^{l-1} P_j^{(m)}(n)\alpha^{n-j}. \tag{4.9}$$

We now claim that for an arbitrary $Q(n) \in \mathcal{Q}_t$, for non-negative integers t and m, and for any integer $n \ge (2t+3m+2)l$, we have

$$\mathcal{F}^{t+2m+1}\mathcal{G}^{t+m+1}\big(Q(n)v_n^{(m)}\big) = 0. \tag{4.10}$$

We prove this claim by a double induction: the external induction is over m, while the inner induction is over t.

We start by proving (4.10) for $m=0$, by doing an induction over t. We put $m=0$ in (4.9), and rewrite the resulting equation in the form

$$\mathcal{F}(v_n^{(0)}) = -\sum_{j=0}^{l-1} P_j^{(0)}(n)\alpha^{n-j}.$$

We apply the operator $\mathcal{G}$ on both sides. By (4.8), this implies

$$\mathcal{F}\mathcal{G}(v_n^{(0)}) = 0,$$

which in turn implies

$$\mathcal{F}\mathcal{G}(Q(n)v_n^{(0)}) = 0$$

for any l-periodic function $Q(n)$, since the operators $\mathcal{F}$ and $\mathcal{G}$ are both polynomials in $\mathcal{N}^l$. This is exactly (4.10) for $m=t=0$.

Now let us assume that (4.10) is proved for $m=0$ and for $t-1$ instead of t. If $Q(n) \in \mathcal{Q}_t$ with $t>0$, then (4.9) implies

$$\mathcal{F}\big(Q(n)v_n^{(0)}\big) = B(Q(n)-Q(n-l))v_{n-l}^{(0)} - \sum_{j=0}^{l-1} Q(n)P_j^{(0)}(n)\alpha^{n-j}.$$

After application of $\mathcal{F}^t\mathcal{G}^{t+1}$ on both sides, we obtain

$$\begin{aligned}\mathcal{F}^{t+1}\mathcal{G}^{t+1}\big(Q(n)v_n^{(0)}\big) = B\mathcal{G}\mathcal{F}^t\mathcal{G}^t\big((Q(n)-Q(n-l))v_{n-l}^{(0)}\big)&\\ -\sum_{j=0}^{l-1}\mathcal{F}^t\mathcal{G}^{t+1}\big(Q(n)P_j^{(0)}(n)\alpha^{n-j}\big)&.\end{aligned}$$

The summands in the sum over j vanish because of (4.8), while the first expression on the right-hand side vanishes because of the induction hypothesis. (Recall that $Q(n)-Q(n-l) \in \mathcal{Q}_{t-1}$.) This proves (4.10) for $m=0$ and arbitrary t.

Now we assume that (4.10) is proved for $0,1,\dots,m-1$ instead of m and arbitrary t. For $m>0$, we write (4.9) as

$$\mathcal{F}v_n^{(m)} = \sum_{\nu=0}^{m-1} \binom{m}{\nu} P_l^{(m-\nu)}(n) v_{n-l}^{(\nu)} - \sum_{j=0}^{l-1} P_j^{(m)}(n)\alpha^{n-j}.$$

We apply $\mathcal{F}^{2m}\mathcal{G}^{m+1}$ on both sides, to obtain

$$\mathcal{F}^{2m+1}\mathcal{G}^{m+1}(v_n^{(m)}) = \sum_{\nu=0}^{m-1}\binom{m}{\nu}\mathcal{F}^{m-\nu-1}\mathcal{F}^{(m-\nu)+2\nu+1}\mathcal{G}^{(m-\nu)+\nu+1}\big(P_l^{(m-\nu)}(n)v_{n-l}^{(\nu)}\big)\\ - \sum_{j=0}^{l-1}\mathcal{F}^{2m}\mathcal{G}^{m+1}\big(P_j^{(m)}(n)\alpha^{n-j}\big).$$

Again, the summands in the sum over j vanish because of (4.8), while the first expression on the right-hand side vanishes because of the induction hypothesis. This establishes (4.10) for m and $t=0$.

