# ON GROUP THEORETICAL ASPECTS AND SYMMETRIES OF ANGULAR MOMENTUM COEFFICIENTS

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Abstract. In this article we present a review of the work on Group theory of ordinary and basic Hypergeometric transformations. We show that an application of the transformations of ordinary hypergeometric series relates the 6-$j$ coefficient to sets of very-well-poised ${}_7F_6(1)$ series.

## 1. Introduction

The fact that the coupling / recoupling coefficients of Quantum Theory of Angular Momentum (QTAM) were related to the hypergeometric functions [1], [2] and the use of a balanced (or Saalschützian) transformation formula to obtain what was claimed to be a *new* symmetry [3] for the Racah (or 6-$j$) coefficient was the starting point for a study of the symmetries of these angular momentum coefficients in terms of generalized hypergeometric functions of unit argument. Early studies [4] in this area of QTAM established that if the 3-$j$ coefficient is to be written as a ${}_3F_2(1)$, a set of six ${}_3F_2(1)$s is necessary and sufficient to account for the 72 symmetries of the 3-$j$ coefficient [5]. In the case of the 6-$j$ coefficient it was shown that there exists a set of three ${}_4F_3(1)$s [6] and an equivalent set of four ${}_4F_3(1)$s [7]. These two sets were shown to be related to each other through the *reversal* of hypergeometric series [8] and this completed an understanding of the symmetries of the 6-$j$ coefficient in terms of the sets of ${}_4F_3(1)$s demonstrating a confirmation of Askey's remark [10] that the 3-$j$ and 6-$j$ coefficients that arise in QTAM are *hypergeometric functions and many of their elementary properties are best understood when considered as such.*

Wu observed [11] that while the 3-$j$ and the 6-$j$ coefficient can be related to a ${}_3F_2(1)$ and ${}_4F_3(1)$, respectively, the 9-$j$ coefficient cannot be related to a ${}_7F_6(1)$ but, that it is related to a *new* hypergeometric function [12], was yet another starting point. This led to the simplest known formula for the 9-$j$ coefficient, a triple sum series due to Ališauskas,

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Jucys and Bandzaitis [13] being related [14] to a triple hypergeometric series, with unit arguments. From a novel way of looking at the symmetries of the 9-$j$ coefficient, the Bailey transform for a Saalschützian ${}_4F_3(1)$ and a transformation of a Kampé de Fériet function into a Saalschützian ${}_4F_3(1)$ or its Bailey transform were derived [15].

The above results were reviewed in an earlier contribution to the *Symmetries in Science* series symposium [16]. Section 2 reviews the Group theoretical aspects of hypergeometric transformations and the 24 Kummer solutions of the Gauß second order ordinary differential equation being related to the symmetries of the cube. Section 4 relates the ${}_7F_6(1)$ forms for the 6-$j$ coefficient with the sets of ${}_4F_3(1)$s using a Whipple transformation and presents an understanding of the symmetries of the 6-$j$ coefficient in terms of these sets. It is interesting to note that, recently, the symmetry groups of hypergeometric series gained considerable attention in irrationality studies of values of the Riemann zeta function at integers (see [17, 18, 19]). To be more precise, what is (independently) investigated (and applied) there (in disguise) is the symmetry group of the ${}_3F_2$ series and of the very-well-poised ${}_7F_6$ series, respectively.

The hypergeometric notations that we shall use throughout the paper are

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!} z^k$$

for the Gauß hypergeometric series ${}_2F_1$, where the Pochhammer symbol $(\alpha)_k$ is defined by $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$, $k>0$, $(\alpha)_0 = 1$, and

$${}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_s)_k\,k!} z^k$$

for the generalized hypergeometric series ${}_rF_s$.
Similarly, the basic hypergeometric notations that we shall use are

$${}_2\phi_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a;q)_k\,(b;q)_k}{(c;q)_k\,(q;q)_k} z^k$$

for the Heine series ${}_2\phi_1$, where the shifted $q$-factorial $(\alpha;q)_k$ is defined by $(\alpha;q)_k = (1-\alpha)(1-q\alpha)(1-q^2\alpha)\cdots(1-q^{k-1}\alpha)$, $k>0$, $(\alpha;q)_0 = 1$, and

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_s;q)_k\,(q;q)_k} z^k$$

for the generalized basic hypergeometric series ${}_r\phi_s$.