In order to prove (4.10) for m and *arbitrary* t, we do again an induction over t. We already know that (4.10) is true for $t=0$. Let us assume that (4.10) is true for $t-1$ instead of t. We multiply both sides of (4.9) by $Q(n)$, and we apply $\mathcal{F}$ on both sides. The resulting equation can then be written in the form

$$\mathcal{F}\big(Q(n)v_n^{(m)}\big) = B(Q(n)-Q(n-l))v_{n-l}^{(m)} + \sum_{\nu=0}^{m-1}\binom{m}{\nu}Q(n)P_l^{(m-\nu)}(n)v_{n-l}^{(\nu)}\\ - \sum_{j=0}^{l-1}Q(n)P_j^{(m)}(n)\alpha^{n-j}.$$

After application of $\mathcal{F}^{t+2m}\mathcal{G}^{t+m+1}$ on both sides, we arrive at

$$\mathcal{F}^{t+2m+1}\mathcal{G}^{t+m+1}\big(Q(n)v_n^{(m)}\big) = B\mathcal{G}\mathcal{F}^{t+2m}\mathcal{G}^{t+m}\big((Q(n)-Q(n-l))v_{n-l}^{(m)}\big)\\ + \sum_{\nu=0}^{m-1}\binom{m}{\nu}\mathcal{F}^{m-\nu-1}\mathcal{F}^{(t+m-\nu)+2\nu+1}\mathcal{G}^{(t+m-\nu)+\nu+1}\big(Q(n)P_l^{(m-\nu)}(n)v_{n-l}^{(\nu)}\big)\\ - \sum_{j=0}^{l-1}\mathcal{F}^{t+2m}\mathcal{G}^{t+m+1}\big(Q(n)P_j^{(m)}(n)\alpha^{n-j}\big).$$

Again, by (4.8) and the induction hypothesis, the right-hand side in this identity vanishes. Thus, (4.10) is completely proved.

We are now ready to treat the Hankel determinant V_n. In fact, we need (4.10) only for $t=0$. (For the proof, it was however necessary to play with t.) What (4.10) for $t=0$ says is that, for $m\ge 1$ and $n\ge(3m-1)l$, the polynomial (in q) $w_{m,n}=(\mathcal{I}-B\mathcal{N}^l)^{2m-1}(\mathcal{I}-\alpha^l\mathcal{N}^l)^m v_n$ satisfies

$$\left.\frac{\partial^j w_{m,n}}{\partial q^j}\right|_{q=\zeta} = 0, \qquad 0\le j\le m-1,$$

and, for any choice of the primitive l-th root of unity ζ. Hence, we have

$$\Phi_l(q)^m \mid w_{m,n}. \tag{4.11}$$

This reasoning also shows that for the polynomial

$$\widetilde{w}_{m,n} = (\mathcal{I}-B\mathcal{N}^l)^{2m}(\mathcal{I}-\alpha^l\mathcal{N}^l)^m v_n = (\mathcal{I}-B\mathcal{N}^l)w_{m,n} = w_{m,n} - Bw_{m,n-l}$$

we have

$$\Phi_l(q)^m \mid \widetilde{w}_{m,n}, \tag{4.12}$$

as long as $n \geq 3ml$.