## 2. Group theoretical aspects of hypergeometric transformations

Pfaff's transformation ([20], p.68) also referred to as Saalschütz's theorem [21]:

$$(1)\qquad {}_2F_1(a,b;c;x) \;=\; (1-x)^{-a}\;{}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right)$$

when iterated generates the well-known Euler transformation formulae [8]:

$$
\begin{aligned}
&(2) && {}_2F_1(a,b;c;x) &=& (1-x)^{-a}\;{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)\\
&(3) && &=& (1-x)^{-b}\;{}_2F_1\left(c-a,b;c;\frac{x}{x-1}\right)\\
&(4) && &=& (1-x)^{c-a-b}\;{}_2F_1(c-a,c-b;c;x),
\end{aligned}
$$

on each of which is superposed the trivial numerator parameter permutation symmetry:

$$ {}_2F_1(a,b;c;x) = {}_2F_1(b,a;c;x), \tag{5}$$

so that we have a set of eight transformations. If we denote the Pfaff transformation by $g_1$, and the numerator parameter permutation of the ${}_2F_1$ by $g_2$, then these two elementary transformations satisfy the properties:

$$g_1^2 = 1, \qquad g_2^2 = 1, \qquad (g_1g_2)^4 = 1,$$

and the group generated by the two elements subjected to these relations is the dihedral group $\mathcal{D}_8$ (sometimes also denoted by $\mathcal{D}_4$, depending on the convention, and denoted $I_2(4)$ in Coxeter group theory [9]), also known as the group of symmetries of the square. It is a subgroup of the symmetric group $S_4$, the permutation group on four elements.

Erdélyi and Weber [22] stated that the recursive use of the Thomae [23] transformation for a ${}_3F_2(1)$ resulted in a new transformation. This was the starting point for a study of the group theory of transformations. Bayer, Louck and Stein [24] showed that the group of transformations of the non-terminating ${}_3F_2(1)$ transformations is $S_5$ and that the group of transformations of the terminating balanced (or Saalschützian) ${}_4F_3(1)$ series is $S_6$. The study of the group theory of the terminating ${}_3F_2(1)$ transformations, obtained by a recursive application of the terminating version of the Thomae transformation was done by Srinivasa Rao et al. [25].

The remark of Hardy [26] that the Thomae transformation for the non-terminating ${}_3F_2(1)$:

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix};1\right) = \frac{\Gamma(d,e,s)}{\Gamma(a,s+b,s+c)}\;{}_3F_2\left(\begin{matrix} d-a,\ e-a,\ s \\ s+b,\ s+c \end{matrix};1\right) \tag{6}$$

where $\Gamma(x,y,\dots) = \Gamma(x)\Gamma(y)\cdots$ and $s = d+e-a-b-c$ is the parameter excess,

*is an expression of the theorem that*

$$\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)} F\left(\begin{matrix} \alpha_1,\alpha_2,\alpha_3 \\ \beta_1,\beta_2 \end{matrix}\right)$$

*is a symmetric function of the five arguments*

$$\beta_1, \beta_2, \beta_1+\beta_2-\alpha_2-\alpha_3, \beta_1+\beta_2-\alpha_3-\alpha_1, \beta_1+\beta_2-\alpha_1-\alpha_2,$$

clearly implies a group theoretical interpretation for the Thomae formula. As in the case of the terminating ${}_3F_2$ series, a recursive use of this transformation (6) results in our obtaining a set of **ten** non-terminating Thomae transformations ([27], Appendix 1). In [28], the following function was constructed:

$$f(x_1,x_2,x_3,x_4,x_5) \;=\; \frac{1}{\Gamma(s,2x_4,2x_5)}\;{}_3F_2\left(\begin{matrix} 2x_1-s,\ 2x_2-s,\ 2x_3-s \\ 2x_4,\quad 2x_5 \end{matrix};1\right) \tag{7}$$

where $s = x_1 + x_2 + x_3 - x_4 - x_5$. That this function is symmetric in all the five variables is a consequence of the Thomae transformation for non-terminating ${}_3F_2(1)$ has been proved as follows: the function $f(\vec{x}) \equiv f(x_1,x_2,x_3,x_4,x_5)$ is manifestly invariant for permutations of $(x_1,x_2,x_3)$ and $(x_4,x_5)$. Consider the permutation $p:\ x_1 \to x_2 \to x_3 \to x_4 \to x_5 \to x_1$, which is a permutation of order 5. Upon relabeling the parameters of the ${}_3F_2(1)$ in $f(\vec{x})$ as ${}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix};1\right)$, it is straight forward to see that corresponding to

$$f(\vec{x}) \;=\; f(p.\vec{x}) \tag{8}$$

we get:

$${}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix};1\right) = \frac{\Gamma(d,s)}{\Gamma(d-a,s-a)}\;{}_3F_2\left(\begin{matrix} e-c,\ e-b,\ a \\ e,\ s+a \end{matrix};1\right), \tag{9}$$

belonging to the set of ten Thomae transformations. Since $f(\vec{x})$ is invariant under this permutation $p$ of order 5, and under the transposition, $x_4 \leftrightarrow x_5$, manifestly, the group generated by these two generators (viz. $p$ and the transposition) is the complete group of permutations on 5 elements, i.e. the symmetric group $S_5$. This result is, obviously, *a succinct, quintessential one-line statement* for the group of Thomae transformations.

From the transformation for the terminating ${}_3F_2$ series:

$${}_3F_2\left(\begin{matrix} a,\ b,\ -n \\ d,\ e \end{matrix};1\right) = \frac{\Gamma(d,d+n-a)}{\Gamma(d+n,d-a}{}_3F_2\left(\begin{matrix} a,\ e-b,\ -n \\ a+a-d-n,\ e \end{matrix};1\right), \tag{10}$$

in [25], the 72-element group $G_T$ of transformations was generated, and its conjugacy classes, irreducible representations and their characters, as well as the invariant subgroups of $G_T$ were derived, and the role of these terminating series for the ${}_3F_2(1)$ forms of the 3-$j$ coefficient were discussed.

The two-term relation for the terminating balanced (or Saalschützian) ${}_4F_3$ of unit argument is:

$$\begin{aligned} {}_4F_3\left(\begin{matrix} A,B,C,-n \\ E,F,G \end{matrix};\ 1\right) &= \frac{(F-C)_n(G-C)_n}{(F)_n(G)_n} \\ &\quad \times {}_4F_3\left(\begin{matrix} E-A,\quad E-B,\quad C\quad,-n \\ E,E+F-A-B,E+G-A-B \end{matrix};\ 1\right). \end{aligned} \tag{11}$$

A recursive use of this terminating ${}_4F_3$ transformation (11) results in a set of **twenty** transformations ([27], Appendix 2). In this case, the function:

$$\begin{aligned} f(x_1,x_2,x_3,x_4,x_5,x_6) &= (x_1+x_2+x_3+x_4, x_1+x_2+x_3+x_5, x_1+x_2+x_3+x_6)_n \\ &\quad \times {}_4F_3\left(\begin{matrix} x_1+x_2, x_2+x_3, x_3+x_1, -n \\ x_1+x_2+x_3+x_4, x_1+x_2+x_3+x_5, x_1+x_2+x_3+x_6 \end{matrix};\ 1\right) \end{aligned} \tag{12}$$

with $(x, y, \ldots)_n = (x)_n (y)_n \cdots$ and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 - n$, for some non-negative integer $n$, being symmetric in all the six variables $x_1, x_2, x_3, x_4, x_5, x_6$ is a consequence of the ${}_4F_3(1)$ transformation.

Hardy's clue enabled us [29] to look for the invariance groups for all the known transformations of basic hypergeometric series. A summary of these results is presented below:

The Heine transformation,

$$ {}_2\phi_1(a, b; c; q, x) = \frac{(a, bz; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1(c/a, z; bz; q, a), \tag{13} $$

when iterated yields a set of 12 transformations [31]. The invariance group is the dihedral group $\mathcal{D}_{12}$ – the group of symmetries of the hexagon (sometimes also denoted as $\mathcal{D}_6$ or $I_2(6)$). We construct the function:

$$ f(x_1, x_2, x_3, x_4, x_5, x_6) = \left( x_1 x_4, \frac{x_2 x_6}{x_1}; q \right)_\infty {}_2\phi_1 \left( \frac{x_1 x_3}{x_2}, \frac{x_1 x_5}{x_6}; x_1 x_4; q, \frac{x_2 x_6}{x_1} \right), \tag{14} $$

whose invariance under the dihedral group acting on the six variables $x_1, \ldots, x_6$ generates the set of 12 Heine transformations.