For $n-1 \geq i \geq 2l$ we apply the operator $(\mathcal{I} - B\mathcal{N}^l)^{2l_i-1}(\mathcal{I} - \alpha^l\mathcal{N}^l)^{l_i}$, where $l_i = \lfloor (i+l)/(3l) \rfloor$ to the i-th row of the Hankel matrix $(v_{i+j})_{0\leq i,j\leq n-1}$, and subsequently, for $n-1 \geq j \geq 3l$, we apply the operator $(\mathcal{I} - B\mathcal{N}^l)^{2m_j}(\mathcal{I} - \alpha^l\mathcal{N}^l)^{m_j}$, where $m_j = \lfloor j/(3l) \rfloor$ to the j-th column. The resulting matrix has entries

$$\begin{cases} \widetilde{w}_{\varepsilon(i,j),i+j} & \text{if } i < 2l \text{ and } j \geq 3l, \\ w_{\varepsilon(i,j),i+j} & \text{otherwise,} \end{cases}$$

where

$$\varepsilon(i,j) = l_i + m_j = \lfloor (i+l)/(3l) \rfloor + \lfloor j/(3l) \rfloor,$$

with the convention that $w_{0,n} = v_n$. As earlier, since the above operations correspond to row and column operations, the determinant of the resulting matrix is still equal to V_n.

In view of (4.11) and (4.12), it remains to observe that, for an arbitrary permutation τ,

$$\sum_{i=0}^{n-1} \varepsilon(i,\tau(i)) = \sum_{i=0}^{n-1} \left(\left\lfloor \frac{i+l}{3l} \right\rfloor + \left\lfloor \frac{\tau(i)}{3l} \right\rfloor \right) = \sum_{i=0}^{n-1} \left(\left\lfloor \frac{i+l}{3l} \right\rfloor + \left\lfloor \frac{i}{3l} \right\rfloor \right) = e_l(n),$$

This completes the proof of the proposition. □

Propositions 1, 2 and 4 may be summarized as follows: The Hankel determinant (2.3) of the sequence (3.1) admits the factorization

$$V_n = \Delta_n(q) \cdot \widetilde{V}_n, \tag{4.13}$$

where

$$\Delta_n(q) = q^{e_0(n)} \prod_{l\geq 1} \Phi_l(q)^{e_l(n)} \tag{4.14}$$

and the exponents $e_0(n), e_1(n), e_2(n), \ldots$ are given by very simple formulas. Since

$$\frac{\log|\Phi_l(q)|}{\log|q|} = \varphi(l) + O(1) \quad \text{as } l \to \infty$$

and

$$e_l(n) = O\left(\frac{n^2}{l}\right) \quad \text{as } n \to \infty \text{ uniformly in } l \geq 1,$$

the asymptotic behaviour of $\Delta_n(q)$ as $n \to \infty$ is governed by the degree of the polynomial,

$$\frac{\log|\Delta_n(q)|}{\log|q|} \sim \deg \Delta_n = e_0(n) + \sum_{l=1}^{\infty} e_l(n)\varphi(l) \quad \text{as } n \to \infty. \tag{4.15}$$

The following lemma enables us to determine the asymptotic behaviour of the sum on the right-hand side of (4.15) as $n \to \infty$.

Lemma 3. *Let a and c be real numbers with $0 \le c < a$. Then, as $n \to \infty$,*

$$\sum_{l\ge1}\varphi(l)\sum_{i=0}^{n}\left\lfloor\frac{i+cl}{al}\right\rfloor=\frac{n^3}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(am-c)^2}+O(n^2\log^2 n).$$

Proof. By interchanging summations, we have

$$\sum_{l=1}^{\infty}\varphi(l)\sum_{i=0}^{n}\left\lfloor\frac{i+cl}{al}\right\rfloor=\sum_{i=0}^{n}\sum_{l=1}^{\infty}\left\lfloor\frac{i+cl}{al}\right\rfloor\varphi(l). \tag{4.16}$$

Writing $\Sigma(x)=\sum_{l\le x}\varphi(l)$, the expression (4.16) can be rewritten in the form

$$\sum_{i=0}^{n}\sum_{m=1}^{\infty}m\left(\Sigma\left(\frac{i}{am-c}\right)-\Sigma\left(\frac{i}{a(m+1)-c}\right)\right)=\sum_{i=0}^{n}\sum_{m=1}^{\infty}\Sigma\left(\frac{i}{am-c}\right).$$