The function:

$$ \begin{aligned} f(x) &= f(x_1, x_2, x_3, x_4, x_5) \\ &= \left( x_1 x_2 x_3 / x_4 x_5, x_4^2, x_5^2; q \right)_\infty \\ &\quad \times {}_3\phi_2 \left( \begin{matrix} x_1 x_4 x_5 / x_2 x_3, x_2 x_4 x_5 / x_1 x_3, x_3 x_4 x_5 / x_1 x_2 \\ x_4^2, x_5^2 \end{matrix} ; q, \frac{x_1 x_2 x_3}{x_4 x_5} \right), \end{aligned} \tag{15} $$

is symmetric in the five variables $x_1, \ldots, x_5$. Let $p$ be the cyclic permutation $p = (14325)$, that is, the permutation $x_1 \to x_4 \to x_3 \to x_2 \to x_5 \to x_1$:

$$ p.(x_1, x_2, x_3, x_4, x_5) = (x_4, x_5, x_2, x_3, x_1). \tag{16} $$

On relabeling the parameters of the ${}_3\phi_2$s for the equation: $f(x) = f(p.x)$, where $x$ is the five component vector, we get the transformation:

$$ {}_3\phi_2 \left( \begin{matrix} a, b, c \\ d, e \end{matrix} ; q, \frac{de}{abc} \right) = (b, de/ab, de/ac)_\infty \; {}_3\phi_2 \left( \begin{matrix} d/b, e/b, s \\ de/ab, de/ac \end{matrix} ; q, b \right), \tag{17} $$

which is the $q$- analogue of the non-terminating Thomae transformation. The defined function is manifestly invariant under the permutations $(x_1, x_2, x_3)$ and $(x_4, x_5)$. $f(x)$ is also invariant under the tansposition $x_4 \leftrightarrow x_5$, as well as the above cyclic permutation $p$ of order 5. Hence, the group generated by $p$ and the transposition is $S_5$ which is thus the invariance group of the $q$-analogue of the Thomae non-terminating transformation.

Let $x_1, \ldots, x_6$ satisfy the condition: $x_1x_2x_3x_4x_5x_6 = q^{1-n}$ for some non-negative integer $n$. Then the function:

$$
\begin{aligned}
f(x) &= f(x_1, x_2, x_3, x_4, x_5, x_6) \\
&= q^{n(n-1)/2}(x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_3x_6; q)_n \frac{1}{(x_1x_2x_3)^n} \\
&\quad \times {}_4\phi_3\left( \begin{matrix} q^{-n}, x_2x_3, x_1x_3, x_1x_2 \\ x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_3x_6 \end{matrix} ; q, q \right),
\end{aligned} \tag{18}
$$

being symmetric in the variables $x_1, \ldots, x_6$ is a manifestation of the $q$-analogue of the terminating balanced (Saalschützian) ${}_4F_3(1)$ transformation. The invariance group is $S_6$ and the proof is along the same lines as that given above.

The above are the $q$-analogues of the results [28] of Bayer, Louck and Stein [24] for ordinary hypergeometric series transformations.

For terminating ${}_3\phi_2$ series, the $q$-analogue of the Whipple transformation obtained by Sears [31] is:

$$
{}_3\phi_2\left( \begin{matrix} q^{-n}, b, c \\ d, e \end{matrix} ; q, q \right) = b^n \frac{(c, de/bc; q)_n}{(d, e; q)_n} \, {}_3\phi_2\left( \begin{matrix} q^{-n}, d/c, b/c \\ de/bc, q^{1-n}/c \end{matrix} ; q, \frac{q}{b} \right), \tag{19}
$$

which when recursively applied yields 72 transformations, as in the case of the terminating ${}_3F_2(1)$ investigated in [27]. This 72-element group cannot be a subgroup of $S_5$ by Cayley's and Lagrange's theorems (since 72 is not a factor of 120). Thus, it is shown in [28] that the invariance group of the terminating ${}_3\phi_2$ series transformations is a 72-element subgroup of $S_6$ generated by the permutations: (24) and (123456).

It was also shown in [28] that the transformations of the very-well-poised ${}_8\phi_7$ series belong to the invariance group (of order 1920), which is a subgroup of signed permutations on 5 elements, coincides with the Weyl group of the root system of type $D_5$ (see [9]).

In 1836, Kummer published a set of six distinct solutions for the second oder ordinary Gauß differential equation characterized by three regular singular points. Each of these six solutions has four forms which are related to each other by Euler's transformations (2)–(4). Thus, there are in all 24 solutions of the Gauß hypergeometric equation which can be found in most classical text books [32]. Recently, Posser [33] related these solutions of Kummer to one another through a finite group (of order 24, or 48 if one includes the mirror symmetries) of transformations. It is remarkable that these 24 solutions have been related [29] to the (rotational) group of symmetries of the cube. This is achieved through the function:

$$
f(x) = {}_2F_1\left( \frac{1}{2} + x_1 + x_2 + x_3, \frac{1}{2} + x_1 + x_2 + x_4; \; \frac{1}{2} + x_1 - x_6; \; -\frac{x_1 + x_6}{x_3 + x_4} \right), \tag{20}
$$

where the six variables $x_1, \ldots, x_6$ (identified with the six faces of a cube labeled as $1, \ldots, 6$ such that labels of the opposite faces of the cube add to 7) satisfy the constraint:

$$\sum_{i=1}^{6} x_i = 0.$$

The symmetries of the cube can then be associated with the 24 permutations of a group $G$ (a subgroup of the 720 permutations belonging to $S_6$) of the indices labeling its faces. Identifying the four arguments of the above ${}_2F_1$ with $a, b, c, z$, if we denote by $g \,\epsilon\, G$ one of these symmetries, then $f(g.x)$ will correspond to one of the Kummer solutions. In [29] it is shown that the 24 complete Kummer solutions (up to a constant) can be related to the permutations which generate the symmetries of the cube and that the mirror symmetries, or the reflection symmetries of the cube, are the ones which correspond to the interchange of the numerator parameters of the ${}_2F_1$ function. The intimate connection between the symmetries of the cube and the 24 Kummer solutions of the Gauß differential equation, is an aesthetically beautiful result discovered 90 years after the discovery of the Gauß equation and 66 years after Kummer established its complete set of solutions.

## 3. 6-$j$ coefficient in terms of sets of very-well-poised ${}_7F_6$ series

In [35], it was shown that the highly symmetric form of the Racah or 6-$j$ angular momentum recoupling coefficient, due to Regge [36], which exhibits the 144 symmetries of the coefficient, can be formally cast into a ${}_5F_4(1)$ form. Though the ${}_5F_4(1)$ exhibits the 144 symmetries, it has the property that for real integral and half integral values of the angular momenta, the numerator and denominator parameters are integers. The termination of the series, which occurs due to the numerator parameter being a negative integer, unfortunately, occurs after the zero due to the denominator parameter! Hence, this formal expansion represents a divergent series and is not useful.

The claim that a new symmetry was found by Minton [37] for the 6-$j$ coefficient which did not satisfy even the triangle inequalities [38] led us to the set I [6] of ${}_4F_3(1)$s for the 6-$j$ coefficient. Racah's [39] achievment was to show that the recoupling coefficient for three angular momenta, $W(abcd;ef)$, can be written as a single sum series (independent of the 3-$j$ coefficients and hence of the projection quantum numbers of angular momenta), viz:

$$\left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\} = (-1)^{a+b+c+d}\, W(abcd;ef) \tag{21}$$

$$= N \sum_P (-1)^P \, (P+1)! \, \{\prod_{i=1}^{4} (P-\alpha_i)! \prod_{j=1}^{3} (P-\beta_j)!\}^{-1}, \tag{22}$$

with

$$N = (-1)^{a+b+c+d} \triangle(a,b,e)\ \triangle(c,d,e)\ \triangle(a,c,f)\ \triangle(b,d,f), \tag{23}$$

$$\triangle(x,y,z) = \left[\frac{(-x+y+z)!(x-y+z)!(x+y-z)!}{(x+y+z+1)!}\right]^{1/2}, \tag{24}$$

$$\begin{gathered} \alpha_1 = a+b+e,\ \alpha_2 = c+d+e,\ \alpha_3 = a+c+f,\ \alpha_4 = b+d+f, \\ \beta_1 = a+b+c+d,\ \beta_2 = a+d+e+f,\ \beta_3 = b+c+e+f \end{gathered} \tag{25}$$

and

$$P_{min} \ \leq \ P \ \leq \ P_{max}, \tag{26}$$

$$P_{min} \ = \ max(\alpha_1,\alpha_2,\alpha_3,\alpha_4), \quad P_{max} \ = \ min(\beta_1,\beta_2,\beta_3). \tag{27}$$