Using Mertens' classical asymptotic formula (cf. [10, p. 268, Theorem 330])

$$\Sigma(x)=\frac{3x^2}{\pi^2}+O(x\log x)\qquad\text{as } x\to\infty,$$

we obtain the following asymptotic estimate for (4.16):

$$\begin{aligned}\sum_{i=\lceil a-c\rceil}^{n}\sum_{m=1}^{\lfloor (i+c)/a\rfloor}&\left(\frac{3i^2}{\pi^2(am-c)^2}+O\left(\frac{i\log n}{m}\right)\right)+O(1)\\&=\sum_{i=\lceil a-c\rceil}^{n}\left(\frac{3i^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(am-c)^2}+O(i\log^2 n)\right)+O(1),\end{aligned}$$

which immediately implies the assertion of the lemma. □

Using Formula (4.1) for $e_l(n)$ and the asymptotics from Lemma 3, we obtain

$$\sum_{l=1}^{\infty}e_l(n)\varphi(l)\approx 0.05301135n^3\quad\text{as } n\to\infty, \tag{4.17}$$

where the exact value of the constant in (4.17) is

$$\frac{1}{54}+\frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(3m-1)^2}=\frac{5}{54}-\frac{\operatorname{Im}\operatorname{Li}_2(e^{2\pi\sqrt{-1}/3})}{\pi^2\sqrt3}.$$

Summarizing our findings from Propositions 1 and 2, as well as from (4.15) and (4.17), we have the following result for the asymptotic degree of $\Delta_n(q)$.

Proposition 5. *Let $\Delta_n(q)$ be defined as in (4.14). Then, as $n\to\infty$, we have*

$$\frac{\log|\Delta_n(q)|}{\log|q|}\sim\deg_q\Delta_n(q)\sim Bn^3,$$

where

$$B=\frac{5}{54}-\frac{\operatorname{Im}\operatorname{Li}_2(e^{2\pi\sqrt{-1}/3})}{\pi^2\sqrt3}+\begin{cases}\dfrac{5}{24} & \text{if } \lambda=0,\\[4pt] \dfrac{1}{6} & \text{if } \lambda\ne0.\end{cases}$$

Remark 7. If λ is a root of unity, expected formulas for the exponents of the cyclotomic factors of V_n obey a different law. To write them down in the (q-exponential) case $\lambda = 1$, we represent the polynomial Δ_n from the factorization (4.13) in the form

$$\Delta_n = q^{e_0(n)} \prod_{l \ge 1} (q^l - 1)^{\widetilde{e}_l(n)}.$$

Then

$$\widetilde{e}_l(n) = 2 \max\{0, n - 2l\}, \tag{4.18}$$

which, together with (3.14), implies that, in the case $\lambda = 1$, we have

$$\deg \Delta_n = \begin{cases} \dfrac{n(n-1)^2}{4} & \text{if } n \text{ is odd,} \\ \dfrac{n^2(n-2)}{4} & \text{if } n \text{ is even.} \end{cases}$$

It is of definite interest to prove also these formulas for the cyclotomic exponents.

5. Arithmetic ingredients

In this section we provide the proof of Theorems 1 and 2. It rests on Propositions 3–5, and two additional auxiliary results, given in Lemmas 4 and 5 below. The first one says that, under certain arithmetic constraints on the complex parameters α, λ and q, if we generalize our sequence v_n to $v_n(x)$, where $v_0(x) = x - 1$ and

$$v_n(x) = v_{n-1}(x) \cdot (q^n - \lambda) - \alpha^n \qquad \text{for} \quad n = 1, 2, \dots, \tag{5.1}$$

then the corresponding Hankel determinant

$$V_n(x) = \det_{0 \le i,j \le n-1} \big(v_{i+j}(x)\big) \tag{5.2}$$

is non-zero infinitely often, while the second establishes a (crude) asymptotic upper bound for it. The reader should note that $v_n(x)$ becomes our previous v_n defined in (3.1) if $x = \mu$, where μ is given by (3.2). Hence, if $x = \mu$, the Hankel determinant $V_n(x)$ becomes our earlier Hankel determinant V_n.