By setting in (22), $s = \beta_k - P$, $k = 1,2,3$, in succession, a set I of three series expansions, and in turn, the set I of the following three ${}_4F_3(1)$s has been obtained in [6]:

$$\begin{aligned} \left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\} \ &= \ (-1)^{E+1} N\ \Gamma(1-E) \\ &\quad \times [\Gamma(1-A,1-B,1-C,1-D,F,G)]^{-1} \\ &\quad \times {}_4F_3(ABCD\ ;\ EFG\ ;\ 1), \end{aligned} \tag{28}$$

where

$$\begin{gathered} A = e-a-b,\ B = e-c-d,\ C = f-a-c,\ D = f-b-d, \\ E = -a-b-c-d-1,\ F = e+f-b-c+1,\ G = e+f-a-d+1, \end{gathered} \tag{29}$$

for $k = 1$ and for $k = 2$ and 3, the numerator and denominator parameter sets are:

$$\begin{gathered} A = a-b-e,\ B = d-c-e,\ C = a-c-f,\ D = d-b-f, \\ E = -b-c-e-f-1,\ F = a+d-b-c+1,\ G = a+d-e-f+1, \end{gathered} \tag{30}$$

and

$$\begin{gathered} A = b-a-e,\ B = c-d-e,\ C = c-a-f,\ D = b-d-f, \\ E = -a-d-e-f-a,\ F = b+c-a-d+1,\ G = b+c-e-f+1. \end{gathered} \tag{31}$$

The Minton procedure was to apply a Saalschützian ${}_4F_3(1)$ transformation to the ${}_4F_3(1)$ in (29) (with the parameters given by (31)) and identify the transformed ${}_4F_3(1)$ with a 6-$j$ coefficient using again (29). We have shown in [5] that this procedure will, *at best*, result in a $j \to -j-1$ substitution for one or more of the six angular momenta in the 6-$j$ coefficient, which violates triangle inequalities [38], though it is a mathematically valid symmetry in quantum theory of angular momentum.

A hypergeometric series is called very-well-poised if the numerator $(a_i)$ and denominator $(b_i)$ parameters of the hypergeometric series are such that:

$$a_i + b_i = 1 + a_0, \quad \text{for} \quad i = 1,2,\ldots,n,$$

and amongst the parameters $a_i$ occurs $1 + a_0/2$. The abbreviated notation for a very-well-poised hypergeometric series with argument 1, which we shall use here,[1] uses its

[1]This notation for the very-well-poised ${}_7F_6$ as ${}_7V_6$ has been introduced by one of us CK in [34], akin to the notation ${}_{p+1}W_p$ used for very-well-poised basic hypergeometric series in [30].

numerator parameters only and it is:

$$
\begin{aligned}
(32)\qquad & {}_{r+1}V_r(a_0;\ a_2, a_3, \ldots, a_r) \\
& = {}_{r+1}F_r\left(\begin{matrix} a_0, 1+\frac{a_0}{2}, & a_2, & a_3, & \ldots, & a_r \\ \frac{a_0}{2}, & 1+a_0-a_2, & 1+a_0-a_3, & \ldots, & 1+a_0-a_r \end{matrix};1\right).
\end{aligned}
$$

The transformation of a balanced or Saalschützian ${}_4F_3(1)$ series into a very-well-poised ${}_7F_6(1)$ series, with the special form of the second parameter, was given by Whipple [40]:

$$
\begin{aligned}
(33)\ {}_4F_3\left(\begin{matrix} x, y, z, -n \\ u, v, w \end{matrix};1\right) &= \Gamma\left(\begin{matrix} v+w-x, 1+y-u, 1+z-u, 1-n-u \\ 1+y-n-u, 1+z-n-u, 1+y+z-u, 1-u \end{matrix}\right) \\
\times {}_7F_6\left(\begin{matrix} a, 1+\frac{a}{2}, w-x, v-x, & y, & z, & -n \\ \frac{a}{2}, \quad v, & w, 1+z-u-n, & 1+y-u-n, & 1+y+z-u \end{matrix};1\right)&,
\end{aligned}
$$

in which $a = y+z-u-n = w+v-x-1$ (and $x+y+z-n+1 = u+v+w$) and the ${}_7F_6(1)$ in the expression above can be written in the shortened notation for the very-well-poised hypergeometric series as:

$$
(34)\qquad {}_7F_6(-;-;1) \equiv {}_7V_6(a;\ w-x, v-x, y, z, -n).
$$

Recently, Ališauskas [41] has claimed that he has found *a new expression for the 6-j coefficient of SU(2)* and it is (with minor changes in the notation):

$$
\begin{aligned}
\left\{\begin{matrix} a & b & e \\ d & c & f \end{matrix}\right\} &= \nabla(a,c,f)\nabla(b,d,f)\nabla(e,d,c)\bigtriangleup(a,b,e) \\
(35)\qquad & \times \sum_j \frac{(-1)^{d+c+f-(a+b+e)/2+j}\ (2j+1)}{\nabla^2(\frac{1}{2}(b+e-a),d,j)\nabla^2(\frac{1}{2}(a-b+e),c,j)\nabla^2(\frac{1}{2}(a+b-e),f,j)},
\end{aligned}
$$

where

$$
\nabla(x,y,z) = \bigtriangleup(x,y,z)\frac{(x+y+z+1)!}{(-x+y+z)!},
$$

and he states that the sum is a very-well-poised ${}_7F_6(1)$. He also obtains another expression for the 6-$j$ coefficient by substituting $a \to -a-1, b \to -b-1$, which is also a very-well-poised ${}_7F_6(1)$.

We use the Whipple transformation for the ${}_4F_3(1)$ forms of the 6-$j$ coefficient, and since in this set of ${}_4F_3(1)$s all the four numerator parameters are negative integers, there are four different ways in which the given transformation can be used to obtain very-well-poised ${}_7V_6(1)$ forms for the 6-$j$ coefficient. In the table given below are the four different permutations and the corresponding ${}_7V_6(1)$ forms we obtain for the 6-$j$ coefficient.

| Parameter permutation | ${}_7V_6$ (parameters) |
|---|---|
| (4231);(231) | ${}_7V_6(A+B+C-E;\ G-D, F-D, A, B, C)$ |
| (3241);(231) | ${}_7V_6(A+B+D-E;\ G-C, F-C, A, B, D)$ |
| (2341);(231) | ${}_7V_6(A+C+D-E;\ G-B, F-B, A, C, D)$ |
| (1234);(231) | ${}_7V_6(B+C+D-E;\ G-A, F-A, B, C, D)$ |

These are the only independent ${}_7V_6$ forms for the 6-$j$ coefficient. Of these, the one corresponding to the second of the three ${}_4F_3(1)$ forms (31) for the 6-$j$ coefficient corresponds to the 'new' expression reported by Ališauskas.

In our approach we have shown [5] that there are two sets of ${}_4F_3(1)$s which are related to each other by 'reversal' of series, which completely maps the set I of three ${}_4F_3(1)$s – explicitly given here in this section – onto the set II of four ${}_4F_3(1)$s. From the table given above, it is clear that we get for each of the ${}_4F_3(1)$s belonging to set I of hypergeometric functions, a set of four ${}_7V_6(1)$s. It is straight forward to then observe that each of these ${}_7V_6(1)$s will account for only 12 of the 144 symmetries of the 6-$j$ coefficient. For instance, the first entry in the table above clearly shows that the symmetries exhibited are due to the permutations of the parameters $A, B, C$ $(S_3)$ and the parameters $F, G$ $(S_2)$. It is also clear from an examination of the table that the 12 symmetries corresponding to each of the four ${}_7V_6(1)$s are distinctly different. Thus, we can conclude, that if we want to express the 6-$j$ coefficient as a ${}_7V_6(1)$, then **a set of 12 ${}_7V_6(1)$s is necessary and sufficient to account for the 144 symmetries** of the 6-$j$-coefficient.

Equivalently, corresponding to the set II of four ${}_4F_3(1)$s, one can write down a set of 16 ${}_7V_6(1)$s, each of which will account for only 9 of the 144 symmetries of the 6-$j$ coefficient, enabling us to draw the conclusion that if the 6-$j$ coefficient is represented as a ${}_7V_6(1)$ derived by the use of the Whipple transformation on the ${}_4F_3(1)$ belonging to set II of ${}_4F_3(1)$s for the 6-$j$ coefficient, then a set of 16 ${}_7V_6(1)$s is necessary and sufficient to account for the 144 symmetries of the 6-$j$ coefficient.

This completes our understanding of the 144 symmetries of the 6-$j$ coefficient in terms of the equivalent sets of 12 or 16 ${}_7V_6(1)$s. The 'new' expression obtained by Ališauskas is just one member of the set of 12 ${}_7V_6(1)$s.

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