Lemma 4. *Let α, λ, q, x be complex numbers with $\alpha \ne 0$, $\lambda \notin q^{\mathbb{Z}_{>0}}$, and $\alpha \notin -\lambda q^{\mathbb{Z}_{>0}}$. Then there are infinitely many positive integers n such that $V_n(x) \ne 0$.*

Proof. Writing the relation (5.1) for the generating series

$$G(z) = G_x(z) = \sum_{n=0}^{\infty} v_n(x) z^n$$

we arrive at

$$\begin{aligned} G(z) &= v_0(x) + z\big(qG(qz) - \lambda G(z)\big) - \sum_{n=1}^{\infty} \alpha^n z^n \\ &= z\big(qG(qz) - \lambda G(z)\big) + x - \frac{1}{1 - \alpha z}. \end{aligned}$$

Equivalently,

$$(1+\lambda z)G(z) - qzG(qz) = x - \frac{1}{1-\alpha z}. \tag{5.3}$$

We claim that this equation does not have a rational function solution unless $\alpha \in -\lambda q^{\mathbb{Z}_{>0}}$. Indeed, if $z = 1/\beta$, $z = 1/(q\beta)$, ..., $z = 1/(q^{k-1}\beta)$ are poles of $G(z)$, for some k, then $z = 1/(q\beta)$, $z = 1/(q^2\beta)$, ..., $z = 1/(q^k\beta)$ are poles of $G(qz)$. Hence, the only way that this is possible in (5.3) is that the factor $1+\lambda z$ cancels the pole $z = 1/\beta$ of $G(z)$, while the term $-1/(1-\alpha z)$ on the right-hand side cancels the pole $z = 1/(q^k\beta)$ of $G(qz)$.

By a result of Kronecker (see [12, pp. 566–567] or [15, Division 7, Problem 24]), the fact that the series $G(z)$ is not a rational function of z implies that infinitely many terms of the sequence $V_n(x)$, where $n = 1, 2, \dots$, do not vanish. $\square$

Lemma 5. *Let $\bar\mu, \bar\alpha, \bar\lambda, q$ be complex numbers with $\bar\alpha \neq 0$ and $|q| > 1$. Define the sequence $(\bar v_n)_{n\ge 0}$ by (5.1) with $x = \bar\mu$, α replaced by $\bar\alpha$, and λ replaced by $\bar\lambda$, and let $\bar V_n = \det_{0\le i,j\le n-1}(\bar v_{i+j})$ be the corresponding Hankel determinant. Then we have*

$$|\bar V_n| \le |q|^{\frac{2}{3}n^3 + o(n^3)}, \qquad \text{as } n \to \infty.$$

Proof. From (2.5) with α replaced by $\bar\alpha$, μ replaced by $\bar\mu$, and with $b_j(q) = q^j - \bar\lambda$, we see that

$$|\bar v_n| \le |q|^{\frac{1}{2}n^2 + o(n^2)}.$$

Hence, we have

$$\begin{aligned}|\bar V_n| &\le n! \max_{\tau\in\mathfrak{S}_n} \prod_{i=0}^{n-1} |\bar v_{i+\tau(i)}| \le n! \max_{\tau\in\mathfrak{S}_n} \prod_{i=0}^{n-1} |q|^{\frac{(i+\tau(i))^2}{2} + o((i+\tau(i))^2)}\\ &\le n! \prod_{i=0}^{n-1} |q|^{\frac{(2i)^2}{2} + o(n^2)} \le |q|^{\frac{2}{3}n^3 + o(n^3)},\end{aligned}$$

as desired. $\square$

We are now finally in the position to prove Theorems 1 and 2. Our proof simplifies the p-adic approach of Bézivin [1] and Choulet [6].

Proof of Theorems 1 and 2. Let $q = \rho/\sigma \in \mathbb{Q}$, $|q| > 1$ and $\rho > 1$. Furthermore, let $\gamma = (\log\rho)/(\log|\sigma|)$ ($\gamma = \infty$ if $q \in \mathbb{Z}$). Let us now assume that all the numbers α, λ and $\mu = F_q(\alpha;\lambda)$ are algebraic and write $K = \mathbb{Q}(\alpha,\lambda,\mu)$, and $d = [K:\mathbb{Q}]$.

In considering V_n we write, as before in (4.13), $V_n = \Delta_n \widetilde V_n$ and note that, as $n \to \infty$, we have

$$|V_n| \le |q|^{-An^3 + o(n^3)}, \tag{5.4}$$

$$\deg_q \Delta_n(q) = Bn^3 + o(n^3), \qquad |\Delta_n(q)| = |q|^{Bn^3 + o(n^3)}, \tag{5.5}$$

where

$$
\begin{aligned}
A &= \frac{1}{2}, \quad B = \frac{65}{216} - \frac{\operatorname{Im}\operatorname{Li}_2(e^{2\pi\sqrt{-1}/3})}{\pi^2\sqrt{3}} \qquad \text{if } \lambda = 0,\\
A &= \frac{1}{3}, \quad B = \frac{7}{27} - \frac{\operatorname{Im}\operatorname{Li}_2(e^{2\pi\sqrt{-1}/3})}{\pi^2\sqrt{3}} \qquad \text{if } \lambda \neq 0,
\end{aligned} \tag{5.6}
$$

by Propositions 3 and 5. On the other hand, by Lemma 5, for all K-conjugates $V_n^{[i]}$ of V_n we have

$$
|V_n^{[i]}| \leq |q|^{Cn^3+o(n^3)}, \qquad i = 1, 2, \dots, d,
$$

where $C = 2/3$. (Of course, for $i = 1$, that is, the case where $V_n^{[i]} = V_n$, we have a better estimate in (5.4).) Clearly, $\Delta_n(q)$ remains invariant under conjugation, whence, by (5.4) and (5.5), we have

$$
|\widetilde{V}_n| = |\widetilde{V}_n^{[1]}| \leq |q|^{-(A+B)n^3+o(n^3)} \qquad \text{as } n \to \infty, \tag{5.7}
$$

and, for $i = 2, 3, \dots, d$,

$$
|\widetilde{V}_n^{[i]}| \leq |q|^{(C-B)n^3+o(n^3)} \qquad \text{as } n \to \infty. \tag{5.8}
$$

We know that V_n is a polynomial in q, α, λ and μ with integer coefficients, hence also $\widetilde{V}_n$. Since the degree of V_n in each of μ, λ, α is at most n^2 (see the paragraph containing (2.6)), the same is also true for $\widetilde{V}_n$. On the other hand, by (2.6), we know that the degree in q of V_n is at most $2n^3/3 + o(n^3)$, whence we are able to find a positive integer $\Omega(n)$, $\log\Omega(n) = o(n^3)$, such that

$$
\sigma^{(C-B)n^3}\Omega(n)\widetilde{V}_n \in \mathbb{Z}_K, \tag{5.9}
$$

where $\mathbb{Z}_K$ denotes the ring of integers of K. If $V_n \neq 0$, then the product of all K-conjugates of the K-integer in (5.9) is a non-zero integer. Therefore, using (5.7) and (5.8),

$$
\begin{aligned}
1 &\leq \left| \prod_{i=1}^{d} \sigma^{(C-B)n^3}\Omega(n)\widetilde{V}_n^{[i]} \right| \\
&\leq |\sigma|^{(C-B)dn^3} \exp(o(n^3))\, |\widetilde{V}_n| \prod_{i=2}^{d} |\widetilde{V}_n^{[i]}| \\
&\leq |\sigma|^{(C-B)dn^3} |q|^{-(A+B)n^3+(C-B)(d-1)n^3} \exp(o(n^3)) \\
&\leq |\sigma|^{(A+C)n^3} \rho^{-(A+C-d(C-B))n^3+o(n^3)} \\
&\leq \rho^{\{(A+C)/\gamma-(A+C-d(C-B))\}n^3+o(n^3)}.
\end{aligned}
$$

If

$$
\frac{1}{\gamma}(A+C) - \bigl(A + C - d(C-B)\bigr) < 0, \tag{5.10}
$$

then the above inequality implies that $V_n = 0$ for all large n, contradicting Lemma 4. The reader should note that (5.10) can only hold if

$$A + C - d(C - B) > 0 \quad \text{or, equivalently,} \quad d < \frac{A+C}{C-B}, \tag{5.11}$$

in which case, we have

$$\gamma > \frac{A+C}{A+C-d(C-B)}. \tag{5.12}$$

From (5.6) it follows that the only values of the degree $d = [K : \mathbb{Q}]$ satisfying (5.11) are $d = 1$ and $d = 2$. Theorems 1 and 2 follow then from (5.12) with $d = 2$ (in Theorem 1) and $d = 1$ (Theorem 2) by using the values of A and B from (5.6), and $C = 2/3$. □

Remark 8. We have a strong feeling that the method used in this work potentially makes it possible to deduce irrationality measures for the values of $F_q(\alpha; \lambda)$ in the cases when the number in question is irrational by Theorem 2. The only problem, which we are not able to overcome, is to establish the required density of non-vanishing of the determinant $V_n(x)$ in (5.2) for a given x. More precisely, in our proof of Theorems 1 and 2 we use the fact (see Lemma 4) that $V_n(x) \neq 0$ infinitely often, and this is (more than) sufficient for a quantitative irrationality, respectively, non-quadraticity result. We expect that a stronger assertion is true, which would then indeed yield irrationality measures for values of $F_q(\alpha; \lambda)$. Namely, for a given $x \in \mathbb{C}$ and the sequence $v_n(x)$ defined in (5.1), there should exist two positive constants c_1 and c_2, $c_1 < c_2$, such that for any $m \geq 1$ one can find an index n in the range $c_1 m < n < c_2 m$, for which the Hankel determinant $V_n(x)$ in (5.2) does not vanish. (In fact, we need this statement only for rational values of x, but this does not seem to be easier than the general case.) The belief in this statement rests upon the fact that the sequence $v_n(x)$ is 'highly structured' (for instance, it is a solution of the simple recurrence relation (2.4) or (5.1) with general x; cf. [1] and [6]); hence $V_n(x)$ should admit a certain structure as well. In fact, it was pointed out by the anonymous referee that in the case $\lambda = 0$ of the Tschakaloff function, $V_n = V_n(\mu)$ is nonzero for all n if $q > 1$ and $\alpha > 0$. This follows from Lemma 2.2 in [1], which provides in this case the expression

$$V_n = \alpha^{n^2-n} \sum_{1 \leq j_1 < \cdots < j_n} w_{j_1} \cdots w_{j_n} \left(\frac{\alpha}{q}\right)^{j_1+\cdots+j_n} \big(V(s_{j_1}, \ldots, s_{j_n})\big)^2,$$

where $w_j = q^{-j(j+1)/2}$, $s_j = q^{-j}$, and $V(s_{j_1}, \ldots, s_{j_n})$ is the Vandermonde determinant built on $s_{j_1}, \ldots, s_{j_n}$. The proof of this result is based on the tail expression (3.21) of $v_n = v_n(\mu)$ and does not work for $V_n(x)$ with x general. This fact clearly supports our expectations above, although it is not enough for irrationality measures.

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